Filomat 39:30 (2025), 10905–10918
https://doi.org/10.2298/FIL2530905K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A numerical iterative approach for solving weakly singular Volterra fractional integral equations

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Abstract. This paper explores a new numerical iterative approach for approximating solutions of fractional Volterra integral equations, based on the successive approximations method and the product integration method. Under certain conditions, we prove the existence and uniqueness of the solution and give error estimates of the approximations. Furthermore, both of the convergence analysis and numerical stability of the suggested method are examined. In the end, some examples confirm the theoretical results and illustrate the accuracy of the approach.

1. Introduction

A lot of phenomena in mathematical physics and chemical reactions including the radiation of heat from a semi-infinite solid [20], stereology [28], the heat conduction, electro-chemistry, superfluidity [27], plasma diagnostics, physical electronics, nuclear physical, optics and astrophysics [25, 43] and many others lead to a fractional integral equations ($\mathbb{FIE}$s).

Fractional calculus is a field dealing with integrals and derivatives of any real or complex order, and their applications in science, engineering and other fields. Since most $\mathbb{FIE}$s cannot be solved analytically, therefore, much attention has been devoted to searching for approximate and numerical techniques to the solution of $\mathbb{FIE}$s. Recently, many numerical methods have been developed by researchers for solving $\mathbb{FIE}$s. In [26] the Haar wavelet method was employed to solve Volterra $\mathbb{FIE}$s ($\mathbb{VFIE}$s). Product integration method has been applied to find the solution of $\mathbb{FIE}$s in References [2, 3, 16, 31]. Collocation methods for the solution of $\mathbb{FIE}$s were introduced in [7, 8]. Fractional multistep method for the solution of $\mathbb{FIE}$s has been applied in [15, 29]. The fractional-order Legendre functions and pseudo spectral method for the solution of $\mathbb{FIE}$s were introduced in [11]. Backward Euler methods for the solution of $\mathbb{FIE}$s were introduced in [5]. Galerkin method and Taylor series expansion have been applied in [32, 34] to solve singular fractional integro-differential equations. The existence (and uniqueness) of the solutions, under various conditions for $\mathbb{FIE}$s can be found in researches [1, 9, 12, 14, 19, 21, 33].

2020 *Mathematics Subject Classification*. Primary 47H09; Secondary 47H10.
Keywords. Iterative method; uniform modulus of continuity; successive approximations; fractional integral equations.
Received: 24 November 2024; Revised: 04 September 2025; Accepted: 28 September 2025
Communicated by Miodrag Spalević
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The authors in [41] studied the following Hammerstein IIEs with two variable delays:

$$z(\mathrm{r}) = f\Big(\mathrm{r}, z(\mathrm{r}), z\big(\tau(\mathrm{r})\big)\Big), \int_a^{\mathrm{r}} g(\mathrm{r}, s)\, K\Big(s, z(s), z\big(\alpha(s)\big)\Big)\, \mathrm{d}s + p(\mathrm{r}),$$

for $\mathrm{r}, s \in J_a := [a, b]$, where $p \in C(J_a, E)$, E is a closed subset of $\mathbb{R}$, $f, h \in C\Big(J_a \times E^2, E\Big)$, with the kernel of the Hammerstein, IIE $g \in C\Big(J_a^2, E\Big)$. Further, they investigated the uniqueness of solutions and Ulam type stabilities of the nonlinear iterative Fredholm and the Volterra IIEs with variable delays, respectively:

$$z(\mathrm{r}) = F\Big(\mathrm{r}, z^{[1]}(\mathrm{r}), z^{[1]}\big(\tau(\mathrm{r})\big), \ldots, \int_a^{\mathrm{r}} K\Big(\mathrm{r}, s, z^{[1]}(s), \ldots, z^{[m]}\big(\alpha(s)\big)\Big)\, \mathrm{d}s\Big),$$

for $\mathrm{r}, s \in J_a$, with mth iterate of the function z, $z^{[m]}$ which is denoted by

$$z^{[1]}(\mathrm{r}) = z(\mathrm{r}), \ldots, z^{[m]}(\mathrm{r}) = \underbrace{z\Big(z(\ldots z(z(t)))\Big)}_{m},$$

where $F \in C\Big(J_a \times E^{2m+1}, E\Big)$, $K \in C\Big(J_a^2 \times E^{2m}, E\Big)$ [40]. For more related works, see [6, 36, 37, 39, 42].

In fact, the Riemann-Liouville integral (RL-FI) of fractional order $\beta \in \mathbb{R}^{>0}$ for function $u \in L^1(a, b)$, $0 \le a < b < \infty$, is showed by

$$\mathbb{I}^{\beta} u(\mathrm{t}) = \tfrac{1}{\Gamma(\beta)} \int_a^{\mathrm{t}} \tfrac{u(s)}{(\mathrm{t}-s)^{1-\beta}}\, \mathrm{d}s, \qquad a < \mathrm{t} < b,$$

with the Euler Gamma function $\Gamma(\cdot)$ [35].

In this study, we present a new iterative procedure according to the successive approximations method for approximating solutions of the next FIIEs

$$z(\mathrm{r}) = p(\mathrm{r}) + \tfrac{\mu}{\Gamma(\beta)} \int_a^{\mathrm{r}} \tfrac{\mathcal{L}(\mathrm{r}, s) z(s)}{(\mathrm{r}-s)^{1-\beta}}\, \mathrm{d}s, \qquad \mathrm{r} \in J_a, \tag{1}$$

where $p \in C(J_a)$, $\mathcal{L} \in C(J_a^2)$ and z is an unknown function on J_a and also, μ is a positive constant. The aim of this paper is to present an iterative method based on product integration method for numerical solution of weakly-singular VIIEs (1). The convergence and numerical stability of the method are proved without smoothness conditions, using only modulus of continuity properties, when the existing numerical methods require smoothness conditions in the proof of convergence. It should also be noted that most numerical techniques used for solving various IIEs ultimately lead to the solution of a system of linear or nonlinear equations. In these cases, there is always the possibility of encountering singular coefficient matrices. Therefore, using successive approximations methods can be very beneficial in such cases [17, 18, 30, 38].

If $\mathcal{L} \equiv 1$ and $0 < \beta < 1$, then Eq. (1) become the so-called Abel's IIE the second kind,

$$z(\mathrm{r}) = p(\mathrm{r}) + \tfrac{1}{\Gamma(\beta)} \int_a^{\mathrm{r}} \tfrac{z(s)}{(\mathrm{r}-s)^{1-\beta}}\, \mathrm{d}s, \qquad \mathrm{r} \in J_a. \tag{2}$$

Abel's IIE is one of the most known equation that repeatedly appears in many models of problems in engineering and mathematical physics [13, 23]. If $\mathcal{L}(\mathrm{r}, s) = a(\mathrm{r})a(s)$ and $0 < \beta < 1$, then Eq. (1) become the following IIE,

$$z(\mathrm{r}) = p(\mathrm{r}) + \tfrac{a(\mathrm{r})}{\Gamma(\beta)} \int_a^{\mathrm{r}} \tfrac{b(s) z(s)}{(\mathrm{r}-s)^{1-\beta}}\, \mathrm{d}s, \qquad \mathrm{r} \in J_a. \tag{3}$$

Existence and uniqueness of the solution of Eq. (3) is presented in [31]. The stability results for 𝔽𝕀𝔼s are found in [43]. The researchers in [44] used the spectral collocation approach for approximating solution of Eq. (3). Also, the authors in [1] used the Simpson's rules to solve special case of 𝕍𝔽𝕀𝔼s.

This work is carry out as follows: In Section 2, some basic definitions, notations, and preliminary for developing our method are given. In Section 3.1, the existence and uniqueness of the solution of 𝕍𝔽𝕀𝔼 (1) is obtained. In Section 3.2, the convergence result of the approach is verified. The numerical stability of the proposed approach is investigated in Section 3.3. In Section 4, numerical experiments are presented. In the end, Section 5 serves as the conclusion for the paper.

2. Mathematical preliminaries

2.1. Modulus of continuity

Hereafter, we consider the Banach space $C(J_a)$ with the standard norm $\|\cdot\|$. Let $u : J_a \to \mathbb{R}$ be bounded function, then the function $\omega^{J_a}(u,\cdot) : \mathbb{R}^{\geq 0}\cup \to \mathbb{R}^{>0}$, which is expressed as form

$$\omega^{J_a}(u,h) = \sup \Big\{|u(\mathfrak{r}_1) - u(\mathfrak{r}_2)| \,:\, \mathfrak{r}_1, \mathfrak{r}_2 \in J_a,\, d(\mathfrak{r}_1, \mathfrak{r}_2) \leq h\Big\},$$

is named the oscillation of u on J_a, and $\omega^{J_a}(u,h)$ is also called uniform modulus of continuity whenever $u \in C(J_a)$. Similarly, for a bounded function $u : J_a \times J_a \to \mathbb{R}$, we denote

$$\overline{\omega}^{J_a}(\mathfrak{t})(u,h) := \sup_{\mathfrak{r}_1,\mathfrak{r}_2\in J_a} \Big\{|u(\mathfrak{r}_1,\mathfrak{t}) - u(\mathfrak{r}_2,\mathfrak{t})| \,:\, d(\mathfrak{r}_1,\mathfrak{r}_2) \leq h\Big\}, \qquad \mathfrak{t} \in J_a,$$

where d is a given metric on $\mathbb{R}$.

Theorem 2.1 ([10, 24]). *Let $u \in C(J_a)$. The following properties hold:*

a) $|u(\mathfrak{r}) - u(\mathfrak{s})| \leq \omega^{J_a}(u, d(\mathfrak{r},\mathfrak{s}))$ *for all* $\mathfrak{r},\mathfrak{s} \in J_a$*;*
b) $\omega^{J_a}(u,h)$ *is an non-decreasing mapping in* h*;*
c) $\omega^{J_a}(u,h_1 + h_2) \leq \omega^{J_a}(u,h_1) + \omega^{J_a}(u,h_2)$ *for any* $h_1,h_2 \geq 0$*;*
d) $\omega^{J_a}(u,nh) \leq n\omega^{J_a}(u,h)$ *for any* $h \geq 0$ *and* $n \in \mathbb{N}$*;*
e) $\omega^{J_a}(u,\lambda h) \leq (\lambda + 1)\omega^{J_a}(u,h)$ *for any* $h,\lambda \geq 0$*;*
f) *If* $J_a \subseteq J'_a$*, then* $\omega^{J_a}(u,h) \leq \omega^{J'_a}(u,h)$ *for all* $h \geq 0$*;*
g) $\omega^{J_a}(u,\cdot)$ *is continuous from the right;*
h) $\omega^{J_a}(u,\cdot)$ *is continuous at* 0 *iff* $u \in C(J_a)$*;*
i) $\overline{\omega}^{J_a}(\cdot)(u,h) \in C(J_a)$.

2.2. Product trapezoidal method

The main idea of the product integration method is introduced by Atkinson for linear integral equations [2, 3]. To solve the Eq. 1, we give a new method called the product trapezoidal rule. Let $m \geq 1$ be an integer, $h = \frac{b-a}{m}$. For general $z \in C(J_a)$, define

$$\Big[\mathcal{L}(\mathfrak{r},\mathfrak{s})z(\mathfrak{r})\Big]_m = \tfrac{1}{h}\Big[(\mathfrak{r}_j - \mathfrak{s})\mathcal{L}(\mathfrak{r},\mathfrak{r}_{j-1})z(\mathfrak{r}_{j-1}) + (\mathfrak{s} - \mathfrak{r}_{j-1})\mathcal{L}(\mathfrak{r},\mathfrak{r}_j)z(\mathfrak{r}_j)\Big], \tag{4}$$

for $\mathfrak{r}_{j-1} \leq \mathfrak{s} \leq \mathfrak{r}_j$, $\mathfrak{r}_j = jh$, $j = 1,\ldots,m$ and $\mathfrak{r} \in J_a$. This is piecewise linear in s, and it interpolation $\mathcal{L}(\mathfrak{r},\mathfrak{s})z(\mathfrak{s})$ at $\mathfrak{s} = \mathfrak{r}_0,\ldots,\mathfrak{r}_n$. Define a numerical approximation to the integral operator in (1) by

$$\Upsilon_m z(\mathfrak{r}) = \tfrac{\mu}{\Gamma(\beta)}\int_a^{\mathfrak{r}} \tfrac{[\mathcal{L}(\mathfrak{r},s)z(s)]_m}{(\mathfrak{r}-s)^{1-\beta}}\,\mathrm{d}s, \qquad \mathfrak{r}, s \in J_a. \tag{5}$$

Using (4), Eq. (5) can be written as

$$\Upsilon_m z(\mathfrak{r}) = \sum_{j=0}^{m} w_j(\mathfrak{r})\mathcal{L}(\mathfrak{r}, \mathfrak{r}_j)z(\mathfrak{r}_j), \qquad \mathfrak{r} \in J_a,\ z \in C(J_a), \tag{6}$$

with weights

$$\begin{aligned}
w_0(\mathfrak{r}) &= \tfrac{1}{h}\tfrac{\mu}{\Gamma(\beta)}\int_{\mathfrak{r}_0}^{\mathfrak{r}_1}(\mathfrak{r}_1 - s)(\mathfrak{r} - s)^{\beta-1}\,\mathrm{d}s,\\
w_j(\mathfrak{r}) &= \tfrac{1}{h}\tfrac{\mu}{\Gamma(\beta)}\int_{\mathfrak{r}_{j-1}}^{\mathfrak{r}_j}(s - \mathfrak{r}_{j-1})(\mathfrak{r} - s)^{\beta-1}\,\mathrm{d}s\\
&\quad + \tfrac{1}{h}\tfrac{\mu}{\Gamma(\beta)}\int_{\mathfrak{r}_j}^{\mathfrak{r}_{j+1}}(\mathfrak{r}_{j+1} - s)(\mathfrak{r} - s)^{\beta-1}\,\mathrm{d}s, \qquad j = 1, \dots, m-1,\\
w_m(\mathfrak{r}) &= \tfrac{1}{h}\tfrac{\mu}{\Gamma(\beta)}\int_{\mathfrak{r}_{m-1}}^{\mathfrak{r}_m}(s - \mathfrak{r}_{m-1})(\mathfrak{r} - s)^{\beta-1}\,\mathrm{d}s.
\end{aligned}$$

To approximate the integral Eq. (1), we use

$$z(\mathfrak{r}) = p(\mathfrak{r}) + \sum_{j=0}^{m} w_j(\mathfrak{r})\mathcal{L}(\mathfrak{r}, \mathfrak{r}_j)z(\mathfrak{r}_j).$$

3. Main results

3.1. The existence and uniqueness theorem

We illustrate the existence and uniqueness theorem for solution of the $\mathbb{FIE}$ (1). We need to the conditions:

a°) Let $p \in C(J_a)$ and $\mathcal{L} \in C(J_a^2)$ such that $\mathcal{L}(\mathfrak{r}, \cdot)^2 \in L^1(J_a)$ for any $\mathfrak{r} \in J_a$;

b°) There exists $\Omega = \frac{\mu M_{\mathcal{L}} b^{\beta}}{\Gamma(\beta+1)} < 1$, $M_{\mathcal{L}} \geq 0$ such that $|\mathcal{L}(\mathfrak{r}, \mathfrak{s})| \leq M_{\mathcal{L}} = \sup_{\mathfrak{r}\in J_a} \|\mathcal{L}(\mathfrak{r}, \mathfrak{s})\|_{L^1}$, $\forall\, \mathfrak{r}, s \in J_a$.

Now, applying the method of Banach contraction principle, we show that the existence and uniqueness of the solution of Eq. (1)

Theorem 3.1. *Eq.* (1), *under conditions* (a°) *and* (b°) *has an unique solution* $z^* \in C(J_a)$. *Moreover, for every* $z_0 \in C(J_a)$, *the sequence of successive approximations*

$$z_l(\mathfrak{r}) := p(\mathfrak{r}) + \frac{\mu}{\Gamma(\beta)}\int_a^{\mathfrak{r}} \frac{\mathcal{L}(\mathfrak{r},s)z_{l-1}(s)}{(\mathfrak{r}-s)^{1-\beta}}\,\mathrm{d}s, \qquad \mathfrak{r} \in J_a,\ l \in \mathbb{N}. \tag{7}$$

with initial value $z_0 := p$ *converges respect to the standard norm* $\|\cdot\|$ *to* z^*. *In addition, the following error estimations hold*

$$\|z^* - z_l\| \leq \frac{\Omega^l}{1-\Omega}\|z_0 - z_1\|, \tag{8}$$

and

$$\|z^* - z_l\| \leq \frac{\Omega}{1-\Omega}\|z_{l-1} - z_l\|, \tag{9}$$

where Ω *is defined by* (b°). *Moreover, choosing* $z_0 = p \in C(J_a)$, *Ineq.* (8) *becomes*

$$\|z^* - z_l\| \leq \frac{\Omega^{l+1}}{1-\Omega}M_0, \tag{10}$$

where $M_0 = \max\left\{|z_0(s)| \,:\, s \in J_a,\ z_0 = p \in \mathbb{R}\right\}$.

Proof. We define the operator $\Lambda : C(J_a) \to C(J_a)$ as follows

$$\Lambda(z)(\mathrm{r}) := p(\mathrm{r}) + \frac{\mu}{\Gamma(\beta)} \int_a^{\mathrm{r}} \frac{\mathcal{L}(\mathrm{r},s)z(s)}{(\mathrm{r}-s)^{1-\beta}}\,\mathrm{d}s, \qquad \mathrm{r} \in J_a. \tag{11}$$

For any $z \in C(J_a)$, we have

$$\begin{aligned}
\omega^{J_a}(\Lambda(z),h) \leq \sup &\left\{|p(\mathrm{r}_2) - p(\mathrm{r}_1)| \,:\, \mathrm{r}_1,\mathrm{r}_2 \in J_a,\ d(\mathrm{r}_2,\mathrm{r}_1) \leq h\right\}\\
&+ \sup \left\{\left|\frac{\mu}{\Gamma(\beta)} \int_a^{\mathrm{r}_2} \frac{\mathcal{L}(\mathrm{r}_2,s)z(s)}{(\mathrm{r}_2-s)^{1-\beta}}\,\mathrm{d}s \right.\right.\\
&\left.\left. - \frac{\mu}{\Gamma(\beta)} \int_a^{\mathrm{r}_1} \frac{\mathcal{L}(\mathrm{r}_1,s)z(s)}{(\mathrm{r}_1-s)^{1-\beta}}\,\mathrm{d}s\right| \,:\, \mathrm{r}_1,\mathrm{r}_2 \in J_a,\ d(\mathrm{r}_2,\mathrm{r}_1) \leq h\right\}\\
&\leq \omega^{J_a}(p,h) + \sup \left\{\left|\frac{\mu}{\Gamma(\beta)} \int_a^{\mathrm{r}_2} \frac{[\mathcal{L}(\mathrm{r}_2,s)-\mathcal{L}(\mathrm{r}_1,s)]z(s)}{(\mathrm{r}_2-s)^{1-\beta}}\,\mathrm{d}s\right| \,:\, \mathrm{r}_1,\mathrm{r}_2 \in J_a,\ d(\mathrm{r}_2,\mathrm{r}_1) \leq h\right\}\\
&+ \sup \left\{\frac{\mu}{\Gamma(\beta)} \int_a^{\mathrm{r}_1} \left|\frac{\mathcal{L}(\mathrm{r}_1,s)}{(\mathrm{r}_2-s)^{1-\beta}} - \frac{\mathcal{L}(\mathrm{r}_1,s)}{(\mathrm{r}_1-s)^{1-\beta}}\right| |z(s)|\,\mathrm{d}s \,:\, \mathrm{r}_1,\mathrm{r}_2 \in J_a,\ d(\mathrm{r}_2,\mathrm{r}_1) \leq h\right\}\\
&+ \sup \left\{\frac{\mu}{\Gamma(\beta)} \int_{\mathrm{r}_1}^{\mathrm{r}_2} |\mathcal{L}(\mathrm{r}_1,s)|\,|z(s)|\,\frac{1}{(\mathrm{r}_2-s)^{1-\beta}}\,\mathrm{d}s \,:\, \mathrm{r}_1,\mathrm{r}_2 \in J_a,\ d(\mathrm{r}_2,\mathrm{r}_1) \leq h\right\}\\
&\leq \omega^{J_a}(p,h) + \frac{\mu\|z\|\overline{\omega}^{J_a}(s)(\mathcal{L},h)(\mathrm{r}_2-a)^{\beta}}{\Gamma(1+\beta)} + \frac{(\mathrm{r}_2-\mathrm{r}_1)^{\beta}\mu M_{\mathcal{L}}\|z\|}{\Gamma(1+\beta)}\\
&\leq \omega^{J_a}(p,h) + \frac{\mu\|z\|h^{\beta}\overline{\omega}^{J_a}(s)(\mathcal{L},h)}{\Gamma(1+\beta)} + \frac{h^{\beta}\mu M_{\mathcal{L}}\|z\|}{\Gamma(1+\beta)}.
\end{aligned}$$

Using Theorem 2.1 and Lebesgue's monotone convergence theorem, we have $\omega^{J_a}(\Lambda(z),h) \to 0$ as $h \to 0$, so Λ maps $C(J_a)$ into itself. Now, we prove that Λ is contraction. In this aim, let $z, y \in C(J_a)$. We obtain

$$\begin{aligned}
|\Lambda(z)(\mathrm{r}) - \Lambda(y)(\mathrm{r})| &= \left|\frac{\mu}{\Gamma(\beta)} \int_a^{\mathrm{r}} \frac{\mathcal{L}(\mathrm{r},s)}{(\mathrm{r}-s)^{1-\beta}}[z(s) - y(s)]\,\mathrm{d}s\right|\\
&\leq \frac{\mu}{\Gamma(\beta)} \int_a^{\mathrm{r}} \frac{|\mathcal{L}(\mathrm{r},s)|}{(\mathrm{r}-s)^{1-\beta}}|z(s) - y(s)|\,\mathrm{d}s\\
&\leq \frac{\mu}{\Gamma(1+\beta)} b^{\beta} \sup_{\mathrm{r}\in J_a} \|\mathcal{L}(\mathrm{r},\cdot)\|_{L_1}\|z - y\|\\
&\leq \frac{\mu}{\Gamma(1+\beta)} b^{\beta} M_{\mathcal{L}}\|z - y\|, \quad \forall\, \mathrm{r} \in J_a,
\end{aligned} \tag{12}$$

thus,

$$\|\Lambda(z) - \Lambda(y)\| \leq \frac{\mu}{\Gamma(1+\beta)} b^{\beta} M_{\mathcal{L}}\|z - y\|.$$

With the condition (b°), we infer that Λ is contraction. Thus, this operator has in $C(J_a)$ an unique fixed point z^*. According to the fixed point principle of Picard-Banach, the estimates (8) and (9) are obtained. □

Now, we present the numerical method to approximate the solution of (7). To this end, let D, defined by

$$D: \quad 0 = \mathrm{r}_0 < \mathrm{r}_1 < \mathrm{r}_2 < \cdots < \mathrm{r}_m = b, \tag{13}$$

be the uniform grid of $J_0 = [0,b]$ with mesh $h = \frac{b}{m}$, and $\mathrm{r}_i = ih$, $i = \overline{0, m}$. On these knots, the iterative procedure (7) becomes:

$$\begin{aligned}
&z_0(\mathrm{r}_i) := p(\mathrm{r}_i), \quad i = \overline{0, m},\\
&z_l(\mathrm{r}_i) := p(\mathrm{r}_i) + \frac{\mu}{\Gamma(\beta)} \int_0^{\mathrm{r}_i} \frac{\mathcal{L}(\mathrm{r}_i,s)z_{l-1}(s)}{(\mathrm{r}_i-s)^{1-\beta}}\,\mathrm{d}s, \quad i = \overline{0, m},\ s \in J_0,
\end{aligned} \tag{14}$$

Using the quadrature rule (6) to the relations (14), we derive the numerical method:

$$\begin{aligned} z_0(\mathfrak{r}_i) &:= p(\mathfrak{r}_i),\\ z_l(\mathfrak{r}_i) &:= p(\mathfrak{r}_i) + \tfrac{\mu}{\Gamma(\beta)} \int_0^{\mathfrak{r}_i} \tfrac{\mathcal{L}(\mathfrak{r}_i,s) z_{l-1}(s)}{(\mathfrak{r}_i - s)^{1-\beta}}\,\mathrm{d}s\\ &= p(\mathfrak{r}_i) + \tfrac{\mu}{\Gamma(\beta)} \sum_{j=1}^{i} \Big[\mathcal{L}(\mathfrak{r}_i,\mathfrak{r}_{j-1}) h \varphi_{2,i}(j-1) z_{l-1}(\mathfrak{r}_{j-1})\\ &\quad + \mathcal{L}(\mathfrak{r}_i,\mathfrak{r}_j) h \varphi_{1,i}(j-1) z_{l-1}(\mathfrak{r}_j)\Big] + E_{l,i}, \qquad i = \overline{0, m},\ \forall\, l \in \mathbb{N}, \end{aligned} \tag{15}$$

where

$$\begin{aligned} \varphi_{1,i}(j) &= \int_0^1 u(\mathfrak{r}_i - (j+u)h)^{\beta-1}\,\mathrm{d}u\\ &= h^{\beta-1} \int_0^1 u(i-j-u)^{\beta-1}\,\mathrm{d}u,\\ \varphi_{2,i}(j) &= \int_0^1 (1-u)(\mathfrak{r}_i - (j+u)h)^{\beta-1}\,\mathrm{d}u\\ &= h^{\beta-1} \int_0^1 (1-u)(i-j-u)^{\beta-1}\,\mathrm{d}u, \qquad i = \overline{0, m},\ j = \overline{1, i}. \end{aligned}$$

The computation of these coefficients is easier with the change of variables $s - \mathfrak{r}_{j-1} = uh$, $0 \le u \le 1$ in the following integrals respectively,

$$\alpha_j(\mathfrak{r}_i) = \int_{\mathfrak{r}_{j-1}}^{\mathfrak{r}_j} \tfrac{(\mathfrak{r}_j - s)}{h} (\mathfrak{r}_i - s)^{\beta-1}\,\mathrm{d}s, \qquad \beta_j(\mathfrak{r}_i) = \int_{\mathfrak{r}_{j-1}}^{\mathfrak{r}_j} \tfrac{(s - \mathfrak{r}_{j-1})}{h} (\mathfrak{r}_i - s)^{\beta-1}\,\mathrm{d}s.$$

Also, note that for any fixed i and j,

$$\varphi_{1,i}(j) + \varphi_{2,i}(j) = \tfrac{h^{\beta-1}}{\beta}\Big[(i-j)^\beta - (i-j-1)^\beta\Big]. \tag{16}$$

Since in (15), we obtain approximated the terms of the sequence of Picard iterations using Lagrange interpolation polynomial,

$$z_l(s) \simeq \tfrac{1}{h}\Big[(\mathfrak{r}_j - s) z_l(\mathfrak{r}_{j-1}) + (s - \mathfrak{r}_{j-1}) z_l(\mathfrak{r}_j)\Big] = Y_l(s), \qquad s \in [\mathfrak{r}_{j-1}, \mathfrak{r}_j],\ j = \overline{1, m},$$

for all $l \in \mathbb{N}$, we infer that

$$\begin{aligned} |Y_l(s) - z_l(s)| &\le \Big|\tfrac{1}{h}\Big[(\mathfrak{r}_j - s) z_l(\mathfrak{r}_{j-1}) + (s - \mathfrak{r}_{j-1}) z_l(\mathfrak{r}_j)\Big] - z_l(s)\Big|\\ &\le \Big|\tfrac{(\mathfrak{r}_j - s)}{h} z_l(\mathfrak{r}_{j-1}) + \tfrac{(s - \mathfrak{r}_{j-1})}{h} z_l(\mathfrak{r}_j) - z_l(s)\Big|\\ &\le \Big|\tfrac{(\mathfrak{r}_j - s)}{h} z_l(\mathfrak{r}_{j-1}) + \tfrac{(s - \mathfrak{r}_{j-1})}{h} z_l(\mathfrak{r}_j) - \big(\tfrac{\mathfrak{r}_j - s}{h} + \tfrac{s - \mathfrak{r}_{j-1}}{h}\big) z_l(s)\Big|\\ &\le \tfrac{(s - \mathfrak{r}_{j-1})}{h} \omega^{[\mathfrak{r}_{j-1},\mathfrak{r}_j]}(z_r, h) + \tfrac{(\mathfrak{r}_j - s)}{h} \omega^{[\mathfrak{r}_{j-1},\mathfrak{r}_j]}(z_l, h)\\ &\le \omega^{J_a}(z_l, h), \qquad s \in [\mathfrak{r}_{j-1}, \mathfrak{r}_j],\ j = \overline{1, m}. \end{aligned}$$

Consequently, we have

$$\begin{aligned} |\bar{z}_l(\mathfrak{r}_i)) - z_l(\mathfrak{r}_i))| &\le \Big|\tfrac{\mu}{\Gamma(\beta)} \int_0^{\mathfrak{r}_i} \tfrac{L(\mathfrak{r}_i,s) Y_{l-1}(s))}{(\mathfrak{r}_i - s)^{1-\beta}}\,\mathrm{d}s - \tfrac{\mu}{\Gamma(\beta)} \int_0^{\mathfrak{r}_i} \tfrac{\mathcal{L}(\mathfrak{r}_i,s) z_{l-1}(s)}{(\mathfrak{r}_i - s)^{1-\beta}}\,\mathrm{d}s\Big|\\ &\le \|E_{l,i}\| \le \tfrac{\mu M_{\mathcal{L}} b^\beta}{\Gamma(\beta+1)} \omega^{J_0}(z_{l-1}, h), \end{aligned} \tag{17}$$

for $i = \overline{0, m}$ and each $l \in \mathbb{N}$. By (14), (15) and (17), we derive the following approximations

$$\begin{aligned} z_l(\mathrm{r}_i) &= p(\mathrm{r}_i) + \tfrac{\mu}{\Gamma(\beta)} \int_0^{\mathrm{r}_i} \tfrac{\mathcal{L}(\mathrm{r}_i,s) z_{l-1}(s)}{(\mathrm{r}_i - s)^{1-\beta}} ds \\ &= p(\mathrm{r}_i) + \tfrac{\mu}{\Gamma(\beta)} \sum_{j=1}^{i} \mathcal{L}(\mathrm{r}_i, \mathrm{r}_j) w_{i,j} z_{l-1}(\mathrm{r}_j) + E_{l,i} \\ &= p(\mathrm{r}_i) + \tfrac{\mu}{\Gamma(\beta)} \sum_{j=1}^{i} L(\mathrm{r}_i, \mathrm{r}_j) w_{i,j} \left(\bar{z}_{l-1}(\mathrm{r}_j) + \bar{E}_{l-1,j} \right) + E_{l,i} = \bar{z}_l(\mathrm{r}_i) + \bar{E}_{l,i}, \end{aligned} \tag{18}$$

where

$$\begin{aligned} w_{i,0} &= \int_{\mathrm{r}_0}^{\mathrm{r}_1} \tfrac{(\mathrm{r}_1 - s)}{h} (\mathrm{r}_i - s)^{\beta-1} \, \mathrm{d}s, \\ w_{i,j} &= \int_{\mathrm{r}_{j-1}}^{\mathrm{r}_j} \tfrac{(s - \mathrm{r}_{j-1})}{h} (\mathrm{r}_i - s)^{\beta-1} \, \mathrm{d}s + \int_{\mathrm{r}_j}^{\mathrm{r}_{j+1}} \tfrac{(\mathrm{r}_{j+1} - s)}{h} (\mathrm{r}_i - s)^{\beta-1} \, \mathrm{d}s, \quad j = 1, \ldots, m-1, \\ w_{i,m} &= \int_{\mathrm{r}_{m-1}}^{\mathrm{r}_m} \tfrac{(s - \mathrm{r}_{m-1})}{h} (\mathrm{r}_i - s)^{\beta-1} \, \mathrm{d}s. \end{aligned} \tag{19}$$

Generally, for $i = \overline{0, m}$, the computed values are

$$\bar{z}_l(\mathrm{r}_i) = p(\mathrm{r}_i) + \tfrac{\mu}{\Gamma(\beta)} \sum_{j=1}^{i} \mathcal{L}(\mathrm{r}_i, \mathrm{r}_j) w_{i,j} \bar{z}_{l-1}(\mathrm{r}_j), \tag{20}$$

and $z_l(\mathrm{r}_i) = \bar{z}_l(\mathrm{r}_i) + \bar{E}_{l,i}$, for each $l \in \mathbb{N}$, $i = \overline{0, m}$.

3.2. *Convergence analysis*

Now, we present the convergence of the suggested approach for solving FIIE (1).

Theorem 3.2. *We consider* FIIE (1) *under the same features as Theorem 3.1. If* $\Omega < 1$*, which was expressed in* (b°)*, then the numerical iterative approach* (20) *converges to the unique solution of* FIIE (1) z^**, and its error estimate is as follows*

$$\|z^* - z_l\|_u \le \tfrac{\Omega^{l+1}}{1-\Omega} M_0 + \tfrac{\Omega}{1-\Omega} \left[\omega^{J_0}(p, h) + \tfrac{\Omega \mu M_{\mathcal{L}} h^\beta \eta}{\Gamma(\beta+1)} \overline{\omega}^{J_0}(s)(\mathcal{L}, h) + \tfrac{\Omega \mu M_{\mathcal{L}} h^\beta \eta}{\Gamma(\beta+1)} \right], \tag{21}$$

where

$$\Gamma_l = \sup_{\mathrm{r} \in J_a} |z_l(\mathrm{r})|, \qquad \eta = \max_{i=\overline{1, l-1}} \left\{ \Gamma_i \right\}. \tag{22}$$

Proof. Using (10), we have

$$\|z^* - \bar{z}_l\| \le \|z^* - z_l\| + \|z_l - \bar{z}_l\| \le \tfrac{\Omega^{l+1}}{1-\Omega} M_0 + \|z_l - \bar{z}_l\|, \tag{23}$$

and hence, it remains to obtain the estimates for $\|z_l - \bar{z}_l\|$. By (18) and (20), for $l = 1$, we can find

$$z_1(\mathrm{r}_i) = p(\mathrm{r}_i) + \tfrac{\mu}{\Gamma(\beta)} \sum_{j=1}^{i} \mathcal{L}(\mathrm{r}_i, \mathrm{r}_j) w_{i,j} z_0(\mathrm{r}_j) + E_{1,i} = \bar{z}_1(\mathrm{r}_i) + \bar{E}_{1,i}, \tag{24}$$

and thanks to (17), we have

$$|z_1(\mathrm{r}) - \bar{z}_1(\mathrm{r})| \le \left\| \bar{E}_{1,i} \right\| \le \tfrac{\mu M_{\mathcal{L}} b^\beta}{\Gamma(\beta+1)} \omega^{J_0}(z_0, h),$$

and

$$\begin{aligned} z_2(\mathfrak{r}_i) &= p(\mathfrak{r}_i) + \tfrac{\mu}{\Gamma(\beta)} \int_a^{\mathfrak{r}_i} \tfrac{\mathcal{L}(\mathfrak{r}_i,s) z_1(s)}{(\mathfrak{r}_i - s)^{1-\beta}}\, \mathrm{d}s \\ &= p(\mathfrak{r}_i) + \tfrac{\mu}{\Gamma(\beta)} \sum_{j=1}^{i} \mathcal{L}(\mathfrak{r}_i, \mathfrak{r}_j) w_{i,j} z_1(\mathfrak{r}_j) + E_{2,i} \\ &= p(\mathfrak{r}_i) + \tfrac{\mu}{\Gamma(\beta)} \sum_{j=1}^{i} \mathcal{L}(\mathfrak{r}_i, \mathfrak{r}_j) w_{i,j} \Big(\bar{z}_1(\mathfrak{r}_j) + \bar{E}_{1,j}\Big) + E_{2,i} = \bar{z}_2(\mathfrak{r}_i) + \bar{E}_{2,i}, \end{aligned} \tag{25}$$

where

$$\bar{z}_2(\mathfrak{r}_j) = p(\mathfrak{r}_i) + \tfrac{\mu}{\Gamma(\beta)} \sum_{j=1}^{i} \mathcal{L}(\mathfrak{r}_i, \mathfrak{r}_j) w_{i,j} \bar{z}_1(\mathfrak{r}_j).$$

This leads us to

$$\begin{aligned} |z_2(\mathfrak{r}) - \bar{z}_2(\mathfrak{r})| &\le \|\bar{E}_{2,i}\| \le \|E_{2,i}\| + \tfrac{\mu}{\Gamma(\beta)} \Big(\sum_{j=1}^{i} \mathcal{L}(\mathfrak{r}_i, \mathfrak{r}_j) w_{i,j} \|\bar{E}_{1,i}\| \Big) \\ &\le \Omega \omega^{J_0}(z_1, h) + \tfrac{\mu M_{\mathcal{L}}}{\Gamma(\beta)} \Omega \omega^{J_0}(z_0, h) \sum_{j=1}^{i} w_{i,j}. \end{aligned}$$

Using (16) and (19), we have

$$\sum_{j=1}^{i} w_{i,j} \le h \sum_{j=1}^{i} (\varphi_{1,i}(j) + \varphi_{2,i}(j)) = \tfrac{h^\beta}{\beta} \sum_{j=1}^{i} \Big[(i-j)^\beta - (i-j-1)^\beta \Big] = \tfrac{(ih)^\beta}{\beta} \le \tfrac{b^\beta}{\beta}.$$

So,

$$|z_2(\mathfrak{r}) - \bar{z}_2(\mathfrak{r})| \le \Omega \left[\omega^{J_0}(z_1, h) + \Omega \omega^{J_0}(p, h) \right]. \tag{26}$$

By induction for $l \ge 3$, using (20), (25) and the estimate (26), we have

$$|z_l(\mathfrak{r}) - \bar{z}_l(\mathfrak{r})| \le \Omega \left[\omega^{J_0}(z_{l-1}, h) + \Omega \omega^{J_0}(z_{l-2}, h) + \cdots + \Omega^{l-1} \omega^{J_0}(p, h) \right]. \tag{27}$$

On the other hand, we see that

$$|z_l(\mathfrak{r}_2) - z_l(\mathfrak{r}_1)| \le |p(\mathfrak{r}_2) - p(\mathfrak{r}_1)| + \left| \int_0^{\mathfrak{r}_2} \tfrac{\mu \mathcal{L}(\mathfrak{r}_2, s) z_{l-1}(s)}{\Gamma(\beta)(\mathfrak{r}_2 - s)^{1-\beta}}\, \mathrm{d}s - \int_0^{\mathfrak{r}_1} \tfrac{\mu \mathcal{L}(\mathfrak{r}_1, s) z_{l-1}(s)}{\Gamma(\beta)(\mathfrak{r}_1 - s)^{1-\beta}}\, \mathrm{d}s \right|.$$

Consequently,

$$\omega^{J_0}(z_l, h) \le \omega^{J_0}(p, h) + \tfrac{\overline{\omega}^{J_0}(s)(\mathcal{L}, h) \mu \Gamma_{l-1} h^\beta}{\Gamma(1+\beta)} + \tfrac{\mu M_{\mathcal{L}} \Gamma_{l-1} h^\beta}{\Gamma(1+\beta)}. \tag{28}$$

Applying (22), (27) and (28), we get

$$\|z_l - \bar{z}_l\| \le \omega^{J_0}(p, h) \Big(\sum_{i=0}^{m-1} \Omega^i \Big) + \tfrac{\mu M_L h^\beta}{\Gamma(\beta+1)} \overline{\omega}^{J_0}(s)(\mathcal{L}, h) \Big(\sum_{i=1}^{m-1} \Omega^i \Gamma_{l-i} \Big) + \tfrac{\mu M_L h^\beta}{\Gamma(\beta+1)} \Big(\sum_{i=1}^{m-1} \Omega^i \Gamma_{l-i} \Big). \tag{29}$$

Using (29), since $\Omega < 1$, we derive

$$\|z_l - \bar{z}_l\| \le \tfrac{\Omega}{1-\Omega} \Big[\omega^{J_0}(p, h) + \tfrac{\Omega \mu M_{\mathcal{L}} h^\beta \eta}{\Gamma(\beta+1)} \overline{\omega}^{J_0}(s)(\mathcal{L}, h) + \tfrac{\Omega \mu M_{\mathcal{L}} h^\beta \eta}{\Gamma(\beta+1)} \Big]. \tag{30}$$

From inequalities (30) and (23), we have

$$\begin{aligned}\|z^* - \bar{z}_l\| &\le \|z^* - z_l\| + \|z_l - \bar{z}_l\| \\ &\le \tfrac{\Omega^{l+1}}{1-\Omega}M_0 + \tfrac{\Omega}{1-\Omega}\Big[\omega^{J_0}(p,h) + \tfrac{\Omega\mu M_{\mathcal{L}} h^\beta \eta}{\Gamma(\beta+1)}\overline{\omega}^{J_0}(s)(\mathcal{L},h) + \tfrac{\Omega\mu M_{\mathcal{L}} h^\beta \eta}{\Gamma(\beta+1)}\Big].\end{aligned}$$

□

Lemma 3.3. *Let $p \in C(J_0)$ and $\mathcal{L} \in C(J_0 \times J_0)$. Then* I) $\lim_{h\to 0} \omega^{J_0}(p,h) = 0$*;* II) $\lim_{h\to 0} \sup_{s\in J_0} \overline{\omega}^{J_0}(s)(\mathcal{L},h) \to 0$*.*

Proof. The function $\mathcal{L}(\mathrm{r},\cdot)$ and p are in $C(J_0)$, for each $\mathrm{r} \in J_0$, so from Theorem 2.1, we have $\lim_{h\to 0} \omega^{J_0}(p,h) = 0$ and $\lim_{h\to 0} \sup_{s\in J_0} \overline{\omega}^{J_0}(s)(\mathcal{L},h) \to 0$. □

Remark 3.4. *Applying Lemma 3.3, the error estimation* (21) *and* $\Omega < 1$*, it is evident that*

$$\lim_{\substack{h\to 0\\ l\to\infty}} \|z^* - z_l\| = 0,$$

it is concluded that z_l convergence to z^*.

3.3. The stability of the numerical results

In order to demonstrate the stability of the numerical results obtained against very small disturbances in the values and initial condition (7), we consider the same previous integral equation with initial condition assumed by another new function, such as $y_0 \in C(J_0, R)$, provided that the relation $\left|y_0(\mathrm{r}) - z_0(\mathrm{r})\right| < \varepsilon$, for each $s \in J_0$ holds. In the case of $y_0(\mathrm{r}) = q(\mathrm{r})$, the new sequence of successive approximations is as follows

$$y_l(\mathrm{r}) = q(\mathrm{r}) + \tfrac{\mu}{\Gamma(\beta)} \int_0^{\mathrm{r}} \tfrac{L(\mathrm{r},s)y_{l-1}(s)}{(\mathrm{r}-s)^{1-\beta}}\,\mathrm{d}s, \qquad \mathrm{r}, s \in J_0,\ l \ge 1. \tag{31}$$

By the same (20) for solving (31), we obtain

$$\begin{aligned}&\bar{y}_0(\mathrm{r}_i) := q(\mathrm{r}_i), \quad i = \overline{0,\, m}, \\ &\bar{y}_l(\mathrm{r}_i) = q(\mathrm{r}_i) + \tfrac{\mu}{\Gamma(\beta)} \sum_{j=1}^{i} \mathcal{L}(\mathrm{r}_i, \mathrm{r}_j) w_{i,j}\, \bar{y}_{l-1}\left(\mathrm{r}_j\right).\end{aligned} \tag{32}$$

Definition 3.5. *We say the iterative algorithm used for* $\mathbb{FIE}$ (1) *is numerically stable with respect to the selection of the initial iterate if and only if there exist constant values $\sigma_1, \sigma_2, \sigma_3 > 0$, independent of h and continuous functions $\psi_p, \psi_q : (0, a] \to [0, \infty)$, with* $\lim_{h\to 0} \psi_p(h) = 0$ *and* $\lim_{h\to 0} \psi_q(h) = 0$ *such that*

$$|\bar{z}_l - \bar{y}_l| < \sigma_1\varepsilon + \sigma_2\psi_p(h) + \sigma_3\psi_q(h), \qquad l \in \mathbb{N} \cup \{0\}.$$

Theorem 3.6. *In the light of the assumptions of Theorem 3.2, the method* (20) *is numerically stable with regard to the choice of the first iteration.*

Proof. Let

$$\Gamma'_l = \sup_{t\in J_a} |y_l(\mathrm{r})|, \qquad \eta' = \max_{i=\overline{1,\, l-1}} \left\{\Gamma'_i\right\}. \tag{33}$$

Similarly as the proof of Theorem 3.2 and (30), we can deduce that

$$|y_l(\mathrm{r}) - \bar{y}_l(\mathrm{r})| \le \tfrac{\Omega}{1-\Omega}\left[\omega^{J_0}(q,h) + \tfrac{\Omega\mu M_{\mathcal{L}} h^\beta \eta'}{\Gamma(\beta+1)}\overline{\omega}^{J_0}(s)(\mathcal{L},h) + \tfrac{\Omega\mu M_{\mathcal{L}} h^\beta \eta'}{\Gamma(\beta+1)}\right]. \tag{34}$$

We have $|z_0(\mathrm{r}) - y_0(\mathrm{r})| < \varepsilon$, for each $\mathrm{r} \in J_0$, and

$$\begin{aligned} \left|z_1(\mathrm{r}) - y_1(\mathrm{r})\right| &\leq |p(\mathrm{r}) - q(\mathrm{r})| + \tfrac{\mu}{\Gamma(\beta)} \int_0^{\mathrm{r}} \tfrac{\mathcal{L}(\mathrm{r},s)}{(\mathrm{r}-s)^{1-\beta}} |z_0(s) - y_0(s)| \, \mathrm{d}s \\ &\leq \varepsilon + \tfrac{M_{\mathcal{L}}\mu\varepsilon}{\Gamma(\beta)} \int_0^{\mathrm{r}} \tfrac{1}{(\mathrm{r}-s)^{1-\beta}} \, \mathrm{d}s < \varepsilon + \Omega\varepsilon. \end{aligned}$$

Also

$$\left|z_2(\mathrm{r}) - y_2(\mathrm{r})\right| \leq \varepsilon + \tfrac{\mu}{\Gamma(\beta)} \int_0^{\mathrm{r}} \tfrac{\mathcal{L}(\mathrm{r},s)}{(\mathrm{r}-s)^{1-\beta}} |z_1(s) - y_1(s)| \, \mathrm{d}s < \varepsilon + \Omega\varepsilon + \Omega^2\varepsilon.$$

By induction, for $l \geq 3$, we obtain

$$|z_l(\mathrm{r}) - y_l(\mathrm{r})| \leq \Big(\sum_{j=0}^{l} \Omega^j \Big) \varepsilon.$$

Since $\Omega < 1$, it is clear to see that,

$$|z_l(\mathrm{r}) - y_l(\mathrm{r})| \leq \tfrac{1}{1-\Omega} \varepsilon. \tag{35}$$

On the other hand, we have

$$\|\bar{z}_l(\mathrm{r}) - \overline{y}_l(\mathrm{r})\| \leq \|\bar{z}_l(\mathrm{r}) - z_l(\mathrm{r})\| + \|z_l(\mathrm{r}) - y_l(\mathrm{r})\| + \|y_l(\mathrm{r}) - \overline{y}_l(\mathrm{r})\|. \tag{36}$$

Now, by the inequalities (30), (34), (35) and (36), we obtain

$$\begin{aligned} |\bar{z}_l(\mathrm{r}) - \overline{y}_l(\mathrm{r})| \leq{}& \tfrac{1}{1-\Omega} \varepsilon + \tfrac{\Omega}{1-\Omega} \Big[\omega^{J_0}(p,h) + \tfrac{\Omega\mu M_{\mathcal{L}} h^\beta \eta}{\Gamma(\beta+1)} \overline{\omega}^{J_0}(s)(\mathcal{L},h) + \tfrac{\Omega\mu M_{\mathcal{L}} h^\beta \eta}{\Gamma(\beta+1)} \Big] \\ &+ \tfrac{\Omega}{1-\Omega} \Big[\omega^{J_0}(q,h) + \tfrac{\Omega\mu M_{\mathcal{L}} h^\beta \eta'}{\Gamma(\beta+1)} \overline{\omega}^{J_0}(s)(\mathcal{L},h) + \tfrac{\Omega\mu M_{\mathcal{L}} h^\beta \eta'}{\Gamma(\beta+1)} \Big], \end{aligned}$$

where $\sigma_1 = \frac{1}{1-\Omega}$, $\sigma_2 = \sigma_2 = \frac{\Omega}{1-\Omega}$ and

$$\begin{aligned} \psi_p(h) &= \tfrac{\Omega}{1-\Omega} \Big[\omega^{J_0}(p,h) + \tfrac{\Omega\mu M_{\mathcal{L}} h^\beta \eta}{\Gamma(\beta+1)} \overline{\omega}^{J_0}(s)(\mathcal{L},h) + \tfrac{\Omega\mu M_{\mathcal{L}} h^\beta \eta}{\Gamma(\beta+1)} \Big], \\ \psi_q(h) &= \tfrac{\Omega}{1-\Omega} \Big[\omega^{J_0}(q,h) + \tfrac{\Omega\mu M_{\mathcal{L}} h^\beta \eta'}{\Gamma(\beta+1)} \overline{\omega}^{J_0}(s)(\mathcal{L},h) + \tfrac{\Omega\mu M_{\mathcal{L}} h^\beta \eta'}{\Gamma(\beta+1)} \Big]. \end{aligned}$$

□

Remark 3.7. *Since $\Omega < 1$, it is evident that $\lim_{h,\varepsilon\to 0} \|\overline{x}_l - \overline{y}_l\| = 0$, this shows the stability of the iterative approach.*

Remark 3.8. *By Ineqs. (9) and (30), we get a practical stopping criterion of the iterative method. For given $\varepsilon' > 0$ (previously chosen), we obtain the first $l \in \mathbb{N}$ with, $|\bar{z}_l(\mathrm{r}_i) - \bar{z}_{l-1}(\mathrm{r}_i)| < \varepsilon'$ and stop at this iterative step. We see*

$$|z^*(\mathrm{r}_i) - \bar{z}_l(\mathrm{r}_i)| \leq |z^*(\mathrm{r}_i) - z_l(\mathrm{r}_i)| + |z_l(\mathrm{r}_j) - \bar{z}_l(\mathrm{r}_j)| \leq \tfrac{\Omega}{1-\Omega} |z_l(\mathrm{r}_i) - z_{l-1}(\mathrm{r}_i)| + \tfrac{\Omega}{1-\Omega} \psi_p(h),$$

and

$$\begin{aligned} |z_l(\mathrm{r}_i) - z_{l-1}(\mathrm{r}_i)| &\leq |z_l(\mathrm{r}_i) - \bar{z}_l(\mathrm{r}_i)| + |\bar{z}_l(\mathrm{r}_i) - \bar{z}_{l-1}(\mathrm{r}_i)| + |\bar{z}_{l-1}(\mathrm{r}_i) - z_{l-1}(\mathrm{r}_i)| \\ &\leq \tfrac{2\Omega}{1-\Omega} \psi_p(h) + |\bar{z}_l(\mathrm{r}_i) - \bar{z}_{l-1}(\mathrm{r}_i)|. \end{aligned}$$

So,

$$|z^*(\mathrm{r}_i) - \bar{z}_l(\mathrm{r}_i)| \leq \tfrac{\Omega(\Omega+1)}{(1-\Omega)^2} \psi_p(h) + \tfrac{\Omega}{1-\Omega} |\bar{z}_l(\mathrm{r}_i) - \bar{z}_{l-1}(\mathrm{r}_i)|,$$

and therefore, for obtain $|z_l^*(\mathfrak{r}_i) - \bar{z}_l(\mathfrak{r}_i)| < \varepsilon$, *we need*

$$\frac{\Omega(\Omega+1)}{(1-\Omega)^2}\psi_p(h) < \frac{\varepsilon}{2}, \qquad \frac{\Omega}{1-\Omega}|\bar{z}_l(\mathfrak{r}_i) - \bar{z}_{l-1}(\mathfrak{r}_i)| < \frac{\varepsilon}{2}.$$

Finally, we have

$$|\bar{z}_l(\mathfrak{r}_i) - \bar{z}_{l-1}(\mathfrak{r}_i)| < \frac{\varepsilon}{2}\cdot\frac{1-\Omega}{\Omega} := \varepsilon'.$$

With these, the inequality $|\bar{z}_l(\mathfrak{r}_i) - \bar{z}_{l-1}(\mathfrak{r}_i)| < \varepsilon'$ *leads to* $|z_l^*(\mathfrak{r}_i) - \bar{z}_l(\mathfrak{r}_i)| < \varepsilon$, *and the desired accuracy* ε *is obtained.*

4. Numerical experiments

Here, we assess the validity of the present method by considering some examples of $\mathbb{FIE}$ (1). The values of absolute errors are calculated as follows $e_m = |z^*(\mathfrak{r}) - \bar{z}(\mathfrak{r})|$, $\mathfrak{r} \in J_0 := [0,1]$. Moreover, the maximum absolute errors

$$\|e_m\|_\infty := \max\left\{|e_m(\mathfrak{r}_j)| \,:\, j = 0,1,2,\dots,m\right\},$$

are computed at points $\mathfrak{r}_j$, $j = 0,1,2,\dots,m$. Also, we introduce the following notations (see [4]),

$$\textbf{\textit{Ratio}} = \frac{\left\|z^* - \bar{z}_l^{(m)}\right\|_\infty}{\left\|z^* - \bar{z}_l^{(2m)}\right\|_\infty} = \frac{\|e_m\|_\infty^{(j)}}{\|e_{2m}\|_\infty^{(j+1)}}, \qquad \textbf{\textit{Order}} = \frac{1}{\log 2}\log\left[\frac{\|e_m\|_\infty^{(j)}}{\|e_{2m}\|_\infty^{(j+1)}}\right],$$

where "**Ratio**" and "**Order**" are rate of convergence and estimate of the convergence order of the presented methods for $\mathbb{FIE}$ (1), respectively.

Example 4.1. *Consider the following* $\mathbb{FIE}$,

$$z(\mathfrak{r}) = 2\sqrt[3]{\mathfrak{r}^2} - \frac{\mathfrak{r}}{2} - \frac{5\sqrt{\pi}}{16}\mathfrak{r}^{7/2} + \frac{4}{15\sqrt{\pi}}\mathfrak{r}^3 + \frac{1}{2\Gamma(1/2)}\int_0^{\mathfrak{r}} \frac{s\sqrt{\mathfrak{r}}z(s)}{(\mathfrak{r}-s)^{1/2}}\,ds, \qquad \mathfrak{r} \in J_0. \tag{37}$$

The exact solution of $\mathbb{FIE}$ (37) *is*

$$z^*(\mathfrak{r}) = 2\sqrt[3]{\mathfrak{r}^2} - \frac{\mathfrak{r}}{2}. \tag{38}$$

In this example, we have $b = 1, \mu = 1, M_{\mathcal{L}} = \frac{1}{2}$ *and* $\beta = \frac{1}{2}$. *Therefore, we have*

$$\Omega = \frac{\mu M_{\mathcal{L}} b^\beta}{\Gamma(\beta+1)} = \frac{1}{\sqrt{\pi}} < 1.$$

By the product trapezoidal scheme for $m = 10, 20, 40$, *we get* $l = 11, 11, 12$, *respectively, whenever* $\varepsilon' = 10^{-15}$. *The results* $\|e_m\|_\infty$ *for* $\varepsilon' = 10^{-15}$ *and* $m = 10, 20, 40$, *respectively, are*

$$7.844115\times10^{-5}, \quad 1.988245\times10^{-5}, \quad 4.987971\times10^{-6}.$$

For more details, please see Tables 1 and 2. Also, the rate and order of convergence applying the algorithm with $m \in \{10, 20, 40, 80\}$, *are given in Table 2.*

Example 4.2 ([22]). *Consider the following* $\mathbb{FIE}$,

$$z(\mathfrak{r}) = \mathfrak{r} - \frac{1}{3} - 0.2615\mathfrak{r}^{9/4} + 0.0981\mathfrak{r}^{5/4} + \frac{1}{3\Gamma(1/4)}\int_0^{\mathfrak{r}} \frac{sz(s)}{(\mathfrak{r}-s)^{3/4}}\,ds, \qquad \mathfrak{r} \in J_0. \tag{39}$$

The exact solution of $\mathbb{FIE}$ (39) *is given by* $z(\mathfrak{r}) = \mathfrak{r} - \frac{1}{3}$. *Tables 3 and 4 display the exact and error results using the product integration method for different value of m and* $\varepsilon' = 10^{-15}$.

Table 1: Numerical results for $m = 10, 20, 40$ in Example 4.1.

r_i	$z^*(r_i)$	e_m		
		$m = 10$	$m = 20$	$m = 40$
0.1	0.380886938	$2.75043301 \times 10^{-5}$	$6.97386177 \times 10^{-6}$	$1.74973101 \times 10^{-6}$
0.3	0.746280949	$3.47364303 \times 10^{-5}$	$8.80691103 \times 10^{-6}$	$2.20955884 \times 10^{-6}$
0.5	1.009921049	$4.32919629 \times 10^{-5}$	$1.09752495 \times 10^{-5}$	$2.75352234 \times 10^{-6}$
0.7	1.226747032	$5.33211542 \times 10^{-5}$	$1.35169165 \times 10^{-5}$	$3.39113155 \times 10^{-6}$
0.9	1.414339503	$6.49821779 \times 10^{-5}$	$1.64719626 \times 10^{-5}$	$4.13243143 \times 10^{-6}$

Table 2: Summery of numerical results of Example 4.1.

m	l	$\|e_m\|_\infty$	***Ratio***	***Order***	CPU time(s)
10	11	7.844115×10^{-5}	–	–	0.013
20	11	1.988245×10^{-5}	3.9452	1.9801	0.825
40	12	4.987971×10^{-6}	3.9861	1.9949	1.971
80	12	1.248273×10^{-6}	3.9959	1.9985	4.015

Table 3: Numerical results for $m = 10, 20, 40$, $\varepsilon' = 10^{-15}$ in Example 4.2.

r_i	$z^*(r_i)$	e_m		
		$m = 10$	$m = 20$	$m = 40$
0.1	−0.2333333	$1.38240357 \times 10^{-5}$	$3.48761189 \times 10^{-6}$	$8.73941333 \times 10^{-7}$
0.3	−0.0333333	$1.77223993 \times 10^{-5}$	$4.47087286 \times 10^{-6}$	$1.12031649 \times 10^{-6}$
0.5	0.16666666	$2.23815626 \times 10^{-5}$	$5.64596963 \times 10^{-6}$	$1.41475640 \times 10^{-6}$
0.7	0.36666666	$2.78931452 \times 10^{-5}$	$7.03599592 \times 10^{-6}$	$1.76304678 \times 10^{-6}$
0.9	0.56666666	$3.43538828 \times 10^{-5}$	$8.66533444 \times 10^{-6}$	$2.17129614 \times 10^{-6}$

Table 4: Summery of numerical results of Example 4.2.

m	l	$\|e_m\|_\infty$	***Ratio***	***Order***	CPU time(s)
10	8	4.186562×10^{-5}	–	–	0.001
20	9	1.055965×10^{-5}	3.9647	1.9872	0.702
40	9	2.645931×10^{-6}	3.9909	1.9967	1.605
80	9	6.620202×10^{-7}	3.9967	1.9988	3.806

5. Conclusions

$\mathbb{FIE}$s are important for solving a large proportion of the problems in many topics in mathematical physics, particularly in chemical reactions. In this article, we used the successive approximations method and product integration method for approximating the solution of $\mathbb{FIE}$ (1). The sufficient conditions (a°) and (b°), for existence and uniqueness solution of $\mathbb{FIE}$ (1) are provided in Theorem 3.1. The convergence of the suggested approach in terms of modulus of continuity for $\mathbb{FIE}$ (1) was proved by providing the error estimate in Theorem 3.2. To demonstrate the convergence of the approach, only modulus of continuity properties are required, smoothness conditions being not necessary. Therefore, the suggested procedure does not need the solution of any system of equations.

Abbreviations

$\mathbb{IE}$: Integral equation
$\mathbb{FIE}$: Fractional integral equation
$\mathbb{VIE}$: Volterra integral equation

$\mathbb{VFIE}$: Volterra fractional integral equation
RL: Riemann-Liouville

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