Filomat 39:30 (2025), 10609–10630
https://doi.org/10.2298/FIL2530609R

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# $q$-Numerical radii of sectorial matrices and $2 \times 2$ operator matrices

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**Abstract.** This article focuses on several significant bounds of $q$-numerical radius $w_q(A)$ for sectorial matrix $A$, which refine and generalize previously established bounds. One of the significant bounds we have derived is as follows:

$$\frac{|q|^2 \cos^2(\alpha)}{2} \|A^*A + AA^*\| \leq w_q^2(A) \leq \frac{\left(\sqrt{(1-|q|^2)\left(1+2\sin^2(\alpha)\right)} + |q|\right)^2}{2} \|A^*A + AA^*\|,$$

where $A$ is a sectorial matrix. Also, upper bounds for commutator and anti-commutator matrices and relations between $w_q(A^t)$ and $w_q^t(A)$ for non-integral power $t \in [0, 1]$ are also obtained. Moreover, a few significant estimations of $q$-numerical radii of off-diagonal $2 \times 2$ operator matrices are developed.

## 1. Introduction

Let $\mathcal{B}(\mathcal{H})$ be the $C^*$ algebra of all bounded linear operators acting on the Hilbert space $(\mathcal{H}, \langle ., . \rangle)$ equipped with the operator norm. For any $T \in \mathcal{B}(\mathcal{H})$, the numerical range $W(T)$, the numerical radius $w(T)$, and the operator norm $\|T\|$ are defined, respectively, by

$$W(T) = \{\langle Tx, x \rangle : x \in \mathcal{H}, \|x\| = 1\},$$

$$w(T) = \sup\{|\langle Tx, x \rangle| : x \in \mathcal{H}, \|x\| = 1\}, \text{ and}$$

$$\|T\| = \sup\{|\langle Tx, y \rangle| : x, y \in \mathcal{H}, \|x\| = \|y\| = 1\}.$$

It is well known that $w(.)$ defines a norm on $\mathcal{B}(\mathcal{H})$, which is equivalent to the usual operator norm $\|T\|$. In fact, for every $T \in \mathcal{B}(\mathcal{H})$, the following relation holds:

$$\frac{\|T\|}{2} \leq w(T) \leq \|T\|. \tag{1}$$

2020 *Mathematics Subject Classification*. Primary 15A60; Secondary 15B48, 47B44, 47A63.
*Keywords*. $q$-numerical radius, Sectorial matrices, Operator matrices, Inequality.
Received: 02 April 2025; Revised: 02 September 2025; Accepted: 16 September 2025
Communicated by Dragan S. Djordjević
The second author gratefully acknowledge the financial support received from the Anusandhan National Research Foundation (ANRF), Government of India (SRG/2023/002420).
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The left-hand side inequality becomes an equality when $T^2 = 0$, and the other inequality becomes an equality when the operator $T$ is normal.

In 2005, Kittaneh [20] obtained one more refinement of inequality (1) as follows

$$\frac{1}{4}\|T^*T + TT^*\| \leq w^2(T) \leq \frac{1}{2}\|T^*T + TT^*\|. \tag{2}$$

For more such inequalities, one may refer to the recent articles [4, 6, 12, 13, 22, 23] along with the references therein and, the books [5, 17].

Recently, studies have been conducted on the numerical radius of a particular class of matrices, known as sectorial matrices. Let $M_n$ be the algebra of $n \times n$ matrices. Every $A \in M_n$ admits the decomposition $A = \mathcal{R}(A) + i\mathcal{I}(A)$, where $\mathcal{R}(A) = \frac{A+A^*}{2}$ and $\mathcal{I}(A) = \frac{A-A^*}{2i}$ are Hermitian matrices. Recall that a matrix $A \in M_n$ is said to be positive semi-definite (positive definite), denoted by $A \geq 0$ ($A > 0$), if $\langle Ax, x\rangle \geq 0$ ($\langle Ax, x\rangle > 0$), for all $x \in \mathbb{C}^n$ ($0 \neq x \in \mathbb{C}^n$). A matrix $A \in M_n$ is said to be accretive if $\mathcal{R}(A) > 0$. Also, $A \in M_n$ is said to be accretive-dissipative if $\mathcal{R}(A) > 0$ and $\mathcal{I}(A) > 0$. In other words, $A$ is accretive if the numerical range $W(A)$ is a subset of the right-half plane. If the numerical range $W(A)$ is a subset of a sector $S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$ in the right half of the complex plane, then $A$ is said to be sectorial where

$$S_\alpha = \{z \in \mathbb{C} : \mathcal{R}z > 0, |\mathcal{I}z| \leq \tan(\alpha)(\mathcal{R}z)\}.$$

Here, $\mathcal{R}z$ and $\mathcal{I}z$ denote the real and imaginary parts of the complex number $z$. The class of all $n \times n$ sectorial matrices where $W(T) \subseteq S_\alpha$ is denoted by $\prod_{s,\alpha}^n$. If $\alpha = 0$, then the sector $S_\alpha$ reduces to the interval $(0, \infty)$, reducing the class $\prod_{s,\alpha}^n$ to the set of all positive definite matrices in $M_n$. If $A > 0$, then $A$ is a sectorial matrix with $\alpha = 0$. The numerical range and radius of sectorial matrices have been explored by several authors. In particular, Samah Abu Sammour et al. [28], Yassine Bedrani et al. [3], and Pintu Bhunia et al. [7] have focused their study on the bounds of the numerical radius of sectorial matrices. However, we explore the concept of more generalized numerical range, namely, the $q$-numerical range of sectorial matrices. The $q$-numerical range of $T \in \mathcal{B}(\mathcal{H})$ is defined by,

$$W_q(T) = \{\langle Tx, y\rangle : x, y \in \mathcal{H}, \|x\| = \|y\| = 1, \langle x, y\rangle = q\},$$

where $|q| \leq 1$. The $q$-numerical radius $w_q(T)$ of $T \in \mathcal{B}(\mathcal{H})$ is

$$w_q(T) = \sup_{w \in W_q(T)} |w|.$$

For any $T \in \mathcal{B}(\mathcal{H})$, we have $T = \mathcal{R}(T) + i\mathcal{I}(T)$, where $\mathcal{R}(T) = \frac{T+T^*}{2}$ and $\mathcal{I}(T) = \frac{T-T^*}{2i}$. The following relations can be easily derived:

$$w_q(\mathcal{R}(T)) \leq w_q(T) \quad \text{and} \quad w_q(\mathcal{I}(T)) \leq w_q(T). \tag{3}$$

Limited research work is available in the literature on the $q$-numerical range. Notable contributions in this direction can be found in [8, 14, 21, 26, 27, 31]. For a comprehensive review, one can refer to [17, p.380]. Recently in [11], for $T \in \mathcal{B}(\mathcal{H})$ and $q \in (0, 1)$, several significant estimations of the $q$-numerical radius has been established, such as

$$\frac{q}{2(2-q^2)}\|T\| \leq w_q(T) \leq \|T\|, \tag{4}$$

$$\frac{1}{4}\left(\frac{q}{2-q^2}\right)^2 \|T^*T + TT^*\| \leq w_q^2(T) \leq \frac{\left(q + 2\sqrt{1-q^2}\right)^2}{2}\|T^*T + TT^*\|. \tag{5}$$

The present work aims to obtain bounds of $q$-numerical radii of sectorial matrices. This leads to the refinement of several results on $q$-numerical radius. Significant results on upper bounds for the $q$-numerical radii of commutator and anti-commutator matrices, and on non-integral powers of matrices, are obtained. Furthermore, this study explores the $q$-numerical radius inequalities associated with $2 \times 2$ block matrices.

## 2. $q$-Numerical Radius Inequalities for Sectorial Matrices

Throughout this paper, the symbol $T$ denotes an element of $\mathcal{B}(\mathcal{H})$, while the letter $A$ is designated specifically for $n \times n$ matrices. We start this section by listing some known outcomes that will be required in our analysis of the principal findings.

**Lemma 2.1.** *[28] Let $A \in \prod_{s,\alpha}^{n}$ for some $\alpha \in [0, \frac{\pi}{2})$. Then*

$$\|\mathcal{I}(A)\| \leq \sin(\alpha) w(A).$$

**Lemma 2.2.** *[28] Let $A \in \prod_{s,\alpha}^{n}$. Then*

$$\|A\| \leq \sqrt{1 + 2\sin^2(\alpha)}\, w(A).$$

The following Lemma represents a relation between $|||\mathcal{R}(A)|||$ and $|||A|||$, where $|||.|||$ is a unitarily invariant norm on $M_n$.

**Lemma 2.3.** *[32] Let $A \in \prod_{s,\alpha}^{n}$ and $|||.|||$ be any unitarily invariant norm on $M_n$. Then*

$$\cos(\alpha)|||A||| \leq |||\mathcal{R}(A)||| \leq |||A|||.$$

The following result states that, if a matrix $A$ is sectorial, then raising it to a fractional power will still preserve its sectoriality.

**Lemma 2.4.** *[10] Let $0 \leq \alpha < \frac{\pi}{2}$, $0 < t < 1$ and $A \in M_n$ is a square matrix with $W(A) \subseteq S_\alpha$. Then $W(A^t) \subseteq S_{t\alpha}$.*

Additionally, it should be noted that $W(A^{-t})$ is a subset of $S_{t\alpha}$. This can be inferred from the fact that $W(A^{-1})$ is a subset of $S_\alpha$ when $W(A)$ is a subset of $S_\alpha$.

Here are some additional important results related to the fractional powers of sectorial matrices:

**Lemma 2.5.** *[9] Let $A \in \prod_{s,\alpha}^{n}$ and $t \in [0, 1]$. Then*

$$\cos^{2t}(\alpha)\mathcal{R}(A^t) \leq (\mathcal{R}(A))^t \leq \mathcal{R}(A^t).$$

**Lemma 2.6.** *[9] Let $A \in \prod_{s,\alpha}^{n}$ and $t \in [-1, 0]$. Then*

$$\mathcal{R}(A^t) \leq (\mathcal{R}(A))^t \leq \cos^{2t}(\alpha)\mathcal{R}(A^t).$$

Let $\mathcal{D}$ be the closed unit disc in the complex plane and $\mathcal{D}' = \mathcal{D} \setminus \{0\}$. Also, if $\dim \mathcal{H} = 1$ then $W_q(A)$ is empty if and only if $|q| < 1$, and $W_q(A)$ is non-empty if $\dim \mathcal{H} \geq 2$ [17, p.380, Proposition 3.1(a)]. Thus, throughout the article we restrict ourselves to the case $\dim \mathcal{H} \geq 2$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be normaloid if $w(T) = \|T\|$. Also, all normal operators satisfy the similar relation. However, similar equality does not hold for $q$-numerical radius. For example, let $T = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}$. Then $\|T\| = 5$ and the $q$-numerical radius $w_q(T) = \frac{9q+1}{2} \neq \|T\|$ for all $q \in [0, 1)$ (i.e., except $q = 1$) follows from Theorem 3.4 [17, p.384]. In this context, Theorem 2.7 provides a relation between $w_q(T)$ and $\|T\|$ for normaloid operator $T$. It is noteworthy that for a normaloid operator $T$, the equality $r(T) = \|T\| = w(T)$ holds, where $r(T)$ is the spectral radius of $T$.

We start our main result with a relation between $w_q(T)$ and $w(T)$ for normaloid operators by using the spectral inclusion relation.

**Theorem 2.7.** *If $T \in \mathcal{B}(\mathcal{H})$ is a normaloid operator and $q \in \mathcal{D}$, then*

$$|q| w(T) \leq w_q(T) \leq w(T).$$

*Proof.* From Proposition 3.1 (h) [17, p.380], we have the following inclusion relation

$$q\sigma(T) \subseteq \overline{W_q(T)} \quad \text{for } |q| \leq 1.$$

It follows that

$$|q|r(T) \leq w_q(T).$$

As $T$ is normaloid, we have

$$|q|w(T) \leq w_q(T).$$

Also, it is well-known that $w_q(T) \leq \|T\|$ for all $T \in \mathcal{B}(\mathcal{H})$. Since $T$ is normaloid, we have

$$w_q(T) \leq \|T\| = w(T).$$

This completes the proof. □

**Remark 2.8.** *A few important observations are mentioned below.*

*(i) For normal $T$, the relation $w(T) = \|T\|$ holds, and subsequently we have $|q|\|T\| \leq w_q(T) \leq \|T\|$ which was recently obtained in [30, Theorem 1.4].*

*(ii) For $q \in (0,1)$, $q \geq \frac{q}{2-q^2}$, thus Theorem 2.7 is a refinement of [11, Theorem 2.1] for normal operator $T$.*

*(iii) Theorem 2.7 is the $q$-numerical radius analogue of the well-known relation* (1) *for normal operators, which reduces to the equality $w(T) = \|T\|$ when $q = 1$.*

*(iv) Both the inequalities of Theorem 2.7 are best possible. Let $S_1$ be the set of all $n \times n$ $(n \geq 2)$ real scalar matrices. Let $S_1 \ni A_1 = \eta I$, where $\eta(\neq 0) \in \mathbb{R}$ and $I$ is $n \times n$ identity matrix. Therefore, $w_q(A_1) = |q|w(A_1)$. Let $S$ be the right shift operator. Then $w_q(S) = w(S) = 1$ [17, p.384].*

For any $T \in \mathcal{B}(\mathcal{H})$, $\mathcal{R}(T)$ and $\mathcal{I}(T)$ are self-adjoint operators. An easy calculation leads to the following corollary.

**Corollary 2.9.** *If $T \in \mathcal{B}(\mathcal{H})$ and $q \in \mathcal{D}$, then we have*

*(a)* $|q|\|\mathcal{R}(T)\| \leq w_q(\mathcal{R}(T)) \leq \|\mathcal{R}(T)\|$,

*(b)* $|q|\|\mathcal{I}(T)\| \leq w_q(\mathcal{I}(T)) \leq \|\mathcal{I}(T)\|$.

The next result provides the equivalence of two norms $\|A\|$ and $w_q(A)$ and extends the inequality (1) for $q$-numerical radii of sectorial matrices.

**Theorem 2.10.** *(a) If $A \in \prod_{s,\alpha}^n$ and $q \in \mathcal{D}'$, then*

$$|q|\cos(\alpha)\|A\| \leq w_q(A) \leq \|A\|.$$

*(b) Let $q \in \mathcal{D}'$. If either $A, A^2 \in \prod_{s,\alpha}^n$ or $A$ is accretive–dissipative, then*

$$\frac{|q|}{\sqrt{2}}\|A\| \leq w_q(A) \leq \|A\|.$$

*Proof.* (a) Using Lemma 2.3, Corollary 2.9, and relation (3) we have

$$\cos(\alpha)\|A\| \leq \|\mathcal{R}(A)\| \leq \frac{1}{|q|}w_q(\mathcal{R}(A)) \leq \frac{1}{|q|}w_q(A) \leq \frac{1}{|q|}\|A\|.$$

Hence, the required result holds.

(b) If $A^2 \in \prod_{s,\alpha}^n$, then Lemma 2.4 implies that

$$W(A) = W((A^2)^{\frac{1}{2}}) \subset S_{\frac{\alpha}{2}} \subset S_{\frac{\pi}{4}}. \tag{6}$$

Also, if $A$ is accretive–dissipative, then

$$W(e^{\frac{-i\pi}{4}} A) \subset S_{\frac{\pi}{4}}. \tag{7}$$

Thus, inequalities (6) and (7), together with part (a), give the desired result.

□

**Remark 2.11.** *(i) If $\alpha \in \left[0, \frac{\pi}{3}\right)$ and $q = 1$, then the lower bound mentioned in Theorem 2.10(a) is a refinement of lower bound mentioned in (1). Moreover, for $\cos(\alpha) \geq \frac{1}{2(2-q^2)}$, $q \in (0,1)$, the lower bound mentioned in Theorem 2.10(a) is a refinement of the lower bound mentioned in relation* (4).

*(ii) Part (b) provides a more precise bound than part (a) of Theorem 2.10 for $\alpha \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.*

The following corollary provides an upper bound for $w_q(AB)$. Recall that for any two matrices $A, B \in M_n$, we have [17, p.125]

$$w(AB) \leq 4w(A)w(B). \tag{8}$$

Also, it is well-known that if $A$ and $B$ are positive semidefinite matrices then

$$w(AB) \leq w(A)w(B). \tag{9}$$

However, the relations (8) and (9) do not hold for $q$-numerical radius. Consider normal matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Then by Theorem 3.4 [17, p.384], we have $w_q(A) = \frac{3q+1}{2}$, $w_q(B) = 2q$ and $w_q(AB) = 3q + 1$. One can check that $w_q(AB) \nleq w_q(A)w_q(B)$ for all $q \in [0,1)$ and $w_q(AB) \nleq 4w_q(A)w_q(B)$ for $q < 0.25$. Our next theorem is an attempt to overcome such situations. Using Theorem 2.10(a) and $w_q(AB) \leq \|A\|\|B\|$, we can derive the following result easily.

**Corollary 2.12.** *For $q \in \mathcal{D}'$, the following hold:*

*(a) If A and B both are sectorial matrices, then*

$$|q|^2 w_q(AB) \leq \sec^2(\alpha) w_q(A) w_q(B).$$

*(b) If A and B both are positive definite matrices, then*

$$|q|^2 w_q(AB) \leq w_q(A) w_q(B).$$

**Remark 2.13.** *If we take $q = 1$, Corollary 2.12(a) implies that $w(AB) \leq \sec^2(\alpha) w(A) w(B)$ when A and B both are sectorial matrices, which is a refinement of inequality (8) when $\alpha \in \left(0, \frac{\pi}{3}\right)$. If we take $q = 1$ in Corollary 2.12(b), we obtain the well-known result $w(AB) \leq w(A)w(B)$ for positive definite matrices A and B.*

Before proceeding to the next theorem, we establish the following construction.

Consider $q \in \mathcal{D}$ and $\dim \mathcal{H} \geq 2$. Let $x \in \mathcal{H}$ be the unit vector. For any $y \in \mathcal{H}$ with $\|y\| = 1$ and $\langle x, y \rangle = q$, using $\mathcal{H} = \text{lin}\{x\} \oplus \{x\}^{\perp}$, we can express $y$ as

$$y = \bar{q}x + \sqrt{1 - |q|^2}\, z, \quad \text{where } z \in \mathcal{H}, \|z\| = 1, \text{ and } \langle x, z \rangle = 0. \tag{10}$$

The converse is also true, i.e., if (10) holds, then $\langle x, y \rangle = q$ and $\|y\| = 1$.

Our next focus is to establish a relation between the $q$-numerical radii and classical numerical radii for sectorial matrices. It is evident that if $A$ is normaloid, then $w_q(A) \leq w(A)$. Take $q \in (0,1)$ and a non-normaloid matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then for $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$, with $\|x\| = \|y\| = 1$ and $\langle x, y\rangle = q$, we have $\langle Ax, y\rangle = x_2 y_1$.

Thus

$$w_q(A) = \sup_{\|x\|=1, \|y\|=1, \langle x,y\rangle = q} |x_2 y_1|. \tag{11}$$

If we take $x = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $y = \left(\frac{q-\sqrt{1-q^2}}{\sqrt{2}}, \frac{q+\sqrt{1-q^2}}{\sqrt{2}}\right)$, then equation (11) implies $w_q(A) \geq \frac{q+\sqrt{1-q^2}}{2}$. Also, $w(A) = \frac{1}{2}$. In this case $w_q(A) > w(A)$, $q \in (0,1)$. In general, there is no relation between classical numerical range and $q$- numerical range. Our next result is an attempt in this direction for sectorial matrices.

**Theorem 2.14.** *Let $A \in \prod_{s,\alpha}^n$ and $q \in \mathcal{D}$. Then*

*(a)* $w_q(A) \leq \left(\sqrt{(1-|q|^2)\left(1+2\sin^2(\alpha)\right)} + |q|\right) w(A)$,

*(b)* $w_q(A) \leq \sqrt{1+\sin^2(\alpha)}\, w(A)$.

*Proof.* (a) Let $x, y \in \mathcal{H}$ such that $\|x\| = \|y\| = 1$ and $\langle x, y\rangle = q$. Then $y$ can be expressed as $y = \overline{q}x + \sqrt{1-|q|^2}z$, where $z \in \mathcal{H}$, $\|z\| = 1$ and $\langle x, z\rangle = 0$. Therefore,

$$\begin{aligned} |\langle Ax, y\rangle| &\leq |\langle Ax, \overline{q}x + \sqrt{1-|q|^2}z\rangle| \\ &\leq |q||\langle Ax, x\rangle| + \sqrt{1-|q|^2}|\langle Ax, z\rangle| \\ &\leq |q||\langle Ax, x\rangle| + \sqrt{1-|q|^2}\|A\|. \end{aligned}$$

Using Lemma 2.2, we get

$$|\langle Ax, y\rangle| \leq \sqrt{(1-|q|^2)(1+2\sin^2(\alpha))}\, w(A) + |q||\langle Ax, x\rangle|.$$

This implies,

$$|\langle Ax, y\rangle| \leq \left(\sqrt{(1-|q|^2)(1+2\sin^2(\alpha))} + |q|\right) w(A).$$

Taking supremum over all $x, y$ with $\|x\| = \|y\| = 1$ and $\langle x, y\rangle = q$, we have

$$w_q(A) \leq \left(\sqrt{(1-|q|^2)(1+2\sin^2(\alpha))} + |q|\right) w(A).$$

(b) Using the Cartesian decomposition $A = \mathcal{R}(A) + i\mathcal{I}(A)$, we obtain

$$\begin{aligned} |\langle Ax, y\rangle| &\leq \sqrt{\langle \mathcal{R}(A)x, y\rangle^2 + \langle \mathcal{I}(A)x, y\rangle^2} \\ &\leq \left(\sqrt{\|\mathcal{R}(A)\|^2 + \|\mathcal{I}(A)\|^2}\right)\|x\|\|y\|. \end{aligned}$$

Using $\|\mathcal{R}(A)\| \leq w(A)$ and Lemma 2.1, we have

$$|\langle Ax, y\rangle \leq \left(\sqrt{w^2(A) + \sin^2(\alpha) w^2(A)}\right)\|x\|\|y\|.$$

Taking supremum over all $x, y$ on both sides with $\|x\| = \|y\| = 1$ and $\langle x, y\rangle = q$, we have

$$w_q(A) \leq \sqrt{1+\sin^2(\alpha)}w(A).$$

□

**Remark 2.15.** *Theorem 2.14 enables us to explore the ratio $\frac{w_q(A)}{w(A)}$ in terms of $q$ and the sectorial index $\alpha$. Also, for different values of $\alpha$ (say $\alpha = 0, \frac{\pi}{4}, \frac{\pi}{2}$) we can compare the bounds in part (a) and part (b) as follows:*

*(i) For $\alpha = \frac{\pi}{2}$, $\sqrt{2} = \sqrt{1+\sin^2(\alpha)} \leq q + \sqrt{3(1-q^2)} = \sqrt{(1-|q|^2)\left(1+2\sin^2(\alpha)\right)} + |q|$ holds when $q \in [0, 0.966]$,*

*(ii) For $\alpha = \frac{\pi}{4}$, $\sqrt{\frac{3}{2}} = \sqrt{1+\sin^2(\alpha)} \leq q + \sqrt{2(1-q^2)} = \sqrt{(1-|q|^2)\left(1+2\sin^2(\alpha)\right)} + |q|$ holds when $q \in [0, 0.986]$,*

*(iii) For $\alpha = 0$, $1 = \sqrt{1+\sin^2(\alpha)} \leq q + \sqrt{1-q^2} = \sqrt{(1-|q|^2)\left(1+2\sin^2(\alpha)\right)} + |q|$ holds when $q \in [0, 1]$.*

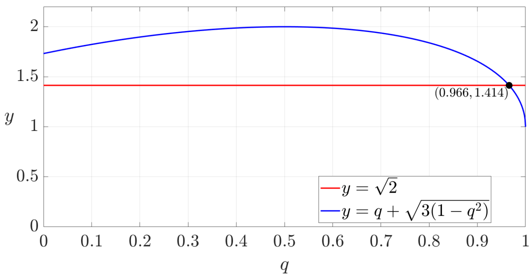


Figure 1: When $\alpha = \frac{\pi}{2}$

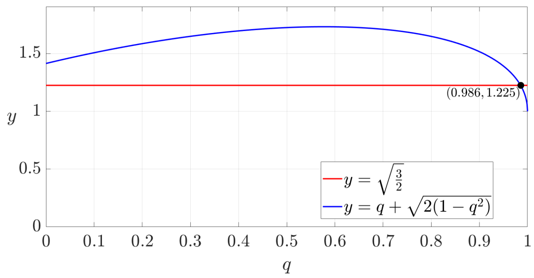


Figure 2: When $\alpha = \frac{\pi}{4}$

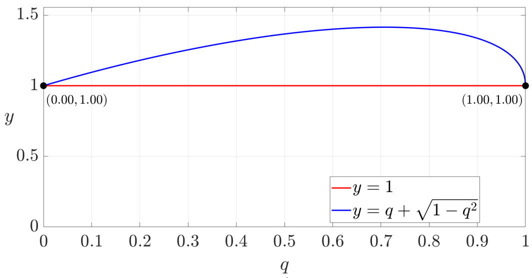


Figure 3: When $\alpha = 0$

*The analysis of the figures demonstrates that the bound presented in part (b) is a refinement of the bound presented in part (a) within a specific range of $q$. As the value of $\alpha$ decreases, the region of $q$ where refinement occurs expands correspondingly. This observation highlights the relationship between the parameter $\alpha$ and the extent of refinement achieved in the specified region of $q$.*

Now we are ready to give $q$-numerical radius version of Theorem 1 [20] for sectorial matrices. Furthermore, this result serves as a refinement of Theorem 3.1 in [11].

**Theorem 2.16.** *Let $A \in \prod_{s,\alpha}^{n}$ and $q \in \mathcal{D}'$. Then*

$$\frac{|q|^2\cos^2\alpha}{2}\|A^*A + AA^*\| \leq w_q^2(A) \leq \left(\sqrt{(1-|q|^2)\left(1+2\sin^2(\alpha)\right)} + |q|\right)^2 \frac{\|A^*A + AA^*\|}{2}.$$

*Proof.* From Lemma 2.1, we have

$$\|\mathcal{I}(A)\| \leq \sin(\alpha) w(A) \leq \sin(\alpha)\|A\|.$$

Using Theorem 2.10($a$), we have

$$\|\mathcal{I}(A)\| \leq \frac{1}{|q|}\tan(\alpha) w_q(A). \tag{12}$$

Also,

$$\begin{aligned}\|A^*A + AA^*\| &= 2\|\mathcal{R}^2(A) + \mathcal{I}^2(A)\| \\ &\leq 2(\|\mathcal{R}(A)\|^2 + \|\mathcal{I}(A)\|^2).\end{aligned}$$

Using equation (12) and Corollary 2.9(a), we have

$$\begin{aligned}\|A^*A + AA^*\| &\leq 2\left(\frac{1}{|q|^2} w_q^2(\mathcal{R}(A)) + \tan^2\alpha \frac{1}{|q|^2} w_q^2(\mathcal{I}(A))\right) \\ &\leq 2\left(\frac{1}{|q|^2} w_q^2(A) + \tan^2\alpha \frac{1}{|q|^2} w_q^2(A)\right).\end{aligned}$$

Thus, we have

$$\frac{|q|^2\cos^2\alpha}{2}\|A^*A + AA^*\| \leq w_q^2(A).$$

For the other part, Theorem 2.14(a) implies,

$$w_q^2(A) \leq \left(\sqrt{(1-|q|^2)\left(1+2\sin^2(\alpha)\right)} + |q|\right)^2 w^2(A).$$

Using relation (2), we obtain

$$w_q^2(A) \leq \left(\sqrt{(1-|q|^2)\left(1+2\sin^2(\alpha)\right)} + |q|\right)^2 \frac{\|A^*A + AA^*\|}{2}.$$

□

**Remark 2.17.** *(i) If $q \in (0,1)$, the upper bound of $w_q^2(A)$ of the aforementioned theorem refines the upper bound of $w_q^2(A)$ given in relation* (5)*. Furthermore, if either $\cos(\alpha) \geq \frac{1}{\sqrt{2}(2-q^2)}$ or $\alpha \in [0, \frac{\pi}{4})$, the lower bound of $w_q^2(A)$ in the aforementioned theorem provides an improvement over the lower bound of $w_q^2(A)$ in relation* (5)*.*

*(ii) If $q = 1$, Theorem 2.16 gives us*

$$\frac{\cos^2(\alpha)}{2}\|A^*A + AA^*\| \leq w^2(A) \leq \frac{\|A^*A + AA^*\|}{2}. \tag{13}$$

*Clearly, if $\alpha \in [0, \frac{\pi}{4})$, then lower bound of above inequality is a refinement of the lower bound of inequality (2), and if $A$ is a positive definite matrix, then inequality (13) gives us $w^2(A) = \frac{\|A^*A+AA^*\|}{2}$.*

Next, we focus on the $q$-numerical radius inequalities of commutator and anti-commutator sectorial matrices.

**Theorem 2.18.** *Let $B, C, D \in M_n$, $A \in \prod_{s,\alpha}^n$ and $q \in \mathcal{D}'$. Then*

$$|q| w_q(ACB \pm BDA) \leq 2\sec(\alpha)\max\{\|C\|, \|D\|\} w_q(A)\|B\|.$$

*Proof.* As the norm satisfies the homogeneity property, we can take $\|C\| \leq 1$ and $\|D\| \leq 1$. Consider,

$$w_q(AC \pm DA) \leq \|AC \pm DA\| \leq 2\|A\|.$$

It follows from Theorem 2.10 that,

$$w_q(AC \pm DA) \leq \frac{2}{|q|} \sec(\alpha) w_q(A). \tag{14}$$

If $C = D = 0$, then the required result holds trivially. Let $\max\{\|C\|, \|D\|\} \neq 0$. Then $\left\|\frac{C}{\max\{\|C\|,\|D\|\}}\right\| \leq 1$ and $\left\|\frac{D}{\max\{\|C\|,\|D\|\}}\right\| \leq 1$.

Therefore, by replacing $C$ with $\frac{C}{\max\{\|C\|,\|D\|\}}$ and $D$ with $\frac{D}{\max\{\|C\|,\|D\|\}}$ in relation (14), we have

$$w_q(AC \pm DA) \leq \frac{2}{|q|} \sec(\alpha) \max\{\|C\|, \|D\|\} w_q(A). \tag{15}$$

Again, replacing $C$ with $CB$ and $D$ with $BD$ in inequality (15), we obtain that

$$\begin{aligned} w_q(ACB \pm BDA) \leq & \frac{2}{|q|} \sec(\alpha) \max\{\|CB\|, \|BD\|\} w_q(A) \\ \leq & \frac{2}{|q|} \sec(\alpha) \max\{\|C\|, \|D\|\} w_q(A) \|B\|. \end{aligned}$$

This completes the proof. □

**Remark 2.19.** *It was obtained in [15, Theorem 11] that*

$$w(AB \pm BA) \leq 2\sqrt{2} w(A) \|B\|. \tag{16}$$

*Take $C = D = I$ in Theorem 2.18, we have*

$$|q| w_q(AB \pm BA) \leq 2 \sec(\alpha) w_q(A) \|B\|. \tag{17}$$

*If $q = 1$, relation (17) implies that*

$$w(AB \pm BA) \leq 2 \sec(\alpha) w(A) \|B\|. \tag{18}$$

*Thus, relation (18) is a refinement of the relation (16) if $\alpha \in \left[0, \frac{\pi}{4}\right)$.*

Replacing $A$ and $B$ in relation (17), we can obtain the following corollary easily.

**Corollary 2.20.** *If $A, B \in \prod_{s,\alpha}^n$ and $q \in \mathcal{D}'$, then*

$$|q| w_q(AB \pm BA) \leq 2 \sec(\alpha) \min\{w_q(A)\|B\|, w_q(B)\|A\|\}.$$

It is noteworthy that the literature does not extensively address non-integral powers of the $q$-numerical radius. In our forthcoming results, we address this gap by establishing a relation between $w_q^t(A)$ and $w_q(A^t)$ for $A \in \prod_{s,\alpha}^n$. It is well-known that $\|A^t\| = \|A\|^t$ for positive semidefinite matrix $A$. The significance of the forthcoming result lies in its ability to establish a relationship between $w_q(A^{-1})$ and $w_q^{-1}(A)$. In general there is no such relation between $w_q(A^{-1})$ and $w_q^{-1}(A)$. For example, let $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $q \in (0, 1)$. Then, $w_q(A) = \frac{3q+1}{2}$ and $w_q(A^{-1}) = \frac{3q+1}{4}$. Here $w_q(A^{-1}) \leq w_q^{-1}(A)$ if $q \in (0, 0.6095]$ and $w_q^{-1}(A) \leq w_q(A^{-1})$ if $q \in [0.6095, 1)$.

**Theorem 2.21.** *If $A \in \prod_{s,\alpha}^n$, $q \in \mathcal{D}'$ and $t \in [0, 1]$, then* *(a)* $|q|^{t+1}\cos^t(\alpha)w_q^t(A) \le |q|^t w_q(A^t) \le \sec^{2t}(\alpha)\sec(t\alpha)w_q^t(A)$,

*(b)* $|q|^{t+1}\cos(t\alpha)\cos^{2t}(\alpha)w_q^{-t}(A) \le w_q(A^{-t})$.

*Proof.* (a) By Lemma 2.3, we have

$$\cos(\alpha)w_q(A) \le \|\mathcal{R}(A)\|.$$

For $t \in [0,1]$, Lemma 2.4 implies,

$$w_q(A^t) \le \sec(t\alpha)\|\mathcal{R}(A^t)\|.$$

Now,

$$\begin{aligned} w_q(A^t) &\le \sec(t\alpha)\sec^{2t}(\alpha)\|\mathcal{R}^t(A)\| \\ &= \sec(t\alpha)\sec^{2t}(\alpha)\|\mathcal{R}(A)\|^t \\ &\le \left(\frac{1}{|q|}\right)^t \sec(t\alpha)\sec^{2t}(\alpha)w_q^t(\mathcal{R}(A)) \\ &\le \left(\frac{1}{|q|}\right)^t \sec(t\alpha)\sec^{2t}(\alpha)w_q^t(A), \end{aligned}$$

where the first inequality follows from Lemma 2.5, while the last two inequalities follow from Corollary 2.9 and, relation (3), respectively.

For the other inequality, we have

$$\begin{aligned} w_q(A^t) &\ge w_q(\mathcal{R}(A^t)) \\ &\ge |q|\|\mathcal{R}(A^t)\| \\ &\ge |q|\|\mathcal{R}^t(A)\| \\ &= |q|\|\mathcal{R}(A)\|^t \\ &\ge |q|\cos^t(\alpha)\|A\|^t \\ &\ge |q|\cos^t(\alpha)w_q^t(A) \end{aligned}$$

where the first, second, third, and fifth inequalities follow from relation (3), Corollary 2.9, Lemma 2.5, and Lemma 2.3, respectively. Finally, we have

$$|q|\cos^t(\alpha)w_q^t(A) \le w_q(A^t) \le \frac{1}{|q|^t}\sec^{2t}(\alpha)\sec(t\alpha)w_q^t(A).$$

(b) Using Theorem 2.10 and Lemma 2.3, we obtain that

$$\begin{aligned} w_q(A^{-t}) &\ge |q|\cos(t\alpha)\|A^{-t}\| \\ &\ge |q|\cos(t\alpha)\|\mathcal{R}(A^{-t})\|. \end{aligned}$$

Using Lemma 2.6, Corollary 2.9(a) and the fact that $\|A\|^{-1} \le \|A^{-1}\|$, where $A$ is invertible matrix, we have

$$\begin{aligned} w_q(A^{-t}) &\ge |q|\cos(t\alpha)\cos^{2t}(\alpha)\|\mathcal{R}^{-t}(A)\| \\ &\ge |q|\cos(t\alpha)\cos^{2t}(\alpha)\|\mathcal{R}(A)\|^{-t} \\ &\ge |q|^{t+1}\cos(t\alpha)\cos^{2t}(\alpha)w_q^{-t}(\mathcal{R}(A)) \\ &\ge |q|^{t+1}\cos(t\alpha)\cos^{2t}(\alpha)w_q^{-t}(A). \end{aligned}$$

Hence the required result.

□

**Remark 2.22.** *(i) For $q = 1$, the lower bound in the part (a) of the aforementioned theorem represents an improvement over the following lower bound, which is mentioned in [3, Theorem 3.1 ]*

$$\cos(t\alpha)\cos^t(\alpha)w^t(A) \leq w(A^t) \leq \sec^{2t}(\alpha)\sec(t\alpha)w^t(A).$$

*(ii) Theorem 2.7 and Theorem 2.21(a) give us the following significant relation,*

$$|q|^{2t+1}\|A\|^t \leq |q|^t w_q(A^t) \leq \|A\|^t, \tag{19}$$

*where $A$ is a positive definite matrix. If $q = 1$, then relation (19) gives us the well-known equality $\|A^t\| = \|A\|^t$, $A$ is positive definite matrix.*

*(iii) The case $t = 1$ in Theorem 2.21(b) provides*

$$|q|^2\cos^3(\alpha)w_q^{-1}(A) \leq w_q(A^{-1}), \tag{20}$$

*and if $A$ is positive definite matrix, relation (20) implies that*

$$|q|^2 w_q^{-1}(A) \leq w_q(A^{-1}).$$

## 3. $q$-Numerical Radius Inequalities of 2 × 2 Operator Matrices

Assuming $\mathcal{H}$ represents a complex Hilbert space equipped with the inner product $\langle .,. \rangle$, the direct sum $\mathcal{H} \oplus \mathcal{H}$ constructs another Hilbert space, and any operator $\mathbf{T} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ can be represented by a $2 \times 2$ operator matrix in the following manner

$$\mathbf{T} = \begin{bmatrix} Z & X \\ Y & W \end{bmatrix}$$

where $Z, X, Y$, and $W$ are in $\mathcal{B}(\mathcal{H})$. In this section, our objective is to analyze the properties of the $q$-numerical radius of operators having off-diagonal representation of the form $\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}$. Since the $q$-numerical radius is weakly unitarily invariant, i.e.

$$W_q(U^*TU) = W_q(T)$$

for any unitary operator $U$ on $\mathcal{H}$, the following relations,

$$w_q\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) = w_q\left(\begin{bmatrix} 0 & X \\ e^{i\theta}Y & 0 \end{bmatrix}\right) \text{ for all } \theta \in \mathbb{R} \tag{21}$$

and

$$w_q\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) = w_q\left(\begin{bmatrix} 0 & Y \\ X & 0 \end{bmatrix}\right), \tag{22}$$

can be easily deduced by considering the unitary operator $U$ as $\begin{bmatrix} I & 0 \\ 0 & e^{-\frac{i\theta}{2}}I \end{bmatrix}$, $\theta \in \mathbb{R}$ and $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$, respectively.

Next, for $q \in [0,1]$, some observations of the $q$-numerical radius of $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$ are mentioned, where $A$ is a Hermitian matrix. Let $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ be the eigenvalues of $A$. Thus by using Theorem 3.4 [17, p.384], we have

$$w_q(A) = \frac{q}{2}|\lambda_1 + \lambda_n| + \frac{1}{2}|\lambda_1 - \lambda_n|.$$

Also, the largest eigenvalue of $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$ is $\lambda_{max} = \max\{-\lambda_n, \lambda_1\}$ and the smallest eigenvalue of $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$ is $\lambda_{min} = \min\{-\lambda_1, \lambda_n\}$. Therefore,

$$w_q\left(\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}\right) = \begin{cases} \frac{q}{2}|\lambda_1 + \lambda_n| + \frac{1}{2}|\lambda_1 - \lambda_n| & \text{if } \lambda_{max} = \lambda_1, \lambda_{min} = \lambda_n \\ \frac{q}{2}|\lambda_1 + \lambda_n| + \frac{1}{2}|\lambda_1 - \lambda_n| & \text{if } \lambda_{max} = -\lambda_n, \lambda_{min} = -\lambda_1 \\ |\lambda_1| & \text{if } \lambda_{max} = \lambda_1, \lambda_{min} = -\lambda_1 \\ |\lambda_n| & \text{if } \lambda_{max} = -\lambda_n, \lambda_{min} = \lambda_n. \end{cases}$$

and $w_q(A) \leq w_q\left(\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}\right)$.

In particular, let $A$ be a positive semidefinite $n \times n$ matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n \geq 0$, we have $\lambda_{max} = \lambda_1$, $\lambda_{min} = -\lambda_1$ and

$$w_q\left(\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}\right) = \|A\| = w(A).$$

Following this, we will establish the bounds for $q$-numerical radius of $\begin{bmatrix} 0 & Y \\ Y & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}$, $X, Y \in \mathcal{B}(\mathcal{H})$. For $Y \in \mathcal{B}(\mathcal{H})$, $\begin{bmatrix} 0 & Y \\ Y & 0 \end{bmatrix}$ satisfies the equality $w\left(\begin{bmatrix} 0 & Y \\ Y & 0 \end{bmatrix}\right) = w(Y)$ [18]. However, a similar assertion may not hold for $q$-numerical radius, as illustrated in the subsequent example.

Let

$$\mathbf{T} = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{bmatrix}.$$

Then $w_q(\mathbf{T}) = 3$ and $w_q\left(\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\right) = \frac{5q}{2} + \frac{1}{2}$. Clearly, $w_q(\mathbf{T}) \neq w_q\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ except $q = 1$ but $w_q\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \leq w_q(\mathbf{T})$.

With this observation, we present the following result.

**Proposition 3.1.** *Let $X, Y \in \mathcal{B}(\mathcal{H})$ and $q \in \mathcal{D}$, we have*

*(a)* $\max\{w_q(X - Y), w_q(X + Y)\} \leq w_q\left(\begin{bmatrix} X & Y \\ Y & X \end{bmatrix}\right) \leq \max\{\|X - Y\|, \|X + Y\|\}$,

*(b)* $\frac{1}{2}\max\{w_q(X + Y), w_q(X - Y)\} \leq w_q\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) \leq \frac{1}{2}(\|X + Y\| + \|X - Y\|)$.

*Proof.* (a) From [24, Remark 4(i)], we have

$$\max\{w_q(X - Y), w_q(X + Y)\} \leq w_q\left(\begin{bmatrix} X - Y & 0 \\ 0 & X + Y \end{bmatrix}\right) \leq \max\{\|X - Y\|, \|X + Y\|\}.$$

As $w_q(U^*TU) = w_q(T)$ holds, taking $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & I \\ -I & I \end{bmatrix}$,

we have $\begin{bmatrix} X & Y \\ Y & X \end{bmatrix}$ is unitarily similar to $\begin{bmatrix} X - Y & 0 \\ 0 & X + Y \end{bmatrix}$. This implies,

$$\max\{w_q(X - Y), w_q(X + Y)\} \leq w_q\left(\begin{bmatrix} X & Y \\ Y & X \end{bmatrix}\right) \leq \max\{\|X - Y\|, \|X + Y\|\}.$$

(b) The triangle inequality of $q$-numerical radius provides

$$w_q\left(\begin{bmatrix} 0 & X+Y \\ X+Y & 0 \end{bmatrix}\right) \leq w_q\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) + w_q\left(\begin{bmatrix} 0 & Y \\ X & 0 \end{bmatrix}\right).$$

From the relation $w_q\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) = w_q\left(\begin{bmatrix} 0 & Y \\ X & 0 \end{bmatrix}\right)$ (equality 22), it follows

$$w_q\left(\begin{bmatrix} 0 & X+Y \\ X+Y & 0 \end{bmatrix}\right) \leq 2w_q\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right).$$

Taking $X = 0$ in part (a), we have $w_q(Y) \leq w_q\left(\begin{bmatrix} 0 & Y \\ Y & 0 \end{bmatrix}\right)$. Therefore, using relation (21), we obtain

$$w_q(X+Y) \leq w_q\left(\begin{bmatrix} 0 & X+Y \\ X+Y & 0 \end{bmatrix}\right) \leq 2w_q\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) = 2w_q\left(\begin{bmatrix} 0 & X \\ e^{i\theta}Y & 0 \end{bmatrix}\right).$$

Replacing $Y$ with $-Y$, and taking $\theta = \pi$, we have

$$w_q(X-Y) \leq w_q\left(\begin{bmatrix} 0 & X-Y \\ X-Y & 0 \end{bmatrix}\right) \leq 2w_q\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right).$$

Thus,

$$w_q\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) \geq \frac{1}{2}\max\{w_q(X+Y), w_q(X-Y)\}.$$

Let $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & -I \\ I & I \end{bmatrix}$. Then

$$\begin{aligned} w_q\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) &= w_q\left(U^*\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}U\right) \\ &= \frac{1}{2}w_q\left(\begin{bmatrix} X+Y & X-Y \\ -(X-Y) & -(X+Y) \end{bmatrix}\right) \\ &\leq \frac{1}{2}w_q\left(\begin{bmatrix} X+Y & 0 \\ 0 & -(X+Y) \end{bmatrix} + \begin{bmatrix} 0 & X-Y \\ -(X-Y) & 0 \end{bmatrix}\right) \\ &\leq \frac{1}{2}\left(\left\|\begin{bmatrix} X+Y & 0 \\ 0 & -(X+Y) \end{bmatrix}\right\| + \left\|\begin{bmatrix} 0 & X-Y \\ -(X-Y) & 0 \end{bmatrix}\right\|\right) \\ &= \frac{1}{2}\left(\|X+Y\| + \|X-Y\|\right). \end{aligned}$$

Hence the required result holds.

□

**Remark 3.2.** *If we take $X = 0$ in part (a) of the aforementioned result, we have*

$$w_q(Y) \leq w_q\left(\begin{bmatrix} 0 & Y \\ Y & 0 \end{bmatrix}\right) \leq \|Y\|. \tag{23}$$

*For $Y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, we have $\|Y\| = w_q(Y) = w_q\left(\begin{bmatrix} 0 & Y \\ Y & 0 \end{bmatrix}\right) = 1$. Thus, the bounds mentioned in relation (23) are best possible.*

**Corollary 3.3.** *Let $X \in \mathcal{B}(\mathcal{H})$ and $q \in \mathcal{D}$. Then*

*(a)* $|q|\max\{\|\mathcal{R}(X)\|, \|\mathcal{I}(X)\|\} \leq w_q\left(\begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix}\right) \leq \|\mathcal{R}(X)\| + \|\mathcal{I}(X)\|,$

*(b)* $\frac{w_q(X)}{2} \leq w_q\left(\begin{bmatrix} 0 & \mathcal{R}(X) \\ \mathcal{I}(X) & 0 \end{bmatrix}\right) \leq \|X\|,$

*(c) if $X$ is Hermitian, then* $|q|\|X\| \leq w_q\left(\begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix}\right) \leq \|X\|,$

*(d) if $X^2 = 0$, then* $|q|\|\mathcal{R}(X)\| \leq w_q\left(\begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix}\right) \leq 2\|\mathcal{R}(X)\|.$

*Proof.* Taking $Y = X^*$ in Proposition 3.1(b) and using Corollary 2.9(a), we can easily obtain the result mentioned in part (a). Replacing $Y$ with $iY$ in Proposition 3.1(b), we have

$$\frac{1}{2}\max\{w_q(X + iY), w_q(X - iY)\} \leq w_q\left(\begin{bmatrix} 0 & X \\ iY & 0 \end{bmatrix}\right) \leq \frac{1}{2}(\|X + iY\| + \|X - iY\|).$$

For $\theta = \frac{\pi}{2}$, from the relation $w_q\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) = w_q\left(\begin{bmatrix} 0 & X \\ e^{i\theta}Y & 0 \end{bmatrix}\right)$(equality (21)), it follows

$$\frac{1}{2}\max\{w_q(X + iY), w_q(X - iY)\} \leq w_q\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) \leq \frac{1}{2}(\|X + iY\| + \|X - iY\|). \tag{24}$$

The inequalities in part (b) follow by replacing $X$ with $\mathcal{R}(X)$ and $Y$ with $\mathcal{I}(X)$ in (24). Moreover, the relations in part (c) and part (d) follow from part (a) when $X$ is Hermitian and $X^2 = 0$ ($\|\mathcal{R}(X)\| = \|\mathcal{I}(X)\|$), respectively. □

Let $T$ be a bounded linear operator, acting on a Hilbert space $\mathcal{H}$. Then there exists a unique complex number $c \in \overline{W(T)}$ such that [29]

$$m(T) = \inf_{\lambda \in \mathbb{C}}\|T - \lambda I\| = \|T - cI\|. \tag{25}$$

Prasanna [25] termed $m(T)$ as the transcendental radius and defined it as

$$m^2(T) = \sup_{\|x\|=1}\left(\|Tx\|^2 - |\langle Tx, x\rangle|^2\right). \tag{26}$$

Consider an orthonormal set $O$ that contains $x$ and $z$. Bessel's inequality implies

$$\sum_{y' \in O}|\langle Tx, y'\rangle|^2 \leq \|Tx\|^2.$$

$$i.e., \quad |\langle Tx, z\rangle|^2 \leq \sum_{y' \in O\setminus\{x\}}|\langle Tx, y'\rangle|^2 \leq \|Tx\|^2 - |\langle Tx, x\rangle|^2.$$

Therefore, relations (25) and (26) imply,

$$|\langle Tx, z\rangle| \leq (\|Tx\|^2 - |\langle Tx, x\rangle|^2)^{\frac{1}{2}} \leq m(T) = \|T - cI\|, \ c \in \overline{W(T)}. \tag{27}$$

Using $m(T)$, the following result gives us an upper bound of $w_q\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right)$ using the concept of non-negative functions $f$ and $g$. The following lemma is used in the next theorem.

**Lemma 3.4.** *[19, Theorem 1] Let $T \in \mathcal{B}(\mathcal{H})$, $f$ and $g$ be non-negative functions on $[0,\infty)$ which are continuous and satisfy the relation $f(t)g(t) = t$ for all $t \in [0,\infty)$. Then $|\langle Tx, y\rangle| \le \|f(|T|)x\|\|g(|T^*|)y\|$ for all $x, y \in \mathcal{H}$.*

**Theorem 3.5.** *Let $f$ and $g$ satisfy the conditions of Lemma 3.4, $q \in \mathcal{D}$, and $c \in \overline{W(T)}$. Then, we have*

*(a)*

$$w_q\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) \le \frac{1}{2}\max\{\|f^2(|Y|) + |q|^2 g^2(|X^*|)\|, \|f^2(|X|) + |q|^2 g^2(|Y^*|)\|\} + \frac{1-|q|^2}{2}\max\{\|g^2(|X^*|)\|, \|g^2(|Y^*|)\|\}$$
$$+ |q|\sqrt{1-|q|^2}\max\{\|g^2(|X^*|) - cI\|, \|g^2(|Y^*|) - cI\|\}.$$

*(b)*

$$w_q^2\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) \le \frac{1}{2}\max\{\|f^4(|Y|) + |q|^2 g^4(|X^*|)\|, \|f^4(|X|) + |q|^2 g^4(|Y^*|)\|\} + \frac{1-|q|^2}{2}\max\{\|g^4(|X^*|)\|, \|g^4(|Y^*|)\|\}$$
$$+ |q|\sqrt{1-|q|^2}\max\{\|g^4(|X^*|) - cI\|, \|g^4(|Y^*|) - cI\|\}.$$

*Proof.* (a) Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{H}$, and $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathcal{H}$ with $\|x\| = \|y\| = 1$ and $\langle x, y\rangle = q$. From Lemma 3.4, we have

$$\begin{aligned}
\left|\left\langle \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} x, y\right\rangle\right| &\le \left\| f\left(\left|\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right|\right)x\right\| \left\| g\left(\left|\begin{bmatrix} 0 & Y^* \\ X^* & 0 \end{bmatrix}\right|\right)y\right\| \\
&\le \left\langle f^2\left|\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right| x, x\right\rangle^{\frac{1}{2}} \left\langle g^2\left|\begin{bmatrix} 0 & Y^* \\ X^* & 0 \end{bmatrix}\right| y, y\right\rangle^{\frac{1}{2}} \\
&\le \left\langle f^2\begin{bmatrix} |Y| & 0 \\ 0 & |X| \end{bmatrix} x, x\right\rangle^{\frac{1}{2}} \left\langle g^2\begin{bmatrix} |X^*| & 0 \\ 0 & |Y^*| \end{bmatrix} y, y\right\rangle^{\frac{1}{2}} \\
&\le \frac{1}{2}\left(\left\langle f^2\begin{bmatrix} |Y| & 0 \\ 0 & |X| \end{bmatrix} x, x\right\rangle + \left\langle g^2\begin{bmatrix} |X^*| & 0 \\ 0 & |Y^*| \end{bmatrix} y, y\right\rangle\right) \\
&\le \frac{1}{2}\left(\left\langle f^2\begin{bmatrix} |Y| & 0 \\ 0 & |X| \end{bmatrix} x, x\right\rangle + \left\langle g^2\begin{bmatrix} |X^*| & 0 \\ 0 & |Y^*| \end{bmatrix} \bar{q}x + \sqrt{1-|q|^2}z, \bar{q}x + \sqrt{1-|q|^2}z\right\rangle\right) \\
&\le \frac{1}{2}\left\langle f^2\begin{bmatrix} |Y| & 0 \\ 0 & |X| \end{bmatrix} x, x\right\rangle + \frac{|q|^2}{2}\left\langle g^2\begin{bmatrix} |X^*| & 0 \\ 0 & |Y^*| \end{bmatrix} x, x\right\rangle + \frac{1-|q|^2}{2}\left\langle g^2\begin{bmatrix} |X^*| & 0 \\ 0 & |Y^*| \end{bmatrix} z, z\right\rangle \\
&+ \frac{1}{2}\left(\bar{q}\sqrt{1-|q|^2}\left\langle g^2\begin{bmatrix} |X^*| & 0 \\ 0 & |Y^*| \end{bmatrix} x, z\right\rangle + q\sqrt{1-|q|^2}\left\langle g^2\begin{bmatrix} |X^*| & 0 \\ 0 & |Y^*| \end{bmatrix} z, x\right\rangle\right) \\
&\le \frac{1}{2}\left\langle f^2\begin{bmatrix} |Y| & 0 \\ 0 & |X| \end{bmatrix} x, x\right\rangle + \frac{|q|^2}{2}\left\langle g^2\begin{bmatrix} |X^*| & 0 \\ 0 & |Y^*| \end{bmatrix} x, x\right\rangle \\
&+ \frac{1-|q|^2}{2}\left\langle g^2\begin{bmatrix} |X^*| & 0 \\ 0 & |Y^*| \end{bmatrix} z, z\right\rangle + \mathcal{R}\left(\bar{q}\sqrt{1-|q|^2}\left\langle g^2\begin{bmatrix} |X^*| & 0 \\ 0 & |Y^*| \end{bmatrix} x, z\right\rangle\right) \\
&\le \frac{1}{2}\left\langle f^2\begin{bmatrix} |Y| & 0 \\ 0 & |X| \end{bmatrix} x, x\right\rangle + \frac{|q|^2}{2}\left\langle g^2\begin{bmatrix} |X^*| & 0 \\ 0 & |Y^*| \end{bmatrix} x, x\right\rangle \\
&+ \frac{1-|q|^2}{2}\left\langle g^2\begin{bmatrix} |X^*| & 0 \\ 0 & |Y^*| \end{bmatrix} z, z\right\rangle + |q|\sqrt{1-|q|^2}\left|\left\langle g^2\begin{bmatrix} |X^*| & 0 \\ 0 & |Y^*| \end{bmatrix} x, z\right\rangle\right|.
\end{aligned}$$

Using relation (27), we have

$$\begin{aligned}
\left|\left\langle\begin{bmatrix}0 & X\\ Y & 0\end{bmatrix}x,y\right\rangle\right| \leq & \frac{1}{2}\left\langle f^2\begin{bmatrix}|Y| & 0\\ 0 & |X|\end{bmatrix}x,x\right\rangle + \frac{|q|^2}{2}\left\langle g^2\begin{bmatrix}|X^*| & 0\\ 0 & |Y^*|\end{bmatrix}x,x\right\rangle\\
&+\frac{1-|q|^2}{2}\left\langle g^2\begin{bmatrix}|X^*| & 0\\ 0 & |Y^*|\end{bmatrix}z,z\right\rangle\\
&+|q|\sqrt{1-|q|^2}\left\|g^2\begin{bmatrix}|X^*| & 0\\ 0 & |Y^*|\end{bmatrix}-c\begin{bmatrix}I & 0\\ 0 & I\end{bmatrix}\right\|, \text{ where } c\in\overline{W(T)}\\
\leq & \frac{1}{2}\left\langle\begin{bmatrix}f^2(|Y|) & 0\\ 0 & f^2(|X|)\end{bmatrix}x,x\right\rangle + \frac{|q|^2}{2}\left\langle\begin{bmatrix}g^2(|X^*|) & 0\\ 0 & g^2(|Y^*|)\end{bmatrix}x,x\right\rangle\\
&+\frac{1-|q|^2}{2}\left\langle\begin{bmatrix}g^2(|X^*|) & 0\\ 0 & g^2(|Y^*|)\end{bmatrix}z,z\right\rangle\\
&+|q|\sqrt{1-|q|^2}\left\|\begin{bmatrix}g^2(|X^*|) & 0\\ 0 & g^2(|Y^*|)\end{bmatrix}-c\begin{bmatrix}I & 0\\ 0 & I\end{bmatrix}\right\|\\
\leq & \frac{1}{2}\left\langle\begin{bmatrix}f^2(|Y|)+|q|^2g^2(|X^*|) & 0\\ 0 & f^2(|X|)+|q|^2g^2(|Y^*|)\end{bmatrix}x,x\right\rangle\\
&+\frac{1-|q|^2}{2}\left\langle\begin{bmatrix}g^2(|X^*|) & 0\\ 0 & g^2(|Y^*|)\end{bmatrix}z,z\right\rangle\\
&+|q|\sqrt{1-|q|^2}\left\|\begin{bmatrix}g^2(|X^*|) & 0\\ 0 & g^2(|Y^*|)\end{bmatrix}-c\begin{bmatrix}I & 0\\ 0 & I\end{bmatrix}\right\|
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left|\left\langle\begin{bmatrix}0 & X\\ Y & 0\end{bmatrix}x,y\right\rangle\right| \leq & \frac{1}{2}\left(\left\|\begin{bmatrix}f^2(|Y|)+|q|^2g^2(|X^*|) & 0\\ 0 & f^2(|X|)+|q|^2g^2(|Y^*|)\end{bmatrix}\right\|\right)\\
&+\frac{1-|q|^2}{2}\left\|\begin{bmatrix}g^2(|X^*|) & 0\\ 0 & g^2(|Y^*|)\end{bmatrix}\right\|\\
&+|q|\sqrt{1-|q|^2}\left\|\begin{bmatrix}g^2(|X^*|) & 0\\ 0 & g^2(|Y^*|)\end{bmatrix}-c\begin{bmatrix}I & 0\\ 0 & I\end{bmatrix}\right\|\\
= & \frac{1}{2}\max\{\|f^2(|Y|)+|q|^2g^2(|X^*|)\|, \|f^2(|X|)+|q|^2g^2(|Y^*|)\|\}\\
&+\frac{1-|q|^2}{2}\max\{\|g^2(|X^*|)\|, \|g^2(|Y^*|)\|\}\\
&+|q|\sqrt{1-|q|^2}\max\{\|g^2(|X^*|)-cI\|, \|g^2(|Y^*|)-cI\|\}.
\end{aligned}$$

Taking supremum for all $x, y\in\mathcal{H}$ with $\|x\|=\|y\|=1$ and $\langle x,y\rangle = q$, we obtain

$$\begin{aligned}
w_q\left(\begin{bmatrix}0 & X\\ Y & 0\end{bmatrix}\right) \leq & \frac{1}{2}\max\{\|f^2(|Y|)+|q|^2g^2(|X^*|)\|, \|f^2(|X|)+|q|^2g^2(|Y^*|)\|\}\\
&+\frac{1-|q|^2}{2}\max\{\|g^2(|X^*|)\|, \|g^2(|Y^*|)\|\}\\
&+|q|\sqrt{1-|q|^2}\max\|\{g^2(|X^*|)-cI\|, \|g^2(|Y^*|)-cI\|\}.
\end{aligned}$$

(b) From Lemma 3.4, it follows

$$\begin{aligned}\left|\left\langle\begin{bmatrix}0 & X\\ Y & 0\end{bmatrix}x,y\right\rangle\right|^2 &\leq \left\|f\left(\left|\begin{bmatrix}0 & X\\ Y & 0\end{bmatrix}\right|\right)x\right\|^2\left\|g\left(\left|\begin{bmatrix}0 & Y^*\\ X^* & 0\end{bmatrix}\right|\right)y\right\|^2\\ &\leq\left\langle f^2\begin{bmatrix}|Y| & 0\\ 0 & |X|\end{bmatrix}x,x\right\rangle\left\langle g^2\begin{bmatrix}|X^*| & 0\\ 0 & |Y^*|\end{bmatrix}y,y\right\rangle\\ &\leq\frac{1}{2}\left(\left\langle f^2\begin{bmatrix}|Y| & 0\\ 0 & |X|\end{bmatrix}x,x\right\rangle^2+\left\langle g^2\begin{bmatrix}|X^*| & 0\\ 0 & |Y^*|\end{bmatrix}y,y\right\rangle^2\right)\\ &\leq\frac{1}{2}\left(\left\langle f^4\begin{bmatrix}|Y| & 0\\ 0 & |X|\end{bmatrix}x,x\right\rangle+\left\langle g^4\begin{bmatrix}|X^*| & 0\\ 0 & |Y^*|\end{bmatrix}y,y\right\rangle\right).\end{aligned}$$

A similar calculation as part (*a*) follows the required result.
□

**Remark 3.6.** *Several upper bounds of* $w_q\left(\begin{bmatrix}0 & X\\ Y & 0\end{bmatrix}\right)$ *follow by choosing particular* $f$ *and* $g$*.*

*(i) Let* $f(t)=t^\gamma$ *and* $g(t)=t^{1-\gamma}$, $0\leq\gamma\leq 1$*. Then Theorem 3.5(a) gives us,*

$$\begin{aligned}w_q\left(\begin{bmatrix}0 & X\\ Y & 0\end{bmatrix}\right)\leq&\frac{1}{2}\max\{\|\,|Y|^{2\gamma}+|q|^2|X^*|^{2(1-\gamma)}\|,\|\,|X|^{2\gamma}+|q|^2|Y^*|^{2(1-\gamma)}\|\}\\ &+\frac{1-|q|^2}{2}\max\{\|\,|X^*|^{2(1-\gamma)}\|,\|\,|Y^*|^{2(1-\gamma)}\|\}\\ &+|q|\sqrt{1-|q|^2}\max\{\|\,|X^*|^{2(1-\gamma)}-cI\|,\|\,|Y^*|^{2(1-\gamma)}-cI\|\}.\end{aligned}$$

*If we take* $\gamma=\frac{1}{2}$ *in the aforementioned inequality, then*

$$\begin{aligned}w_q\left(\begin{bmatrix}0 & X\\ Y & 0\end{bmatrix}\right)\leq&\frac{1}{2}\max\{\|\,|Y|+|q|^2|X^*|\,\|,\|\,|X|+|q|^2|Y^*|\,\|\}\\ &+\frac{1-|q|^2}{2}\max\{\|\,|X^*|\,\|,\|\,|Y^*|\,\|\}\\ &+|q|\sqrt{1-|q|^2}\max\{\|\,|X^*|-cI\|,\|\,|Y^*|-cI\|\}.\end{aligned}$$

*When* $q=1$*, we have*

$$w\left(\begin{bmatrix}0 & X\\ Y & 0\end{bmatrix}\right)\leq\frac{1}{2}\max\{\|f^2(|Y|)+g^2(|X^*|)\|,\|f^2(|X|)+g^2(|Y^*|)\|\}$$

*which is mentioned in [2, Theorem 1].*

*(ii) If we take* $f(t)=\frac{t}{1+t}$ *and* $g(t)=1+t$ *in Theorem 3.5(a), we have*

$$\begin{aligned}w_q\left(\begin{bmatrix}0 & X\\ Y & 0\end{bmatrix}\right)\leq&\frac{1}{2}\max\left\{\left\|\left(\frac{|Y|}{I+|Y|}\right)^2+|q|^2\,(I+|X^*|)^2\right\|,\left\|\left(\frac{|X|}{I+|X|}\right)^2+|q|^2\,(I+|Y^*|)^2\right\|\right\}\\ &+\frac{1-|q|^2}{2}\max\{\|(I+|X^*|)^2\|,\|(I+|Y^*|)^2\|\}\\ &+|q|\sqrt{1-|q|^2}\max\|\{(I+|X^*|)^2-cI\|,\|(I+|Y^*|)^2-cI\|\}.\end{aligned}$$

The following result provides a lower bound of the $q$-numerical radius of $\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}$.

**Theorem 3.7.** *Let $X, Y \in \mathcal{B}(\mathcal{H})$ and $q \in \mathcal{D}$, we have*

$$w_q\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) \geq \frac{|q|}{2}\max\{\|X\|, \|Y\|\} + \frac{|q|}{4}\left|\, \|X + Y^*\| - \|X - Y^*\| \,\right|.$$

*Proof.* Corollary 2.9 implies that $w_q\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) \geq |q| \left\|\mathcal{R}\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right)\right\|$ and $w_q\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) \geq |q| \left\|\mathcal{I}\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right)\right\|$. This implies,

$$\begin{aligned} w_q\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) \geq |q| \left\|\mathcal{R}\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right)\right\| &= \frac{|q|}{2}\left\|\left(\begin{bmatrix} 0 & X + Y^* \\ Y + X^* & 0 \end{bmatrix}\right)\right\| \\ &= \frac{|q|}{2}\max\{\|X + Y^*\|, \|Y + X^*\|\} \\ &= \frac{|q|}{2}\|X + Y^*\|. \end{aligned}$$

Also,

$$\begin{aligned} w_q\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) \geq |q| \left\|\mathcal{I}\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right)\right\| &= \frac{|q|}{2}\left\|\left(\begin{bmatrix} 0 & X - Y^* \\ Y - X^* & 0 \end{bmatrix}\right)\right\| \\ &= \frac{|q|}{2}\max\{\|X - Y^*\|, \|Y - X^*\|\} \\ &= \frac{|q|}{2}\|X - Y^*\|. \end{aligned}$$

Finally,

$$\begin{aligned} w_q\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) &\geq \frac{|q|}{2}\max\{\|X + Y^*\|, \|X - Y^*\|\} \\ &= \frac{|q|}{4}(\|X + Y^*\| + \|X - Y^*\|) + \frac{|q|}{4}\left|\|X + Y^*\| - \|X - Y^*\|\right| \\ &\geq \frac{|q|}{4}(\|(X + Y^*) \pm (X - Y^*)\|) + \frac{|q|}{4}\left|\|X + Y^*\| - \|X - Y^*\|\right| \\ &\geq \frac{|q|}{2}\max\{\|X\|, \|Y\|\} + \frac{|q|}{4}\left|\|X + Y^*\| - \|X - Y^*\|\right|. \end{aligned}$$

□

Taking $X = Y$ in the aforementioned result, we obtain the following corollary.

**Corollary 3.8.** *If $X \in \mathcal{B}(\mathcal{H})$ and $q \in \mathcal{D}$, then we have*

$$w_q\left(\begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix}\right) \geq \frac{|q|}{2}\|X\| + \frac{|q|}{2}\left|\|\mathcal{R}(X)\| - \|\mathcal{I}(X)\|\right|. \quad (28)$$

**Remark 3.9.** *For $q = 1$, (28) gives us a refinement of inequality (1). Also, if $X^2 = 0$, then relation (28) gives us a significant result as follows*

$$\frac{|q|}{2}\|X\| \leq w_q\left(\begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix}\right) \leq \|X\|.$$

In our next result, we extend the following well-known equality for $q$-numerical radius.

**Lemma 3.10.** *[1] Let $T, S \in \mathcal{B}(\mathcal{H})$ be positive definite operators. Then*

$$w\left(\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}\right) = \frac{1}{2}\|T + S\|.$$

To prove this, we need the following relation.

$$|q| \sup_{\theta\in\mathbb{R}} \|\mathcal{R}(e^{i\theta}T)\| \leq \sup_{\theta\in\mathbb{R}} w_q(\mathcal{R}(e^{i\theta}T)) \leq \sup_{\theta\in\mathbb{R}} w_q(e^{i\theta}T) = w_q(T).$$

Hence, for $T \in \mathcal{B}(\mathcal{H})$, we have

$$w_q(T) \geq |q| \sup_{\theta\in\mathbb{R}} \|\mathcal{R}(e^{i\theta}T)\|.$$

**Theorem 3.11.** *Let $X, Y \in \mathcal{B}(\mathcal{H})$, $q \in \mathcal{D}$ and $0 \leq \gamma \leq 1$, we have*

$$\frac{|q|}{2} \sup_{\theta\in\mathbb{R}} \left\{\|e^{i\theta}X + e^{-i\theta}Y^*\|\right\} \leq w_q\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) \leq \frac{|q|}{2} \||X|^{2\gamma} + |Y^*|^{2(1-\gamma)}\|^{\frac{1}{2}} \||X^*|^{2(1-\gamma)} + |Y|^{2\gamma}\|^{\frac{1}{2}} + \sqrt{1-|q|^2} \max\{\|X\|, \|Y\|\}.$$

*Proof.* From the relation $w_q(T) \geq |q| \sup_{\theta\in\mathbb{R}} \|\mathcal{R}(e^{i\theta}T)\|$, we have

$$\begin{aligned} w_q\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) &\geq |q| \sup_{\theta\in\mathbb{R}} \left\|\mathcal{R}\left(e^{i\theta}\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right)\right\| \\ &\geq \frac{|q|}{2} \sup_{\theta\in\mathbb{R}} \left\|\begin{bmatrix} 0 & e^{i\theta}X + e^{-i\theta}Y^* \\ e^{i\theta}Y + e^{-i\theta}X^* & 0 \end{bmatrix}\right\| \\ &= \frac{|q|}{2} \sup_{\theta\in\mathbb{R}} \|e^{i\theta}X + e^{-i\theta}Y^*\|. \end{aligned}$$

Now, to prove the second part, let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{H}$, and $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathcal{H}$ with $\|x\| = \|y\| = 1$ and $\langle x, y\rangle = q$. Then we can take $y = \bar{q}x + \sqrt{1-|q|^2}z$, where $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in \mathcal{H}$, $\|z\| = 1$ and $\langle x, z\rangle = 0$. Thus, $y_1 = \bar{q}x_1 + \sqrt{1-|q|^2}z_1$ and $y_2 = \bar{q}x_2 + \sqrt{1-|q|^2}z_2$.

Hence,

$$\begin{aligned} &\left|\left\langle\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right\rangle\right| \\ &\leq |\langle Xx_2, y_1\rangle| + |\langle Yx_1, y_2\rangle| \\ &= |\langle Xx_2, \bar{q}x_1 + \sqrt{1-|q|^2}z_1\rangle| + |\langle Yx_1, \bar{q}x_2 + \sqrt{1-|q|^2}z_2\rangle| \\ &\leq |q|(|\langle Xx_2, x_1\rangle| + |\langle Yx_1, x_2\rangle|) + \sqrt{1-|q|^2}(|\langle Xx_2, z_1\rangle| + |\langle Yx_1, z_2\rangle|). \end{aligned}$$

Using Theorem 1[16] and the Cauchy-Schwarz inequality, respectively, we have

$$\begin{aligned}
&\left|\left\langle \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle\right| \\
\leq & |q| (\langle |X|^{2\gamma} x_2, x_2 \rangle^{\frac{1}{2}} \langle |X^*|^{2(1-\gamma)} x_1, x_1 \rangle^{\frac{1}{2}} + \langle |Y|^{2\gamma} x_1, x_1 \rangle^{\frac{1}{2}} \langle |Y^*|^{2(1-\gamma)} x_2, x_2 \rangle^{\frac{1}{2}}) \\
&+ \sqrt{1-|q|^2} (\|X\| \|x_2\| \|z_1\| + \|Y\| \|x_1\| \|z_2\|) \\
\leq & |q| (\langle |X|^{2\gamma} x_2, x_2 \rangle + \langle |Y^*|^{2(1-\gamma)} x_2, x_2 \rangle)^{\frac{1}{2}} (\langle |X^*|^{2(1-\gamma)} x_1, x_1 \rangle + \langle |Y|^{2\gamma} x_1, x_1 \rangle)^{\frac{1}{2}} \\
&+ \sqrt{1-|q|^2} (\|X\| \|x_2\| \|z_1\| + \|Y\| \|x_1\| \|z_2\|) \\
\leq & |q| \| |X|^{2\gamma} + |Y^*|^{2(1-\gamma)} \|^{\frac{1}{2}} \| |X^*|^{2(1-\gamma)} + |Y|^{2\gamma} \|^{\frac{1}{2}} \|x_1\| \|x_2\| \\
&+ \sqrt{1-|q|^2} (\|X\| \|x_2\| \|z_1\| + \|Y\| \|x_1\| \|z_2\|).
\end{aligned}$$

Take $\|x_1\| = \sin(\theta)$, $\|x_2\| = \cos(\theta)$, $\|z_1\| = \cos(\phi)$ and $\|z_2\| = \sin(\phi)$, where $\theta, \phi \in \mathbb{R}$.

$$\begin{aligned}
\left|\left\langle \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle\right| \leq & |q| \left( \| |X|^{2\gamma} + |Y^*|^{2(1-\gamma)} \|^{\frac{1}{2}} \| |X^*|^{2(1-\gamma)} + |Y|^{2\gamma} \|^{\frac{1}{2}} \right) \cos(\theta) \sin(\theta) \\
&+ \sqrt{1-|q|^2} (\|X\| \cos(\theta) \cos(\phi) + \|Y\| \sin(\theta) \sin(\phi)) \\
\leq & \frac{|q| \sin(2\theta)}{2} \| |X|^{2\gamma} + |Y^*|^{2(1-\gamma)} \|^{\frac{1}{2}} \| |X^*|^{2(1-\gamma)} + |Y|^{2\gamma} \|^{\frac{1}{2}} \\
&+ \sqrt{1-|q|^2} \max\{\|X\|, \|Y\|\}.
\end{aligned}$$

Hence,

$$w_q\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) \leq \frac{|q|}{2} \| |X|^{2\gamma} + |Y^*|^{2(1-\gamma)} \|^{\frac{1}{2}} \| |X^*|^{2(1-\gamma)} + |Y|^{2\gamma} \|^{\frac{1}{2}} + \sqrt{1-|q|^2} \max\{\|X\|, \|Y\|\}.$$

□

**Remark 3.12.** *If $X$ and $Y$ are positive definite operators, then for $\gamma = 1/2$ it follows from the above theorem that*

$$\frac{|q|}{2} \|X+Y\| \leq w_q\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) \leq \frac{|q|}{2} \|X+Y\| + \sqrt{1-|q|^2} \max\{\|X\|, \|Y\|\}.$$

For $q = 1$, it follows $w\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) = \frac{1}{2}\|X+Y\|$, which is mentioned in Lemma 3.10.

In our final result, we assume $X$ and $Y$ both are in $\prod_{s,\alpha}^n$.

**Theorem 3.13.** *Let $X, Y \in \prod_{s,\alpha}^n$ and $q \in \mathcal{D}'$.*

*(a) If $\alpha \neq 0$, then we have*

$$\begin{aligned}
w_q\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) \geq & \frac{|q|}{4} \max\{\|(1+\cot(\alpha))X + (1-\cot(\alpha))Y^*\|, \|(1-\cot(\alpha))X + (1+\cot(\alpha))Y^*\|\} \\
&+ \frac{|q|}{4} \left| \|X+Y^*\| - \cot(\alpha)\|X-Y^*\| \right|.
\end{aligned}$$

*(b) If $\alpha = 0$, then we have*

$$w_q\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) \geq \frac{|q|}{2} \max\{\|X\|, \|Y\|\} + \frac{|q|}{4} \left| \|X+Y\| - \|X-Y\| \right|.$$

*Proof.* (a) From Theorem 2.10 and Lemma 2.1, the following relations

$$\|\mathcal{I}(T)\| \leq \sin(\alpha) w(T) \leq \sin(\alpha)\|T\| \leq \frac{\tan(\alpha)}{|q|} w_q(T)$$

hold for any $T \in \mathcal{B}(\mathcal{H})$. Hence, for $T \in \mathcal{B}(\mathcal{H})$, we have

$$w_q(T) \geq |q|\|\mathcal{R}(T)\| \text{ and } w_q(T) \geq |q| \cot(\alpha)\|\mathcal{I}(T)\|. \tag{29}$$

Taking $T = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}$ in inequality (29), we have

$$w_q\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) \geq \frac{|q|}{2}\|X + Y^*\| \text{ and } w_q\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) \geq \frac{|q|}{2} \cot(\alpha)\|X - Y^*\|.$$

Hence,

$$\begin{aligned} w_q\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) &\geq \frac{|q|}{2} \max\{\|X + Y^*\|, \cot(\alpha)\|X - Y^*\|\} \\ &= \frac{|q|}{4} \left(\|X + Y^*\| + \cot(\alpha)\|X - Y^*\| + \big|\|X + Y^*\| - \cot(\alpha)\|X - Y^*\|\big|\right) \\ &\geq \frac{|q|}{4} \left(\|(X + Y^*) \pm \cot(\alpha)(X - Y^*)\| + \big|\|X + Y^*\| - \cot(\alpha)\|X - Y^*\|\big|\right). \end{aligned}$$

Thus,

$$\begin{aligned} w_q\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) \geq \frac{|q|}{4}\Big( &\max\{\|(1 + \cot(\alpha))X + (1 - \cot(\alpha))Y^*\|, \|(1 - \cot(\alpha))X + (1 + \cot(\alpha))Y^*\|\} \\ &+ \big|\|X + Y^*\| - \cot(\alpha)\|X - Y^*\|\big| \Big). \end{aligned}$$

(b) If $\alpha = 0$, then $X$ and $Y$ are positive definite matrices. We have

$$w_q(T) \geq |q|\|\mathcal{R}(T)\| \text{ and } w_q(T) \geq |q|\|\mathcal{I}(T)\|.$$

By using similar calculations as Theorem 3.7, we have

$$w_q\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) \geq \frac{|q|}{2} \max\{\|X\|, \|Y^*\|\} + \frac{|q|}{4}\big|\|X + Y^*\| - \|X - Y^*\|\big|.$$

As $X$ and $Y$ are positive definite so $X = X^*$ and $Y = Y^*$,we have

$$w_q\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) \geq \frac{|q|}{2} \max\{\|X\|, \|Y\|\} + \frac{|q|}{4}\big|\|X + Y\| - \|X - Y\|\big|.$$

□

**Remark 3.14.** *One notable point is that if $X \in \prod_{s,\alpha}^n$, then $X^* \in \prod_{s,\alpha}^n$. If we take $Y = X^*$, then Theorem 3.13(a) gives us*

$$|q|\|X\| \leq w_q\left(\begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix}\right) \leq \|X\|.$$

*The lower bound mentioned in the aforementioned inequality is better as compared to the lower bound in Corollary 3.3(a).*

## Declarations:

*Acknowledgement*

The authors are grateful to the anonymous referee for insightful comments that helped improve the manuscript.

*Author Contributions*

All authors contributed equally to this research article.

*Competing Interests*

The authors have no competing interests.

*Funding*

The second author gratefully acknowledge the financial support received from the Anusandhan National Research Foundation (ANRF), Government of India (SRG/2023/002420).

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