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Certain novel inequalities for monotonic and convex functions involving modified unified generalized fractional integral operators with extended unified Mittag-Leffler functions and related results

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Abstract. In this study, based on the modified unified generalized fractional integral operators (MUGFIOs) with extended unified Mittag–Leffler functions (EUMLFs), we investigate certain new MUGFIO inequalities for some positive continuous monotonic functions and convex functions. Furthermore, some novel Chebyshev type inequalities and reversed Minkowski inequalities for MUGFIOs with EUMLFs are established. The presented results constitute a significant extension of classical inequalities in the literature, unifying and generalizing multiple existing frameworks through the proposed MUGFIOs with EUMLFs.

1. Introduction

In 2009, Liu et al. [16] presented the following integral inequality: assume that $\mathfrak{f}$ and $\mathfrak{g}$ are two positive monotonic decreasing and increasing functions on $[a, b]$, respectively, then

$$\frac{\int_a^b \mathfrak{f}^{\kappa}(x)dx}{\int_a^b \mathfrak{f}^{\vartheta}(x)dx} \geqslant \frac{\int_a^b \mathfrak{g}^{\varpi}(x)\mathfrak{f}^{\kappa}(x)dx}{\int_a^b \mathfrak{g}^{\varpi}(x)\mathfrak{f}^{\vartheta}(x)dx} \tag{1}$$

holds for positive constants $\varpi > 0$ and $\kappa \geqslant \vartheta > 0$. If $\mathfrak{f}$ is increasing, then the reverse of inequality (1) holds. In the same paper [16], the integral inequality involving convex function was also presented: assume that $\mathfrak{f}, \mathfrak{g}$ and $\mathfrak{h}$ are three positive continuous functions with $\mathfrak{f}(x) \leqslant \mathfrak{g}(x)$ for $x \in [a, b]$ such that $\mathfrak{f}/\mathfrak{g}$ and $\mathfrak{f}, \mathfrak{h}$ are decreasing and increasing functions on $[a, b]$, respectively. By setting different parameters, the inequality (1) can reduce to some interesting results. Furthermore, let ψ be a convex function with $\psi(0) = 0$. Then the following inequality holds

$$\frac{\int_a^b \mathfrak{f}(x)dx}{\int_a^b \mathfrak{g}(x)dx} \geqslant \frac{\int_a^b \mathfrak{h}(x)\psi(\mathfrak{f}(x))dx}{\int_a^b \mathfrak{h}(x)\psi(\mathfrak{g}(x))dx}. \tag{2}$$

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Fractional integral inequalities (FIIs) have attracted intensive attentions and great interests from many scholars both domestically and internationally since they play a significant role in discussions of the quantitative and qualitative behavior of solutions to fractional differential equations. For the past few decades, there are A large number of FIIs involving all kinds of fractional integral operators (FIOs), we can refer the reader to see [4, 9, 10, 14, 29, 32, 36] and the reference quoted therein. For instance, using the Riemann–Liouville (RL) FIOs, Mohammed and Abdeljawad [17] studied the modified Hermite–Hadamard inequality and related FIIs. In [15], Khurshid et al. derived some new conformable fractional Hermite-Hadamard type inequalities containing GG- and GA-convex functions.

In 2014, based on the RLFIOs, Dahmani [8] obtained the following result: let $(\mathfrak{f}_k)_{k=1,2,\dots,n}$ be n be positive continuous and monotonic decreasing functions on $[a,b]$. Then the following FII

$$\frac{J^{\alpha}[\mathfrak{f}_p^{\kappa}(x)\prod_{k\neq p}^{n}\mathfrak{f}_k^{\vartheta_k}(x)]}{J^{\alpha}[\prod_{k=1}^{n}\mathfrak{f}_k^{\vartheta_k}(x)]}\geqslant\frac{J^{\alpha}[(x-a)^{\varpi}\mathfrak{f}_p^{\kappa}(x)\prod_{k\neq p}^{n}\mathfrak{f}_k^{\vartheta_k}(x)]}{J^{\alpha}[(x-a)^{\varpi}\prod_{k=1}^{n}\mathfrak{f}_k^{\vartheta_k}(x)]} \tag{3}$$

holds for any $x\in[a,b]$, $\alpha,\varpi>0$, $\kappa\geqslant\vartheta_p>0$ and a fixed integer number $p\in\{1,2,\dots,n\}$, where J^{α} denotes the RLFIO of order α. By means of generalized k-conformable and unified weighted FIOs, Qi et al. [22] and Rahman et al. [26] established the FIIs similar to the inequality (3), respectively. The inequality (3) can be the fractional order generalization of the inequality (1).

In 2020, by taking advantage of Hadamard proportional (HP) FIOs, Rahman et al. [25] obtained the following conclusion: for any $\varpi>0$ and $\kappa\geqslant\vartheta>0$, let $\mathfrak{f},\mathfrak{g}$ and $\mathfrak{h}$ be three positive continuous functions on $[1,\infty)$ satisfying

$$(\mathfrak{g}^{\varpi}(\mathfrak{s})\mathfrak{f}^{\varpi}(\mathfrak{t})-\mathfrak{g}^{\varpi}(\mathfrak{t})\mathfrak{f}^{\varpi}(\mathfrak{s}))(\mathfrak{f}^{\kappa-\vartheta}(\mathfrak{t})-\mathfrak{f}^{\kappa-\vartheta}(\mathfrak{s}))\geqslant 0,\quad \mathfrak{s},\mathfrak{t}\in(1,x),\quad x>1. \tag{4}$$

Then, for all $x>1$, the following HPFII holds

$$\frac{\mathcal{H}_{1,x}^{\alpha,v}[\mathfrak{h}(x)\mathfrak{f}^{\kappa+\varpi}(x)]}{\mathcal{H}_{1,x}^{\alpha,v}[\mathfrak{h}(x)\mathfrak{f}^{\vartheta+\varpi}(x)]}\geqslant\frac{\mathcal{H}_{1,x}^{\alpha,v}[\mathfrak{h}(x)\mathfrak{f}^{\kappa}(x)\mathfrak{g}^{\varpi}(x)]}{\mathcal{H}_{1,x}^{\alpha,v}[\mathfrak{h}(x)\mathfrak{f}^{\vartheta}(x)\mathfrak{g}^{\varpi}(x)]}, \tag{5}$$

where $\mathcal{H}_{1,x}^{\alpha,v}$ stands for HPFIO of order α. Based on the generalized proportional (GP) FIOs, Abdeljawad *et al.* [1] etablished the GPFII similar to the inequality (5). In the paper [18, 31], Nale et al. and Tassaddiq et al. presented the special case of the inequality (5) by utilizing the HP and multiple Erdélyi–Kober FIOs, respectively.

In 2021, by using the FIOs with extended generalized (EG) Bessel function, Hussain et al. [13] observed the interesting inequality: let $\mathfrak{f},\mathfrak{g}$ and $\mathfrak{h}$ be three positive continuous functions with $\mathfrak{f}(x)\leqslant\mathfrak{g}(x)$ for $x\in[a,b]$ such that $\mathfrak{f}/\mathfrak{g}$ and $\mathfrak{f},\mathfrak{h}$ are decreasing and increasing functions on $[a,b]$, respectively. Furthermore, suppose that ψ be any convex function with $\psi(0)=0$. Then the following FII holds

$$\frac{J_{\theta_1^+;\vartheta,q}^{\omega,\zeta,\delta,c}[\mathfrak{f}(x);p]}{J_{\theta_1^+;\vartheta,q}^{\omega,\zeta,\delta,c}[\mathfrak{g}(x);p]}\geqslant\frac{J_{\theta_1^+;\vartheta,q}^{\omega,\zeta,\delta,c}[\mathfrak{h}(x)\psi(\mathfrak{f}(x));p]}{J_{\theta_1^+;\vartheta,q}^{\omega,\zeta,\delta,c}[\mathfrak{h}(x)\psi(\mathfrak{g}(x));p]}, \tag{6}$$

where $J_{\theta_1^+;\vartheta,q}^{\omega,\zeta,\delta,c}$ denotes the FIO containing the EG Bessel function in the kernel. By employing the generalized Katugampola, GP, generalized-type, weighted-type and generalized weighted FIOs, Nale *et al.* [19], Neamah and Ibrahim [20] and Neamah et al. [21], Yıldız and Gürbüz [37] and Yıldız et al. [38] derived some FIIs involving convex functions similar to the inequality (6), respectively. The inequality (6) can be the fractional order generalization of the inequality (2).

To the best of my knowledge, there have been no relevant achievements by using modified unified generalized (MUG) FIOs with extended unified Mittag–Leffler functions (EUMLFs) to study the inequalities mentioned earlier. Motivated by the mentioned paper above, based on the MUGFIOs with EUMLFs, we will consider the corresponding inequalities involving some positive monotonic functions and convex functions. Then, we will present some novel Chebyshev type inequalities and reversed Minkowski inequalities for

MUGFIOs with EUMLFs. Finally, by exploiting the known important FIOs, some special cases of the obtained main results will be given.

The remaining part of this paper is organized as follows. In Section 2, we will introduce some related definitions of MUGFIOs with EUMLFs. In Section 3, some MUGFIIs involving some positive continuous monotonic functions and convex functions will be acquired. Then some new Chebyshev type inequalities and reversed Minkowski inequalities via the MUGFIOs with EUMLFs will gained in Section 4 and Section 5, respectively. Finally, we will sum up the conclusion in Section 6.

2. Preliminaries

In 2022, Abubakar et al. [2] introduced the definition of EUMLF based on the modified extended beta function as follows.

Definition 1 ([2]). Assume that $\underline{\mathfrak{a}} = (\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_n)$, $\underline{\mathfrak{b}} = (\mathfrak{b}_1, \mathfrak{b}_2, \cdots, \mathfrak{b}_n)$, $\underline{\mathfrak{c}} = (\mathfrak{c}_1, \mathfrak{c}_2, \cdots, \mathfrak{c}_n)$, where $\mathfrak{a}_i, \mathfrak{b}_i, \mathfrak{c}_i \in \mathbb{C}$, $i = 1, 2, \ldots, n$, such that $\forall i$, $\mathscr{R}(\mathfrak{a}_i), \mathscr{R}(\mathfrak{b}_i), \mathscr{R}(\mathfrak{c}_i) > 0$, where $\mathscr{R}(\cdot)$ denotes the real part of complex number. Moreover, suppose that $\alpha, \beta, \gamma, \delta, \mu, \nu, \rho, \theta, \mathfrak{s} \in \mathbb{C}$, $\min\{\mathscr{R}(\alpha), \mathscr{R}(\beta), \mathscr{R}(\gamma), \mathscr{R}(\delta), \mathscr{R}(\lambda), \mathscr{R}(\theta)\} > 0$, $k \in (0, 1) \cup \mathbb{N}$, and $k + \mathscr{R}(\rho) < \mathscr{R}(\delta + \nu + \alpha)$ with $\mathscr{I}(\rho) = \mathscr{I}(\delta + \nu + \alpha)$, where $\mathscr{I}(\cdot)$ denotes the imaginary part of complex number. Then we define the following EUMLF

$$ {}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\mathfrak{s}; \underline{\mathfrak{a}}, \underline{\mathfrak{b}}, \underline{\mathfrak{c}}, \varkappa) = \sum_{l=0}^{\infty} \frac{\prod_{i=1}^{n} B^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}(\mathfrak{b}_i, \mathfrak{a}_i, \varkappa)(\lambda)_{\rho l}(\theta)_{kl}\mathfrak{s}^l}{\prod_{i=1}^{n} B(\mathfrak{c}_i, \mathfrak{a}_i)(\gamma)_{\delta l}(\mu)_{\nu l}\Gamma(\alpha l + \beta)}, \tag{7} $$

where $(\lambda)_{\rho l}$, $\Gamma(\cdot)$, $B(\cdot, \cdot)$ and $B^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}(\cdot, \cdot, \cdot)$ stand for the well-known generalized Pochhammer symbol, gamma function, beta function and modified extended beta function, respectively. Here

$$ (\lambda)_{\rho l} = \frac{\Gamma(\lambda + \rho l)}{\Gamma(\lambda)}, \quad B^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}(\tau_1, \tau_2, \varkappa) = \int_0^1 \mathfrak{s}^{\tau_1 - 1}(1 - \mathfrak{s})^{\tau_2 - 1}\varkappa^{\left(-\frac{\sigma_1}{\mathfrak{s}^{\varrho_1}} - \frac{\sigma_2}{(1-\mathfrak{s})^{\varrho_2}}\right)}d\mathfrak{s}, \tag{8} $$

for $\min\{\mathscr{R}(\tau_1), \mathscr{R}(\tau_2), \mathscr{R}(\sigma_1), \mathscr{R}(\sigma_2), \mathscr{R}(\varrho_1), \mathscr{R}(\varrho_2)\} > 0$, $\varkappa \in (0, \infty)\backslash\{1\}$.

According to the EUMLF, the author [35] presented the definition of MUGFIOs as follows.

Definition 2 ([35]). Let $\underline{\mathfrak{a}} = (\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_n)$, $\underline{\mathfrak{b}} = (\mathfrak{b}_1, \mathfrak{b}_2, \cdots, \mathfrak{b}_n)$, $\underline{\mathfrak{c}} = (\mathfrak{c}_1, \mathfrak{c}_2, \cdots, \mathfrak{c}_n)$, where $\mathfrak{a}_i, \mathfrak{b}_i, \mathfrak{c}_i \in \mathbb{C}$, $i = 1, 2, \ldots, n$, such that $\forall i$, $\mathscr{R}(\mathfrak{a}_i), \mathscr{R}(\mathfrak{b}_i), \mathscr{R}(\mathfrak{c}_i) > 0$. Let $\psi, \xi \in C[\mathfrak{u}, \mathfrak{v}]$ such that $\psi \in L_1[\mathfrak{u}, \mathfrak{v}]$ is positive and ξ is a strictly monotonically increasing differentiable function. Also let ϕ be a positive continuous function such that ϕ/x is a monotonically increasing function on $[\mathfrak{u}, +\infty)$. Furthermore, let $\omega, \alpha, \beta, \gamma, \delta, \mu, \nu, \rho, \theta, t \in \mathbb{C}$, $\min\{\mathscr{R}(\alpha), \mathscr{R}(\beta), \mathscr{R}(\gamma), \mathscr{R}(\delta), \mathscr{R}(\lambda), \mathscr{R}(\theta)\} > 0$, $k \in (0, 1) \cup \mathbb{N}$, and $k + \mathscr{R}(\rho) < \mathscr{R}(\delta + \nu + \alpha)$ with $\mathscr{I}(\rho) = \mathscr{I}(\delta + \nu + \alpha)$. Then, for $x \in [\mathfrak{u}, \mathfrak{v}]$, we introduce the left and right-side MUGFIOs $({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\psi)(x; \varkappa)$ and $({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{v}^-}\psi)(x; \varkappa)$ with the EUMLF (7) as follows

$$ ({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\psi)(x) = ({}^{\phi,\varrho_1,\varrho_2}_{\xi,\sigma_1,\sigma_2}\Theta^{\omega,\lambda,\rho,\theta,k,n}_{\mathfrak{u}^+,\alpha,\beta,\gamma,\mu,\nu}\psi)(x; \varkappa) = \aleph^{-1}(x)\int_{\mathfrak{u}}^{x} \aleph(\mathfrak{s})\mathfrak{M}^{\mathfrak{s}}_{x}({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}, \xi, \phi)\psi(\mathfrak{s})d(\xi(\mathfrak{s})), \tag{9} $$

$$ ({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{v}^-}\psi)(x) = ({}^{\phi,\varrho_1,\varrho_2}_{\xi,\sigma_1,\sigma_2}\Theta^{\omega,\lambda,\rho,\theta,k,n}_{\mathfrak{v}^-,\alpha,\beta,\gamma,\mu,\nu}\psi)(x; \varkappa) = \aleph^{-1}(x)\int_{x}^{\mathfrak{v}} \aleph(\mathfrak{s})\mathfrak{M}^{x}_{\mathfrak{s}}({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}, \xi, \phi)\psi(\mathfrak{s})d(\xi(\mathfrak{s})), \tag{10} $$

where $\aleph(\mathfrak{s})$ expresses a weighted function satisfying $\aleph(\mathfrak{s}) > 0$ for all $\mathfrak{s} \in [\mathfrak{u}, \mathfrak{v}]$ and $\mathfrak{M}^{\mathfrak{s}}_{x}(\cdot, \cdot, \cdot)$ stands for the kernel function defined by

$$ \mathfrak{M}^{\mathfrak{s}}_{x}({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}, \xi, \phi) = \frac{\phi(\xi(x) - \xi(\mathfrak{s}))}{\xi(x) - \xi(\mathfrak{s})}\, {}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\xi(x) - \xi(\mathfrak{s}))^{\alpha}; \underline{\mathfrak{a}}, \underline{\mathfrak{b}}, \underline{\mathfrak{c}}, \varkappa), \tag{11} $$

which satisfies the following inequality

$$ \mathfrak{M}^{\mathfrak{s}}_{x}({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}, \xi, \phi) \leqslant \mathfrak{M}^{\mathfrak{u}}_{x}({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}, \xi, \phi). \tag{12} $$

Remark 1. The EUMLF (7) can be seen as a generalization of the gamma function, beta function, hypergeometric MLF, generalized Q function, extended MLF and unified MLF. Therefore, FIOs with generalized Q function [40], FIOs with EGMLF [39], FIOs with generalized unified MLF [3], more general FIOs with EGMLF [27], unified FIOs with EGMLF [11] and unified FIOs with generalized unified MLF [12] can be considered as special cases of the MUGFIOs (9) and (10). Furthermore, it follows from [33, Remarks 9 and 10] and [34, Remarks 2.2 and 2.3] that the foregoing MUGFIOs (9) and (10) can acquire a tremendous amount of noteworthy existent FIOs in accordance with different settings of the adjustable parameters and functions.

Nextly, we will give the upper bounds of the MUGFIOs (9) and (10) in variable form.

Theorem 1. *For $x \in [\mathfrak{u}, \mathfrak{v}]$, then we have*

$$({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\psi)(x) \leqslant \aleph^{-1}(x)\phi(\xi(x)-\xi(\mathfrak{u}))\,{}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\xi(x)-\xi(\mathfrak{u}))^{\alpha};\underline{\mathfrak{a}},\underline{\mathfrak{b}},\underline{\mathfrak{c}},\varkappa)\|\psi\|^{\aleph}_{[\mathfrak{u},x]}, \tag{13}$$

$$({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{v}^-}\psi)(x) \leqslant \aleph^{-1}(x)\phi(\xi(\mathfrak{v})-\xi(x))\,{}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\xi(\mathfrak{v})-\xi(x))^{\alpha};\underline{\mathfrak{a}},\underline{\mathfrak{b}},\underline{\mathfrak{c}},\varkappa)\|\psi\|^{\aleph}_{[x,\mathfrak{v}]}, \tag{14}$$

where $\|\psi\|^{\aleph}_{[\mathfrak{u},x]} = \sup_{\mathfrak{s}\in[\mathfrak{u},x]}|\aleph(\mathfrak{s})\psi(\mathfrak{s})|$ and $\|\psi\|^{\aleph}_{[x,\mathfrak{v}]} = \sup_{\mathfrak{s}\in[x,\mathfrak{v}]}|\aleph(\mathfrak{s})\psi(\mathfrak{s})|$.

Proof. Since ξ is a strictly monotonically increasing differentiable function and ϕ/x is a monotonically increasing function on $[\mathfrak{u},+\infty)$, then, from (11), we have

$$\frac{\phi(\xi(x)-\xi(\mathfrak{s}))}{\xi(x)-\xi(\mathfrak{s})}\,{}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\xi(x)-\xi(\mathfrak{s}))^{\alpha};\underline{\mathfrak{a}},\underline{\mathfrak{b}},\underline{\mathfrak{c}},\varkappa) \leqslant \frac{\phi(\xi(x)-\xi(\mathfrak{u}))}{\xi(x)-\xi(\mathfrak{u})}\,{}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\xi(x)-\xi(\mathfrak{s}))^{\alpha};\underline{\mathfrak{a}},\underline{\mathfrak{b}},\underline{\mathfrak{c}},\varkappa), \quad \mathfrak{s}\in[\mathfrak{u},x], \tag{15}$$

$$\frac{\phi(\xi(\mathfrak{s})-\xi(x))}{\xi(t)-\xi(x)}\,{}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\xi(\mathfrak{s})-\xi(x))^{\alpha};\underline{\mathfrak{a}},\underline{\mathfrak{b}},\underline{\mathfrak{c}},\varkappa) \leqslant \frac{\phi(\xi(\mathfrak{v})-\xi(x))}{\xi(\mathfrak{v})-\xi(x)}\,{}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\xi(\mathfrak{s})-\xi(x))^{\alpha};\underline{\mathfrak{a}},\underline{\mathfrak{b}},\underline{\mathfrak{c}},\varkappa), \quad \mathfrak{s}\in[x,\mathfrak{v}]. \tag{16}$$

Multiplying both sides of (15) and (16) by $\aleph^{-1}(x)\aleph(\mathfrak{s})\psi(\mathfrak{s})\xi'(\mathfrak{s})$ and integrating with respect to $\mathfrak{s}$ both sides of the obtained results from $\mathfrak{u}$ to x and x to $\mathfrak{v}$ simultaneously, respectively, then we can observe

$$\begin{aligned}
({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\psi)(x) \leqslant & \aleph^{-1}(x)\frac{\phi(\xi(x)-\xi(\mathfrak{u}))}{\xi(x)-\xi(\mathfrak{u})}\int_{\mathfrak{u}}^{x}{}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\xi(x)-\xi(\mathfrak{s}))^{\alpha};\underline{\mathfrak{a}},\underline{\mathfrak{b}},\underline{\mathfrak{c}},\varkappa)\aleph(\mathfrak{s})\psi(\mathfrak{s})\xi'(\mathfrak{s})d\mathfrak{s}\\
\leqslant & \aleph^{-1}(x)\frac{\phi(\xi(x)-\xi(\mathfrak{u}))}{\xi(x)-\xi(\mathfrak{u})}\|\psi\|^{\aleph}_{[\mathfrak{u},x]}\sum_{l=0}^{\infty}\frac{\prod_{i=1}^{n}B^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}(\mathfrak{b}_i,\mathfrak{a}_i,\varkappa)(\lambda)_{\rho l}(\theta)_{kl}}{\prod_{i=1}^{n}B(\mathfrak{c}_i,\mathfrak{a}_i)(\gamma)_{\delta l}(\mu)_{\nu l}\Gamma(\alpha l+\beta)}\int_{\mathfrak{u}}^{x}\left(\omega(\xi(x)-\xi(\mathfrak{s}))^{\alpha}\right)^{l}\xi'(\mathfrak{s})d\mathfrak{s}\\
\leqslant & \aleph^{-1}(x)\phi(\xi(x)-\xi(\mathfrak{u}))\,{}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\xi(x)-\xi(\mathfrak{u}))^{\alpha};\underline{\mathfrak{a}},\underline{\mathfrak{b}},\underline{\mathfrak{c}},\varkappa)\|\psi\|^{\aleph}_{[\mathfrak{u},x]},
\end{aligned} \tag{17}$$

$$\begin{aligned}
({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{v}^-}\psi)(x) \leqslant & \aleph^{-1}(x)\frac{\phi(\xi(\mathfrak{v})-\xi(x))}{\xi(\mathfrak{v})-\xi(x)}\int_{x}^{\mathfrak{v}}{}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\xi(\mathfrak{s})-\xi(x))^{\alpha};\underline{\mathfrak{a}},\underline{\mathfrak{b}},\underline{\mathfrak{c}},\varkappa)\aleph(\mathfrak{s})\psi(\mathfrak{s})\xi'(\mathfrak{s})d\mathfrak{s}\\
\leqslant & \aleph^{-1}(x)\frac{\phi(\xi(\mathfrak{v})-\xi(x))}{\xi(\mathfrak{v})-\xi(x)}\|\psi\|^{\aleph}_{[x,\mathfrak{v}]}\sum_{l=0}^{\infty}\frac{\prod_{i=1}^{n}B^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}(\mathfrak{b}_i,\mathfrak{a}_i,\varkappa)(\lambda)_{\rho l}(\theta)_{kl}}{\prod_{i=1}^{n}B(\mathfrak{c}_i,\mathfrak{a}_i)(\gamma)_{\delta l}(\mu)_{\nu l}\Gamma(\alpha l+\beta)}\int_{x}^{\mathfrak{v}}\left(\omega(\xi(\mathfrak{s})-\xi(x))^{\alpha}\right)^{l}\xi'(\mathfrak{s})d\mathfrak{s}\\
\leqslant & \aleph^{-1}(x)\phi(\xi(\mathfrak{v})-\xi(x))\,{}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\xi(\mathfrak{v})-\xi(x))^{\alpha};\underline{\mathfrak{a}},\underline{\mathfrak{b}},\underline{\mathfrak{c}},\varkappa)\psi\|^{\aleph}_{[x,\mathfrak{v}]},
\end{aligned} \tag{18}$$

which imply that (13) and (14) hold using the inequality $1/(\alpha l+1) \leqslant 1$. This completes the proof of Theorem 1. □

For the convenience of narration, we always suppose that all the MUGFIOs mentioned later in this paper exist. Meanwhile, we introduce the following notation $\mathfrak{i}(x) = x$, $\mathfrak{f}\mathfrak{g}(x) = \mathfrak{f}(x)\mathfrak{g}(x)$, $\mathfrak{f}/\mathfrak{g}(x) = \mathfrak{f}(x)/\mathfrak{g}(x)$ and $\mathfrak{f}(\mathfrak{g})(x) = \mathfrak{f}(\mathfrak{g}(x))$.

3. MUGFIIs for Monotonic and Convex Functions

In this section, by utilizing the MUGFIOs with EUMLFs, we will present some new FIIs for some positive continuous monotonic functions and convex functions.

Theorem 2. *Let* $\mathfrak{f}, \mathfrak{g}$ *and* $\mathfrak{h}$ *be three positive continuous functions on* $[\mathfrak{u}, \mathfrak{v}]$ *satisfying*

$$(\mathfrak{g}^{\varpi}(\mathfrak{t})\mathfrak{f}^{\varpi}(\mathfrak{s}) - \mathfrak{g}^{\varpi}(\mathfrak{s})\mathfrak{f}^{\varpi}(\mathfrak{t}))(\mathfrak{f}^{\kappa-\vartheta}(\mathfrak{s}) - \mathfrak{f}^{\kappa-\vartheta}(\mathfrak{t})) \geqslant 0, \quad \forall \mathfrak{s}, \mathfrak{t} \in [\mathfrak{u}, \mathfrak{v}], \tag{19}$$

where $\varpi > 0$ *and* $\kappa \geqslant \vartheta > 0$*. Then, for* $x \in [\mathfrak{u}, \mathfrak{v}]$*, the following FII holds*

$$\frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\varpi+\kappa})(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\varpi+\vartheta})(x)} \geqslant \frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\kappa}\mathfrak{g}^{\varpi})(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\vartheta}\mathfrak{g}^{\varpi})(x)}. \tag{20}$$

Proof. It follows from (19) that

$$\mathfrak{g}^{\varpi}(\mathfrak{t})\mathfrak{f}^{\varpi+\kappa-\vartheta}(\mathfrak{s}) + \mathfrak{g}^{\varpi}(\mathfrak{s})\mathfrak{f}^{\varpi+\kappa-\vartheta}(\mathfrak{t}) \geqslant \mathfrak{g}^{\varpi}(\mathfrak{t})\mathfrak{f}^{\varpi}(\mathfrak{s})\mathfrak{f}^{\kappa-\vartheta}(\mathfrak{t}) + \mathfrak{g}^{\varpi}(\mathfrak{s})\mathfrak{f}^{\varpi}(\mathfrak{t})\mathfrak{f}^{\kappa-\vartheta}(\mathfrak{s}). \tag{21}$$

Multiplying both sides of (21) by $\aleph^{-1}(x)\aleph(\mathfrak{s})\mathfrak{M}^{\mathfrak{s}}_{x}({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}, \xi, \phi)\mathfrak{h}(\mathfrak{s})\mathfrak{f}^{\vartheta}(\mathfrak{s})\xi'(\mathfrak{s})$ and integrating with respect to $\mathfrak{s}$ both sides of the obtained inequality from $\mathfrak{u}$ to x, then we have

$$\mathfrak{g}^{\varpi}(\mathfrak{t})({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\varpi+\kappa})(x) + \mathfrak{f}^{\varpi+\kappa-\vartheta}(\mathfrak{t})({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\vartheta}\mathfrak{g}^{\varpi})(x) \geqslant \mathfrak{g}^{\varpi}(\mathfrak{t})\mathfrak{f}^{\kappa-\vartheta}(\mathfrak{t})({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\varpi+\vartheta})(x) + \mathfrak{f}^{\varpi}(\mathfrak{t})({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\kappa}\mathfrak{g}^{\varpi})(x). \tag{22}$$

Multiplying both sides of (22) by $\aleph^{-1}(x)\aleph(\mathfrak{t})\mathfrak{M}^{\mathfrak{t}}_{x}({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}, \xi, \phi)\mathfrak{h}(\mathfrak{t})\mathfrak{f}^{\vartheta}(\mathfrak{t})\xi'(\mathfrak{t})$ and integrating with respect to $\mathfrak{t}$ both sides of the obtained inequality from $\mathfrak{u}$ to x, then we achieve

$$\begin{aligned}({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\vartheta}\mathfrak{g}^{\varpi})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\varpi+\kappa})(x) &+ ({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\varpi+\kappa})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\vartheta}\mathfrak{g}^{\varpi})(x)\\ &\geqslant ({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\kappa}\mathfrak{g}^{\varpi})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\varpi+\vartheta})(x) + ({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\varpi+\vartheta})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\kappa}\mathfrak{g}^{\varpi})(x),\end{aligned} \tag{23}$$

which implies the required inequality (20). This completes the proof of Theorem 2. □

Theorem 3. *Assume that all the conditions of Theorem 2 hold. Then, for* $x \in [\mathfrak{u}, \mathfrak{v}]$*, we have the following FII*

$$\frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\varpi+\kappa})(x)({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\vartheta}\mathfrak{g}^{\varpi})(x) + ({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\vartheta}\mathfrak{g}^{\varpi})(x)({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\varpi+\kappa})(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\varpi+\vartheta})(x)({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\kappa}\mathfrak{g}^{\varpi})(x) + ({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\kappa}\mathfrak{g}^{\varpi})(x)({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\varpi+\vartheta})(x)} \geqslant 1. \tag{24}$$

Proof. Multiplying both sides of (23) by $\aleph^{-1}(x)\aleph(\mathfrak{t})\mathfrak{M}^{\mathfrak{t}}_{x}({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\varsigma,\beta,\gamma,\delta,\mu,\nu}, \xi, \phi)\mathfrak{h}(\mathfrak{t})\mathfrak{f}^{\vartheta}(\mathfrak{t})\xi'(\mathfrak{t})$ and integrating with respect to $\mathfrak{t}$ both sides of the obtained result from $\mathfrak{u}$ to x, then we get

$$\begin{aligned}({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\vartheta}\mathfrak{g}^{\varpi})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\varpi+\kappa})(x) &+ ({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\varpi+\kappa})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\vartheta}\mathfrak{g}^{\varpi})(x)\\ &\geqslant ({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\kappa}\mathfrak{g}^{\varpi})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\varpi+\vartheta})(x) + ({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\varpi+\vartheta})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\kappa}\mathfrak{g}^{\varpi})(x),\end{aligned} \tag{25}$$

which indicates the desired assertion (24). This completes the proof of Theorem 3. □

Remark 2. Employing Theorem 3 for $\varsigma = \alpha$, we achieve Theorem 2.

Corollary 1. *Suppose that one of the following assumptions holds for* $\varpi > 0$ *and* $\kappa \geqslant \vartheta > 0$, (a) $\mathfrak{f}$ *and* $\mathfrak{g}$ *are two positive continuous monotonic increasing and decreasing functions on* $[\mathfrak{u}, \mathfrak{v}]$*, respectively;* (b) $\mathfrak{f}$ *and* $\mathfrak{g}$ *are two positive continuous monotonic decreasing and increasing functions on* $[\mathfrak{u}, \mathfrak{v}]$*, respectively;* (c) $\mathfrak{f}^{\kappa-\vartheta}$ *and* $\mathfrak{f}^{\varpi}/\mathfrak{g}^{\varpi}$ *are two continuous synchronous functions on* $[\mathfrak{u}, \mathfrak{v}]$*. Then the FIIs* (20) *and* (24) *still hold true.*

Proof. If one of the previous assumptions holds true, then we can derive the inequality (19). Furthermore, we obtain the FIIs (20) and (24) similar to the proofs of Theorem 2 and Theorem 3. This completes the proof of Corollary 1. □

Theorem 4. *Let* $\mathfrak{f},\mathfrak{g}$ *and* $\mathfrak{h}$ *be three positive continuous functions on* $[\mathfrak{u},\mathfrak{v}]$ *satisfying*

$$(\mathfrak{g}^{\varpi}(\mathfrak{t})-\mathfrak{g}^{\varpi}(\mathfrak{s}))(\mathfrak{f}^{\kappa-\vartheta}(\mathfrak{s})-\mathfrak{f}^{\kappa-\vartheta}(\mathfrak{t}))\geqslant 0,\quad \forall \mathfrak{s},\mathfrak{t}\in[\mathfrak{u},\mathfrak{v}],\tag{26}$$

where $\varpi>0$ *and* $\kappa\geqslant\vartheta>0$. *Then, for* $x\in[\mathfrak{u},\mathfrak{v}]$, *the following FII holds*

$$\frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\kappa})(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\vartheta})(x)}\geqslant\frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\kappa}\mathfrak{g}^{\varpi})(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\vartheta}\mathfrak{g}^{\varpi})(x)}.\tag{27}$$

Proof. It follows from (26) that

$$\mathfrak{g}^{\varpi}(\mathfrak{t})\mathfrak{f}^{\kappa-\vartheta}(\mathfrak{s})+\mathfrak{g}^{\varpi}(\mathfrak{s})\mathfrak{f}^{\kappa-\vartheta}(\mathfrak{t})\geqslant\mathfrak{g}^{\varpi}(\mathfrak{t})\mathfrak{f}^{\kappa-\vartheta}(\mathfrak{t})+\mathfrak{g}^{\varpi}(\mathfrak{s})\mathfrak{f}^{\kappa-\vartheta}(\mathfrak{s}).\tag{28}$$

Multiplying both sides of (28) by $\aleph^{-1}(x)\aleph(\mathfrak{s})\mathfrak{M}^{\mathfrak{s}}_{x}({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu},\xi,\phi)\mathfrak{h}(\mathfrak{s})\mathfrak{f}^{\vartheta}(\mathfrak{s})\xi'(\mathfrak{s})$ and integrating with respect to $\mathfrak{s}$ both sides of the obtained result from $\mathfrak{u}$ to x, then we gain

$$\mathfrak{g}^{\varpi}(\mathfrak{t})({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\kappa})(x)+\mathfrak{f}^{\kappa-\vartheta}(\mathfrak{t})({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\vartheta}\mathfrak{g}^{\varpi})(x)\geqslant\mathfrak{g}^{\varpi}(\mathfrak{t})\mathfrak{f}^{\kappa-\vartheta}(\mathfrak{t})({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\vartheta})(x)+({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\kappa}\mathfrak{g}^{\varpi})(x).\tag{29}$$

Multiplying both sides of (29) by $\aleph^{-1}(x)\aleph(\mathfrak{t})\mathfrak{M}^{\mathfrak{t}}_{x}({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu},\xi,\phi)\mathfrak{h}(\mathfrak{t})\mathfrak{f}^{\vartheta}(\mathfrak{t})\xi'(\mathfrak{t})$ and integrating with respect to $\mathfrak{t}$ both sides of the obtained inequality from $\mathfrak{u}$ to x, then we get

$$\begin{aligned}&({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\vartheta}\mathfrak{g}^{\varpi})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\kappa})(x)+({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\kappa})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\vartheta}\mathfrak{g}^{\varpi})(x)\\&\qquad\geqslant({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\kappa}\mathfrak{g}^{\varpi})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\vartheta})(x)+({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\vartheta})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\kappa}\mathfrak{g}^{\varpi})(x),\end{aligned}\tag{30}$$

which reveals the desired inequality (27). This completes the proof of Theorem 4. □

Theorem 5. *Assume that all the conditions of Theorem 4 hold. Then, for* $x\in[\mathfrak{u},\mathfrak{v}]$, *we obtain the following FII*

$$\frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\kappa})(x)({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\vartheta}\mathfrak{g}^{\varpi})(x)+({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\vartheta}\mathfrak{g}^{\varpi})(x)({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\kappa})(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\vartheta})(x)({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\kappa}\mathfrak{g}^{\varpi})(x)+({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\kappa}\mathfrak{g}^{\varpi})(x)({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\vartheta})(x)}\geqslant 1.\tag{31}$$

Proof. Multiplying both sides of (29) by $\aleph^{-1}(x)\aleph(\mathfrak{t})\mathfrak{M}^{\mathfrak{t}}_{x}({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\varsigma,\beta,\gamma,\delta,\mu,\nu},\xi,\phi)\mathfrak{h}(\mathfrak{t})\mathfrak{f}^{\vartheta}(\mathfrak{t})\xi'(\mathfrak{t})$ and integrating with respect to $\mathfrak{t}$ both sides of the obtained results from $\mathfrak{u}$ to x, then we claim

$$\begin{aligned}&({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\vartheta}\mathfrak{g}^{\varpi})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\kappa})(x)+({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\kappa})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\vartheta}\mathfrak{g}^{\varpi})(x)\\&\qquad\geqslant({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\kappa}\mathfrak{g}^{\varpi})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\vartheta})(x)+({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\vartheta})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\kappa}\mathfrak{g}^{\varpi})(x),\end{aligned}\tag{32}$$

which displays the required assertion (31). This completes the proof of Theorem 5. □

Remark 3. Employing Theorem 5 for $\varsigma=\alpha$, we have Theorem 4.

Corollary 2. *Suppose that one of the following assumptions holds for* $\varpi>0$ *and* $\kappa\geqslant\vartheta>0$, (a) $\mathfrak{f}$ *and* $\mathfrak{g}$ *are two positive continuous monotonic increasing and decreasing functions on* $[\mathfrak{u},\mathfrak{v}]$, *respectively;* (b) $\mathfrak{f}$ *and* $\mathfrak{g}$ *are two positive continuous monotonic decreasing and increasing functions on* $[\mathfrak{u},\mathfrak{v}]$, *respectively;* (c) $\mathfrak{f}^{\kappa-\vartheta}$ *and* $\mathfrak{g}^{\varpi}$ *are two continuous asynchronous functions on* $[\mathfrak{u},\mathfrak{v}]$. *Then the FIIs (27) and (31) still hold true.*

Proof. If one of the previous assumptions holds true, then we can derive the inequality (26). Furthermore, we obtain the FIIs (27) and (31) similar to the proofs of Theorem 4 and Theorem 5. This completes the proof of Corollary 2. □

Theorem 6. *Let* $(\mathfrak{f}_k)_{k=1,2,\ldots,n}$, $\mathfrak{g}$ *and* $\mathfrak{h}$ *be positive continuous functions on* $[\mathfrak{u},\mathfrak{v}]$ *satisfying*

$$(\mathfrak{g}^{\varpi}(\mathfrak{t})\mathfrak{f}_p^{\varpi}(\mathfrak{s})-\mathfrak{g}^{\varpi}(\mathfrak{s})\mathfrak{f}_p^{\varpi}(\mathfrak{t}))(\mathfrak{f}_p^{\kappa-\vartheta_p}(\mathfrak{s})-\mathfrak{f}_p^{\kappa-\vartheta_p}(\mathfrak{t}))\geqslant 0,\quad \forall \mathfrak{s},\mathfrak{t}\in[\mathfrak{u},\mathfrak{v}], \tag{33}$$

where $\varpi>0$ *and* $\kappa\geqslant\vartheta_p>0(p=1,2,\ldots,n)$. *Then, for any fixed integer number* $p\in\{1,2,\ldots,n\}$ *and* $x\in[\mathfrak{u},\mathfrak{v}]$, *the following FII holds*

$$\frac{({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}\mathfrak{f}_p^{\varpi+\kappa}\prod_{k\neq p}\mathfrak{f}_k^{\vartheta_k})(x)}{({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}\mathfrak{f}_p^{\varpi}\prod_{k=1}\mathfrak{f}_k^{\vartheta_k})(x)}\geqslant\frac{({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}\mathfrak{g}^{\varpi}\mathfrak{f}_p^{\kappa}\prod_{k\neq p}\mathfrak{f}_k^{\vartheta_k})(x)}{({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}\mathfrak{g}^{\varpi}\prod_{k=1}^{n}\mathfrak{f}_k^{\vartheta_k})(x)}. \tag{34}$$

Proof. It follows from (33) that for $p=1,2,\ldots,n$,

$$\mathfrak{g}^{\varpi}(\mathfrak{t})\mathfrak{f}_p^{\varpi+\kappa-\vartheta_p}(\mathfrak{s})+\mathfrak{g}^{\varpi}(\mathfrak{s})\mathfrak{f}_p^{\varpi+\kappa-\vartheta_p}(\mathfrak{t})\geqslant\mathfrak{g}^{\varpi}(\mathfrak{t})\mathfrak{f}_p^{\varpi}(\mathfrak{s})\mathfrak{f}_p^{\kappa-\vartheta_p}(\mathfrak{t})+\mathfrak{g}^{\varpi}(\mathfrak{s})\mathfrak{f}_p^{\varpi}(\mathfrak{t})\mathfrak{f}_p^{\kappa-\vartheta_p}(\mathfrak{s}). \tag{35}$$

Multiplying both sides of (35) by $\aleph^{-1}(x)\aleph(\mathfrak{s})\mathfrak{M}_x^{\mathfrak{s}}({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu},\xi,\phi)\mathfrak{h}(\mathfrak{s})\prod_{k=1}^{n}\mathfrak{f}_k^{\vartheta_k}(\mathfrak{s})\xi'(\mathfrak{s})$ and integrating with respect to $\mathfrak{s}$ both sides of the obtained result from $\mathfrak{u}$ to x, then we derive

$$\begin{aligned}&\mathfrak{g}^{\varpi}(\mathfrak{t})\Big({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}\mathfrak{f}_p^{\varpi+\kappa}\prod_{k\neq p}\mathfrak{f}_k^{\vartheta_k}\Big)(x)+\mathfrak{f}_p^{\varpi+\kappa-\vartheta_p}(\mathfrak{t})\Big({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}\mathfrak{g}^{\varpi}\prod_{k=1}^{n}\mathfrak{f}_k^{\vartheta_k}\Big)(x)\\&\qquad\geqslant\mathfrak{g}^{\varpi}(\mathfrak{t})\mathfrak{f}_p^{\kappa-\vartheta_p}(\mathfrak{t})\Big({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}\mathfrak{f}_p^{\varpi}\prod_{k=1}\mathfrak{f}_k^{\vartheta_k}\Big)(x)+\mathfrak{f}_p^{\varpi}(\mathfrak{t})\Big({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}\mathfrak{g}^{\varpi}\mathfrak{f}_p^{\kappa}\prod_{k\neq p}\mathfrak{f}_k^{\vartheta_k}\Big)(x).\end{aligned} \tag{36}$$

Multiplying both sides of (36) by $\aleph^{-1}(x)\aleph(\mathfrak{t})\mathfrak{M}_x^{\mathfrak{t}}({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu},\xi,\phi)\mathfrak{h}(\mathfrak{t})\prod_{k=1}^{n}\mathfrak{f}_k^{\vartheta_k}(\mathfrak{t})\xi'(\mathfrak{t})$ and integrating with respect to $\mathfrak{t}$ both sides of the obtained inequality from $\mathfrak{u}$ to x, then we achieve

$$\begin{aligned}&\Big({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}\mathfrak{g}^{\varpi}\prod_{k=1}^{n}\mathfrak{f}_k^{\vartheta_k}\Big)(x)\Big({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}\mathfrak{f}_p^{\varpi+\kappa}\prod_{k\neq p}\mathfrak{f}_k^{\vartheta_k}\Big)(x)+\Big({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}\mathfrak{f}_p^{\varpi+\kappa}\prod_{k\neq p}\mathfrak{f}_k^{\vartheta_k}\Big)(x)\Big({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}\mathfrak{g}^{\varpi}\prod_{k=1}^{n}\mathfrak{f}_k^{\vartheta_k}\Big)(x)\\&\quad\geqslant\Big({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}\mathfrak{g}^{\varpi}\mathfrak{f}_p^{\kappa}\prod_{k\neq p}\mathfrak{f}_k^{\vartheta_k}\Big)(x)\Big({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}\mathfrak{f}_p^{\varpi}\prod_{k=1}\mathfrak{f}_k^{\vartheta_k}\Big)(x)+\Big({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}\mathfrak{f}_p^{\varpi}\prod_{k=1}\mathfrak{f}_k^{\vartheta_k}\Big)(x)\Big({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}\mathfrak{g}^{\varpi}\mathfrak{f}_p^{\kappa}\prod_{k\neq p}\mathfrak{f}_k^{\vartheta_k}\Big)(x),\end{aligned} \tag{37}$$

which exhibits the required inequality (34). This completes the proof of Theorem 6. □

Theorem 7. *Assume that all the conditions of Theorem 6 hold. Then, for any fixed integer number* $p\in\{1,2,\ldots,n\}$ *and* $x\in[\mathfrak{u},\mathfrak{v}]$, *we have the following FII*

$$\frac{({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}\mathfrak{f}_p^{\varpi+\kappa}\prod_{k\neq p}\mathfrak{f}_k^{\vartheta_k})(x)({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\varsigma}\mathfrak{h}\mathfrak{g}^{\varpi}\prod_{k=1}^{n}\mathfrak{f}_k^{\vartheta_k})(x)+({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\varsigma}\mathfrak{h}\mathfrak{f}_p^{\varpi+\kappa}\prod_{k\neq p}\mathfrak{f}_k^{\vartheta_k})(x)({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}\mathfrak{g}^{\varpi}\prod_{k=1}^{n}\mathfrak{f}_k^{\vartheta_k})(x)}{({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}\mathfrak{f}_p^{\varpi}\prod_{k=1}\mathfrak{f}_k^{\vartheta_k})(x)({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\varsigma}\mathfrak{h}\mathfrak{g}^{\varpi}\mathfrak{f}_p^{\kappa}\prod_{k\neq p}\mathfrak{f}_k^{\vartheta_k})(x)+({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\varsigma}\mathfrak{h}\mathfrak{f}_p^{\varpi}\prod_{k=1}\mathfrak{f}_k^{\vartheta_k})(x)({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}\mathfrak{g}^{\varpi}\mathfrak{f}_p^{\kappa}\prod_{k\neq p}\mathfrak{f}_k^{\vartheta_k})(x)}\geqslant 1. \tag{38}$$

Proof. Multiplying both sides of (36) by $\aleph^{-1}(x)\aleph(\mathfrak{t})\mathfrak{M}_x^{\mathfrak{t}}({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\varsigma,\beta,\gamma,\delta,\mu,\nu},\xi,\phi)\mathfrak{h}(\mathfrak{t})\prod_{k=1}^{n}\mathfrak{f}_k^{\vartheta_k}(\mathfrak{t})\cdot\xi'(\mathfrak{t})$ and integrating with respect to $\mathfrak{t}$ both sides of the obtained inequality from $\mathfrak{u}$ to x, then we obtain

$$\begin{aligned}&\Big({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\varsigma}\mathfrak{h}\mathfrak{g}^{\varpi}\prod_{k=1}^{n}\mathfrak{f}_k^{\vartheta_k}\Big)(x)\Big({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}\mathfrak{f}_p^{\varpi+\kappa}\prod_{k\neq p}\mathfrak{f}_k^{\vartheta_k}\Big)(x)+\Big({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\varsigma}\mathfrak{h}\mathfrak{f}_p^{\varpi+\kappa}\prod_{k\neq p}\mathfrak{f}_k^{\vartheta_k}\Big)(x)\Big({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}\mathfrak{g}^{\varpi}\prod_{k=1}^{n}\mathfrak{f}_k^{\vartheta_k}\Big)(x)\\&\quad\geqslant\Big({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\varsigma}\mathfrak{h}\mathfrak{g}^{\varpi}\mathfrak{f}_p^{\kappa}\prod_{k\neq p}\mathfrak{f}_k^{\vartheta_k}\Big)(x)\Big({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}\mathfrak{f}_p^{\varpi}\prod_{k=1}\mathfrak{f}_k^{\vartheta_k}\Big)(x)+\Big({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\varsigma}\mathfrak{h}\mathfrak{f}_p^{\varpi}\prod_{k=1}\mathfrak{f}_k^{\vartheta_k}\Big)(x)\Big({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}\mathfrak{g}^{\varpi}\mathfrak{f}_p^{\kappa}\prod_{k\neq p}\mathfrak{f}_k^{\vartheta_k}\Big)(x),\end{aligned} \tag{39}$$

which bespeaks the desired assertion (38). This completes the proof of Theorem 7. □

Remark 4. Employing Theorem 7 for $\varsigma = \alpha$, we obtain Theorem 6.

Corollary 3. *Suppose that one of the following assumptions holds for $\varpi > 0$, $\kappa \geqslant \vartheta > 0$ and $p = 1, 2, \dots, n$,* (a) $\mathfrak{f}_p$ *and* $\mathfrak{g}$ *are two positive continuous monotonic increasing and decreasing functions on* $[\mathfrak{u}, \mathfrak{v}]$*, respectively;* (b) $\mathfrak{f}_p$ *and* $\mathfrak{g}$ *are two positive continuous monotonic decreasing and increasing functions on* $[\mathfrak{u}, \mathfrak{v}]$*, respectively;* (c) $\mathfrak{f}_p^{\kappa-\vartheta}$ *and* $\mathfrak{f}_p^{\varpi}/\mathfrak{g}^{\varpi}$ *are two continuous synchronous functions on* $[\mathfrak{u}, \mathfrak{v}]$*. Then the FIIs* (34) *and* (38) *still hold true.*

Proof. If one of the previous assumptions holds true, then we can derive the inequality (33). Furthermore, we obtain the FIIs (34) and (38) similar to the proofs of Theorem 6 and Theorem 7. This completes the proof of Corollary 3. □

Theorem 8. *Let* $(\mathfrak{f}_k)_{k=1,2,\dots,n}$, $\mathfrak{g}$ *and* $\mathfrak{h}$ *be positive continuous functions on* $[\mathfrak{u}, \mathfrak{v}]$ *satisfying*

$$(\mathfrak{g}^{\varpi}(\mathfrak{t}) - \mathfrak{g}^{\varpi}(\mathfrak{s}))(\mathfrak{f}_p^{\kappa-\vartheta_p}(\mathfrak{s}) - \mathfrak{f}_p^{\kappa-\vartheta_p}(\mathfrak{t})) \geqslant 0, \quad \forall \mathfrak{s}, \mathfrak{t} \in [\mathfrak{u}, \mathfrak{v}], \tag{40}$$

where $\varpi > 0$ and $\kappa \geqslant \vartheta_p > 0 (p = 1, 2, \dots, n)$. Then, for any fixed integer number $p \in \{1, 2, \dots, n\}$ and $x \in [\mathfrak{u}, \mathfrak{v}]$, the following FII holds

$$\frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}_p^{\kappa}\prod_{k\neq p}\mathfrak{f}_k^{\vartheta_k})(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\prod_{k=1}^{n}\mathfrak{f}_k^{\vartheta_k})(x)} \geqslant \frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}^{\varpi}\mathfrak{f}_p^{\kappa}\prod_{k\neq p}\mathfrak{f}_k^{\vartheta_k})(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}^{\varpi}\prod_{k=1}^{n}\mathfrak{f}_k^{\vartheta_k})(x)}. \tag{41}$$

Proof. It follows from (40) that for $p = 1, 2, \dots, n$,

$$\mathfrak{g}^{\varpi}(\mathfrak{t})\mathfrak{f}_p^{\kappa-\vartheta_p}(\mathfrak{s}) + \mathfrak{g}^{\varpi}(\mathfrak{s})\mathfrak{f}_p^{\kappa-\vartheta_p}(\mathfrak{t}) \geqslant \mathfrak{g}^{\varpi}(\mathfrak{t})\mathfrak{f}_p^{\kappa-\vartheta_p}(\mathfrak{t}) + \mathfrak{g}^{\varpi}(\mathfrak{s})\mathfrak{f}_p^{\kappa-\vartheta_p}(\mathfrak{s}). \tag{42}$$

Multiplying both sides of (42) by $\aleph^{-1}(x)\aleph(\mathfrak{s})\mathfrak{M}^{\mathfrak{s}}_{x}({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}, \xi, \phi)\mathfrak{h}(\mathfrak{s})\prod_{k=1}^{n}\mathfrak{f}_k^{\vartheta_k}(\mathfrak{s})\xi'(\mathfrak{s})$ and integrating with respect to $\mathfrak{s}$ both sides of the obtained result from $\mathfrak{u}$ to x, then we have

$$\begin{aligned}&\mathfrak{g}^{\varpi}(\mathfrak{t})\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}_p^{\kappa}\prod_{k\neq p}\mathfrak{f}_k^{\vartheta_k}\Big)(x) + \mathfrak{f}_p^{\kappa-\vartheta_p}(\mathfrak{t})\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}^{\varpi}\prod_{k=1}^{n}\mathfrak{f}_k^{\vartheta_k}\Big)(x)\\ &\qquad \geqslant \mathfrak{g}^{\varpi}(\mathfrak{t})\mathfrak{f}_p^{\kappa-\vartheta}(\mathfrak{t})\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\prod_{k=1}^{n}\mathfrak{f}_k^{\vartheta_k}\Big)(x) + \Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}^{\varpi}\mathfrak{f}_p^{\kappa}\prod_{k\neq p}\mathfrak{f}_k^{\vartheta_k}\Big)(x).\end{aligned} \tag{43}$$

Multiplying both sides of (43) by $\aleph^{-1}(x)\aleph(\mathfrak{t})\mathfrak{M}^{\mathfrak{t}}_{x}({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}, \xi, \phi)\mathfrak{h}(\mathfrak{t})\prod_{k=1}^{n}\mathfrak{f}_k^{\vartheta_k}(\mathfrak{t})\xi'(\mathfrak{t})$ and integrating with respect to $\mathfrak{t}$ both sides of the obtained result from $\mathfrak{u}$ to x, then we get

$$\begin{aligned}&\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}^{\varpi}\prod_{k=1}^{n}\mathfrak{f}_k^{\vartheta_k}\Big)(x)\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}_p^{\kappa}\prod_{k\neq p}\mathfrak{f}_k^{\vartheta_k}\Big)(x) + \Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}_p^{\kappa}\prod_{k\neq p}\mathfrak{f}_k^{\vartheta_k}\Big)(x)\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}^{\varpi}\prod_{k=1}^{n}\mathfrak{f}_k^{\vartheta_k}\Big)(x)\\ &\qquad \geqslant \Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}^{\varpi}\mathfrak{f}_p^{\kappa}\prod_{k\neq p}\mathfrak{f}_k^{\vartheta_k}\Big)(x)\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\prod_{k=1}^{n}\mathfrak{f}_k^{\vartheta_k}\Big)(x) + \Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\prod_{k=1}^{n}\mathfrak{f}_k^{\vartheta_k}\Big)(x)\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}^{\varpi}\mathfrak{f}_p^{\kappa}\prod_{k\neq p}\mathfrak{f}_k^{\vartheta_k}\Big)(x),\end{aligned} \tag{44}$$

which manifests the desired inequality (41). This completes the proof of Theorem 8. □

Theorem 9. *Assume that all the conditions of Theorem 8 hold. Then, for any fixed integer number $p \in \{1, 2, \dots, n\}$ and $x \in [\mathfrak{u}, \mathfrak{v}]$, the following FII holds*

$$\frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}_p^{\kappa}\prod_{k\neq p}\mathfrak{f}_k^{\vartheta_k})(x)({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}^{\varpi}\prod_{k=1}^{n}\mathfrak{f}_k^{\vartheta_k})(x) + ({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}_p^{\kappa}\prod_{k\neq p}\mathfrak{f}_k^{\vartheta_k})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}^{\varpi}\prod_{k=1}^{n}\mathfrak{f}_k^{\vartheta_k})(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\prod_{k=1}^{n}\mathfrak{f}_k^{\vartheta_k})(x)({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}^{\varpi}\mathfrak{f}_p^{\kappa}\prod_{k\neq p}\mathfrak{f}_k^{\vartheta_k})(x) + ({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\prod_{k=1}^{n}\mathfrak{f}_k^{\vartheta_k})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}^{\varpi}\mathfrak{f}_p^{\kappa}\prod_{k\neq p}\mathfrak{f}_k^{\vartheta_k})(x)} \geqslant 1. \tag{45}$$

Proof. Multiplying both sides of (43) by $\aleph^{-1}(x)\aleph(\mathfrak{t})\mathfrak{M}^{\mathfrak{t}}_{x}(^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\varsigma,\beta,\gamma,\delta,\mu,\nu},\xi,\phi)\mathfrak{h}(\mathfrak{t})\prod_{k=1}^{n}\mathfrak{f}_k^{\vartheta_k}(\mathfrak{t})\cdot\xi'(\mathfrak{t})$ and integrating with respect to $\mathfrak{t}$ both sides of the obtained inequality from $\mathfrak{u}$ to x, then we claim

$$\Big({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}^{\varpi}\prod_{k=1}^{n}\mathfrak{f}_k^{\vartheta_k}\Big)(x)\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}_p^{\kappa}\prod_{k\neq p}\mathfrak{f}_k^{\vartheta_k}\Big)(x)+\Big({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}_p^{\kappa}\prod_{k\neq p}\mathfrak{f}_k^{\vartheta_k}\Big)(x)\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}^{\varpi}\prod_{k=1}^{n}\mathfrak{f}_k^{\vartheta_k}\Big)(x)$$
$$\geqslant\Big({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}^{\varpi}\mathfrak{f}_p^{\kappa}\prod_{k\neq p}\mathfrak{f}_k^{\vartheta_k}\Big)(x)\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\prod_{k=1}^{n}\mathfrak{f}_k^{\vartheta_k}\Big)(x)+\Big({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\prod_{k=1}^{n}\mathfrak{f}_k^{\vartheta_k}\Big)(x)\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}^{\varpi}\mathfrak{f}_p^{\kappa}\prod_{k\neq p}\mathfrak{f}_k^{\vartheta_k}\Big)(x),\quad(46)$$

which emanates the desired assertion (44). This completes the proof of Theorem 9. □

Remark 5. Employing Theorem 9 for $\varsigma=\alpha$, we confirm Theorem 8.

Corollary 4. *Suppose that one of the following assumptions holds for $\varpi>0$, $\kappa\geqslant\vartheta>0$ and $p=1,2,\ldots,n$, (a) $\mathfrak{f}_p$ and $\mathfrak{g}$ are two positive continuous monotonic increasing and decreasing functions on $[\mathfrak{u},\mathfrak{v}]$, respectively; (b) $\mathfrak{f}_p$ and $\mathfrak{g}$ are two positive continuous monotonic decreasing and increasing functions on $[\mathfrak{u},\mathfrak{v}]$, respectively; (c) $\mathfrak{f}_p^{\kappa-\vartheta}$ and $\mathfrak{g}^{\varpi}$ are two continuous asynchronous functions on $[\mathfrak{u},\mathfrak{v}]$. Then the FIIs (41) and (45) still hold true.*

Proof. If one of the previous assumptions holds true, then we can derive the inequality (40). Furthermore, we obtain the FIIs (41) and (45) similar to the proofs of Theorem 8 and Theorem 9. This completes the proof of Corollary 4. □

Theorem 10. *Let $\mathfrak{e}$, $(\mathfrak{f}_k)_{k=1,2,\ldots,n}$, $\mathfrak{g}$ and $\mathfrak{h}$ be positive continuous functions on $[\mathfrak{u},\mathfrak{v}]$ satisfying*

$$(\mathfrak{g}(\mathfrak{t})-\mathfrak{g}(\mathfrak{s}))\Big(\frac{\mathfrak{e}(\mathfrak{s})}{\mathfrak{f}_p(\mathfrak{s})}-\frac{\mathfrak{e}(\mathfrak{t})}{\mathfrak{f}_p(\mathfrak{t})}\Big)\geqslant 0,\quad\forall\mathfrak{s},\mathfrak{t}\in[\mathfrak{u},\mathfrak{v}],\quad(47)$$

where $p=1,2,\ldots,n$. Then, for any fixed integer number $p\in\{1,2,\ldots,n\}$ and $x\in[\mathfrak{u},\mathfrak{v}]$, the following FII holds

$$\frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e}\prod_{k\neq p}\mathfrak{f}_k)(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\prod_{k=1}^{n}\mathfrak{f}_k)(x)}\geqslant\frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e}\mathfrak{g}\prod_{k\neq p}\mathfrak{f}_k)(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}\prod_{k=1}^{n}\mathfrak{f}_k)(x)}.\quad(48)$$

Proof. It follows from (47) that for $p=1,2,\ldots,n$,

$$\mathfrak{g}(\mathfrak{t})\frac{\mathfrak{e}(\mathfrak{s})}{\mathfrak{f}_p(\mathfrak{s})}+\mathfrak{g}(\mathfrak{s})\frac{\mathfrak{e}(\mathfrak{t})}{\mathfrak{f}_p(\mathfrak{t})}\geqslant\mathfrak{g}(\mathfrak{t})\frac{\mathfrak{e}(\mathfrak{t})}{\mathfrak{f}_p(\mathfrak{t})}+\mathfrak{g}(\mathfrak{s})\frac{\mathfrak{e}(\mathfrak{s})}{\mathfrak{f}_p(\mathfrak{s})}.\quad(49)$$

Multiplying both sides of (49) by $\aleph^{-1}(x)\aleph(\mathfrak{s})\mathfrak{M}^{\mathfrak{s}}_{x}(^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu},\xi,\phi)\mathfrak{h}(\mathfrak{s})\prod_{k=1}^{n}\mathfrak{f}_k(\mathfrak{s})\xi'(\mathfrak{s})$ and integrating with respect to $\mathfrak{s}$ both sides of the obtained result from $\mathfrak{u}$ to x, then we have

$$\mathfrak{g}(\mathfrak{t})\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e}\prod_{k\neq p}\mathfrak{f}_k\Big)(x)+\frac{\mathfrak{e}(\mathfrak{t})}{\mathfrak{f}_p(\mathfrak{t})}\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}\prod_{k=1}^{n}\mathfrak{f}_k\Big)(x)\geqslant\mathfrak{g}(\mathfrak{t})\frac{\mathfrak{e}(\mathfrak{t})}{\mathfrak{f}_p(\mathfrak{t})}\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\prod_{k=1}^{n}\mathfrak{f}_k\Big)(x)+\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e}\mathfrak{g}\prod_{k\neq p}\mathfrak{f}_k\Big)(x).\quad(50)$$

Multiplying both sides of (50) by $\aleph^{-1}(x)\aleph(\mathfrak{t})\mathfrak{M}^{\mathfrak{t}}_{x}(^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu},\xi,\phi)\mathfrak{h}(\mathfrak{t})\prod_{k=1}^{n}\mathfrak{f}_k(\mathfrak{t})\xi'(\mathfrak{t})$ and integrating with respect to $\mathfrak{t}$ both sides of the obtained result from $\mathfrak{u}$ to x, then we get

$$\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}\prod_{k=1}^{n}\mathfrak{f}_k\Big)(x)\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e}\prod_{k\neq p}\mathfrak{f}_k\Big)(x)+\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e}\prod_{k\neq p}\mathfrak{f}_k\Big)(x)\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}\prod_{k=1}^{n}\mathfrak{f}_k\Big)(x)$$
$$\geqslant\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e}\mathfrak{g}\prod_{k\neq p}\mathfrak{f}_k\Big)(x)\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\prod_{k=1}^{n}\mathfrak{f}_k\Big)(x)+\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\prod_{k=1}^{n}\mathfrak{f}_k\Big)(x)\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e}\mathfrak{g}\prod_{k\neq p}\mathfrak{f}_k\Big)(x),\quad(51)$$

which manifests the desired inequality (48). This completes the proof of Theorem 10. □

Theorem 11. *Assume that all the conditions of Theorem 10 hold. Then, for any fixed integer number $p \in \{1, 2, \ldots, n\}$ and $x \in [\mathfrak{u}, \mathfrak{v}]$, the following FII holds*

$$\frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e}\prod_{k\neq p}\mathfrak{f}_k)(x)({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}\prod_{k=1}^{n}\mathfrak{f}_k)(x) + ({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e}\prod_{k\neq p}\mathfrak{f}_k)(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}\prod_{k=1}^{n}\mathfrak{f}_k)(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\prod_{k=1}^{n}\mathfrak{f}_k)(x)({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e}\mathfrak{g}\prod_{k\neq p}\mathfrak{f}_k)(x) + ({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\prod_{k=1}^{n}\mathfrak{f}_k)(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e}\mathfrak{g}\prod_{k\neq p}\mathfrak{f}_k)(x)} \geqslant 1. \tag{52}$$

Proof. Multiplying both sides of (50) by $\aleph^{-1}(x)\aleph(\mathfrak{t})\mathfrak{M}^{\mathfrak{t}}_{x}({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\varsigma,\beta,\gamma,\delta,\mu,\nu}, \xi, \phi)\mathfrak{h}(\mathfrak{t})\prod_{k=1}^{n}\mathfrak{f}_k(\mathfrak{t}) \cdot \xi'(\mathfrak{t})$ and integrating with respect to $\mathfrak{t}$ both sides of the obtained inequality from $\mathfrak{u}$ to x, then we claim

$$\begin{aligned}&\Big({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}\prod_{k=1}^{n}\mathfrak{f}_k\Big)(x)\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e}\prod_{k\neq p}\mathfrak{f}_k\Big)(x) + \Big({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e}\prod_{k\neq p}\mathfrak{f}_k\Big)(x)\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}\prod_{k=1}^{n}\mathfrak{f}_k\Big)(x)\\ &\geqslant \Big({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e}\mathfrak{g}\prod_{k\neq p}\mathfrak{f}_k\Big)(x)\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\prod_{k=1}^{n}\mathfrak{f}_k\Big)(x) + \Big({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\prod_{k=1}^{n}\mathfrak{f}_k\Big)(x)\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e}\mathfrak{g}\prod_{k\neq p}\mathfrak{f}_k\Big)(x),\end{aligned} \tag{53}$$

which emanates the desired assertion (52). This completes the proof of Theorem 11. □

Remark 6. Employing Theorem 11 for $\varsigma = \alpha$, we get Theorem 10.

Corollary 5. *Suppose that one of the following assumptions holds for $\varpi > 0$, $\kappa \geqslant \vartheta > 0$ and $p = 1, 2, \ldots, n$,* (a) $\mathfrak{f}_p, \mathfrak{g}$ *and* $\mathfrak{e}$ *are two positive continuous monotonic increasing and decreasing functions on* $[\mathfrak{u}, \mathfrak{v}]$, *respectively;* (b) $\mathfrak{f}_p, \mathfrak{g}$ *and* $\mathfrak{e}$ *are two positive continuous monotonic decreasing and increasing functions on* $[\mathfrak{u}, \mathfrak{v}]$, *respectively;* (c) $\mathfrak{e}/\mathfrak{f}_p$ *and* $\mathfrak{g}$ *are two continuous asynchronous functions on* $[\mathfrak{u}, \mathfrak{v}]$. *Then the FIIs* (48) *and* (52) *still hold true.*

Proof. If one of the previous assumptions holds true, then we can derive the inequality (47). Furthermore, we obtain the FIIs (48) and (52) similar to the proofs of Theorem 10 and Theorem 11. This completes the proof of Corollary 5. □

Corollary 6. *Let $\mathfrak{e}$, $(\mathfrak{f}_k)_{k=1,2,\ldots,n}$, $\mathfrak{g}$ and $\mathfrak{h}$ be positive continuous functions on $[\mathfrak{u}, \mathfrak{v}]$.*

(a) *Assume that $\mathfrak{g}$ is monotonic increasing, $\mathfrak{e}$ and $(\mathfrak{f}_k)_{k=1,2,\ldots,n}$ are differentiable and there exist positive constants $(M_k)_{k=1,2,\ldots,n}$ such that $M_k = \sup_{x\in[\mathfrak{u},\mathfrak{v}]}(e/f_k)'(x)$. Then, for any fixed integer number $p \in \{1, 2, \ldots, n\}$ and $x \in [\mathfrak{u}, \mathfrak{v}]$, the following FII holds*

$$\frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e}\prod_{k\neq p}\mathfrak{f}_k)(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}\prod_{k=1}^{n}\mathfrak{f}_k)(x) + M_p({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\prod_{k=1}^{n}\mathfrak{f}_k)(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{i}\mathfrak{g}\prod_{k=1}^{n}\mathfrak{f}_k)(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\prod_{k=1}^{n}\mathfrak{f}_k)(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e}\mathfrak{g}\prod_{k\neq p}\mathfrak{f}_k)(x) + M_p({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}\prod_{k=1}^{n}\mathfrak{f}_k)(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{i}\prod_{k=1}^{n}\mathfrak{f}_k)(x)} \geqslant 1. \tag{54}$$

(b) *Assume that $\mathfrak{g}$ is monotonic decreasing, $\mathfrak{e}$ and $(\mathfrak{f}_k)_{k=1,2,\ldots,n}$ are differentiable and there exist positive constants $(m_k)_{k=1,2,\ldots,n}$ such that $m_k = \inf_{x\in[\mathfrak{u},\mathfrak{v}]}(e/f_k)'(x)$. Then, for any fixed integer number $p \in \{1, 2, \ldots, n\}$ and $x \in [\mathfrak{u}, \mathfrak{v}]$, the following FII holds*

$$\frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e}\prod_{k\neq p}\mathfrak{f}_k)(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}\prod_{k=1}^{n}\mathfrak{f}_k)(x) + m_p({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\prod_{k=1}^{n}\mathfrak{f}_k)(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{i}\mathfrak{g}\prod_{k=1}^{n}\mathfrak{f}_k)(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\prod_{k=1}^{n}\mathfrak{f}_k)(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e}\mathfrak{g}\prod_{k\neq p}\mathfrak{f}_k)(x) + m_p({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}\prod_{k=1}^{n}\mathfrak{f}_k)(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{i}\prod_{k=1}^{n}\mathfrak{f}_k)(x)} \geqslant 1. \tag{55}$$

Proof. (a) In Theorem 10, we substitute $U_p(x) = \mathfrak{e}/\mathfrak{f}_p(x) - M_p x$ for $\mathfrak{e}/\mathfrak{f}_p(x)$, $x \in [\mathfrak{u}, \mathfrak{v}]$. According to the hypothesis, it is obvious to see that U_p is decreasing. Then U_p and $\mathfrak{g}$ are two continuous asynchronous functions on $[\mathfrak{u}, \mathfrak{v}]$. It follows from (47) and Corollary 5 that by means of a simple computation, we can claim (54).

(b) In Theorem 10, we substitute $V_p(x) = \mathfrak{e}/\mathfrak{f}_p(x) - m_p x$ for $\mathfrak{e}/\mathfrak{f}_p(x)$, $x \in [\mathfrak{u}, \mathfrak{v}]$. According to the hypothesis, it is obvious to see that U_p is increasing. Then V_p and $\mathfrak{g}$ are two continuous asynchronous functions on $[\mathfrak{u}, \mathfrak{v}]$. It follows from (47) and Corollary 5 that by means of a simple computation, we can achieve (55). This completes the proof of Corollary 6. □

Corollary 7. *Let* $\mathfrak{e}, (\mathfrak{f}_k)_{k=1,2,\ldots,n}, \mathfrak{g}$ *and* $\mathfrak{h}$ *be positive continuous functions on* $[\mathfrak{u}, \mathfrak{v}]$.

(a) *Under the hypothesis* (a) *of* Corollary 6, *then, for any fixed integer number* $p \in \{1, 2, \ldots, n\}$ *and* $x \in [\mathfrak{u}, \mathfrak{v}]$, *the following FII still holds true*

$$\frac{\mathfrak{S}_{\alpha,\varsigma,M_p}(x)}{\mathfrak{T}_{\alpha,\varsigma,M_p}(x)} \geqslant 1; \tag{56}$$

(b) *Under the hypothesis* (b) *of* Corollary 6, *then, for any fixed integer number* $p \in \{1, 2, \ldots, n\}$ *and* $x \in [\mathfrak{u}, \mathfrak{v}]$, *the following FII still holds true*

$$\frac{\mathfrak{S}_{\alpha,\varsigma,m_p}(x)}{\mathfrak{T}_{\alpha,\varsigma,m_p}(x)} \geqslant 1, \tag{57}$$

where

$$\begin{aligned}\mathfrak{S}_{\alpha,\varsigma,\mathfrak{d}}(x) =& \Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e}\prod_{k\neq p}\mathfrak{f}_k\Big)(x)\Big({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}\prod_{k=1}^{n}\mathfrak{f}_k\Big)(x) + \Big({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e}\prod_{k\neq p}\mathfrak{f}_k\Big)(x)\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}\prod_{k=1}^{n}\mathfrak{f}_k\Big)(x)\\ &+ \mathfrak{d}\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\prod_{k=1}^{n}\mathfrak{f}_k\Big)(x)\Big({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{i}\mathfrak{g}\prod_{k=1}^{n}\mathfrak{f}_k\Big)(x) + \mathfrak{d}\Big({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\prod_{k=1}^{n}\mathfrak{f}_k\Big)(x)\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{i}\mathfrak{g}\prod_{k=1}^{n}\mathfrak{f}_k\Big)(x),\end{aligned} \tag{58}$$

$$\begin{aligned}\mathfrak{T}_{\alpha,\varsigma,\mathfrak{d}}(x) =& \Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\prod_{k=1}^{n}\mathfrak{f}_k\Big)(x)\Big({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e}\mathfrak{g}\prod_{k\neq p}\mathfrak{f}_k\Big)(x) + \Big({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\prod_{k=1}^{n}\mathfrak{f}_k\Big)(x)\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e}\mathfrak{g}\prod_{k\neq p}\mathfrak{f}_k\Big)(x)\\ &+ \mathfrak{d}\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}\prod_{k=1}^{n}\mathfrak{f}_k\Big)(x)\Big({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{i}\prod_{k=1}^{n}\mathfrak{f}_k\Big)(x) + \mathfrak{d}\Big({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}\prod_{k=1}^{n}\mathfrak{f}_k\Big)(x)\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{i}\prod_{k=1}^{n}\mathfrak{f}_k\Big)(x).\end{aligned} \tag{59}$$

Proof. Similar to the proof of Corollary 6, by applying Theorem 11 and Corollary 5, then we can get Corollary 7. □

Remark 7. Applying Theorem 10 and Theorem 11 for $n = 1$, we have the following FIIs

$$\frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e})(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)} \geqslant \frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}\mathfrak{e})(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}\mathfrak{f})(x)}, \tag{60}$$

$$\frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e})(x)({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}\mathfrak{f})(x) + ({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}\mathfrak{f})(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}\mathfrak{e})(x) + ({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}\mathfrak{e})(x)} \geqslant 1. \tag{61}$$

Moreover, suppose that $\mathfrak{e}, \mathfrak{f}, \mathfrak{h}$ are three positive continuous functions with $\mathfrak{e} \leqslant \mathfrak{f}$ such that $\mathfrak{e}/\mathfrak{f}$ is decreasing and $\mathfrak{e}$ is increasing on $[\mathfrak{u}, \mathfrak{v}]$. It follows from (60) and (61) that for $\varpi \geqslant 1$, then we obtain

$$\frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e})(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)} \geqslant \frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e}^{\varpi})(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\varpi})(x)}, \tag{62}$$

$$\frac{({}^{\phi}_{\xi}\theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e})(x)({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\varpi})(x) + ({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\varpi})(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e}^{\varpi})(x) + ({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e}^{\varpi})(x)} \geqslant 1. \tag{63}$$

Remark 8. By setting the different parameters and functions in (9), the MUGFIIs with EUMLFs in the front can reduce to the standard Riemann integral inequalities [16], RLFIIs [8], generalized k-conformable type FIIs [22], unified weighted type FIIs [26], HPFIIs [18, 25], GPFIIs [1], multiple Erdélyi–Kober type FIIs [31] and more generalized type FIIs [27] for positive monotonic functions.

Theorem 12. *Let $\mathfrak{e}, \mathfrak{f}$ and $\mathfrak{h}$ be three positive continuous functions with $\mathfrak{e} \leqslant \mathfrak{f}$ such that $\mathfrak{e}/\mathfrak{f}$ is decreasing and $\mathfrak{e}$ is increasing on $[\mathfrak{u}, \mathfrak{v}]$. Moreover, let φ be a convex function with $\varphi(0) = 0$. Then, for $x \in [\mathfrak{u}, \mathfrak{v}]$, the following FII holds*

$$\frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e})(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)} \geqslant \frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\varphi(\mathfrak{e}))(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\varphi(\mathfrak{f}))(x)}. \tag{64}$$

Proof. Since φ is a convex function with $\varphi(0) = 0$, then $\varphi(v)/v$ is increasing. Furthermore, the function $\varphi(\mathfrak{e}(v))/\mathfrak{e}(v)$ is also increasing as $\mathfrak{e}$ is increasing. And because the function $\mathfrak{e}/\mathfrak{f}$ is decreasing. We have the following inequality

$$\Big(\frac{\varphi(\mathfrak{e}(\mathfrak{t}))}{\mathfrak{e}(\mathfrak{t})} - \frac{\varphi(\mathfrak{e}(\mathfrak{s}))}{\mathfrak{e}(\mathfrak{s})}\Big)\Big(\frac{\mathfrak{e}(\mathfrak{s})}{\mathfrak{f}(\mathfrak{s})} - \frac{\mathfrak{e}(\mathfrak{t})}{\mathfrak{f}(\mathfrak{t})}\Big) \geqslant 0 \quad \text{for } \forall \mathfrak{s}, \mathfrak{t} \in [\mathfrak{u}, \mathfrak{v}]. \tag{65}$$

Multiplying both sides of (65) by $\mathfrak{f}(\mathfrak{s})\mathfrak{f}(\mathfrak{t}) > 0$, then we claim

$$\frac{\varphi(\mathfrak{e}(\mathfrak{t}))}{\mathfrak{e}(\mathfrak{t})}\mathfrak{e}(\mathfrak{s})\mathfrak{f}(\mathfrak{t}) + \frac{\varphi(\mathfrak{e}(\mathfrak{s}))}{\mathfrak{e}(\mathfrak{s})}\mathfrak{e}(\mathfrak{t})\mathfrak{f}(\mathfrak{s}) \geqslant \varphi(\mathfrak{e}(\mathfrak{t}))\mathfrak{f}(\mathfrak{s}) + \varphi(\mathfrak{e}(\mathfrak{s}))\mathfrak{f}(\mathfrak{t}) \quad \text{for } \forall \mathfrak{s}, \mathfrak{t} \in [\mathfrak{u}, \mathfrak{v}]. \tag{66}$$

Multiplying both sides of (66) by $\aleph^{-1}(x)\aleph(\mathfrak{s})\mathfrak{M}^{\mathfrak{s}}_{x}({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}, \xi, \phi)\mathfrak{h}(\mathfrak{s})\xi'(\mathfrak{s})$ and integrating with respect to $\mathfrak{s}$ both sides of the obtained result from $\mathfrak{u}$ to x, then we have

$$\frac{\varphi(\mathfrak{e}(\mathfrak{t}))}{\mathfrak{e}(\mathfrak{t})}\mathfrak{f}(\mathfrak{t})({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e})(x) + \mathfrak{e}(\mathfrak{t})({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}\varphi(\mathfrak{e})/\mathfrak{e})(x) \geqslant \varphi(\mathfrak{e}(\mathfrak{t}))({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x) + \mathfrak{f}(\mathfrak{t})({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\varphi(\mathfrak{e}))(x). \tag{67}$$

Multiplying both sides of (67) by $\aleph^{-1}(x)\aleph(\mathfrak{s})\mathfrak{M}^{\mathfrak{t}}_{x}({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}, \xi, \phi)\mathfrak{h}(\mathfrak{t})\xi'(\mathfrak{t})$ and integrating with respect to $\mathfrak{t}$ both sides of the obtained result from $\mathfrak{u}$ to x, then we observe

$$\begin{aligned}&({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}\varphi(\mathfrak{e})/\mathfrak{e})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e})(x) + ({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}\varphi(\mathfrak{e})/\mathfrak{e})(x)\\ &\qquad \geqslant ({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\varphi(\mathfrak{e}))(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x) + ({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\varphi(\mathfrak{e}))(x).\end{aligned} \tag{68}$$

It follows from (68) that

$$\frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e})(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)} \geqslant \frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\varphi(\mathfrak{e}))(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}\varphi(\mathfrak{e})/\mathfrak{e})(x)}. \tag{69}$$

On the other hand, since $\varphi(v)/v$ is increasing and $\mathfrak{e} \leqslant \mathfrak{f}$ on $[\mathfrak{u}, \mathfrak{v}]$, then, we get

$$\frac{\varphi(\mathfrak{e}(\mathfrak{s}))}{\mathfrak{e}(\mathfrak{s})} \leqslant \frac{\varphi(\mathfrak{f}(\mathfrak{s}))}{\mathfrak{f}(\mathfrak{s})} \quad \text{for } \forall \mathfrak{s} \in [\mathfrak{u}, \mathfrak{v}]. \tag{70}$$

Multiplying both sides of (70) by $\aleph^{-1}(x)\aleph(\mathfrak{s})\mathfrak{M}^{\mathfrak{s}}_{x}({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}, \xi, \phi)\mathfrak{h}(\mathfrak{s})\mathfrak{f}(\mathfrak{s})\xi'(\mathfrak{s})$ and integrating with respect to $\mathfrak{s}$ both sides of the obtained result from $\mathfrak{u}$ to x, then we derive

$$({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}\varphi(\mathfrak{e})/\mathfrak{e})(x) \leqslant ({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\varphi(\mathfrak{f}))(x). \tag{71}$$

It follows from (69) and (71) that we give the desired inequality (64). This completes the proof of Theorem 12. □

Theorem 13. *Assume that all the conditions of Theorem 12 hold. Then, for $x \in [\mathfrak{u}, \mathfrak{v}]$, we have*

$$\frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e})(x)({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\varphi(\mathfrak{f}))(x) + ({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\varphi(\mathfrak{f}))(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\varphi(\mathfrak{e}))(x) + ({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\varphi(\mathfrak{e}))(x)} \geqslant 1. \tag{72}$$

Proof. Multiplying both sides of (67) by $\aleph^{-1}(x)\aleph(\mathfrak{s})\mathfrak{M}_x^{\mathfrak{t}}({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\varsigma,\beta,\gamma,\delta,\mu,\nu},\xi,\phi)\mathfrak{h}(\mathfrak{t})\xi'(\mathfrak{t})$ and integrating with respect to $\mathfrak{t}$ both sides of the obtained result from $\mathfrak{u}$ to x, then we observe

$$({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}\varphi(\mathfrak{e})/\mathfrak{e})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e})(x)+({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}\varphi(\mathfrak{e})/\mathfrak{e})(x)$$
$$\geqslant ({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\varphi(\mathfrak{e}))(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)+({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\varphi(\mathfrak{e}))(x). \quad (73)$$

On the other hand, multiplying both sides of (70) by $\aleph^{-1}(x)\aleph(\mathfrak{s})\mathfrak{M}_x^{\mathfrak{s}}({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\varsigma,\beta,\gamma,\delta,\mu,\nu},\xi,\phi)\ \mathfrak{h}(\mathfrak{s})\mathfrak{f}(\mathfrak{s})\xi'(\mathfrak{s})$ and integrating with respect to $\mathfrak{s}$ both sides of the obtained result from $\mathfrak{u}$ to x, then we derive

$$({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}\varphi(\mathfrak{e})/\mathfrak{e})(x)\leqslant({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\varphi(\mathfrak{f}))(x). \quad (74)$$

It follows from (71), (73) and (74) that we present the desired inequality (72). This completes the proof of Theorem 13. □

Remark 9. Employing Theorem 12 and Theorem 13 for $\varphi(\upsilon)=\upsilon^{\varpi}$ with $\varpi\geqslant 1$, then FIIs (62) and (70) reduce to FIIs (61) and (62), respectively. Applying Theorem 13 for $\varsigma=\alpha$, we get Theorem 12.

Theorem 14. *Let $\mathfrak{e},\mathfrak{f},\mathfrak{g}$ and $\mathfrak{h}$ be four positive continuous functions with $\mathfrak{e}\leqslant\mathfrak{f}$ such that $\mathfrak{e}/\mathfrak{f}$ is decreasing and $\mathfrak{e},\mathfrak{g}$ are increasing on $[\mathfrak{u},\mathfrak{v}]$. Moreover, let φ be a convex function with $\varphi(0)=0$. Then, for $x\in[\mathfrak{u},\mathfrak{v}]$, the following FII holds*

$$\frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e})(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)}\geqslant\frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}\varphi(\mathfrak{e}))(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}\varphi(\mathfrak{f}))(x)}. \quad (75)$$

Proof. Since φ is a convex function with $\varphi(0)=0$, then $\varphi(\upsilon)/\upsilon$ is increasing. Furthermore, the function $\mathfrak{g}(\upsilon)\varphi(\mathfrak{e}(\upsilon))/\mathfrak{e}(\upsilon)$ is also increasing as $\mathfrak{e},\mathfrak{g}$ are increasing. And because the function $\mathfrak{e}/\mathfrak{f}$ is decreasing. We have the following inequality

$$\Big(\frac{\varphi(\mathfrak{e}(\mathfrak{t}))}{\mathfrak{e}(\mathfrak{t})}\mathfrak{g}(\mathfrak{t})-\frac{\varphi(\mathfrak{e}(\mathfrak{s}))\mathfrak{g}(\mathfrak{s})}{\mathfrak{e}(\mathfrak{s})}\Big)\Big(\frac{\mathfrak{e}(\mathfrak{s})}{\mathfrak{f}(\mathfrak{s})}-\frac{\mathfrak{e}(\mathfrak{t})}{\mathfrak{f}(\mathfrak{t})}\Big)\geqslant 0\quad\text{for }\forall\mathfrak{s},\mathfrak{t}\in[\mathfrak{u},\mathfrak{v}]. \quad (76)$$

Multiplying both sides of (76) by $\mathfrak{f}(\mathfrak{s})\mathfrak{f}(\mathfrak{t})>0$, then, for $\forall\mathfrak{s},\mathfrak{t}\in[\mathfrak{u},\mathfrak{v}]$, we claim

$$\frac{\varphi(\mathfrak{e}(\mathfrak{t}))}{\mathfrak{e}(\mathfrak{t})}\mathfrak{e}(\mathfrak{s})\mathfrak{f}(\mathfrak{t})\mathfrak{g}(\mathfrak{t})+\frac{\varphi(\mathfrak{e}(\mathfrak{s}))}{\mathfrak{e}(\mathfrak{s})}\mathfrak{e}(\mathfrak{t})\mathfrak{f}(\mathfrak{s})\mathfrak{g}(\mathfrak{s})\geqslant\varphi(\mathfrak{e}(\mathfrak{t}))\mathfrak{f}(\mathfrak{s})\mathfrak{g}(\mathfrak{t})+\varphi(\mathfrak{e}(\mathfrak{s}))\mathfrak{f}(\mathfrak{t})\mathfrak{g}(\mathfrak{s}). \quad (77)$$

Multiplying both sides of (77) by $\aleph^{-1}(x)\aleph(\mathfrak{s})\mathfrak{M}_x^{\mathfrak{s}}({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu},\xi,\phi)\mathfrak{h}(\mathfrak{s})\xi'(\mathfrak{s})$ and integrating with respect to $\mathfrak{s}$ both sides of the obtained result from $\mathfrak{u}$ to x, then we have

$$\frac{\varphi(\mathfrak{e}(\mathfrak{t}))}{\mathfrak{e}(\mathfrak{t})}\mathfrak{f}(\mathfrak{t})\mathfrak{g}(\mathfrak{t})({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e})(x)+\mathfrak{e}(\mathfrak{t})({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}\mathfrak{g}\varphi(\mathfrak{e})/\mathfrak{e})(x)\geqslant\varphi(\mathfrak{e}(\mathfrak{t}))\mathfrak{g}(\mathfrak{t})({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)+\mathfrak{f}(\mathfrak{t})({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}\varphi(\mathfrak{e}))(x). \quad (78)$$

Multiplying both sides of (78) by $\aleph^{-1}(x)\aleph(\mathfrak{s})\mathfrak{M}_x^{\mathfrak{t}}({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu},\xi,\phi)\mathfrak{h}(\mathfrak{t})\xi'(\mathfrak{t})$ and integrating with respect to $\mathfrak{t}$ both sides of the obtained result from $\mathfrak{u}$ to x, then we observe

$$({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}\mathfrak{g}\varphi(\mathfrak{e})/\mathfrak{e})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e})(x)+({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}\mathfrak{g}\varphi(\mathfrak{e})/\mathfrak{e})(x)$$
$$\geqslant({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}\varphi(\mathfrak{e}))(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)+({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}\varphi(\mathfrak{e}))(x). \quad (79)$$

It follows from (79) that

$$\frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e})(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)}\geqslant\frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}\varphi(\mathfrak{e}))(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}\mathfrak{g}\varphi(\mathfrak{e})/\mathfrak{e})(x)}. \quad (80)$$

On the other hand, multiplying both sides of (70) by $\aleph^{-1}(x)\aleph(\mathfrak{s})\mathfrak{M}_x^{\mathfrak{s}}({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu},\xi,\phi)\ \mathfrak{h}(\mathfrak{s})\mathfrak{f}(\mathfrak{s})\mathfrak{g}(\mathfrak{s})\xi'(\mathfrak{s})$ and integrating with respect to $\mathfrak{s}$ both sides of the obtained result from $\mathfrak{u}$ to x, then we derive

$$({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}\mathfrak{g}\varphi(\mathfrak{e})/\mathfrak{e})(x) \leqslant ({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}\varphi(\mathfrak{f}))(x). \tag{81}$$

It follows from (80) and (81) that we give the desired inequality (75). This completes the proof of Theorem 14. □

Theorem 15. *Assume that all the conditions of Theorem 14 hold. Then, for $x \in [\mathfrak{u},\mathfrak{v}]$, we have*

$$\frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e})(x)({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}\varphi(\mathfrak{f}))(x) + ({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}\varphi(\mathfrak{f}))(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}\varphi(\mathfrak{e}))(x) + ({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}\varphi(\mathfrak{e}))(x)} \geqslant 1. \tag{82}$$

Proof. Multiplying both sides of (78) by $\aleph^{-1}(x)\aleph(\mathfrak{s})\mathfrak{M}_x^{\mathfrak{t}}({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\varsigma,\beta,\gamma,\delta,\mu,\nu},\xi,\phi)\mathfrak{h}(\mathfrak{t})\xi'(\mathfrak{t})$ and integrating with respect to $\mathfrak{t}$ both sides of the obtained result from $\mathfrak{u}$ to x, then we observe

$$\begin{aligned}({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}\mathfrak{f}\varphi(\mathfrak{e})/\mathfrak{e})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e})(x) + ({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{e})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}\mathfrak{g}\varphi(\mathfrak{e})/\mathfrak{e})(x)\\ \geqslant ({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}\varphi(\mathfrak{e}))(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x) + ({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}\varphi(\mathfrak{e}))(x).\end{aligned} \tag{83}$$

On the other hand, multiplying both sides of (70) by $\aleph^{-1}(x)\aleph(\mathfrak{s})\mathfrak{M}_x^{\mathfrak{s}}({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\varsigma,\beta,\gamma,\delta,\mu,\nu},\xi,\phi)\ \mathfrak{h}(\mathfrak{s})\mathfrak{f}(\mathfrak{s})\mathfrak{g}(\mathfrak{s})\xi'(\mathfrak{s})$ and integrating with respect to $\mathfrak{s}$ both sides of the obtained result from $\mathfrak{u}$ to x, then we derive

$$({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}\mathfrak{g}\varphi(\mathfrak{e})/\mathfrak{e})(x) \leqslant ({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}\varphi(\mathfrak{f}))(x). \tag{84}$$

It follows from (81), (83) and (84) that we present the desired inequality (82). This completes the proof of Theorem 15. □

Remark 10. Applying Theorem 15 for $\varsigma = \alpha$, then we get Theorem 14.

Remark 11. By setting the different parameters and functions in (9), the MUGFIIs in Theorem 12–Theorem 15 can reduce to the FIIs with Bessel function in the kernel [13], generalized Katugampola FIIs [19], GPFIIs [20], (k,s)-type FIIs [21], weighted type FIIs [37], generalized weighted type FIIs [38], GPFIIs [28], Hadamard type FIIs [6], Saigo type FIIs [7], GP Hadamard type FIIs [23] and tempered type FIIs [24] for some convex functions, respectively.

4. Chebyshev type Inequalities for MUGFIOs

In this section, by utilizing the MUGFIOs with EUMLFs, we will presented some new Chebyshev type FIIs.

Similar to the proof of Theorem 4 and Theorem 5, we will give the next theorem without proof.

Theorem 16. *Let $\mathfrak{f}$, $\mathfrak{g}$ and $\mathfrak{h}$ be three positive continuous functions such that $\mathfrak{f}$ and $\mathfrak{g}$ are two synchronous functions on $[\mathfrak{u},\mathfrak{v}]$, that is, $(\mathfrak{f}(\mathfrak{s})-\mathfrak{f}(\mathfrak{t}))(\mathfrak{g}(\mathfrak{s})-\mathfrak{g}(\mathfrak{t})) \geqslant 0$ for any $\mathfrak{s},\mathfrak{t}]in[\mathfrak{u},\mathfrak{v}]$. Then, for $x \in [\mathfrak{u},\mathfrak{v}]$, we have the following FIIs*

$$({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}\mathfrak{g})(x) \geqslant ({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g})(x), \tag{85}$$

$$({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h})(x)({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}\mathfrak{g})(x) + ({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}\mathfrak{g})(x) \geqslant ({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g})(x) + ({}^{\phi}_{\xi}\Theta^{\varsigma}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g})(x). \tag{86}$$

Theorem 17. *Let* $\mathfrak{f}, (\mathfrak{g}_k)_{k=1,2,\dots,n}$ *and* $\mathfrak{h}$ *be positive continuous monotonic increasing functions on* $[\mathfrak{u},\mathfrak{v}]$*. Then, for* $x \in [\mathfrak{u},\mathfrak{v}]$*, we have the following FII*

$$({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h})^n(x)\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}\prod_{k=1}^{n}\mathfrak{g}_k\Big)(x) \geqslant ({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)\prod_{k=1}^{n}({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}_k)(x). \tag{87}$$

Proof. We will present the proof by taking advantage of the principle of mathematical induction. When $n = 1$, $\mathfrak{f}$ and $\mathfrak{g}_1$ are positive continuous monotonic increasing functions, then $\mathfrak{f}$ and $\mathfrak{g}_1$ are two synchronous functions. According to the inequality (85) in Theorem 16, we have the next FII

$$({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}\mathfrak{g}_1)(x) \geqslant ({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}_1)(x), \tag{88}$$

Now we assume that the FII (87) holds for some $n = \mathfrak{n}-1$, $\mathfrak{n} \in \mathbb{N}$, that is,

$$({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h})^{\mathfrak{n}-1}(x)\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}\prod_{k=1}^{\mathfrak{n}-1}\mathfrak{g}_k\Big)(x) \geqslant ({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)\prod_{k=1}^{\mathfrak{n}-1}({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}_k)(x). \tag{89}$$

The function $\mathfrak{f}\prod_{k=1}^{\mathfrak{n}-1}\mathfrak{g}_k$ is also increasing as $\mathfrak{f}$ and $(\mathfrak{g}_k)_{k=1,2,\dots,\mathfrak{n}}$ are increasing. Applying the inequality (85) in Theorem 16 for $\hat{\mathfrak{f}} = \mathfrak{f}\prod_{k=1}^{\mathfrak{n}-1}\mathfrak{g}_k$ and $\hat{\mathfrak{g}} = \mathfrak{g}_{\mathfrak{n}}$, then, from (89), we observe

$$\begin{aligned}\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}\prod_{k=1}^{\mathfrak{n}}\mathfrak{g}_k\Big)(x) &= ({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\hat{\mathfrak{f}}\hat{\mathfrak{g}})(x) \geqslant ({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\hat{\mathfrak{f}})(x)\frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\hat{\mathfrak{g}})(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h})(x)}\\ &= \Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}\prod_{k=1}^{\mathfrak{n}-1}\mathfrak{g}_k\Big)(x)\frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}_{\mathfrak{n}})(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h})(x)} \geqslant \frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h})^{\mathfrak{n}}(x)}\prod_{k=1}^{\mathfrak{n}}({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}_k)(x),\end{aligned} \tag{90}$$

which manifests the desired FII (87). This completes the proof of Theorem 17. □

Theorem 18. *Let* $\mathfrak{f}, \mathfrak{g}$ *and* $\mathfrak{h}$ *be three positive continuous functions on* $[\mathfrak{u},\mathfrak{v}]$ *such that* $\mathfrak{f}$ *is increasing and* $\mathfrak{g}$ *is differentiable with* $m = \inf_{x\in[\mathfrak{u},\mathfrak{v}]}\mathfrak{g}'(x)$*. Then, for* $x \in [\mathfrak{u},\mathfrak{v}]$*, we have the following FII*

$$\frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}\mathfrak{g})(x) + m({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{i})(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g})(x) + m({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}\mathfrak{i})(x)} \geqslant 1. \tag{91}$$

Proof. Let $\hat{\mathfrak{g}}(v) = \mathfrak{g}(v) - m\mathfrak{i}(v)$ with $\mathfrak{i}(v) = v$, then $\hat{\mathfrak{g}}$ is increasing on $[\mathfrak{u},\mathfrak{v}]$. According to the inequality (85) in Theorem 16, we obtain

$$({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}\hat{\mathfrak{g}})(x) \geqslant ({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\hat{\mathfrak{g}})(x) = ({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g})(x) - m({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{i})(x). \tag{92}$$

From (92) and $({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}\hat{\mathfrak{g}})(x) = ({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}\mathfrak{g})(x) - m({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}\mathfrak{i})(x)$, we get

$$({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}\mathfrak{g})(x) - m({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}\mathfrak{i})(x) \geqslant ({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g})(x) - m({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{i})(x). \tag{93}$$

which implies the desired FII (91). This completes the proof of Theorem 18. □

Corollary 8. *Let* $\mathfrak{f}, \mathfrak{g}$ *and* $\mathfrak{h}$ *be three positive continuous functions on* $[\mathfrak{u},\mathfrak{v}]$ *such that* $\mathfrak{f}$ *is decreasing and* $\mathfrak{g}$ *is differentiable with* $M = \sup_{x\in[\mathfrak{u},\mathfrak{v}]}\mathfrak{g}'(x)$*. Then, for* $x \in [\mathfrak{u},\mathfrak{v}]$*, we have the following FII*

$$\frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}\mathfrak{g})(x) + M({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{i})(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g})(x) + M({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}\mathfrak{i})(x)} \geqslant 1. \tag{94}$$

Proof. Let $\tilde{\mathfrak{g}}(v) = \mathfrak{g}(v) - M\mathfrak{i}(v)$ with $\mathfrak{i}(v) = v$, then $\tilde{\mathfrak{g}}$ is decreasing on $[\mathfrak{u}, \mathfrak{v}]$. Similar to the proof of Theorem 18, we claim the desired FII (94). □

Corollary 9. *Let $\mathfrak{f}, \mathfrak{g}$ and $\mathfrak{h}$ be three positive continuous functions on $[\mathfrak{u}, \mathfrak{v}]$ such that $\mathfrak{f}$ and $\mathfrak{g}$ are differentiable with $m_1 = \inf_{x\in[\mathfrak{u},\mathfrak{v}]} \mathfrak{f}'(x)$ and $m_2 = \inf_{x\in[\mathfrak{u},\mathfrak{v}]} \mathfrak{g}'(x)$. Then, for $x \in [\mathfrak{u}, \mathfrak{v}]$, we have the following FII*

$$\frac{\mathfrak{U}_{\alpha,m_1,m_2}(x)}{\mathfrak{V}_{\alpha,m_1,m_2}(x)} \geqslant 1, \tag{95}$$

where

$$\begin{aligned}\mathfrak{U}_{\alpha,\vartriangle,\triangledown}(x) =&({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}\mathfrak{g})(x) + \vartriangle ({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{i})(x)\\ &+ \triangledown({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{i})(x) + \vartriangle \triangledown({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{i}^2)(x),\end{aligned} \tag{96}$$

$$\begin{aligned}\mathfrak{V}_{\alpha,\vartriangle,\triangledown}(x) =&({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g})(x) + \vartriangle ({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}\mathfrak{i})(x)\\ &+ \triangledown({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h})(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}\mathfrak{i})(x) + \vartriangle \triangledown({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{i})^2(x).\end{aligned} \tag{97}$$

Proof. Let $\mathfrak{f}^*(v) = \mathfrak{f}(v) - m_1\mathfrak{i}(v)$ and $\mathfrak{g}^*(v) = \mathfrak{g}(v) - m_2\mathfrak{i}(v)$ with $\mathfrak{i}(v) = v$, then $\mathfrak{f}^*$ and $\mathfrak{g}^*$ are increasing on $[\mathfrak{u}, \mathfrak{v}]$. Similar to the proof of Theorem 18, we obtain the FII (95). □

Corollary 10. *Let $\mathfrak{f}, \mathfrak{g}$ and $\mathfrak{h}$ be three positive continuous functions on $[\mathfrak{u}, \mathfrak{v}]$ such that $\mathfrak{f}$ and $\mathfrak{g}$ are differentiable with $M_1 = \sup_{x\in[\mathfrak{u},\mathfrak{v}]} \mathfrak{f}'(x)$ and $M_2 = \sup_{x\in[\mathfrak{u},\mathfrak{v}]} \mathfrak{g}'(x)$. Then, for $x \in [\mathfrak{u}, \mathfrak{v}]$, we have the following FII*

$$\frac{\mathfrak{U}_{\alpha,M_1,M_2}(x)}{\mathfrak{V}_{\alpha,M_1,M_2}(x)} \geqslant 1. \tag{98}$$

Proof. Let $\mathfrak{f}^\star(v) = \mathfrak{f}(v) - M_1\mathfrak{i}(v)$ and $\mathfrak{g}^\star(v) = \mathfrak{g}(v) - M_2\mathfrak{i}(v)$ with $\mathfrak{i}(v) = v$, then $\mathfrak{f}^\star$ and $\mathfrak{g}^\star$ are decreasing on $[\mathfrak{u}, \mathfrak{v}]$. Similar to the proof of Theorem 18, we claim the FII (98). □

Remark 12. By setting the different parameters and functions in (9), the modified unified generalized Chebyshev type FIIs in Theorem 16–Theorem 18 can reduce to the generalized FIIs with Wright function in the kernel [30, Thereom 1-Theorem 3].

5. Reversed Minkowski Inequalities for MUGFIOs

In this section, by utilizing the MUGFIOs with EUMLFs, we will presented some new reversed Minkowski FIIs. It follows from [35] that we have the next FIIs: let $\mathfrak{f}, \mathfrak{g}$ and $\mathfrak{h}$ are three positive continuous functions on $[\mathfrak{u}, \mathfrak{v}]$ and φ be a convex function, then, for $1/\mathfrak{p} + 1/\mathfrak{q} =$ with $\mathfrak{p} > 1$, we have

$$({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}\mathfrak{g})(x) \leqslant ({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\mathfrak{p}})^{\frac{1}{\mathfrak{p}}}(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}^{\mathfrak{q}})^{\frac{1}{\mathfrak{q}}}(x) \quad \text{(Hölder's inequality)}, \tag{99}$$

$$\varphi\left(\frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h})(x)}\right) \leqslant \frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\varphi(\mathfrak{f}))(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h})(x)} \quad \text{(Jensen's inequality)}. \tag{100}$$

It follows from (99) that for $0 < \mathfrak{p} \leqslant \mathfrak{q}$, we get the following inequality

$$({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\mathfrak{p}})^{\frac{1}{\mathfrak{p}}}(x) \leqslant ({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h})^{\frac{\mathfrak{q}-\mathfrak{p}}{\mathfrak{p}\mathfrak{q}}}(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\mathfrak{q}})^{\frac{1}{\mathfrak{q}}}(x). \tag{101}$$

It follows from (100) that for $\varphi(v) = v^{\hbar}$, we get the following inequalities

$$\left(\frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h})(x)}\right)^{\hbar} \leqslant \frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\hbar})(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h})(x)} \quad (\hbar \geqslant 1), \tag{102}$$

$$\left(\frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f})(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h})(x)}\right)^{\hbar} \geqslant \frac{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\hbar})(x)}{({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h})(x)} \quad (0 < \hbar \leqslant 1). \tag{103}$$

Theorem 19. *Let* $\mathfrak{f}, \mathfrak{g}$ *and* $\mathfrak{h}$ *be three positive continuous functions satisfying* $0 < \kappa < \mathbb{m} \leqslant \vartheta\mathfrak{f}(\mathfrak{s})/\mathfrak{g}(\mathfrak{s}) \leqslant \mathbb{M}$ *for all* $\mathfrak{s} \in [\mathfrak{u}, \mathfrak{v}]$. *Then, for* $0 < \mathbb{p} \leqslant \mathbb{q}$ *and* $x \in [\mathfrak{u}, \mathfrak{v}]$, *we have the following FII*

$$\frac{\mathbb{M}+\vartheta}{\vartheta(\mathbb{M}-\kappa)}({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h})^{\frac{\mathbb{p}-\mathbb{q}}{\mathbb{p}\mathbb{q}}}(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}(\vartheta\mathfrak{f}-\kappa\mathfrak{g})^{\mathbb{p}})^{\frac{1}{\mathbb{p}}}(x) \leqslant ({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\mathbb{q}})^{\frac{1}{\mathbb{q}}}(x) + ({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}^{\mathbb{q}})^{\frac{1}{\mathbb{q}}}(x) \leqslant \frac{\mathbb{m}+\vartheta}{\vartheta(\mathbb{m}-\kappa)}({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}(\vartheta\mathfrak{f}-\kappa\mathfrak{g})^{\mathbb{q}})^{\frac{1}{\mathbb{q}}}(x). \quad (104)$$

Proof. From the hypothesis $0 < \kappa < \mathbb{m} \leqslant \vartheta\mathfrak{f}(\mathfrak{s})/\mathfrak{g}(\mathfrak{s}) \leqslant \mathbb{M}$, we have

$$\frac{1}{\kappa}-\frac{1}{\mathbb{m}} \leqslant \frac{1}{\kappa}-\frac{\mathfrak{g}(\mathfrak{s})}{\vartheta\mathfrak{f}(\mathfrak{s})} \leqslant \frac{1}{\kappa}-\frac{1}{\mathbb{M}} \Rightarrow \frac{\mathbb{M}}{\mathbb{M}-\kappa} \leqslant \frac{\vartheta\mathfrak{f}(\mathfrak{s})}{\vartheta\mathfrak{f}(\mathfrak{s})-\kappa\mathfrak{g}(\mathfrak{s})} \leqslant \frac{\mathbb{m}}{\mathbb{m}-\kappa}, \quad (105)$$

$$0 < \mathbb{m}-\kappa \leqslant \frac{\vartheta\mathfrak{f}(\mathfrak{s})-\kappa\mathfrak{g}(\mathfrak{s})}{\mathfrak{g}(\mathfrak{s})} \leqslant \mathbb{M}-\kappa \Rightarrow \frac{\vartheta\mathfrak{f}(\mathfrak{s})-\kappa\mathfrak{g}(\mathfrak{s})}{\mathbb{M}-\kappa} \leqslant \mathfrak{g}(\mathfrak{s}) \leqslant \frac{\vartheta\mathfrak{f}(\mathfrak{s})-\kappa\mathfrak{g}(\mathfrak{s})}{\mathbb{m}-\kappa}. \quad (106)$$

It follows from the inequality (105) that for $0 < \mathbb{p} \leqslant \mathbb{q}$, we obtain

$$\Big(\frac{\mathbb{M}}{\vartheta(\mathbb{M}-\kappa)}(\vartheta\mathfrak{f}(\mathfrak{s})-\kappa\mathfrak{g}(\mathfrak{s}))\Big)^{\mathbb{p}} \leqslant \mathfrak{f}^{\mathbb{p}}(\mathfrak{s}), \quad \mathfrak{f}^{\mathbb{q}}(\mathfrak{s}) \leqslant \Big(\frac{\mathbb{m}}{\vartheta(\mathbb{m}-\kappa)}(\vartheta\mathfrak{f}(\mathfrak{s})-\kappa\mathfrak{g}(\mathfrak{s}))\Big)^{\mathbb{q}}. \quad (107)$$

Multiplying both sides of (107) by $\aleph^{-1}(x)\aleph(\mathfrak{s})\mathfrak{M}^{\mathfrak{s}}_{x}({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}, \xi, \phi)\mathfrak{h}(\mathfrak{s})\xi'(\mathfrak{s})$ and integrating with respect to $\mathfrak{s}$ both sides of the obtained results from $\mathfrak{u}$ to x, then we observe

$$\frac{\mathbb{M}}{\vartheta(\mathbb{M}-\kappa)}({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}(\vartheta\mathfrak{f}-\kappa\mathfrak{g})^{\mathbb{p}})^{\frac{1}{\mathbb{p}}}(x) \leqslant ({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\mathbb{p}})^{\frac{1}{\mathbb{p}}}(x), \quad (108)$$

$$({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\mathbb{q}})^{\frac{1}{\mathbb{q}}}(x) \leqslant \frac{\mathbb{m}}{\vartheta(\mathbb{m}-\kappa)}({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}(\vartheta\mathfrak{f}-\kappa\mathfrak{g})^{\mathbb{q}})^{\frac{1}{\mathbb{q}}}(x). \quad (109)$$

According to the inequalities (101) and (108), we get the next FII

$$\frac{\mathbb{M}}{\vartheta(\mathbb{M}-\kappa)}({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h})^{\frac{\mathbb{p}-\mathbb{q}}{\mathbb{p}\mathbb{q}}}(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}(\vartheta\mathfrak{f}-\kappa\mathfrak{g})^{\mathbb{p}})^{\frac{1}{\mathbb{p}}}(x) \leqslant ({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\mathbb{q}})^{\frac{1}{\mathbb{q}}}(x). \quad (110)$$

It follows from the inequality (106) that for $0 < \mathbb{p} \leqslant \mathbb{q}$, we obtain

$$\Big(\frac{1}{\mathbb{M}-\kappa}(\vartheta\mathfrak{f}(\mathfrak{s})-\kappa\mathfrak{g}(\mathfrak{s}))\Big)^{\mathbb{p}} \leqslant \mathfrak{g}^{\mathbb{p}}(\mathfrak{s}), \quad \mathfrak{g}^{\mathbb{q}}(\mathfrak{s}) \leqslant \Big(\frac{1}{\mathbb{m}-\kappa}(\vartheta\mathfrak{f}(\mathfrak{s})-\kappa\mathfrak{g}(\mathfrak{s}))\Big)^{\mathbb{q}}. \quad (111)$$

Multiplying both sides of (111) by $\aleph^{-1}(x)\aleph(\mathfrak{s})\mathfrak{M}^{\mathfrak{s}}_{x}({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}, \xi, \phi)\mathfrak{h}(\mathfrak{s})\xi'(\mathfrak{s})$ and integrating with respect to $\mathfrak{s}$ both sides of the obtained results from $\mathfrak{u}$ to x, then we acuqire

$$\frac{1}{\mathbb{M}-\kappa}({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}(\vartheta\mathfrak{f}-\kappa\mathfrak{g})^{\mathbb{p}})^{\frac{1}{\mathbb{p}}}(x) \leqslant ({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}^{\mathbb{p}})^{\frac{1}{\mathbb{p}}}(x), \quad (112)$$

$$({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}^{\mathbb{q}})^{\frac{1}{\mathbb{q}}}(x) \leqslant \frac{1}{\mathbb{m}-\kappa}({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}(\vartheta\mathfrak{f}-\kappa\mathfrak{g})^{\mathbb{q}})^{\frac{1}{\mathbb{q}}}(x). \quad (113)$$

According to the inequalities (101) and (112), we get the next FII

$$\frac{1}{\mathbb{M}-\kappa}({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h})^{\frac{\mathbb{p}-\mathbb{q}}{\mathbb{p}\mathbb{q}}}(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}(\vartheta\mathfrak{f}-\kappa\mathfrak{g})^{\mathbb{p}})^{\frac{1}{\mathbb{p}}}(x) \leqslant ({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}^{\mathbb{q}})^{\frac{1}{\mathbb{q}}}(x). \quad (114)$$

By means of the inequalities (109), (110), (113) and (114), we acquire the desired FII (104). This completes the proof of Theorem 19. □

Theorem 20. *Let* $\mathfrak{f}, \mathfrak{g}$ *and* $\mathfrak{h}$ *be three positive continuous functions satisfying* $0 < \kappa < \mathbb{m} \leqslant \vartheta\mathfrak{f}(\mathfrak{s})/\mathfrak{g}(\mathfrak{s}) \leqslant \mathbb{M}$ *for all* $\mathfrak{s} \in [\mathfrak{u}, \mathfrak{v}]$*. Then, for* $x \in [\mathfrak{u}, \mathfrak{v}]$ *and* $0 < \mathbb{q}$*, we have the following FIIs*

$$\frac{\mathbb{M}+\vartheta}{\vartheta(\mathbb{M}-\kappa)}({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h})^{\frac{1-\mathbb{p}}{\mathbb{q}}}(x)({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}(\vartheta\mathfrak{f}-\kappa\mathfrak{g})^{\frac{\mathbb{q}}{\mathbb{p}}})^{\frac{\mathbb{p}}{\mathbb{q}}}(x) \leqslant ({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}\mathfrak{f}^{\mathbb{q}})^{\frac{1}{\mathbb{q}}}(x) + ({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}\mathfrak{g}^{\mathbb{q}})^{\frac{1}{\mathbb{q}}}(x) \leqslant \frac{\mathbb{m}+\vartheta}{\vartheta(\mathbb{m}-\kappa)}({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}(\vartheta\mathfrak{f}-\kappa\mathfrak{g})^{\mathbb{q}})^{\frac{1}{\mathbb{q}}}(x) \quad (1 \leqslant \mathbb{p}), \tag{115}$$

$$\frac{\mathbb{M}+\vartheta}{\vartheta(\mathbb{M}-\kappa)}({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}(\vartheta\mathfrak{f}-\kappa\mathfrak{g})^{\mathbb{q}})^{\frac{1}{\mathbb{q}}}(x) \leqslant ({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}\mathfrak{f}^{\mathbb{q}})^{\frac{1}{\mathbb{q}}}(x) + ({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}\mathfrak{g}^{\mathbb{q}})^{\frac{1}{\mathbb{q}}}(x) \leqslant \frac{\mathbb{m}+\vartheta}{\vartheta(\mathbb{m}-\kappa)}({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h})^{\frac{1-\mathbb{p}}{\mathbb{q}}}(x)({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}(\vartheta\mathfrak{f}-\kappa\mathfrak{g})^{\frac{\mathbb{q}}{\mathbb{p}}})^{\frac{\mathbb{p}}{\mathbb{q}}}(x) \quad (0 < \mathbb{p} \leqslant 1). \tag{116}$$

Proof. It follows from the inequality (105) that for $1 \leqslant \mathbb{p}$, we obtain

$$\Big(\frac{\mathbb{M}}{\vartheta(\mathbb{M}-\kappa)}(\vartheta\mathfrak{f}(\mathfrak{s})-\kappa\mathfrak{g}(\mathfrak{s}))\Big)^{\frac{\mathbb{q}}{\mathbb{p}}} \leqslant \mathfrak{f}^{\frac{\mathbb{q}}{\mathbb{p}}}(\mathfrak{s}). \tag{117}$$

Multiplying both sides of (107) by $\aleph^{-1}(x)\aleph(\mathfrak{s})\mathfrak{M}_x^{\mathfrak{s}}({}_{\sigma_1,\sigma_2}^{\varrho_1,\varrho_2}\mathbf{M}_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}, \xi, \phi)\mathfrak{h}(\mathfrak{s})\xi'(\mathfrak{s})$ and integrating with respect to $\mathfrak{s}$ both sides of the obtained result from $\mathfrak{u}$ to x, then we derive

$$\Big(\frac{\mathbb{M}}{\vartheta(\mathbb{M}-\kappa)}\Big)^{\frac{\mathbb{q}}{\mathbb{p}}}({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}(\vartheta\mathfrak{f}-\kappa\mathfrak{g})^{\frac{\mathbb{q}}{\mathbb{p}}})(x) \leqslant ({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}\mathfrak{f}^{\frac{\mathbb{q}}{\mathbb{p}}})(x). \tag{118}$$

Applying the inequality (103) with $\hbar = 1/\mathbb{p} \leqslant 1$ to the right side of (118), then we have

$$({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}\mathfrak{f}^{\frac{\mathbb{q}}{\mathbb{p}}})(x) \leqslant ({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h})^{\frac{\mathbb{p}-1}{\mathbb{p}}}(x)({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}\mathfrak{f}^{\mathbb{q}})^{\frac{1}{\mathbb{p}}}(x). \tag{119}$$

According to the inequalities (118) and (119), we get the next FII

$$\frac{\mathbb{M}}{\vartheta(\mathbb{M}-\kappa)}({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h})^{\frac{1-\mathbb{p}}{\mathbb{q}}}(x)({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}(\vartheta\mathfrak{f}-\kappa\mathfrak{g})^{\frac{\mathbb{q}}{\mathbb{p}}})^{\frac{\mathbb{p}}{\mathbb{q}}}(x) \leqslant ({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}\mathfrak{f}^{\mathbb{q}})^{\frac{1}{\mathbb{q}}}(x). \tag{120}$$

It follows from the inequality (106) that for $1 \leqslant \mathbb{p}$, we obtain

$$\Big(\frac{1}{\mathbb{M}-\kappa}(\vartheta\mathfrak{f}(\mathfrak{s})-\kappa\mathfrak{g}(\mathfrak{s}))\Big)^{\frac{\mathbb{q}}{\mathbb{p}}} \leqslant \mathfrak{g}^{\frac{\mathbb{q}}{\mathbb{p}}}(\mathfrak{s}). \tag{121}$$

Multiplying both sides of (121) by $\aleph^{-1}(x)\aleph(\mathfrak{s})\mathfrak{M}_x^{\mathfrak{s}}({}_{\sigma_1,\sigma_2}^{\varrho_1,\varrho_2}\mathbf{M}_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}, \xi, \phi)\mathfrak{h}(\mathfrak{s})\xi'(\mathfrak{s})$ and integrating with respect to $\mathfrak{s}$ both sides of the obtained result from $\mathfrak{u}$ to x, then we observe

$$\Big(\frac{1}{\mathbb{M}-\kappa}\Big)^{\frac{\mathbb{q}}{\mathbb{p}}}({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}(\vartheta\mathfrak{f}-\kappa\mathfrak{g})^{\frac{\mathbb{q}}{\mathbb{p}}})(x) \leqslant ({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}\mathfrak{g}^{\frac{\mathbb{q}}{\mathbb{p}}})(x). \tag{122}$$

Applying the inequality (103) with $\hbar = 1/\mathbb{p} \leqslant 1$ to the right side of (122), then we have

$$({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}\mathfrak{g}^{\frac{\mathbb{q}}{\mathbb{p}}})(x) \leqslant ({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h})^{\frac{\mathbb{p}-1}{\mathbb{p}}}(x)({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}\mathfrak{g}^{\mathbb{q}})^{\frac{1}{\mathbb{p}}}(x). \tag{123}$$

According to the inequalities (122) and (123), we get the next FII

$$\frac{1}{\mathbb{M}-\kappa}({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h})^{\frac{1-\mathbb{p}}{\mathbb{q}}}(x)({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}(\vartheta\mathfrak{f}-\kappa\mathfrak{g})^{\frac{\mathbb{q}}{\mathbb{p}}})^{\frac{\mathbb{p}}{\mathbb{q}}}(x) \leqslant ({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\mathfrak{h}\mathfrak{g}^{\mathbb{q}})^{\frac{1}{\mathbb{q}}}(x). \tag{124}$$

By means of the inequalities (109), (113), (120) and (124), we acquire the desired FII (115).

Next, it follows from the inequality (105) that for $0 < \mathbb{p} \leqslant 1$, we obtain

$$\Big(\frac{\mathbb{M}}{\vartheta(\mathbb{M}-\kappa)}(\vartheta\mathfrak{f}(\mathfrak{s})-\kappa\mathfrak{g}(\mathfrak{s}))\Big)^{\mathbb{q}} \leqslant \mathfrak{f}^{\mathbb{q}}(\mathfrak{s}), \quad \mathfrak{f}^{\frac{\mathbb{q}}{\mathbb{p}}}(\mathfrak{s}) \leqslant \Big(\frac{\mathbb{m}}{\vartheta(\mathbb{m}-\kappa)}(\vartheta\mathfrak{f}(\mathfrak{s})-\kappa\mathfrak{g}(\mathfrak{s}))\Big)^{\frac{\mathbb{q}}{\mathbb{p}}}. \tag{125}$$

Multiplying both sides of (125) by $\aleph^{-1}(x)\aleph(\mathfrak{s})\mathfrak{M}_x^{\mathfrak{s}}({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu},\xi,\phi)\mathfrak{h}(\mathfrak{s})\xi'(\mathfrak{s})$ and integrating with respect to $\mathfrak{s}$ both sides of the obtained results from $\mathfrak{u}$ to x, then we observe

$$\frac{\mathbb{M}}{\vartheta(\mathbb{M}-\kappa)}({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}(\vartheta\mathfrak{f}-\kappa\mathfrak{g})^{\mathbb{q}})^{\frac{1}{\mathbb{q}}}(x) \leqslant ({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\mathbb{q}})^{\frac{1}{\mathbb{q}}}(x), \tag{126}$$

$$({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\frac{\mathbb{q}}{\mathbb{p}}})(x) \leqslant \Big(\frac{\mathbb{m}}{\vartheta(\mathbb{m}-\kappa)}\Big)^{\frac{\mathbb{q}}{\mathbb{p}}}({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}(\vartheta\mathfrak{f}-\kappa\mathfrak{g})^{\frac{\mathbb{q}}{\mathbb{p}}})(x). \tag{127}$$

Applying the inequality (102) with $\hbar = 1/\mathbb{p} \geqslant 1$ to the left side of (127), then we have

$$({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h})^{\frac{\mathbb{p}-1}{\mathbb{p}}}(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\mathbb{q}})^{\frac{1}{\mathbb{p}}}(x) \leqslant ({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\frac{\mathbb{q}}{\mathbb{p}}})(x). \tag{128}$$

According to the inequalities (127) and (128), we get the next FII

$$({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{f}^{\mathbb{q}})^{\frac{1}{\mathbb{q}}}(x) \leqslant \frac{\mathbb{m}}{\vartheta(\mathbb{m}-\kappa)}({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h})^{\frac{1-\mathbb{p}}{\mathbb{q}}}(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}(\vartheta\mathfrak{f}-\kappa\mathfrak{g})^{\frac{\mathbb{q}}{\mathbb{p}}})^{\frac{\mathbb{p}}{\mathbb{q}}}(x). \tag{129}$$

It follows from the inequality (106) that for $0 < \mathbb{p} \leqslant 1$, we obtain

$$\Big(\frac{1}{\mathbb{M}-\kappa}(\vartheta\mathfrak{f}(\mathfrak{s})-\kappa\mathfrak{g}(\mathfrak{s}))\Big)^{\mathbb{q}} \leqslant \mathfrak{g}^{\mathbb{q}}(\mathfrak{s}), \quad \mathfrak{g}^{\frac{\mathbb{q}}{\mathbb{p}}}(\mathfrak{s}) \leqslant \Big(\frac{1}{\mathbb{m}-\kappa}(\vartheta\mathfrak{f}(\mathfrak{s})-\kappa\mathfrak{g}(\mathfrak{s}))\Big)^{\frac{\mathbb{q}}{\mathbb{p}}}. \tag{130}$$

Multiplying both sides of (130) by $\aleph^{-1}(x)\aleph(\mathfrak{s})\mathfrak{M}_x^{\mathfrak{s}}({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu},\xi,\phi)\mathfrak{h}(\mathfrak{s})\xi'(\mathfrak{s})$ and integrating with respect to $\mathfrak{s}$ both sides of the obtained results from $\mathfrak{u}$ to x, then we acuqire

$$\frac{1}{\mathbb{M}-\kappa}({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}(\vartheta\mathfrak{f}-\kappa\mathfrak{g})^{\mathbb{q}})^{\frac{1}{\mathbb{q}}}(x) \leqslant ({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}^{\mathbb{q}})^{\frac{1}{\mathbb{q}}}(x), \tag{131}$$

$$({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}^{\frac{\mathbb{q}}{\mathbb{p}}})(x) \leqslant \frac{1}{\mathbb{m}-\kappa}({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}(\vartheta\mathfrak{f}-\kappa\mathfrak{g})^{\frac{\mathbb{q}}{\mathbb{p}}})(x). \tag{132}$$

Applying the inequality (102) with $\hbar = 1/\mathbb{p} \geqslant 1$ to the left side of (132), then we derive

$$({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h})^{\frac{\mathbb{p}-1}{\mathbb{p}}}(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}^{\mathbb{q}})^{\frac{1}{\mathbb{p}}}(x) \leqslant ({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}^{\frac{\mathbb{q}}{\mathbb{p}}})(x). \tag{133}$$

According to the inequalities (132) and (133), we get the next FII

$$({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}\mathfrak{g}^{\mathbb{q}})^{\frac{1}{\mathbb{q}}}(x) \leqslant \frac{1}{\mathbb{m}-\kappa}({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h})^{\frac{1-\mathbb{p}}{\mathbb{q}}}(x)({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\mathfrak{h}(\vartheta\mathfrak{f}-\kappa\mathfrak{g})^{\frac{\mathbb{q}}{\mathbb{p}}})^{\frac{\mathbb{p}}{\mathbb{q}}}(x). \tag{134}$$

By means of the inequalities (126), (129), (131) and (134), we acquire the desired FII (116). This completes the proof of Theorem 20. □

Remark 13. Applying Theorem 19 for $\mathbb{p} = \mathbb{q}$ and $\vartheta = 1$ and Theorem 20 for $\mathbb{p} = 1$ and $\vartheta = 1$, respectively, then Theorem 19 and Theorem 20 can degenerate into the result given by the author [35, Theorem 13]. By setting the different parameters and functions in (9), Theorem 19 and Theorem 20 can degenerate into the inequalities [5, Theorems 1 and 2].

Theorem 21. *Let $\mathfrak{f}_k, \mathfrak{g}_k$ and $\mathfrak{h}_k$ be positive continuous functions satisfying $0 < \kappa < \mathbb{m} \leqslant \vartheta\mathfrak{f}_k(\mathfrak{s})/\mathfrak{g}_k(\mathfrak{s}) \leqslant \mathbb{M}$ for all $\mathfrak{s} \in [\mathfrak{u}, \mathfrak{v}]$ and $k = 1, 2, \ldots, n$. Then, for $0 < \mathbb{p} \leqslant \mathbb{q}$ and $x \in [\mathfrak{u}, \mathfrak{v}]$, we have the following FII*

$$\frac{(\mathbb{M}+\vartheta)n^{\frac{1}{\mathbb{q}}}}{\vartheta(\mathbb{M}-\kappa)}\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\Big)^{\frac{\mathbb{p}-\mathbb{q}}{\mathbb{p}\mathbb{q}}}(x)\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}(\vartheta\mathfrak{f}_k-\kappa\mathfrak{g}_k)^{\frac{\mathbb{p}}{n}}\Big)^{\frac{1}{\mathbb{p}}}(x) \leqslant \Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\sum_{k=1}^{n}\mathfrak{h}_k\mathfrak{f}_k^{\mathbb{q}}\Big)^{\frac{1}{\mathbb{q}}}(x)$$
$$+\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\sum_{k=1}^{n}\mathfrak{h}_k\mathfrak{g}_k^{\mathbb{q}}\Big)^{\frac{1}{\mathbb{q}}}(x) \leqslant \frac{\mathbb{m}+\vartheta}{\vartheta(\mathbb{m}-\kappa)}\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\sum_{k=1}^{n}\mathfrak{h}_k(\vartheta\mathfrak{f}_k-\kappa\mathfrak{g}_k)^{\mathbb{q}}\Big)^{\frac{1}{\mathbb{q}}}(x). \tag{135}$$

Proof. From the hypothesis $0 < \kappa < \mathbbm{m} \leqslant \vartheta\mathfrak{f}_k(\mathfrak{s})/\mathfrak{g}_k(\mathfrak{s}) \leqslant \mathbb{M}$, we get

$$\frac{1}{\kappa} - \frac{1}{\mathbbm{m}} \leqslant \frac{1}{\kappa} - \frac{\mathfrak{g}_k(\mathfrak{s})}{\vartheta\mathfrak{f}_k(\mathfrak{s})} \leqslant \frac{1}{\kappa} - \frac{1}{\mathbb{M}} \Rightarrow \frac{\mathbb{M}}{\mathbb{M}-\kappa} \leqslant \frac{\vartheta\mathfrak{f}_k(\mathfrak{s})}{\vartheta\mathfrak{f}_k(\mathfrak{s}) - \kappa\mathfrak{g}_k(\mathfrak{s})} \leqslant \frac{\mathbbm{m}}{\mathbbm{m}-\kappa}, \tag{136}$$

$$0 < \mathbbm{m} - \kappa \leqslant \frac{\vartheta\mathfrak{f}_k(\mathfrak{s}) - \kappa\mathfrak{g}_k(\mathfrak{s})}{\mathfrak{g}_k(\mathfrak{s})} \leqslant \mathbb{M} - \kappa \Rightarrow \frac{\vartheta\mathfrak{f}_k(\mathfrak{s}) - \kappa\mathfrak{g}_k(\mathfrak{s})}{\mathbb{M}-\kappa} \leqslant \mathfrak{g}_k(\mathfrak{s}) \leqslant \frac{\vartheta\mathfrak{f}_k(\mathfrak{s}) - \kappa\mathfrak{g}_k(\mathfrak{s})}{\mathbbm{m}-\kappa}. \tag{137}$$

It follows from the inequality (136) that for $0 < \mathbbm{p} \leqslant \mathbbm{q}$, we obtain

$$\prod_{k=1}^{n} \mathfrak{h}_k^{\frac{1}{n}}(\mathfrak{s})\Big(\frac{\mathbb{M}}{\vartheta(\mathbb{M}-\kappa)}(\vartheta\mathfrak{f}_k(\mathfrak{s}) - \kappa\mathfrak{g}_k(\mathfrak{s}))\Big)^{\frac{\mathbbm{p}}{n}} \leqslant \prod_{k=1}^{n} \mathfrak{h}_k^{\frac{1}{n}}(\mathfrak{s})\mathfrak{f}_k^{\frac{\mathbbm{p}}{n}}(\mathfrak{s}), \tag{138}$$

$$\sum_{k=1}^{n} \mathfrak{h}_k(\mathfrak{s})\mathfrak{f}_k^{\mathbbm{q}}(\mathfrak{s}) \leqslant \sum_{k=1}^{n} \mathfrak{h}_k(\mathfrak{s})\Big(\frac{\mathbbm{m}}{\vartheta(\mathbbm{m}-\kappa)}(\vartheta\mathfrak{f}_k(\mathfrak{s}) - \kappa\mathfrak{g}_k(\mathfrak{s}))\Big)^{\mathbbm{q}}. \tag{139}$$

Multiplying both sides of (138) and (139) by $\aleph^{-1}(x)\aleph(\mathfrak{s})\mathfrak{M}_x^{\mathfrak{s}}({}_{\sigma_1,\sigma_2}^{\varrho_1,\varrho_2}\mathbf{M}_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}, \xi, \phi)\xi'(\mathfrak{s})$ and integrating with respect to $\mathfrak{s}$ both sides of the obtained results from $\mathfrak{u}$ to x, respectively, then we observe

$$\frac{\mathbb{M}}{\vartheta(\mathbb{M}-\kappa)}\Big({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}(\vartheta\mathfrak{f}_k - \kappa\mathfrak{g}_k)^{\frac{\mathbbm{p}}{n}}\Big)^{\frac{1}{\mathbbm{p}}}(x) \leqslant \Big({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\mathfrak{f}_k^{\frac{\mathbbm{p}}{n}}\Big)^{\frac{1}{\mathbbm{p}}}(x), \tag{140}$$

$$\Big({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\sum_{k=1}^{n}\mathfrak{h}_k\mathfrak{f}_k^{\mathbbm{q}}\Big)^{\frac{1}{\mathbbm{q}}}(x) \leqslant \frac{\mathbbm{m}}{\vartheta(\mathbbm{m}-\kappa)}\Big({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\sum_{k=1}^{n}\mathfrak{h}_k(\vartheta\mathfrak{f}_k - \kappa\mathfrak{g}_k)^{\mathbbm{q}}\Big)^{\frac{1}{\mathbbm{q}}}(x). \tag{141}$$

Applying the inequalities (101) to the right side of (140), we get the next FII

$$\frac{\mathbb{M}}{\vartheta(\mathbb{M}-\kappa)}\Big({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\Big)^{\frac{\mathbbm{p}-\mathbbm{q}}{\mathbbm{p}\mathbbm{q}}}(x)\Big({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}(\vartheta\mathfrak{f}_k - \kappa\mathfrak{g}_k)^{\frac{\mathbbm{p}}{n}}\Big)^{\frac{1}{\mathbbm{p}}}(x) \leqslant \Big({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\mathfrak{f}^{\frac{\mathbbm{q}}{n}}\Big)^{\frac{1}{\mathbbm{q}}}(x). \tag{142}$$

It follows from the inequality (137) that for $0 < \mathbbm{p} \leqslant \mathbbm{q}$, we obtain

$$\Big(\frac{1}{\mathbb{M}-\kappa}\Big)^{\mathbbm{p}}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}(\mathfrak{s})(\vartheta\mathfrak{f}_k(\mathfrak{s}) - \kappa\mathfrak{g}_k(\mathfrak{s}))^{\frac{\mathbbm{p}}{n}} \leqslant \prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}(\mathfrak{s})\mathfrak{g}_k^{\mathbbm{p}}(\mathfrak{s}), \tag{143}$$

$$\sum_{k=1}^{n}\mathfrak{h}_k(\mathfrak{s})\mathfrak{g}_k^{\mathbbm{q}}(\mathfrak{s}) \leqslant \Big(\frac{1}{\mathbbm{m}-\kappa}\Big)^{\mathbbm{q}}\sum_{k=1}^{n}\mathfrak{h}_k(\mathfrak{s})(\vartheta\mathfrak{f}_k(\mathfrak{s}) - \kappa\mathfrak{g}_k(\mathfrak{s})))^{\mathbbm{q}}. \tag{144}$$

Multiplying both sides of (143) and (144) by $\aleph^{-1}(x)\aleph(\mathfrak{s})\mathfrak{M}_x^{\mathfrak{s}}({}_{\sigma_1,\sigma_2}^{\varrho_1,\varrho_2}\mathbf{M}_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}, \xi, \phi)\xi'(\mathfrak{s})$ and integrating with respect to $\mathfrak{s}$ both sides of the obtained results from $\mathfrak{u}$ to x, respectively, then we acquire

$$\frac{1}{\mathbb{M}-\kappa}\Big({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}(\vartheta\mathfrak{f}_k - \kappa\mathfrak{g}_k)^{\frac{\mathbbm{p}}{n}}\Big)^{\frac{1}{\mathbbm{p}}}(x) \leqslant \Big({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\mathfrak{g}_k^{\frac{\mathbbm{p}}{n}}\Big)^{\frac{1}{\mathbbm{p}}}(x), \tag{145}$$

$$\Big({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\sum_{k=1}^{n}\mathfrak{h}_k\mathfrak{g}_k^{\mathbbm{q}}\Big)^{\frac{1}{\mathbbm{q}}}(x) \leqslant \frac{1}{\mathbbm{m}-\kappa}\Big({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\sum_{k=1}^{n}\mathfrak{h}_k(\vartheta\mathfrak{f}_k - \kappa\mathfrak{g}_k)^{\mathbbm{q}}\Big)^{\frac{1}{\mathbbm{q}}}(x). \tag{146}$$

According to the inequalities (101) and (112), we get the next FII

$$\frac{1}{\mathbb{M}-\kappa}\Big({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\Big)^{\frac{\mathbbm{p}-\mathbbm{q}}{\mathbbm{p}\mathbbm{q}}}(x)\Big({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}(\vartheta\mathfrak{f}_k - \kappa\mathfrak{g}_k)^{\frac{\mathbbm{p}}{n}}\Big)^{\frac{1}{\mathbbm{p}}}(x) \leqslant \Big({}_{\xi}^{\phi}\Theta_{\mathfrak{u}^+}^{\alpha}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\mathfrak{g}_k^{\frac{\mathbbm{q}}{n}}\Big)^{\frac{1}{\mathbbm{q}}}(x). \tag{147}$$

Adding (142) and (147) yields the following result

$$\frac{\mathbb{M}+\vartheta}{\vartheta(\mathbb{M}-\kappa)}\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\Big)^{\frac{\mathfrak{p}-\mathfrak{q}}{\mathfrak{p}\mathfrak{q}}}(x)\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}(\vartheta\mathfrak{f}_k-\kappa\mathfrak{g}_k)^{\frac{\mathfrak{p}}{n}}\Big)^{\frac{1}{\mathfrak{p}}}(x)$$
$$\leqslant\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\mathfrak{f}^{\frac{\mathfrak{q}}{n}}\Big)^{\frac{1}{\mathfrak{q}}}(x)+\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\mathfrak{g}_k^{\frac{\mathfrak{q}}{n}}\Big)^{\frac{1}{\mathfrak{q}}}(x). \tag{148}$$

Applying the arithmetic mean-geometric mean inequality $\big(\prod_{k=1}^{n}\Xi_k\big)^{1/n}\leqslant\sum_{k=1}^{n}\Xi_k/n$ to the right side of the inequality (148), then we derive

$$\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\mathfrak{f}^{\frac{\mathfrak{q}}{n}}\Big)^{\frac{1}{\mathfrak{q}}}(x)+\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\mathfrak{g}_k^{\frac{\mathfrak{q}}{n}}\Big)^{\frac{1}{\mathfrak{q}}}(x)\leqslant\frac{1}{n^{\frac{1}{\mathfrak{q}}}}\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\sum_{k=1}^{n}\mathfrak{h}_k\mathfrak{f}_k^{\mathfrak{q}}\Big)^{\frac{1}{\mathfrak{q}}}(x)+\frac{1}{n^{\frac{1}{\mathfrak{q}}}}\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\sum_{k=1}^{n}\mathfrak{h}_k\mathfrak{g}_k^{\mathfrak{q}}\Big)^{\frac{1}{\mathfrak{q}}}(x). \tag{149}$$

By taking advantage of the inequalities (141), (146), (148) and (149), we acquire the desired FII (135). This completes the proof of Theorem 21. □

Theorem 22. *Let* $\mathfrak{f},\mathfrak{g}$ *and* $\mathfrak{h}$ *be three positive continuous functions satisfying* $0<\kappa<\mathfrak{m}\leqslant\vartheta\mathfrak{f}(\mathfrak{s})/\mathfrak{g}(\mathfrak{s})\leqslant\mathbb{M}$ *for all* $\mathfrak{s}\in[\mathfrak{u},\mathfrak{v}]$. *Then, for* $x\in[\mathfrak{u},\mathfrak{v}]$ *and* $0<\mathfrak{q}$, *we have the following FIIs*

$$\frac{(\mathbb{M}+\vartheta)n^{\frac{1}{\mathfrak{q}}}}{\vartheta(\mathbb{M}-\kappa)}\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\Big)^{\frac{1-\mathfrak{p}}{\mathfrak{q}}}(x)\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}(\vartheta\mathfrak{f}_k-\kappa\mathfrak{g}_k)^{\frac{\mathfrak{q}}{n\mathfrak{p}}}\Big)^{\frac{\mathfrak{p}}{\mathfrak{q}}}(x)\leqslant\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\sum_{k=1}^{n}\mathfrak{h}_k\mathfrak{f}_k^{\mathfrak{q}}\Big)^{\frac{1}{\mathfrak{q}}}(x)$$
$$+\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\sum_{k=1}^{n}\mathfrak{h}_k\mathfrak{g}_k^{\mathfrak{q}}\Big)^{\frac{1}{\mathfrak{q}}}(x)\leqslant\frac{\mathfrak{m}+\vartheta}{\vartheta(\mathfrak{m}-\kappa)}\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\sum_{k=1}^{n}\mathfrak{h}_k(\vartheta\mathfrak{f}_k-\kappa\mathfrak{g}_k)^{\mathfrak{q}}\Big)^{\frac{1}{\mathfrak{q}}}(x)\quad(1\leqslant\mathfrak{p}), \tag{150}$$

$$\frac{(\mathbb{M}+\vartheta)n^{\frac{\mathfrak{p}}{\mathfrak{q}}}}{\vartheta(\mathbb{M}-\kappa)}\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\Big)^{\frac{\mathfrak{p}-1}{\mathfrak{q}}}(x)\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}(\vartheta\mathfrak{f}_k-\kappa\mathfrak{g}_k)^{\frac{\mathfrak{q}}{n}}\Big)^{\frac{1}{\mathfrak{q}}}(x)\leqslant\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\sum_{k=1}^{n}\mathfrak{h}_k\mathfrak{f}_k^{\frac{\mathfrak{q}}{\mathfrak{p}}}\Big)^{\frac{\mathfrak{p}}{\mathfrak{q}}}(x)$$
$$+\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\sum_{k=1}^{n}\mathfrak{h}_k\mathfrak{g}_k^{\frac{\mathfrak{q}}{\mathfrak{p}}}\Big)^{\frac{\mathfrak{p}}{\mathfrak{q}}}(x)\leqslant\frac{\mathfrak{m}+\vartheta}{\vartheta(\mathfrak{m}-\kappa)}\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\sum_{k=1}^{n}\mathfrak{h}_k(\vartheta\mathfrak{f}_k-\kappa\mathfrak{g}_k)^{\frac{\mathfrak{q}}{\mathfrak{p}}}\Big)^{\frac{\mathfrak{p}}{\mathfrak{q}}}(x)\quad(0<\mathfrak{p}\leqslant1). \tag{151}$$

Proof. It follows from the inequality (136) that for $1\leqslant\mathfrak{p}$, we obtain

$$\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}(\mathfrak{s})\Big(\frac{\mathbb{M}}{\vartheta(\mathbb{M}-\kappa)}(\vartheta\mathfrak{f}_k(\mathfrak{s})-\kappa\mathfrak{g}_k(\mathfrak{s}))\Big)^{\frac{\mathfrak{q}}{n\mathfrak{p}}}\leqslant\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}(\mathfrak{s})\mathfrak{f}_k^{\frac{\mathfrak{q}}{n\mathfrak{p}}}(\mathfrak{s}). \tag{152}$$

Multiplying both sides of (152) by $\aleph^{-1}(x)\aleph(\mathfrak{s})\mathfrak{M}^{\mathfrak{s}}_{x}\big({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu},\xi,\phi\big)\xi'(\mathfrak{s})$ and integrating with respect to $\mathfrak{s}$ both sides of the obtained result from $\mathfrak{u}$ to x, then we derive

$$\Big(\frac{\mathbb{M}}{\vartheta(\mathbb{M}-\kappa)}\Big)^{\frac{\mathfrak{q}}{\mathfrak{p}}}\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}(\vartheta\mathfrak{f}_k-\kappa\mathfrak{g}_k)^{\frac{\mathfrak{q}}{n\mathfrak{p}}}\Big)(x)\leqslant\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\mathfrak{f}_k^{\frac{\mathfrak{q}}{n\mathfrak{p}}}\Big)(x). \tag{153}$$

Applying the inequality (103) with $\hbar=1/\mathfrak{p}\leqslant1$ to the right side of (153), then we have

$$\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\mathfrak{f}_k^{\frac{\mathfrak{q}}{n\mathfrak{p}}}\Big)(x)\leqslant\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\Big)^{\frac{\mathfrak{p}-1}{\mathfrak{p}}}(x)\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\mathfrak{f}_k^{\frac{\mathfrak{q}}{n}}\Big)^{\frac{1}{\mathfrak{p}}}(x). \tag{154}$$

According to the inequalities (153) and (154), we get the next FII

$$\frac{\mathbb{M}}{\vartheta(\mathbb{M}-\kappa)}\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\Big)^{\frac{1-\mathfrak{p}}{\mathfrak{q}}}(x)\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}(\vartheta\mathfrak{f}_k-\kappa\mathfrak{g}_k)^{\frac{\mathfrak{q}}{n\mathfrak{p}}}\Big)(x)^{\frac{\mathfrak{p}}{\mathfrak{q}}}(x)\leqslant\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\mathfrak{f}_k^{\frac{\mathfrak{q}}{n}}\Big)^{\frac{1}{\mathfrak{q}}}(x). \tag{155}$$

It follows from the inequality (137) that for $1 \leqslant \mathbb{p}$, we obtain

$$\prod_{k=1}^{n} \mathfrak{h}_k^{\frac{1}{n}}(\mathfrak{s})\Big(\frac{1}{\mathbb{M}-\kappa}(\vartheta\mathfrak{f}_k(\mathfrak{s})-\kappa\mathfrak{g}_k(\mathfrak{s}))\Big)^{\frac{\mathbb{q}}{n\mathbb{p}}} \leqslant \prod_{k=1}^{n} \mathfrak{h}_k^{\frac{1}{n}}(\mathfrak{s})\mathfrak{g}_k^{\frac{\mathbb{q}}{n\mathbb{p}}}(\mathfrak{s}). \tag{156}$$

Multiplying both sides of (156) by $\aleph^{-1}(x)\aleph(\mathfrak{s})\mathfrak{M}_x^{\mathfrak{s}}({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu},\xi,\phi)\xi'(\mathfrak{s})$ and integrating with respect to $\mathfrak{s}$ both sides of the obtained result from $\mathfrak{u}$ to x, then we observe

$$\Big(\frac{1}{\mathbb{M}-\kappa}\Big)^{\frac{\mathbb{q}}{\mathbb{p}}}\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n} \mathfrak{h}_k^{\frac{1}{n}}(\vartheta\mathfrak{f}_k-\kappa\mathfrak{g}_k)^{\frac{\mathbb{q}}{n\mathbb{p}}}\Big)(x) \leqslant \Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n} \mathfrak{h}_k^{\frac{1}{n}}\mathfrak{g}_k^{\frac{\mathbb{q}}{n\mathbb{p}}}\Big)(x). \tag{157}$$

Applying the inequality (103) with $\hbar = 1/\mathbb{p} \leqslant 1$ to the right side of (157), then we have

$$\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n} \mathfrak{h}_k^{\frac{1}{n}}\mathfrak{g}_k^{\frac{\mathbb{q}}{n\mathbb{p}}}\Big)(x) \leqslant \Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n} \mathfrak{h}_k^{\frac{1}{n}}\Big)^{\frac{\mathbb{p}-1}{\mathbb{p}}}(x)\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n} \mathfrak{h}_k^{\frac{1}{n}}\mathfrak{g}_k^{\frac{\mathbb{q}}{n}}\Big)^{\frac{1}{\mathbb{p}}}(x). \tag{158}$$

According to the inequalities (157) and (158), we get the next FII

$$\frac{1}{\mathbb{M}-\kappa}\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n} \mathfrak{h}_k^{\frac{1}{n}}\Big)^{\frac{1-\mathbb{p}}{\mathbb{q}}}(x)\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n} \mathfrak{h}_k^{\frac{1}{n}}(\vartheta\mathfrak{f}_k-\kappa\mathfrak{g}_k)^{\frac{\mathbb{q}}{n\mathbb{p}}}\Big)(x)^{\frac{\mathbb{p}}{\mathbb{q}}}(x) \leqslant \Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n} \mathfrak{h}_k^{\frac{1}{n}}\mathfrak{g}_k^{\frac{\mathbb{q}}{n}}\Big)^{\frac{1}{\mathbb{q}}}(x). \tag{159}$$

Adding (155) and (159) results in the following FII

$$\frac{\mathbb{M}+\vartheta}{\vartheta(\mathbb{M}-\kappa)}\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n} \mathfrak{h}_k^{\frac{1}{n}}\Big)^{\frac{1-\mathbb{p}}{\mathbb{q}}}(x)\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n} \mathfrak{h}_k^{\frac{1}{n}}(\vartheta\mathfrak{f}_k-\kappa\mathfrak{g}_k)^{\frac{\mathbb{q}}{n\mathbb{p}}}\Big)(x)^{\frac{\mathbb{p}}{\mathbb{q}}}(x)$$
$$\leqslant \Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n} \mathfrak{h}_k^{\frac{1}{n}}\mathfrak{f}_k^{\frac{\mathbb{q}}{n}}\Big)^{\frac{1}{\mathbb{q}}}(x) + \Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n} \mathfrak{h}_k^{\frac{1}{n}}\mathfrak{g}_k^{\frac{\mathbb{q}}{n}}\Big)^{\frac{1}{\mathbb{q}}}(x). \tag{160}$$

By means of the inequalities (141), (146), (149) and (160), we acquire the desired FII (150).

Next, it follows from the inequality (136) that for $0 < \mathbb{p} \leqslant 1$, we obtain

$$\Big(\frac{\mathbb{M}}{\vartheta(\mathbb{M}-\kappa)}\Big)^{\mathbb{q}}\prod_{k=1}^{n} \mathfrak{h}_k^{\frac{1}{n}}(\mathfrak{s})(\vartheta\mathfrak{f}_k(\mathfrak{s})-\kappa\mathfrak{g}_k(\mathfrak{s}))^{\frac{\mathbb{q}}{n}} \leqslant \prod_{k=1}^{n} \mathfrak{h}_k^{\frac{1}{n}}(\mathfrak{s})\mathfrak{f}_k^{\frac{\mathbb{q}}{n}}(\mathfrak{s}), \tag{161}$$

$$\sum_{k=1}^{n} \mathfrak{h}_k(\mathfrak{s})\mathfrak{f}_k^{\frac{\mathbb{q}}{\mathbb{p}}}(\mathfrak{s}) \leqslant \sum_{k=1}^{n} \mathfrak{h}_k(\mathfrak{s})\Big(\frac{\mathbb{m}}{\vartheta(\mathbb{m}-\kappa)}(\vartheta\mathfrak{f}_k(\mathfrak{s})-\kappa\mathfrak{g}_k(\mathfrak{s}))\Big)^{\frac{\mathbb{q}}{\mathbb{p}}}. \tag{162}$$

Multiplying both sides of (161) and (162) by $\aleph^{-1}(x)\aleph(\mathfrak{s})\mathfrak{M}_x^{\mathfrak{s}}({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu},\xi,\phi)\xi'(\mathfrak{s})$ and integrating with respect to $\mathfrak{s}$ both sides of the obtained results from $\mathfrak{u}$ to x, respectively, then we observe

$$\frac{\mathbb{M}}{\vartheta(\mathbb{M}-\kappa)}\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n} \mathfrak{h}_k^{\frac{1}{n}}(\vartheta\mathfrak{f}_k-\kappa\mathfrak{g}_k)^{\frac{\mathbb{q}}{n}}\Big)^{\frac{1}{\mathbb{q}}}(x) \leqslant \Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n} \mathfrak{h}_k^{\frac{1}{n}}(\mathfrak{s})\mathfrak{f}_k^{\frac{\mathbb{q}}{n}}\Big)^{\frac{1}{\mathbb{q}}}(x), \tag{163}$$

$$\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\sum_{k=1}^{n} \mathfrak{h}_k\mathfrak{f}_k^{\frac{\mathbb{q}}{\mathbb{p}}}\Big)^{\frac{\mathbb{p}}{\mathbb{q}}}(x) \leqslant \frac{\mathbb{m}}{\vartheta(\mathbb{m}-\kappa)}\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\sum_{k=1}^{n} \mathfrak{h}_k(\vartheta\mathfrak{f}_k-\kappa\mathfrak{g}_k)^{\frac{\mathbb{q}}{\mathbb{p}}}\Big)^{\frac{\mathbb{p}}{\mathbb{q}}}(x). \tag{164}$$

Applying the inequality (102) with $\hbar = 1/\mathbb{p} \geqslant 1$ to the right side of (163), then we have

$$\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n} \mathfrak{h}_k^{\frac{1}{n}}\mathfrak{f}_k^{\frac{\mathbb{q}}{n}}\Big)^{\frac{1}{\mathbb{p}}}(x) \leqslant \Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n} \mathfrak{h}_k^{\frac{1}{n}}\Big)^{\frac{1-\mathbb{p}}{\mathbb{p}}}(x)\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n} \mathfrak{h}_k^{\frac{1}{n}}\mathfrak{f}_k^{\frac{\mathbb{q}}{n\mathbb{p}}}\Big)(x)$$

$$\Rightarrow \Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\mathfrak{f}_k^{\frac{\mathfrak{q}}{n}}\Big)^{\frac{1}{\mathfrak{q}}}(x) \leqslant \Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\Big)^{\frac{1-\mathfrak{p}}{\mathfrak{q}}}(x)\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\mathfrak{f}_k^{\frac{\mathfrak{q}}{n\mathfrak{p}}}\Big)^{\frac{\mathfrak{p}}{\mathfrak{q}}}(x). \tag{165}$$

According to the inequalities (163) and (165), we get the next FII

$$\frac{\mathbb{M}}{\vartheta(\mathbb{M}-\kappa)}\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\Big)^{\frac{\mathfrak{p}-1}{\mathfrak{q}}}(x)\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}(\vartheta\mathfrak{f}_k-\kappa\mathfrak{g}_k)^{\frac{\mathfrak{q}}{n}}\Big)^{\frac{1}{\mathfrak{q}}}(x) \leqslant \Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\mathfrak{f}_k^{\frac{\mathfrak{q}}{n\mathfrak{p}}}\Big)^{\frac{\mathfrak{p}}{\mathfrak{q}}}(x). \tag{166}$$

It follows from the inequality (137) that for $0 < \mathfrak{p} \leqslant 1$, we obtain

$$\Big(\frac{1}{\mathbb{M}-\kappa}\Big)^{\mathfrak{q}}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}(\mathfrak{s})(\vartheta\mathfrak{f}(\mathfrak{s})-\kappa\mathfrak{g}(\mathfrak{s}))^{\frac{\mathfrak{q}}{n}} \leqslant \prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}(\mathfrak{s})\mathfrak{g}^{\frac{\mathfrak{q}}{n}}(\mathfrak{s}), \tag{167}$$

$$\sum_{k=1}^{n}\mathfrak{h}_k(\mathfrak{s})\mathfrak{g}^{\frac{\mathfrak{q}}{\mathfrak{p}}}(\mathfrak{s}) \leqslant \sum_{k=1}^{n}\mathfrak{h}_k(\mathfrak{s})\Big(\frac{1}{\mathfrak{m}-\kappa}(\vartheta\mathfrak{f}(\mathfrak{s})-\kappa\mathfrak{g}(\mathfrak{s}))\Big)^{\frac{\mathfrak{q}}{\mathfrak{p}}}. \tag{168}$$

Multiplying both sides of (167) and (168) by $\aleph^{-1}(x)\aleph(\mathfrak{s})\mathfrak{M}^{\mathfrak{s}}_{x}({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}\mathbf{M}^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu},\xi,\phi)\xi'(\mathfrak{s})$ and integrating with respect to $\mathfrak{s}$ both sides of the obtained results from $\mathfrak{u}$ to x, respectively, then we acquire

$$\frac{1}{\mathbb{M}-\kappa}\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}(\vartheta\mathfrak{f}_k-\kappa\mathfrak{g}_k)^{\frac{\mathfrak{q}}{n}}\Big)^{\frac{1}{\mathfrak{q}}}(x) \leqslant \Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\mathfrak{g}_k^{\frac{\mathfrak{q}}{n}}\Big)^{\frac{1}{\mathfrak{q}}}(x), \tag{169}$$

$$\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\sum_{k=1}^{n}\mathfrak{h}_k\mathfrak{g}_k^{\frac{\mathfrak{q}}{\mathfrak{p}}}\Big)^{\frac{\mathfrak{p}}{\mathfrak{q}}}(x) \leqslant \frac{1}{\mathfrak{m}-\kappa}\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\sum_{k=1}^{n}\mathfrak{h}_k(\vartheta\mathfrak{f}_k-\kappa\mathfrak{g}_k)^{\frac{\mathfrak{q}}{\mathfrak{p}}}\Big)^{\frac{\mathfrak{p}}{\mathfrak{q}}}(x). \tag{170}$$

Applying the inequality (102) with $\hbar = 1/\mathfrak{p} \geqslant 1$ to the right side of (169), then we derive

$$\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\mathfrak{g}_k^{\frac{\mathfrak{q}}{n}}\Big)^{\frac{1}{\mathfrak{p}}}(x) \leqslant \Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\Big)^{\frac{1-\mathfrak{p}}{\mathfrak{p}}}(x)\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\mathfrak{g}_k^{\frac{\mathfrak{q}}{n\mathfrak{p}}}\Big)(x)$$

$$\Rightarrow \Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\mathfrak{g}_k^{\frac{\mathfrak{q}}{n}}\Big)^{\frac{1}{\mathfrak{q}}}(x) \leqslant \Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\Big)^{\frac{1-\mathfrak{p}}{\mathfrak{q}}}(x)\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\mathfrak{g}_k^{\frac{\mathfrak{q}}{n\mathfrak{p}}}\Big)^{\frac{\mathfrak{p}}{\mathfrak{q}}}(x). \tag{171}$$

According to the inequalities (169) and (171), we get the next FII

$$\frac{1}{\mathbb{M}-\kappa}\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\Big)^{\frac{\mathfrak{p}-1}{\mathfrak{q}}}(x)\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}(\vartheta\mathfrak{f}_k-\kappa\mathfrak{g}_k)^{\frac{\mathfrak{q}}{n}}\Big)^{\frac{1}{\mathfrak{q}}}(x) \leqslant \Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\mathfrak{g}_k^{\frac{\mathfrak{q}}{n\mathfrak{p}}}\Big)^{\frac{\mathfrak{p}}{\mathfrak{q}}}(x). \tag{172}$$

Adding (166) and (172) leads to the following result

$$\frac{\mathbb{M}+\vartheta}{\vartheta(\mathbb{M}-\kappa)}\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\Big)^{\frac{\mathfrak{p}-1}{\mathfrak{q}}}(x)\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}(\vartheta\mathfrak{f}_k-\kappa\mathfrak{g}_k)^{\frac{\mathfrak{q}}{n}}\Big)^{\frac{1}{\mathfrak{q}}}(x)$$

$$\leqslant \Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\mathfrak{f}_k^{\frac{\mathfrak{q}}{n\mathfrak{p}}}\Big)^{\frac{\mathfrak{p}}{\mathfrak{q}}}(x) + \Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\mathfrak{g}_k^{\frac{\mathfrak{q}}{n\mathfrak{p}}}\Big)^{\frac{\mathfrak{p}}{\mathfrak{q}}}(x). \tag{173}$$

Applying the arithmetic mean-geometric mean inequality $\Big(\prod_{k=1}^{n}\Xi_k\Big)^{1/n} \leqslant \sum_{k=1}^{n}\Xi_k/n$ to the right side of the inequality (173), then we derive

$$\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\mathfrak{f}_k^{\frac{\mathfrak{q}}{n\mathfrak{p}}}\Big)^{\frac{\mathfrak{p}}{\mathfrak{q}}}(x) + \Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\prod_{k=1}^{n}\mathfrak{h}_k^{\frac{1}{n}}\mathfrak{g}_k^{\frac{\mathfrak{q}}{n\mathfrak{p}}}\Big)^{\frac{\mathfrak{p}}{\mathfrak{q}}}(x) \leqslant \frac{1}{n^{\frac{\mathfrak{p}}{\mathfrak{q}}}}\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\sum_{k=1}^{n}\mathfrak{h}_k\mathfrak{f}_k^{\frac{\mathfrak{q}}{\mathfrak{p}}}\Big)^{\frac{\mathfrak{p}}{\mathfrak{q}}}(x) + \frac{1}{n^{\frac{\mathfrak{p}}{\mathfrak{q}}}}\Big({}^{\phi}_{\xi}\Theta^{\alpha}_{\mathfrak{u}^+}\sum_{k=1}^{n}\mathfrak{h}_k\mathfrak{g}_k^{\frac{\mathfrak{q}}{\mathfrak{p}}}\Big)^{\frac{\mathfrak{p}}{\mathfrak{q}}}(x). \tag{174}$$

By means of the inequalities (164), (170), (173) and (174), we acquire the desired FII (151). This completes the proof of Theorem 22. □

Remark 14. Applying (135) and (150) for $n = 1$, then the inequalities (135) and (150) reduce to the inequalities (104) and (115), respectively.

6. Conclusion

According to some known inequalities involving some positive continuous functions and convex functions, we have established certain new MUGFIIs for some monotonic functions and convex functions. Furthermore, some novel Chebyshev type inequalities and reversed Minkowski inequalities have been obtained based on the MUGFIOs with EUMLFs. By setting the different parameters and functions in MUGFIOs, the previous main results can degenerate into a great deal of FIIs with kinds of known FIOs. Hence, some existing FIOs in the references can be seen as the special cases of the main results in this paper. Following the main results in this paper, we will investigate other related MUGFIIs with EUMLFs in the future.

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