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New characterizations of subclasses of A-selfadjoint, A-normal and A-partial isometry operators on semi-Hilbertian spaces

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Abstract. In this paper subclasses of A-selfadjoint, A-normal and A-partial isometry are characterizated in terms of operators inequalities by using the **arithmetic-geometric mean inequality**. Some properties of this subclasses are also presented.

1. Introduction

Throughout this manuscript H denotes a complex Hilbert space with inner product $\langle .,.\rangle$ and the operator norm $\|.\|$. $L(H)$ be the algebra of all bounded linear operators on H. This paper is devoted to the study of the following characterizations of some distinguished classes of bounded linear operators acting on H, namely, the selfadjoint operators, the normal operators, and the unitary operators, in terms of operator inequalities when an additional seminorm is consider on a complex Hilbert space H, which denoted $\|.\|_A$ when A be a nonzero positive operator which defines a positive semi-definite sesquilinear form

$$\langle .,.\rangle_A : H \times H \longrightarrow \mathbb{C}, \langle x,y\rangle \longmapsto \langle x,y\rangle_A = \langle Ax,y\rangle.$$

The seminorm induced by $\langle .,.\rangle_A$ denoted by $\|.\|_A$, is given by $\|x\|_A = \sqrt{\langle x,x\rangle_A}$, for every $x \in H$. This makes H into a semi-Hilbertian space. It is obvious that $\|x\|_A = 0$ if and only if $x \in \mathcal{N}(A)$. Then $\|.\|_A$ is a norm if only if A is injective operator. In addition the semi-hilbert space $(H, \|x\|_A)$ is complete if, and only if A has a closed range.
The author in [16] proved that

- The class of selfadjoint operators with closed range $S \in L(H)$ is characterized by each of the following properties

$$\forall X, S \in \mathcal{B}(H),\ \|SXS^+ + S^+XS\| = \|S^*XS^+ + S^+XS^*\|,$$
$$\forall X \in \mathcal{B}(H),\ \|SXS^+ + S^+XS\| \geq \|S^*XS^+ + S^+XS^*\|$$
$$\forall X \in \mathcal{B}(H),\ \|SXS^+ + S^+XS\| \geq 2\|SS^+XS^+S\|,$$
$$\forall X \in \mathcal{B}(H),\ \|S^2X + XS^2\| \geq 2\|SXS\|.$$

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- The class of all normal operators with closed range $S \in L(H)$ is characterized by each of the following properties

$$\forall X \in \mathcal{B}(H),\ \|SXS^{+}\| + \|S^{+}XS\| = \|S^{*}XS^{+}\| + \|S^{+}XS^{*}\|, \quad (S \in \mathscr{R}(H)),$$
$$\forall X \in \mathcal{B}(H),\ \|SXS^{+}\| + \|S^{+}XS\| \geq \|S^{*}XS^{+}\| + \|S^{+}XS^{*}\|, \quad (S \in \mathscr{R}(H)),$$
$$\forall X \in \mathcal{B}(H),\ \|SXS^{+}\| + \|S^{+}XS\| \leq \|S^{*}XS^{+}\| + \|S^{+}XS^{*}\|, \quad (S \in \mathscr{R}(H)),$$
$$\forall X \in \mathcal{B}(H),\ \|SXS^{+}\| + \|S^{+}XS\| \geq 2\|S^{+}XS^{+}S\|, \quad (S \in \mathscr{R}(H)),$$
$$\forall X \in \mathcal{B}(H),\ \|S^{2}X\| + \|XS^{2}\| \geq 2\|SXS\|, \quad (S \in \mathscr{R}(H)).$$

And the author in [13] proved the following. For all $T \in L(H)$ with closed range the following statements are equivalent:

(i) $\dfrac{A}{\|A\|}$ is a partial isometry,
(ii) $\|A\| \cdot \|A^{+}\| = 1$,
(iii) $\forall X \in \mathcal{B}(H),\ \|AXA^{+}\| = \|A^{+}AXA^{+}A\|$,
(iv) $\forall X \in \mathcal{B}(H),\ \|A^{+}XA\| = \|AA^{+}XAA^{+}\|$.

My main purpose of this survey paper is to present these characterizations when we consider this seminorm and when we replace the operator uniform norm by the following seminorm on a subset of $L(H)$:

$$\|T\|_{A} = \sup_{x \in \overline{\mathcal{R}(A)}, x \neq 0} \frac{\|Tx\|_{A}}{\|x\|_{A}} = \sup\left\{\|Tx\|_{A} : x \in H, \|x\|_{A} = 1\right\}.$$

So in this work, we shall explore, among other things, new characterizations for some subclasses of A-bounded operators acting on a complex semi-Hilbertian space H (namely A-selfadjoint, A-normal, and A-partial isometry).
The paper is organized as follows. In section 3 we prove some results related to **The arithmetic-geometric mean inequality** for A-selfadjoint operators with closed range. In section 4 we give a new characterizations of Subclasses of A-normal operators. Finally in section 5 we prove a new characterizations of Subclasses of A-partial isometry.

2. Preliminaries

Throughout this manuscript $L(H)^{+}$ is the cone of positive (semidefinite) operators of $L(H)$, i.e., $L(H)^{+} = \{T \in L(H) : \quad \langle Tx, x\rangle \geq 0 \quad \forall x \in H\}$. In all that follows, by the range and the null space of an operator T are denoted by $\mathcal{R}(T)$ and $\mathcal{N}(T)$, respectively, and its adjoint by T^{*} and its spectrum by $\sigma(T)$. $\overline{\mathcal{R}(T)}$ denotes the closure of $\mathcal{R}(T)$ with respect to the norm topology of H. $L_{cr}(H)$ is the subset of $L(H)$ of all operators with closed range. $\mathbb{Q} = \{Q \in B(H) : Q^{2} = Q\}$ is the subset of $L(H)$ of all projection. $P = \{P \in \mathbb{Q} : P^{*} = P\}$ is the subset of $L(H)$ of all orthogonal projection.
From now on, every positive operator is assumed to be non-zero.
We denote by $r(T)$ and $\gamma(T)$ the spectral radius and the minimum modulus of T respectively. They are defined as

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

and

$$\gamma(T) = \inf\{\|Tx\|, x \in \mathcal{N}(T)^{\perp} \text{ and } \|x\| = 1\}.$$

Given $T \in L_{cr}(H)$, the Moore-Penrose inverse of T , denoted by T^{+}, is defined as the unique linear tranformation from $D(T^{+}) = \mathcal{R}(T) \oplus \mathcal{R}(T)^{\perp}$ to H. T^{+} is the unique solution of the four **Moore-Penrose equations**:

$$T^{+}TT^{+} = T^{+}, \quad TT^{+}T = T, \quad TT^{+} = P_{\overline{\mathcal{R}(T)}}|D(T^{+}), \quad T^{+}T = P_{\mathcal{N}(T)^{\perp}}.$$

Note that $T^{+} \in L(H)$ if, and only if $\mathcal{R}(T)$ is closed (see [14] for its proof)(In general, T^{+} is not bounded).
It is easy to check that for every $T \in L_{cr}(H)$, we have

- $(T^+)^+ = T$.
- $(T^+)^* = (T^*)^+$.
- $T^* = T^*TT^+ = T^+TT^*$.
- $T^+ = (T^*T)^+T^* = T^*(TT^*)^+$.

For every bounded linear densely defined operator T there exists a unique bounded linear extension $\overline{T} \in L(H)$ of T. In the next proposition, we present some properties of $\overline{T}$

Proposition 2.1. *Let T and R be bounded densely defined linear operators. Then:*

1. $\overline{(T)^*} = \overline{T}^* = T^*$.
2. *If* $T = R^*R$ *then* $\overline{T} = \overline{R^*R}$.

The following main theorem is Known as the Douglas theorem (see [2] or [3] for its proof)

Theorem 2.2. *Let* $B, C \in L(H)$. *The following conditions are equivalent:*

1. $\mathcal{R}(C) \subset \mathcal{R}(B)$.
2. *There is a positive number* λ *such that* $CC^* \leq \lambda BB^*$.
3. *There is* $D \in L(H)$ *such that* $BD = C$.

If one of these conditions holds then there exists a unique operator $E \in L(H)$ such that $BE = C$ and $\mathcal{R}(E) \subseteq \overline{\mathcal{R}(B^*)}$ and $\mathcal{N}(D) = \mathcal{N}(E) = \mathcal{N}(C)$ (E known as the Douglas Solution).

We denote
$$L^A(H) = \{T \in L(H) : \|T\|_A < \infty\}.$$

Proposition 2.3. *[8] Let* $A \in L(H)^+$ *and* $T \in L(H)$. *Then the following conditions are equivalent:*

1. $T \in L^A(H)$.
2. $A^{\frac{1}{2}}T(A^{\frac{1}{2}})^+$ *and* $T^0 = (A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}}$ *are bounded operators.*
3. $\mathcal{R}(A^{\frac{1}{2}}T^*A^{\frac{1}{2}}) \subseteq \mathcal{R}(A)$.

Moreover, if one of these conditions holds then

$$\|T\|_A = \|A^{\frac{1}{2}}T(A^{\frac{1}{2}})^+\| = \|(A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}}\|.$$

If $M \subseteq H$ then we denote by $M^{\perp_A} = \{x \in H\ \langle x, y\rangle_A = 0 \text{ for every } \quad y \in M\}$.

Definition 2.4. *[9] Let* $T \in L(H)$. *An operator* $R \in L(H)$ *is called an* ***A-adjoint*** *of T if for every* $x, y \in H$ $\langle Tx, y\rangle_A = \langle x, Ry\rangle_A$, *i.e., if* $AR = T^*A$.
T is called ***A-selfadjoint*** *if* $AT = T^*A$.

Remark 2.5. *Not every* $T \in L(H)$ *admits an A-adjoint. By Douglas theorem, T admits an A-adjoint if and only if* $\mathcal{R}(T^*A) \subseteq \mathcal{R}(A)$.

Throughout this paper.

- $L_A(H)$ denotes the set of all operators $T \in L(H)$ which admit an A-adjoint. Then
$$L_A(H) = \{T \in L(H) : T^*\mathcal{R}(A) \subseteq \mathcal{R}(A)\}.$$
- $L_{A^{\frac{1}{2}}}(H)$ is the set of all operators $T \in L(H)$ which admit an $A^{\frac{1}{2}}$-adjoint (called also A-bounded operator). Then
$$L_{A^{\frac{1}{2}}}(H) = \{T \in L(H) : T^*\mathcal{R}(A^{\frac{1}{2}}) \subseteq \mathcal{R}(A^{\frac{1}{2}})\}.$$

By Douglas Theorem , it is obvious that

$$L_{A^{\frac{1}{2}}}(H) = \{T \in L(H) : \exists c > 0, \|Tx\|_A \leq c\|x\|_A, \forall x \in H\}.$$

The relationship between the obove sets is proved in [[6], Theorem 5.1]).

Proposition 2.6. *[6] let $A \in L(H)^+$ then $L_A(H) \subseteq L_{A^{\frac{1}{2}}}(H) \subseteq L^A(H)$.*

It easy to check that if A has a closed range, then $L_A(H) = L_{A^{\frac{1}{2}}}(H)$.
Let $T \in L_A(H)$ then the reduced solution of the equation $AX = T^*A$ is a distinguished A-adjoint operator of T. We denote this operator by $T^\sharp$. Note that, $T^\sharp = A^+T^*A$.
The main properties of $T^\sharp$ are

$$AT^\sharp = T^*A, \quad \mathcal{R}(T^\sharp) \subseteq \overline{\mathcal{R}(A)}, \quad \mathcal{N}(T^\sharp) = \mathcal{N}(T^*A).$$

If $T \in L_{A^{\frac{1}{2}}}(H)$, then the reduced solution of the equation $A^{\frac{1}{2}}X = T^*A^{\frac{1}{2}}$ is a distinguished $A^{\frac{1}{2}}$-adjoint operator of T, which will be denoted by T^0. Note that, $T^0 = (A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}}$.
The folllowing proposition gives some properties of $T^\sharp$ and its relationship with the semi-norm $\|T\|_A$ which we shall use along this work. (for its proof and more details we refer the reader to see [9] and [8].

Proposition 2.7. *[9],[8] Let $T \in L_A(H)$. Then the following statement hold:*

1. *$(A^t)^\sharp = A^t$ for every $t > 0$.*
2. *If $AT = T^*A$, then $(A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}}$ is selfadjoint.*
3. *$T^\sharp \in L_A(H)$, $(T^\sharp)^\sharp = PTP$ and $((T^\sharp)^\sharp)^\sharp = T^\sharp$.*
4. *$TT^\sharp$ and $T^\sharp T$ are A-selfadjoint.*
5. *$\|T\|_A = \|T^\sharp\|_A = \|T^\sharp T\|_A^{\frac{1}{2}}$.*
6. *If $W \in L_A(H)$, then $TW \in L_A(H)$ and $(TW)^\sharp = W^\sharp T^\sharp$.*
7. *If $W \in L_A(H)$, then $\|TW\|_A = \|WT\|_A$.*

Given $A \in L_A(H)^+$ and a closed subspace S, we denote by $P(A, S)$ the set of A-selfadjoint projections with fixed range S:

$$P(A, S) = \{Q \in L_A(H) : \quad Q^2 = Q, \quad R(Q) = S, \quad and \quad AQ = Q^*A\}.$$

The classes of normal, isometries and partial isometries and contractions on hilbert spaces have been generalized to semi-hilbert spaces by many authors in [9],[15]. The following definition gives this class of operators.

Definition 2.8. *Let $T \in L_A(H)$. Then T is*

1. *normal if $T^\sharp T = TT^\sharp$ or equivanlently if $\|Tx\|_A = \|T^\sharp x\|_A$ for every $x \in H$.*
2. *A-contraction if $\|Tx\|_A \leq \|x\|_A$ for every $x \in H$ or equivanlently if $T^*AT \leq A$.*
3. *A-partial isometry if $\|Tx\|_A = \|x\|_A$ for every $x \in \mathcal{N}(A)^{\perp_A}$.*

3. Main results

Proposition 3.1. *(**(Heinz´s inequality**[7]) For every two positive operators S and T $\in B(H)$, and for every $\alpha \in [0, 1]$, the following operator inequality holds*

$$\forall X \in B(H) \quad \|SX + XT\| \geq \|S^\alpha XT^{1-\alpha} + S^{1-\alpha}XT^\alpha\|.$$

*It is well known that the Heinz inequality is equivalent to the following form operator of the so-called **The arithmetic-geometric mean inequality***

$$\forall S, X, T \in B(H) \quad \|SS^*X + XT^*T\| \geq 2\|SXT\|. \tag{1}$$

3.1. New Caracterizations of subclasses of A-Selfadjoint operators

We start our results with the main proposition

Proposition 3.2. *Let $S, T \in L_A(H)$ be $A-$selfadjoint operators and $X \in L_{A^{\frac{1}{2}}}(H)$. Then*

$$\|(T^\sharp)^2X + X(S^\sharp)^2\|_A \geq 2\|T^\sharp XS^\sharp\|_A.$$

Proof. Assume that $S, T \in L_A(H)$ be $A-$selfadjoint operators and $X \in L_{A^{\frac{1}{2}}}(H)$. Then

$$\begin{aligned}
\|(T^\sharp)^2X + X(S^\sharp)^2\|_A &= \|(A^+T^*A)^2X + X(A^+S^*A)^2\|_A \\
&= \|A^+T^*AA^+T^*AX + XA^+S^*AA^+S^*A\|_A \\
&= \|A^+(T^*)^2AX + XA^+(S^*)^2A\|_A \\
&= \|A^{\frac{1}{2}}A^+(T^*)^2AX(A^{\frac{1}{2}})^+ + A^{\frac{1}{2}}XA^+(S^*)^2A(A^{\frac{1}{2}})^+\| \\
&= \|(A^{\frac{1}{2}})^+(T^*)^2A^{\frac{1}{2}}A^{\frac{1}{2}}X(A^{\frac{1}{2}})^+ + A^{\frac{1}{2}}X(A^{\frac{1}{2}})^+(A^{\frac{1}{2}})^+(S^*)^2A^{\frac{1}{2}}\|.
\end{aligned}$$

Let $T^0 = (A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}}$, $X^0 = A^{\frac{1}{2}}X(A^{\frac{1}{2}})^+$ and $S^0 = (A^{\frac{1}{2}})^+S^*A^{\frac{1}{2}}$. Then we obtain

$$\|(T^\sharp)^2X + X(S^\sharp)^2\|_A = \|(T^0)^2X^0 + X^0(S^0)^2\|.$$

Since $S, T \in L_A(H)$ be $A-$selfadjoint operators and $X \in L_{A^{\frac{1}{2}}}(H)$. Then S^0, X^0 and T^0 are bounded operators (By using proposition 2.3) and S^0, T^0 are selfadjoints. Moreover

$$\begin{aligned}
\|(T^\sharp)^2X + X(S^\sharp)^2\|_A &= \|(T^0)^2X^0 + X^0(S^0)^2\| \\
&\geq 2\|T^0X^0S^0\| \qquad \textbf{(by the The arithmetic-geometric mean inequality)} \\
&= \|(A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}}A^{\frac{1}{2}}X(A^{\frac{1}{2}})^+(A^{\frac{1}{2}})^+S^*A^{\frac{1}{2}}\| \\
&= \|A^{\frac{1}{2}}A^+T^*AXA^+S^*A(A^{\frac{1}{2}})^+\| \\
&= \|A^{\frac{1}{2}}T^\sharp XS^\sharp A(A^{\frac{1}{2}})^+\| \\
&= \|T^\sharp XS^\sharp\|_A.
\end{aligned}$$

□

Lemma 3.3. *Let $T \in L_{cl}(H)$ such that $T, T^+ \in L_A(H)$, TT^+, T^+T are A-selfadjoint operators. Then*

a) $((A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}})^+ = (A^{\frac{1}{2}})^+(T^*)^+A^{\frac{1}{2}} = (A^{\frac{1}{2}})^+(T^+)^*A^{\frac{1}{2}}$.

b) $R((A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}})$ *is closed.*

Proof. a) Let $T \in L_A(H)$ and $T^0 = (A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}}$. Then

1) $T^0(T^0)^+T^0 = (A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}}(A^{\frac{1}{2}})^+(T^*)^+A^{\frac{1}{2}}(A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}} = (A^{\frac{1}{2}})^+T^*(T^*)^+T^*A^{\frac{1}{2}} = (A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}} = T^0$.

2) $(T^0)^+T^0(T^0)^+ = (A^{\frac{1}{2}})^+(T^*)^+A^{\frac{1}{2}}(A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}}(A^{\frac{1}{2}})^+(T^*)^+A^{\frac{1}{2}} = (A^{\frac{1}{2}})^+(T^*)^+S^*(T^*)^+A^{\frac{1}{2}} = (T^0)^+$.

3) $T^0(T^0)^+ = A^{\frac{1}{2}}T^+T(A^{\frac{1}{2}})^+$

Let's prove that $T^0(T^0)^+$ is an orthogonal projection.

$$T^0(T^0)^+T^0(T^0)^+ = A^{\frac{1}{2}}T^+T(A^{\frac{1}{2}})^+A^{\frac{1}{2}}T^+T(A^{\frac{1}{2}})^+ = A^{\frac{1}{2}}(T^+T)^2(A^{\frac{1}{2}})^+ = A^{\frac{1}{2}}T^+T(A^{\frac{1}{2}})^+ = T^0(T^0)^+$$

Since T^+T is A-self adjoint then we obtain that $T^0(T^0)^+ = A^{\frac{1}{2}}T^+T(A^{\frac{1}{2}})^+$ is a self adjoint operator. So $T^0(T^0)^+$ is an orthogonal projection

4) By the same argument we find that $(T^0)^+T^0 = A^{\frac{1}{2}}TT^+(A^{\frac{1}{2}})^+$ and $(T^0)^+T^0$ is an orthogonal projection.

Finally $((A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}})^+ = (A^{\frac{1}{2}})^+(T^*)^+A^{\frac{1}{2}}$.

b) Since $T^+ \in L_A(H)$ then $(A^{\frac{1}{2}})^+(T^+)^*A^{\frac{1}{2}} = ((A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}})^+ = (A^{\frac{1}{2}})^+(T^*)^+A^{\frac{1}{2}}) \in B(H)$ then $R((A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}})$ is closed.

□

Remark 3.4. *If $T \in L_{cl}(H)$ such that $T, T^+ \in L_A(H)$, TT^+, T^+T are A-selfadjoint operators. Thus by using Lemma (3.3), we obtain*

$$(T^\sharp)^+ = (A^+T^*A)^+ = ((A^{\frac{1}{2}})^+(A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}}A^{\frac{1}{2}})^+ = (A^{\frac{1}{2}})^+(A^{\frac{1}{2}})^+(T^*)^+A^{\frac{1}{2}}A^{\frac{1}{2}} = (T^+)^\sharp.$$

Proposition 3.5. *Let $S, T \in L_{cl}(H)$ such that $S, S^+, T, T^+ \in L_A(H)$. If S, T, SS^+, S^+S, TT^+, T^+T are A-selfadjoint operators. Then*

$$\forall X \in L_{A^{\frac{1}{2}}}(H): \quad \|T^\sharp X(S^+)^\sharp + (T^+)^\sharp XS^\sharp\|_A \geq 2\|(T^+T)^\sharp X(SS^+)^\sharp\|_A.$$

Proof. Assume that $S, T \in L_{cl}(H)$ such that $S, S^+, T, T^+ \in L_A(H)$, and S, T, SS^+, S^+S, TT^+, T^+T are A-selfadjoint operators. For every $X \in L_{A^{\frac{1}{2}}}(H)$ we obtain

$$\begin{aligned}
\|T^\sharp X(S^+)^\sharp + (T^+)^\sharp XS^\sharp\|_A &= \|A^{\frac{1}{2}}T^\sharp X(S^+)^\sharp(A^{\frac{1}{2}})^+ + A^{\frac{1}{2}}(T^+)^\sharp XS^\sharp(A^{\frac{1}{2}})^+\| \\
&= \|A^{\frac{1}{2}}A^+T^*AXA^+(S^+)^*A(A^{\frac{1}{2}})^+ + A^{\frac{1}{2}}A^+(T^+)^*AXA^+S^*A(A^{\frac{1}{2}})^+\| \\
&= \|(A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}}A^{\frac{1}{2}}X(A^{\frac{1}{2}})^+(A^{\frac{1}{2}})^+(S^+)^*A^{\frac{1}{2}} + (A^{\frac{1}{2}})^+(T^+)^*A^{\frac{1}{2}}A^{\frac{1}{2}}X(A^{\frac{1}{2}})^+(A^{\frac{1}{2}})^+S^*A^{\frac{1}{2}}\| \\
&= \|T^0X^0(S^0)^+ + (T^0)^+X^0S^0\|.
\end{aligned}$$

AS S and T are A-selfadjoint, S^0 and T^0 are selfadjoints. So

$$\begin{aligned}
\|T^\sharp X(S^+)^\sharp + (T^+)^\sharp XS^\sharp\|_A &= \|T^0X^0(S^0)^+ + (T^0)^+X^0S^0\| \\
&= \|(T^0)^*X^0(S^0)^+ + (T^0)^+X^0(S^0)^*\|.
\end{aligned}$$

From the properties of the Moore-penrose inverse, we obtain $(T^0)^* = (T^0)^*T^0(T^0)^+$ and $(S^0)^* = (S^0)^+S^0(S^0)^*$. Then we have

$$\begin{aligned}
\|T^\sharp X(S^+)^\sharp + (T^+)^\sharp XS^\sharp\|_A &= \|(T^0)^*T^0(T^0)^+X^0(S^0)^+ + (T^0)^+X^0(S^0)^+S^0(S^0)^*\| \\
&\geq 2\|T^0(T^0)^+X^0(S^0)^+S^0\| \text{ (by \textbf{The arithmetic-geometric mean inequality})} \\
&= \|(A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}}(A^{\frac{1}{2}})^+(T^+)^*A^{\frac{1}{2}}A^{\frac{1}{2}}X(A^{\frac{1}{2}})^+(A^{\frac{1}{2}})^+(S^+)^*A^{\frac{1}{2}}(A^{\frac{1}{2}})^+S^*A^{\frac{1}{2}}\| \\
&= \|(A^{\frac{1}{2}})^+(T^+T)^*AXA^+(SS^+)^*A^{\frac{1}{2}}\| \\
&= \|A^{\frac{1}{2}}A^+(T^+T)^*AXA^+(SS^+)^*A^+(A^{\frac{1}{2}})^+\| \\
&= \|(T^+T)^\sharp X(SS^+)^\sharp\|_A.
\end{aligned}$$

□

Corollary 3.6. *Let $S, T \in L_{cl}(H)$ such that $S, S^+, T, T^+ \in L_A(H)$ and S, T, SS^+, S^+S, TT^+, T^+T are A-selfadjoint operators. For every $X \in L_{A^{\frac{1}{2}}}(H)$ if T is injective, and S is surjective, then*

$$\|T^\sharp X(S^+)^\sharp + (T^+)^\sharp XS^\sharp\|_A \geq 2\|X\|_A.$$

Proof. .

- Since T is injective, then $T^+T = I$, So $(T^+T)^\sharp = P_{\overline{\mathcal{R}(A)}}$.
- Since S is surjective, then $SS^+ = I$, So $(SS^+)^\sharp = P_{\overline{\mathcal{R}(A)}}$.

Thus by by using proposition (3.5) we obtain

$$\|T^\sharp X(S^+)^\sharp + (T^+)^\sharp XS^\sharp\|_A \geq 2\|X\|_A.$$

□

Definition 3.7. *[1] Let $T \in L_A(H)$ be a non-zero operator. T is said to be A-invertible in $L_A(H)$ if there exists a non-zero operator $S \in L_A(H)$ such that $ATS = AST = A$. Here S is called an A-inverse of T in $L_A(H)$.*

Proposition 3.8. *[1] Let $T \in L_A(H)$ be an A-invertible operator in $T \in L_A(H)$ with an A-inverse $S \in L_A(H)$. Ten the following statements are equivalent:*

1. $A = ATS = AST$.
2. $PTS = PST = P$.
3. $A^{\frac{1}{2}} = A^{\frac{1}{2}}TS = A^{\frac{1}{2}}ST$.

Remark 3.9. *[11] $T \in L_A(H)$ is A-invertible operator in $L_A(H)$ with an A-inverse $S \in L_A(H)$ if and only if $T^\sharp$ is A-invertible operator in $L_A(H)$ with an A-inverse $S^\sharp \in L_A(H)$.*

Lemma 3.10. *Let $T \in L_{cl}(H)$ such that $T, T^+ \in L_A(H)$. If T is an A-invertible operator with an A-inverse $S \in L_A(H)$. Then*

a) $((A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}})^+ = (A^{\frac{1}{2}})^+S^*A^{\frac{1}{2}}$

b) if T is A-selfadjoint, then S is A-selfadjoint.

Proof. Let $T \in L_A(H)$ such that T is $A-$inversible with an A-inverse $S \in L_A(H)$. Then

a)

$$
\begin{aligned}
1)\ ((A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}})((A^{\frac{1}{2}})^+S^*A^{\frac{1}{2}}) &= A^{\frac{1}{2}}A^+T^*A(A^{\frac{1}{2}})^+(A^{\frac{1}{2}})^+S^*A(A^{\frac{1}{2}})^+\\
&= A^{\frac{1}{2}}T^\sharp S^\sharp(A^{\frac{1}{2}})^+\\
&= (A^{\frac{1}{2}})^+AT^\sharp S^\sharp(A^{\frac{1}{2}})^+\\
&= (A^{\frac{1}{2}})^+A(A^{\frac{1}{2}})^+\\
&= A^{\frac{1}{2}}(A^{\frac{1}{2}})^+.
\end{aligned}
$$

So $((A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}})((A^{\frac{1}{2}})^+S^*A^{\frac{1}{2}})$ is an orthogonal projection.

$$
\begin{aligned}
2)\ ((A^{\frac{1}{2}})^+S^*A^{\frac{1}{2}})((A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}}) &= A^{\frac{1}{2}}A^+S^*A(A^{\frac{1}{2}})^+(A^{\frac{1}{2}})^+T^*A(A^{\frac{1}{2}})^+\\
&= A^{\frac{1}{2}}S^\sharp T^\sharp(A^{\frac{1}{2}})^+\\
&= (A^{\frac{1}{2}})^+AS^\sharp T^\sharp(A^{\frac{1}{2}})^+\\
&= (A^{\frac{1}{2}})^+A(A^{\frac{1}{2}})^+\\
&= (A^{\frac{1}{2}})^+A^{\frac{1}{2}}.
\end{aligned}
$$

So $((A^{\frac{1}{2}})^+S^*A^{\frac{1}{2}})((A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}})$ is an orthogonal projection.
By using item (1), we obtain

$$
\begin{aligned}
3)\ ((A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}})((A^{\frac{1}{2}})^+S^*A^{\frac{1}{2}})((A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}}) &= A^{\frac{1}{2}}(A^{\frac{1}{2}})^+(A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}}\\
&= A^{\frac{1}{2}}A^+T^*A^{\frac{1}{2}}\\
&= (A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}}.
\end{aligned}
$$

By using item (2), we obtain

$$
\begin{aligned}
4)\ ((A^{\frac{1}{2}})^+S^*A^{\frac{1}{2}})((A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}})((A^{\frac{1}{2}})^+S^*A^{\frac{1}{2}}) &= (A^{\frac{1}{2}})^+A^{\frac{1}{2}}(A^{\frac{1}{2}})^+S^*A^{\frac{1}{2}}\\
&= (A^{\frac{1}{2}})^+S^*A^{\frac{1}{2}}.
\end{aligned}
$$

Finally, we obtain that $((A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}})^+ = (A^{\frac{1}{2}})^+S^*A^{\frac{1}{2}}$.

b) Since T is A-invertible with A-inverse S, then $T^\sharp$ is A-invertible with A-inverse $S^\sharp$. Then we have

$$A = AT^\sharp S^\sharp = AS^\sharp T^\sharp \quad \Leftrightarrow \quad A = T^*S^*A = S^*T^*A \tag{2}$$

Assume that T is A-selfadjoint with A-inverse S. We have

$$
\begin{aligned}
S^*A &= S^*ATS \qquad \text{(S is the A-inverse of T)}\\
&= S^*T^*AS \qquad \text{(T is A-selfadjoint)}\\
&= AS \qquad (S^\sharp \text{ is the A-inverse of } T^\sharp, \text{by using equation 2}).
\end{aligned}
$$

Then S is A-selfadjoint. □

Proposition 3.11. *Let $T \in L_{cl}(H)$ such that $T, T^+ \in L_A(H)$. If T is an A-invertible operator with an A-inverse $S \in L_A(H)$ and if T is A-selfadjoint, we obtain*

$$\forall X \in L_{A^{\frac{1}{2}}}(H): \quad \|T^\sharp X (T^+)^\sharp + (T^+)^\sharp X T^\sharp\|_A \geq 2\|X\|_A.$$

Proof. Let $T \in L_{cl}(H)$ such that $T, T^+ \in L_A(H)$. If T is A-invertible operator with an A-inverse $S \in L_A(H)$, and if T is A-selfadjoint then S is A-selfadjoint. By using proposition 3.5 and, we obtain

$$\begin{aligned}
\|T^\sharp X (T^+)^\sharp + (T^+)^\sharp X T^\sharp\|_A &\geq 2\|(T^+T)^\sharp X (TT^+)^\sharp\|_A \\
&= 2\|(ST)^\sharp X (TS)^\sharp\|_A \quad \text{by using Lemma (3.10)} \\
&= 2\|A^{\frac{1}{2}}(ST)^\sharp X (TS)^\sharp (A^{\frac{1}{2}})^+\| \\
&= 2\|A^{\frac{1}{2}} T^\sharp S^\sharp X S^\sharp T^\sharp (A^{\frac{1}{2}})^+\| \\
&= 2\|(A^{\frac{1}{2}})^+ A T^\sharp S^\sharp X A^+ A S^\sharp T^\sharp (A^{\frac{1}{2}})^+\| \\
&= 2\|(A^{\frac{1}{2}})^+ A X A^+ A (A^{\frac{1}{2}})^+\| \quad (\text{T is A-invertvible iff } T^\sharp \text{ is also A-invertible }) \\
&= 2\|A^{\frac{1}{2}} X (A^{\frac{1}{2}})^+\| \\
&= 2\|X\|_A.
\end{aligned}$$

□

3.2. New characterizations of subclasses of A-normal operators

The following results can be found in [8].

Proposition 3.12. *[8] Let $A \in L(H)^+$ and $Q \in L(H)$ such that $S = R(Q)$ is a closed subspace of $\overline{R(A)}$.*

1. *If $Q \in \mathbb{Q} \cap L^A(H)$ then $\overline{A^{\frac{1}{2}} Q (A^{\frac{1}{2}})^+}$ is a projection.*
2. *The following conditions are equivalent:*
   (a) *$Q \in P(A, S)$.*
   (b) *$Q \in L_A(H)$ and $\overline{A^{\frac{1}{2}} Q (A^{\frac{1}{2}})^+}$ is an orthogonal projection.*
   In one of these conditions holds then $\|Q\|_A = \|\overline{A^{\frac{1}{2}} Q (A^{\frac{1}{2}})^+}\| = 1$.

For all $x, y \in H$ the operator $x \otimes y$ is called a rank-one operator in $L(H)$. The author in [17] define the so-called A-rank one operator in $L_{A^{\frac{1}{2}}}(H)$.

Definition 3.13. *[17] Let $A \in L_A(H)^+$. The A-rank one operator denoted by $x \otimes_A y$ is defined as follows for all $x, y \in H$*

$$(x \otimes_A y)z = \langle z, y \rangle_A x.$$

Proposition 3.14. *[17] Let $x, y \in H$ and $T \in L_{A^{\frac{1}{2}}}(H)$. Then $x \otimes_A y \in L_{A^{\frac{1}{2}}}(H)$ and the next assertions hold:*

1. *$\|x \otimes_A y\|_A = \|x\|_A \|y\|_A$ and $(x \otimes_A y)^\sharp = y \otimes_A x$.*
2. *$T(x \otimes_A y) = Tx \otimes_A y$ and $(x \otimes_A y)T = x \otimes_A T^\sharp y$.*

Proposition 3.15. *[16] LetT $\in L(H)$. Then following properties are equivalent*

1. *T is normal operator.*
2. *$\forall S \in B(H) \quad \|TS\| = \|T^*S\|$.*
3. *$\forall S \in B(H) \quad \|ST\| = \|ST^*\|$.*

A useful characterizations of the class of A-normal operators is given in the following proposition.

Proposition 3.16. *Let $T \in L_A(H)$. Then the following inequalities are equivalent:*

1. *T is A-normal.*
2. *$\forall S \in L_{A^{\frac{1}{2}}}(H), \quad \|TS\|_A = \|T^\sharp S\|_A$.*
3. *$\forall S \in L_{A^{\frac{1}{2}}}(H), \quad \|ST\|_A = \|ST^\sharp\|_A$.*

Proof. • (1) ⇒ (2). Suppose that T is A-normal. Then for every $x \in H$, we obtain

$$\begin{aligned}\|Tx\|_A^2 = \|T^\sharp x\|_A^2 \quad &\Leftrightarrow \quad \langle ATx, Tx\rangle = \langle AT^\sharp x, T^\sharp x\rangle \\ &\Leftrightarrow \quad \langle T^*ATx, x\rangle = \langle (T^\sharp)^* AT^\sharp x, x\rangle \\ &\Leftrightarrow \quad T^*AT = (T^\sharp)^* AT^\sharp.\end{aligned}$$

For every $x \in H$, and $S \in L_A(H)$ we have

$$\begin{aligned}\|TSx\|_A^2 &= \langle ATSx, TSx\rangle \\ &= \langle T^*ATSx, Sx\rangle \\ &= \langle (T^\sharp)^* AT^\sharp Sx, Sx\rangle \\ &= \langle AT^\sharp Sx, T^\sharp Sx\rangle \\ &= \|T^\sharp Sx\|_A^2.\end{aligned}$$

By taking the supremum over $x \in H$ with $\|x\|_A = 1$, we obtain

$$\|TS\|_A = \|T^\sharp S\|_A.$$

• (2) ⇒ (1). For every $x, y \in H$, let ($S = x \otimes_A y$) in item 3. So by using proposition 3.14 we obtain that $S \in L_{A^{\frac{1}{2}}}(H)$ and

$$\begin{aligned}\|T^\sharp S\|_A &= \|T^\sharp(x \otimes_A y)\|_A \\ &= \|(T^\sharp x) \otimes_A y\|_A \\ &= \|T^\sharp x\|_A \|y\|_A.\end{aligned}$$

and

$$\begin{aligned}\|TS\|_A &= \|T(x \otimes_A y)\|_A \\ &= \|(Tx) \otimes_A y\|_A \\ &= \|Tx\|_A \|y\|_A.\end{aligned}$$

Since $\|TS\|_A = \|T^\sharp S\|_A$, we have

$$\|Tx\|_A \|y\|_A = \|T^\sharp x\|_A \|y\|_A.$$

By taking the supremum over $y \in H$ with $\|y\|_A = 1$, we obtain

$$\|Tx\|_A = \|T^\sharp x\|_A.$$

then T is A-normal.

• (2) ⇒ (3). Suppose that $\|TS\|_A = \|T^\sharp S\|_A$. Then for every $T \in L_A(H)$ and $S \in L_{A^{\frac{1}{2}}}(H)$, we obtain

$$\begin{aligned}\|ST^\sharp\|_A &= \|(ST^\sharp)^\sharp\|_A \quad && \textit{By proposition 2.7 item (5)} \\ &= \|(T^\sharp)^\sharp S^\sharp\|_A \quad && \textit{By proposition 2.7 item (6)} \\ &= \|T^\sharp S^\sharp\|_A \quad && \text{(By hypothesis} \\ &= \|(ST)^\sharp\|_A \quad && \textit{By proposition 2.7 item (6)} \\ &= \|ST\|_A \quad && \textit{By proposition 2.7 item (5).}\end{aligned}$$

- (3) ⇒ (2). Let $\|ST\|_A = \|ST^\sharp\|_A$, then

$$\begin{aligned}
\|T^\sharp S\|_A &= \|(T^\sharp S)^\sharp\|_A \quad \textit{By proposition 2.7 item (5)} \\
&= \|S^\sharp (T^\sharp)^\sharp\|_A \quad \textit{By proposition 2.7 item (6)} \\
&= \|S^\sharp T^\sharp\|_A \quad \text{(By hypothesis)} \\
&= \|(TS)^\sharp\|_A \quad \textit{By proposition 2.7 item (6)} \\
&= \|TS\|_A \quad \textit{By proposition 2.7 item (5).}
\end{aligned}$$

□

Corollary 3.17. *Let $T \in L_A(H)$. Then the following properties are equivalent:*

1. *T is A-normal.*
2. *$(A^{\frac{1}{2}})^+ T^* A^{\frac{1}{2}}$ is normal operator.*

Proof. .

- Assume that T is A-normal operator. By using proposition (3.16), we obtain that $\|TS\|_A = \|T^\sharp S\|_A$ for every $S \in L_{A^{\frac{1}{2}}}(H)$. So

$$\begin{aligned}
\|TS\|_A &= \|A^{\frac{1}{2}} TS (A^{\frac{1}{2}})^+\| \\
&= \|A^{\frac{1}{2}} T (A^{\frac{1}{2}})^+ A^{\frac{1}{2}} S (A^{\frac{1}{2}})^+\|.
\end{aligned}$$

and

$$\begin{aligned}
\|T^\sharp S\|_A &= \|A^{\frac{1}{2}} T^\sharp S (A^{\frac{1}{2}})^+\| \\
&= \|A^{\frac{1}{2}} A^+ T^* A S (A^{\frac{1}{2}})^+\| \\
&= \|(A^{\frac{1}{2}})^+ T^* A^{\frac{1}{2}} A^{\frac{1}{2}} S (A^{\frac{1}{2}})^+\| \\
&= \|\overline{A^{\frac{1}{2}} T (A^{\frac{1}{2}})^+} A^{\frac{1}{2}} S (A^{\frac{1}{2}})^+\|.
\end{aligned}$$

Then

$$\|A^{\frac{1}{2}} T (A^{\frac{1}{2}})^+ A^{\frac{1}{2}} S (A^{\frac{1}{2}})^+\| = \|\overline{A^{\frac{1}{2}} T (A^{\frac{1}{2}})^+} A^{\frac{1}{2}} S (A^{\frac{1}{2}})^+\|. \tag{3}$$

Since $T \in L_A(H)$, and $S \in L_{A^{\frac{1}{2}}}(H)$ then $(A^{\frac{1}{2}})^+ T^* A^{\frac{1}{2}}$, $A^{\frac{1}{2}} S (A^{\frac{1}{2}})^+ \in L(H)$. Therefore $(A^{\frac{1}{2}})^+ T^* A^{\frac{1}{2}}$ and $A^{\frac{1}{2}} T (A^{\frac{1}{2}})^+$ are normal by using proposition (3.15).

- Conversely, assume that $(A^{\frac{1}{2}})^+ T^* A^{\frac{1}{2}}$ is normal operator. Then by using proposition (3.15), we obtain

$$\begin{aligned}
\|T^\sharp S\|_A &= \|A^{\frac{1}{2}} A^+ T^* A S (A^{\frac{1}{2}})^+\| \\
&= \|(A^{\frac{1}{2}})^+ T^* A^{\frac{1}{2}} A^{\frac{1}{2}} S (A^{\frac{1}{2}})^+\| \\
&= \|\overline{A^{\frac{1}{2}} T (A^{\frac{1}{2}})^+} A^{\frac{1}{2}} S (A^{\frac{1}{2}})^+\| \quad (\textit{because} \quad (A^{\frac{1}{2}})^+ T^* A^{\frac{1}{2}} \quad \textit{is} \quad \textit{normal}) \\
&= \|\overline{A^{\frac{1}{2}} T (A^{\frac{1}{2}})^+ A^{\frac{1}{2}} S (A^{\frac{1}{2}})^+}\| \\
&= \|\overline{A^{\frac{1}{2}} TS (A^{\frac{1}{2}})^+}\| \\
&= \|A^{\frac{1}{2}} TS (A^{\frac{1}{2}})^+\| \\
&= \|TS\|_A.
\end{aligned}$$

Therefore T is A-normal by using proposition (3.16).

□

Example 3.18. 1. *Let* $H = \mathbb{C}^2$, $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \in L(H)^+$, $T = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \in L(H)^+$. *It is easy to check that* $T \in L_A$ *and* $T^\sharp = A^+T^*A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $TT^\sharp = T^\sharp T = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. *So T is A−normal.*
By direct computation we find that $\left((A^{\frac12})^+T^*A^{\frac12}\right)\left((A^{\frac12})^+T^*A^{\frac12}\right)^* = \left((A^{\frac12})^+T^*A^{\frac12}\right)^*\left((A^{\frac12})^+T^*A^{\frac12}\right) = \begin{pmatrix} 64 & 0 \\ 0 & 64 \end{pmatrix}$.
So $A^{\frac12}T^*(A^{\frac12})^+$ *is a normal operator.*

2. *Let* $A = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \in L(H)^+$, $T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in L_A(H)$. *So* $T^\sharp = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}$. *By a simple computation we find that* $TT^\sharp \neq T^\sharp T$. *So T is not A−normal operator.*
It easy to see that $\left((A^{\frac12})^+T^*A^{\frac12}\right)\left((A^{\frac12})^+T^*A^{\frac12}\right)^* \neq \left((A^{\frac12})^+T^*A^{\frac12}\right)^*\left((A^{\frac12})^+T^*A^{\frac12}\right)$. *So* $(A^{\frac12})^+T^*A^{\frac12}$ *is not normal.*

The following proposition will be useful in the proof of the next result.

Proposition 3.19. *[16] Let* $T \in L(H)$. *Then the following properties are equivalent:*

1. *T is normal.*
2. $\forall X \in B(H), \quad \|T^2X\| + \|XT^2\| \geq 2\|TXT\|.$

In the following proposition we give our main result in this section.

Proposition 3.20. *Let* $T \in L_{cl}(H)$ *such that* $T, T^+ \in L_A(H)$, T, TT^+, T^+T *are A-selfadjoint operators. Then the following conditions are equivalent:*

1. *T is A-normal.*
2. $\forall X \in L_{A^{\frac12}}(H)$, $\|T^\sharp X(T^+)^\sharp\|_A + \|(T^+)^\sharp XT^\sharp\|_A = \|TX(T^+)^\sharp\|_A + \|(T^+)^\sharp XT\|_A$.
3. $\forall X \in L_{A^{\frac12}}(H)$, $\|T^\sharp X(T^+)^\sharp\|_A + \|(T^+)^\sharp XT^\sharp\|_A \geq \|TX(T^+)^\sharp\|_A + \|(T^+)^\sharp XT\|_A$.
4. $\forall X \in L_{A^{\frac12}}(H)$, $\|T^\sharp X(T^+)^\sharp\|_A + \|(T^+)^\sharp XT^\sharp\|_A \geq 2\|(T^+T)^\sharp X(TT^+)^\sharp\|_A$.
5. $\forall X \in L_{A^{\frac12}}(H)$, $\|(T^\sharp)^2X\|_A + \|X(T^\sharp)^2\|_A \geq 2\|T^\sharp XT^\sharp\|_A$.

Proof. .

- (1) ⇒ (2) assume that $T \in L_A(H)$ be an A-normal operator with closed range. Let $X \in L_{A^{\frac12}}(H)$. By using proposition (3.16), we obtain
$$\|T^\sharp X(T^+)^\sharp\|_A = \|TX(T^+)^\sharp\|_A.$$
and
$$\|(T^+)^\sharp XT^\sharp\|_A = \|(T^+)^\sharp XT\|_A.$$
Thus
$$\|T^\sharp X(T^+)^\sharp\|_A + \|(T^+)^\sharp XT^\sharp\|_A = \|TX(T^+)^\sharp\|_A + \|(T^+)^\sharp XT\|_A.$$

- The implication (2) ⇒ (3) is trivial.

- (3) ⇒ (4). From the triangle inequality, we obtain
$$\begin{aligned} \|T^\sharp X(T^+)^\sharp\|_A + \|(T^+)^\sharp XT^\sharp\|_A &\geq \|T^\sharp X(T^+)^\sharp + (T^+)^\sharp XT^\sharp\|_A \\ &\geq 2\|(T^+T)^\sharp X(TT^+)^\sharp\|_A \qquad \text{(by using proposition (3.5)).} \end{aligned}$$

- (4) ⇒ (5). Replace X by $T^\sharp XT^\sharp$ in (4), we obtain

$$\begin{aligned} \|T^\sharp T^\sharp XT^\sharp (T^+)^\sharp\|_A + \|(T^+)^\sharp T^\sharp XT^\sharp T^\sharp\|_A &\geq 2\|(T^+T)^\sharp T^\sharp XT^\sharp (TT^+)^\sharp\|_A \\ &= 2\|T^\sharp (T^+)^\sharp T^\sharp XT^\sharp (T^+)^\sharp T^\sharp\|_A \\ &= 2\|T^\sharp (T^\sharp)^+ T^\sharp XT^\sharp (T^\sharp)^+ T^\sharp\|_A \quad \text{(by using remark (3.4))} \\ &= 2\|T^\sharp XT^\sharp\|_A. \end{aligned}$$

Therefore

$$\|(T^\sharp)^2 X(T^+T)^\sharp\|_A + \|(TT^+)^\sharp X(T^\sharp)^2\|_A \geq 2\|T^\sharp XT^\sharp\|_A$$

By using triangle inequality, we obtain

$$\|(T^+T)^\sharp\|_A\|(T^\sharp)^2X\|_A + \|(TT^+)^\sharp\|_A\|X(T^\sharp)^2\|_A \geq \|(T^\sharp)^2X(T^+T)^\sharp\|_A + \|(TT^+)^\sharp X(T^\sharp)^2\|_A \geq 2\|T^\sharp XT^\sharp\|_A.$$

Since T^+T and TT^+ are A-selfadjoint projection, we have $\|(T^+T)^\sharp\|_A = \|T^+T\|_A = 1$ and $\|(TT^+)^\sharp\|_A = \|TT^+\|_A = 1$ (by using proposition 3.12), it holds

$$\|(T^\sharp)^2X\|_A + \|X(T^\sharp)^2\|_A \geq 2\|T^\sharp XT^\sharp\|_A.$$

- (5) ⇒ (1). Since $T \in L_{cl}(H)$, $T, T^+ \in L_A(H)$, and TT^+, T^+T are A-selfadjoint operators, we obtain

$$\begin{aligned} \|(T^\sharp)^2X\|_A + \|X(T^\sharp)^2\|_A &= \|A^{\frac12}(T^\sharp)^2X(A^{\frac12})^+\| + \|A^{\frac12}X(T^\sharp)^2(A^{\frac12})^+\| \\ &= \|(A^{\frac12})^+(T^*)^2A^{\frac12}A^{\frac12}X(A^{\frac12})^+\| + \|A^{\frac12}X(A^{\frac12})^+(A^{\frac12})^+(T^*)^2A^{\frac12}\| \\ &= \|((A^{\frac12})^+T^*A^{\frac12})^2A^{\frac12}X(A^{\frac12})^+\| + \|A^{\frac12}X(A^{\frac12})^+((A^{\frac12})^+T^*A^{\frac12})^2\|. \end{aligned}$$

and

$$\|T^\sharp XT^\sharp\|_A = \|A^{\frac12}T^\sharp XT^\sharp(A^{\frac12})^+\| = \|(A^{\frac12})^+T^*A^{\frac12}A^{\frac12}X(A^{\frac12})^+(A^{\frac12})^+T^*A^{\frac12}\|.$$

From (5), we obtain

$$\|((A^{\frac12})^+T^*A^{\frac12})^2A^{\frac12}X(A^{\frac12})^+\| + \|A^{\frac12}X(A^{\frac12})^+((A^{\frac12})^+T^*A^{\frac12})^2\| \geq 2\|(A^{\frac12})^+T^*A^{\frac12}A^{\frac12}X(A^{\frac12})^+(A^{\frac12})^+T^*A^{\frac12}\|. \quad (4)$$

Since $(A^{\frac12})^+T^*A^{\frac12} \in L(H)$, and $A^{\frac12}X(A^{\frac12})^+ \in L(H)$. So from proposition (3.19), we obtain that $(A^{\frac12})^+T^*A^{\frac12}$ is normal. Therefore T is A- normal (By corollary 3.17).

□

Corollary 3.21. *Let $T \in L_{cl}(H)$ such that $T, T^+ \in L_A(H)$, T, TT^+, T^+T are A-selfadjoint operators. Then, the following conditions are equivalent*

1. *T is A-normal.*
2. *$\forall X \in L_{A^{\frac12}}(H)$, $\|(T^\sharp)^2X\|_A\|X(T^\sharp)^2\|_A \geq \|T^\sharp XT^\sharp\|_A^2$.*

Proof. .

- (1) ⇒ (2). Assume (1) holds. Let $X \in L_{A^{\frac12}}(H)$, then we obtain

$$\|(T^\sharp)^2X\|_A = \|T^\sharp T^\sharp X\|_A = \|TT^\sharp X\|_A \qquad \text{(Using proposition 3.16).}$$

and

$$\|X(T^\sharp)^2\|_A = \|XT^\sharp T^\sharp\|_A = \|XT^\sharp T\|_A \qquad \text{(Using proposition 3.16).}$$

Then

$$\begin{aligned}
\|(T^{\sharp})^2X\|_A\|X(T^{\sharp})^2\|_A &= \|TT^{\sharp}X\|_A\|XT^{\sharp}T\|_A \\
&= \|A^{\frac{1}{2}}TT^{\sharp}X(A^{\frac{1}{2}})^{+}\|\|A^{\frac{1}{2}}XT^{\sharp}T(A^{\frac{1}{2}})^{+}\| \\
&= \|A^{\frac{1}{2}}T(A^{\frac{1}{2}})^{+}(A^{\frac{1}{2}})^{+}T^{*}A^{\frac{1}{2}}A^{\frac{1}{2}}X(A^{\frac{1}{2}})^{+}\|\|A^{\frac{1}{2}}X(A^{\frac{1}{2}})^{+}(A^{\frac{1}{2}})^{+}T^{*}A^{\frac{1}{2}}A^{\frac{1}{2}}T(A^{\frac{1}{2}})^{+}\| \\
&= \|\overline{(T^0)^*}T^0X^0\|\|X^0T^0\overline{(T^0)^*}\| \\
&= \|\overline{(T^0)^*}T^0X^0\|\|\overline{T^0(T^0)^*(X^0)^*}\| \qquad \textit{because } \|VW\| = \|WV\| \textit{ for every } V,W\in L(H) \\
&\geq \|\overline{(T^0)^*}T^0X^0\overline{T^0(T^0)^*(X^0)^*}\| \qquad \textit{because } \|VW\| \geq \|W\|\|V\| \textit{ for every } V,W\in L(H) \\
&\geq r(\overline{(T^0)^*}T^0X^0\overline{T^0(T^0)^*(X^0)^*}) \qquad \textit{because } \|VW\| \geq r(WV) \textit{ for every } V,W\in L(H) \\
&= r(\overline{(T^0)^*(X^0)^*(T^0)^*}T^0X^0\overline{T^0}) \qquad (\text{r(VW)=r(WV) for every V,W} \in \text{L(H)}) \\
&= r(\overline{(T^0)^*(X^0)^*(T^0)^*}\ \overline{T^0X^0T^0}) \\
&= r(\overline{(T^0X^0T^0)^*}\ \overline{T^0X^0T^0}) \\
&= \|T^0X^0T^0\|^2 \\
&= \|(A^{\frac{1}{2}})^{+}T^{*}A^{\frac{1}{2}}A^{\frac{1}{2}}X(A^{\frac{1}{2}})^{+}(A^{\frac{1}{2}})^{+}T^{*}A^{\frac{1}{2}}\|^2 \\
&= \|A^{\frac{1}{2}}A^{+}T^{*}AXA^{+}T^{*}A(A^{\frac{1}{2}})^{+}\|^2 \\
&= \|A^{\frac{1}{2}}T^{\sharp}XT^{\sharp}(A^{\frac{1}{2}})^{+}\|^2 \\
&= \|T^{\sharp}XT^{\sharp}\|_A^2.
\end{aligned}$$

- (2) ⇒ (1)

$$\begin{aligned}
\frac{\|(T^{\sharp})^2X\|_A + \|X(T^{\sharp})^2\|_A}{2} &\geq \sqrt{\|(T^{\sharp})^2X\|_A\|X(T^{\sharp})^2\|_A} \quad (\textbf{ By Arithmetic-geometric mean inequality}) \\
&\geq \|T^{\sharp}XT^{\sharp}\|.
\end{aligned}$$

By using proposition (3.20), we obtain that T is A-normal.

□

3.3. New characterizations of Subclasses of A-Partial isometry operators

In the following section, we give some preliminary characterizations of A-partial isometries. Before starting our results we need the following proposition.

Proposition 3.22. *[9] Let $A \in L(H)^+$ with closed range and $T \in L(H)$. The following statements are equivalent:*

1. *T is A-partial isometry.*
2. *$T \in L_A(H)$ and $T^*AT = AP_{\overline{AR(T^{\sharp}T)}}$.*
3. *$T \in L_A(H)$ and $T^{\sharp}T = P_{\overline{AR(T^{\sharp}T)}}$.*

Our first result in this section is stated as follows:

Proposition 3.23. *Let $T \in L_A(H)$. The following conditions are equivalent:*

1. *T is a nonzero A-partial isometry.*
2. *$(A^{\frac{1}{2}})^{+}T^{*}A^{\frac{1}{2}}$ is a nonzero partial isometry.*

Proof. .

- (1) ⇒ (2). Assume (1) hold. Since $D((A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}}) = H$, we obtain that

$$((A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}})^* = \overline{(A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}})^*} = \overline{A^{\frac{1}{2}}T(A^{\frac{1}{2}})^+}.$$

and

$$\begin{aligned}
\overline{T^0(T^0)^*T^0(T^0)^*} &= \overline{(A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}}A^{\frac{1}{2}}T(A^{\frac{1}{2}})^+(A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}}A^{\frac{1}{2}}T(A^{\frac{1}{2}})^+} \\
&= \overline{A^{\frac{1}{2}}A^+T^*ATA^+T^*AT(A^{\frac{1}{2}})^+} \\
&= \overline{A^{\frac{1}{2}}T^\sharp TT^\sharp T(A^{\frac{1}{2}})^+} \\
&= \overline{A^{\frac{1}{2}}T^\sharp T(A^{\frac{1}{2}})^+} \qquad (T \text{ is A-partial isometry, } T^\sharp T \text{ is projection by proppositin 3.22 }) \\
&= \overline{A^{\frac{1}{2}}A^+T^*AT(A^{\frac{1}{2}})^+} \\
&= \overline{(A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}}A^{\frac{1}{2}}T(A^{\frac{1}{2}})^+} \\
&= \overline{T^0(T^0)^*}.
\end{aligned}$$

Then $T^0(T^0)^*$ is an orthogonal projection, Therefore $T^0 = (A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}}$ is partial isometry.

- (2) ⇒ (1). Let T^0 be a partial isometry. Then $\overline{T^0(T^0)^*} = \overline{A^{\frac{1}{2}}T^\sharp T(A^{\frac{1}{2}})^+}$ is an orthogonal projection.
  So since $T^\sharp T \in L_A(H)$ is A-selfadjoint and $\overline{A^{\frac{1}{2}}T^\sharp T(A^{\frac{1}{2}})^+}$ is an orthogonal projection, then by using proposition (3.12), we obtain that $T^\sharp T = P(A, \overline{R(T^\sharp T)})$. Then by using proposition 3.22 we obtain that T is A-partial isometry.

□

Example 3.24. *Let $H = \mathbb{C}^2$, $A = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \in L(H)^+$, $T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. It is easy to check that $T \in L_A(H)$ and $T^\sharp = A^+T^*A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $T^\sharp T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $T^\sharp T = P_{\overline{AR(T^\sharp T)}}$. So T is A−partial isometry.*
With a simple computation we find that

$$\left((A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}}\right)\left((A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}}\right)^*\left((A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}}\right) = (A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

So $A^{\frac{1}{2}}T^*(A^{\frac{1}{2}})^+$ is a partial isometry.

In 2004, Mbekhta has given the following characterization of partial isometries in Hilbert spaces.

Proposition 3.25. *[12] Let $T \in L_{cl}(H)$). Then T is a nonzero partial isometry if and only if $\|T\| = \|T^+\| = 1$*

Now, let us study this characterization in semi-Hilbertian spaces.

Proposition 3.26. *Let $T \in L_{cl}(H)$ such that $T, T^+ \in L_A(H)$, TT^+, T^+T are A-selfadjoint operators. The following conditions are equivalent:*

1. *T is a nonzero A-partial isometry.*
2. $\|T\|_A = \|T^+\|_A = 1$.

Proof.
- (1) ⇒ (2). Since T is A-partial isometry. Then from proposition (3.23), we obtain that $(A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}}$ is partial isometry. So by using proposition 3.25 it holds

$$1 = \|(A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}}\| = \|\overline{A^{\frac{1}{2}}T(A^{\frac{1}{2}})^+}\| = \|A^{\frac{1}{2}}T(A^{\frac{1}{2}})^+\| = \|T\|_A.$$

and

$$1 = \|((A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}})^+\| = \|(A^{\frac{1}{2}})^+(T^+)^*A^{\frac{1}{2}}\| = \|\overline{A^{\frac{1}{2}}T^+(A^{\frac{1}{2}})^+}\| = \|A^{\frac{1}{2}}T^+(A^{\frac{1}{2}})^+\| = \|T^+\|_A.$$

- (2) ⇒ (1) Let $\|T\|_A = \|T^+\|_A = 1$. Then

$$1 = \|T\|_A = \|T^\sharp\|_A \Rightarrow \|A^{\frac{1}{2}}T(A^{\frac{1}{2}})^+\| = \|(A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}}\| = 1.$$

and

$$1 = \|T^+\|_A = \|(T^+)^\sharp\|_A \Rightarrow \|A^{\frac{1}{2}}T^+(A^{\frac{1}{2}})^+\| = \|(A^{\frac{1}{2}})^+(T^+)^*A^{\frac{1}{2}}\| = 1.$$

By using proposition (3.25), we obtain that $(A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}}$ is partial isometry, So T is A-partial isometry (by using proposition (3.23)).
□

Proposition 3.27. *Let $T \in L_{cl}(H)$ such that $T, T^+ \in L_A(H)$, TT^+, T^+T are A-selfadjoint operators. The following conditions are equivalent:*

1. $\|T\|_A\|T^+\|_A = 1$.
2. $\dfrac{T}{\|T\|_A}$ *is a nonzero A-partial isometry.*

Proof.
- (1) ⇒ (2). Assume (1) holds. Let $S = \dfrac{T}{\|T\|_A}$ then $\|S\|_A = 1$ and $S^+ = T^+\|T\|_A$. It is easy to see that $\|S^+\|_A = \|T^+\|_A\|T\|_A = 1$. So $\|S\|_A = \|S^+\|_A = 1$ By using proposition (3.26), we obtain $S = \dfrac{T}{\|T\|_A}$ is A-partial isometry.

- (2) ⇒ (1). If $S = \dfrac{T}{\|T\|_A}$ is A-partial isometry. Then by using proposition (3.26), we obtain

$$\|S\|_A = \frac{\|T\|_A}{\|T\|_A} = 1.$$

and

$$\|S^+\|_A = \|T\|_A\|T^+\|_A = 1.$$

□

Our main result in this section is stated as follows.

Proposition 3.28. *Let $T \in L_{cl}(H)$ such that $T, T^+ \in L_A(H)$, TT^+, T^+T are A-selfadjoint operators. The following conditions are equivalent:*

1. $\dfrac{T}{\|T\|_A}$ *is a nonzero A-partial isometry.*
2. $\forall X \in L_{A^{\frac{1}{2}}}(H)$, $\quad \|T^\sharp X(T^+)^\sharp\|_A = \|(TT^+)^\sharp X(TT^+)^\sharp\|_A$.
3. $\forall X \in L_{A^{\frac{1}{2}}}(H)$, $\quad \|(T^+)^\sharp XT^\sharp\|_A = \|(T^+T)^\sharp X(T^+T)^\sharp\|_A$.

Proof.
- (1) ⇒ (2) Asume (1) holds. By using proposition (3.27), we obtain

$$\|T\|_A\|T^+\|_A = \|T^0\|\|(T^0)^+\| = 1.$$

And

$$\begin{aligned}
\|T^\sharp X(T^+)^\sharp\|_A &= \|A^{\frac{1}{2}}A^+T^*AXA^+(T^+)^*A(A^{\frac{1}{2}})^+\| \\
&= \|(A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}}A^{\frac{1}{2}}X(A^{\frac{1}{2}})^+(A^{\frac{1}{2}})^+(T^+)^*A^{\frac{1}{2}}\| \\
&= \|T^0X^0(T^0)^+\| \\
&= \|T^0(T^0)^+T^0X^0(T^0)^+T^0(T^0)^+\| \\
&\leq \|T^0\|\|(T^0)^+T^0X^0(T^0)^+T^0\|\|(T^0)^+\| \\
&= \|(T^0)^+T^0X^0(T^0)^+T^0\| \qquad (\text{ because } \|T^0\|\|(T^0)^+\| = 1) \\
&= \|(A^{\frac{1}{2}})^+(T^+)^*A^{\frac{1}{2}}(A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}}A^{\frac{1}{2}}X(A^{\frac{1}{2}})^+(A^{\frac{1}{2}})^+(T^+)^*A^{\frac{1}{2}}(A^{\frac{1}{2}})^+T^*A^{\frac{1}{2}}\| \\
&= \|A^{\frac{1}{2}}A^+(T^+)^*T^*AXA^+(T^+)^*T^*A(A^{\frac{1}{2}})^+\| \\
&= \|A^{\frac{1}{2}}A^+(TT^+)^*AXA^+(TT^+)^*A(A^{\frac{1}{2}})^+\| \\
&= \|(TT^+)^\sharp X(TT^+)^\sharp\|_A.
\end{aligned}$$

Then

$$\|T^\sharp X(T^+)^\sharp\|_A \leq \|(TT^+)^\sharp X(TT^+)^\sharp\|_A.$$

From another side, since $\|(T^+)^\sharp\|_A\|T^\sharp\|_A = \|T^+\|_A\|T\|_A = 1$, then we have

$$\|(TT^+)^\sharp X(TT^+)^\sharp\|_A = \|(T^+)^\sharp T^\sharp X(T^+)^\sharp T^\sharp\|_A \leq \|(T^+)^\sharp\|_A\|T^\sharp X(T^+)^\sharp\|_A\|T^\sharp\|_A = \|T^\sharp X(T^+)^\sharp\|_A.$$

Consequently, $\|T^\sharp X(T^+)^\sharp\|_A = \|(TT^+)^\sharp X(TT^+)^\sharp\|_A$.

(2) ⇒ (3). If we replace X by $(T^+)^\sharp XT^\sharp$ in item (2), we obtain

$$\begin{aligned}
\|T^\sharp(T^+)^\sharp XT^\sharp(T^+)^\sharp\|_A &= \|(TT^+)^\sharp(T^+)^\sharp XT^\sharp(TT^+)^\sharp\|_A \\
&= \|(T^+)^\sharp T^\sharp(T^+)^\sharp XT^\sharp(T^+)^\sharp T^\sharp\|_A \\
&= \|(T^\sharp)^+T^\sharp(T^\sharp)^+XT^\sharp(T^\sharp)^+T^\sharp\|_A \qquad (\text{Using remark (3.4)}) \\
&= \|(T^\sharp)^+XT^\sharp\|_A \\
&= \|(T^+)^\sharp XT^\sharp\|_A.
\end{aligned}$$

(3) ⇒ (1). Let $X = x \otimes_A y$ in item (3), then we obtain

$$\begin{aligned}
\|(T^+)^\sharp XT^\sharp\|_A &= \|(T^+)^\sharp(x \otimes_A y)T^\sharp\|_A \\
&= \|((T^+)^\sharp x) \otimes_A ((T^\sharp)^\sharp y)\|_A \\
&= \|(T^+)^\sharp x\|_A\|(T^\sharp)^\sharp y\|_A.
\end{aligned}$$

by taking the supremum over $\|x\|_A = \|y\|_A = 1$, we obtain

$$\begin{aligned}
\|(T^+)^\sharp XT^\sharp\|_A &= \|(T^+)^\sharp\|_A\|(T^\sharp)^\sharp\|_A \\
&= \|T^+\|_A\|T^\sharp\|_A \\
&= \|T^+\|_A\|T\|_A.
\end{aligned}$$

Since $\|(T^+)^\sharp XT^\sharp\|_A = \|(T^+T)^\sharp X(T^+T)^\sharp\|_A$, then by taking $X = x \otimes_A y$, we obtain

$$\begin{aligned}
\|T^+\|_A\|T\|_A &= \|(T^+)^\sharp XT^\sharp\|_A \\
&= \|(T^+T)^\sharp X(T^+T)^\sharp\|_A \qquad (Using \quad item \quad 3) \\
&= \|(T^+T)^\sharp(x \otimes_A y)(T^+T)^\sharp\|_A \\
&= |(T^+T)^\sharp x \otimes_A ((T^+T)^\sharp)^\sharp y\|_A \\
&= \|(T^+T)^\sharp x\|_A\|((T^+T)^\sharp)^\sharp y\|_A.
\end{aligned}$$

by taking the supremum over $\|x\|_A = \|y\|_A = 1$, we obtain

$$\|T^+\|_A\|T\|_A = \|(T^+T)^\sharp\|_A\|((T^+T)^\sharp)^\sharp\|_A = \|T^+T\|_A\|T^+T\|_A.$$

Since T^+T is A-selfadjoint projection, then by using proposition (3.12), we obtain $\|T^+T\|_A = 1$. Consequently

$$\|T^+\|_A\|T\|_A = 1.$$

By using proposition (3.27), we obtain $\dfrac{T}{\|T\|_A}$ is A-partial isometry. □

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