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Strait AND product operations on strait soft sets and their reduced soft matrices with decision making applications

Akın Osman Atagün[a]

[a]*Department of Mathematics, Kırşehir Ahi Evran University, 40100 Kırşehir, Türkiye*

Abstract. In this study, firstly, a new soft set type, common image strait soft sets, which has a special place in the strait soft set structure, is introduced. Common image strait soft sets are strait soft sets whose image is the same partition set of a universal set. A new operation, Strait AND product is defined on both common image strait soft sets and their reduced soft matrices. A multi-criteria group decision-making (MCGDM) problem on performance analysis is described, and a decision-making algorithm is proposed to solve this type of problem, using the Strait AND product of reduced soft matrices of common image strait soft sets. Then the operation Strait AND product defined for common image strait soft sets is generalized for arbitrary strait soft sets. A decision-making algorithm is proposed using the Strait AND product of reduced matrices of arbitrary strait soft sets, and an MCGDM problem is created on market analysis to determine the effect of battery quality of electric cars on sales volume. The validity, advantages and disadvantages of the decision-making method presented in this study are discussed in comparison with the solution of a multi-attribute decision-making (MADM) problem.

1. Introduction

Since its introduction by Molodtsov [33] in 1999, soft set theory has proven to be an important tool for uncertainty modeling in theoretical and applied sciences. In order to provide more effective solutions to different types of problems involving uncertainty, different subtypes of soft sets, such as inverse soft sets [13], covering, partition and full soft sets [17], bijective soft sets [18], and strait soft sets [6], have been studied.

The structure of a strait soft set, which was first introduced in 2023, represents the parameterized state of the sets that form a partition of a universal set U. Although strait soft sets, by definition, appear to be a narrower version of classical soft sets, they seek to solve problems related to rough set theory [35] from a broader perspective. When examined from this perspective, the structure of a strait soft set can be positioned between the structures of a soft set and a rough soft set, introduced in [17]. In [6], the structures of a strait soft set and a strait rough set were first introduced. Since strait soft sets parameterize a partition of a set U, the strait rough set structure arises naturally. Because, as is known, a partition of a set U corresponds to an equivalence relation on U. In [7], a new strait structure, strait fuzzy sets, was first introduced. The strait fuzzy set structure has emerged by expressing the fuzzy membership degree of an

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Communicated by Miodrag Spalević
Email address: aosman.atagun@ahievran.edu.tr (Akın Osman Atagün)
ORCID iD: https://orcid.org/0000-0002-2131-9980 (Akın Osman Atagün)

object or point in the sub-intervals that form a partition of the closed interval $[0, 1]$. As in strait soft sets, the mutually entailing relationship between the partition set and equivalence relation on a set naturally allowed the emergence of the strait fuzzy rough set structure.

In [1, 2], comparisons of soft sets with fuzzy sets [49] and rough sets were given, and the concept of a soft group was defined. In addition, in [3, 8, 31, 36, 42–44], soft set operations were focused on, and their algebraic properties were examined. In [16, 19–21, 25, 41], algebraic structures built on soft sets and soft operations on these structures and also the relations of these soft algebraic structures with each other were examined. In [11], soft matrices, soft matrix operations, and their properties were introduced, and a decision-making method produced by using the four different multiplication operations defined is proposed; moreover, these studies have pioneered the production of many new decision-making methods [5, 8, 11, 24, 32, 37].

In real life, many problems involving uncertainty naturally use partitioning of a set. In fact, rough set theory deals with such problems. Strait soft sets directly parameterize these partitioning sets, thus eliminating the necessity of establishing an equivalence relation required in rough sets. Dubois and Prade's original idea about rough fuzzy sets, given in [15], inspired the determination of lower and upper approximations in hybrid structures containing rough sets. To take advantage of soft sets in a Pawlak approximation space, the hybrid structure of rough soft sets was introduced in [17]. In order to produce more efficient solutions to the problems considered in rough set theory, many different hybrid structures have been defined. In [45], where bipolar soft set and rough set theories were used together, the modified rough bipolar soft set *MRBS*-set is an example of such hybrid structures. In [39], the concepts of rough approximations, type-2 soft sets, and fuzzy sets were combined to introduce the concepts of rough fuzzy soft set, rough type-2 soft set and rough fuzzy type-2 soft set. Also, many algebraic results based on rough soft sets were obtained. In [4], the concepts of an approximation space depending on parameters in soft sets, and an approximation space depending on a soft set were studied. As a generalized concept of rough hemiring and soft hemiring, rough soft hemiring was introduced in [50]. Contributing to the algebraic study of hybrid structures of rough sets and soft sets, rough soft rings and rough idealistic soft rings were introduced in [51].

Various methods have been proposed to solve different types of decision-making problems using hybrid structures created with soft sets and rough sets. The decision-making model given in [6] is built using strait soft sets and strait rough sets, which are primary outputs of a multi-attribute decision-making model. In [40], the concept of rough approximations was associated with different algebraic structures built on soft sets, and also, rough soft relations, rough soft graphs, and rough soft hypergraphs were introduced. At the same time, the importance of soft and rough information was explained by proposing a group decision-making algorithm. In [30], two types of decision-making methods for rough soft sets were proposed, and their corresponding applied examples were given. In [28], it was stated that the two decision-making methods proposed in [30] were limited to rough soft sets, and their improved versions and a group decision-making method were proposed. In both the methods introduced in [28] and [30], inverse soft sets were used. In addition, similarity-based decision-making methods are also a field of study that researchers have focused on in recent years and produced various decision-making methods thanks to the studies on new similarity measures on uncertainty modeling [9, 10, 14, 22, 23, 26, 27, 29, 34, 38, 46–48, 52]. In [7], two new similarity-based decision-making methods were proposed using strait fuzzy sets and strait fuzzy rough sets.

Objectives of the study:

In this study, the structure of a strait soft set is examined by dividing it into two parts as common image strait soft sets and arbitrary strait soft sets. Considering that a strait soft set is a soft set that parameterizes a partition of a set U, this study first focuses on two or more strait soft sets that parameterize the same partition of a set U. The main purpose of studying common image strait soft sets specifically is that a common partition set is used in solving many real-life problems such as market analysis, employee performance measurement, personnel recruitment, determination of drug effectiveness, etc. In addition, the ease of operation in common image strait soft sets and the practicality of producing solutions to problems also support the main purpose. The presented definitions and obtained results are given both on the strait soft set structure and their reduced soft matrices, thus aiming to pave the way for both theoretical and applied studies. Later, in order to expand the range of problems to be solved, the obtained results are extended to arbitrary strait soft sets and their reduced matrices. In [11], the AND-product operation on soft matrices was defined, but since this definition is valid on soft matrices of the same type, this restriction was generalized in [5] and made valid on soft matrices of different types. Inspired by these definitions, in this study, the Strait AND-product operation, which reveals the interactions of the parameters with each other, is defined for common image strait soft sets and their matrices and is also generalized to arbitrary strait soft sets and their matrices.

The authentic life motivation about common image strait soft sets and arbitrary strait soft sets:

Score	Letter Grade	Coefficient	Success Status	Grade Average
88-100	AA	4.00	Successful	Attends
82-87	BA	3.50	Successful	Attends
76-81	BB	3.00	Successful	Attends
66-75	CB	2.50	Successful	Attends
60-65	CC	2.00	Successful	Attends
55-59	DC	1.50	Conditionally Successful	Attends
45-54	DD	1.00	Unsuccessful	Attends
30-44	FD	0.50	Unsuccessful	Attends
0-29	FF	0.00	Unsuccessful	Attends
	F1		Unsuccessful, absent	Attends
	F2		Unsuccessful, not attend to exam	Attends
	F3		Unsuccessful, make-up exam	Attends
	F4		Unsuccessful, retaking the course	Attends
	B		Successful for non-credit courses	Not attends
	K		Unsuccessful for non-credit courses	Not attends
	M		Exempt	Not attends

Figure 1: Kırşehir Ahi Evran University (KAEU) undergraduate grading system

Letter Grade	Coefficient	Score intervals	Letter Grade	Standing
AA	4,00	90-l00	S	Successful
BA	3,50	85-89	U	Unsuccessful
BB	3,00	80-84	EX	Exempt
CB	2,50	75-79	I	Incomplete
CC	2,00	70-74	W	Withdrawn
DC	1,50	65-69		
DD	1,00	60-64		
FD	0,50	50-59		
FF	0,00	0-49		
NA	0,00	*		

Figure 2: Middle East Technical University (METU) undergraduate grading system

The grading systems of two different universities and their letter equivalents are given in Figure 1 and Figure 2.

When comparing the grades of two students taking the Linear Algebra course in the (KAEU) Mathematics Department, the grading system in Figure 1 is used. Since the same partition set of the closed interval $[0,100]$ will be used here, this situation is an example of common image strait soft sets.
When Figure 1 and Figure 2 are examined, it is seen that two different partitions of $U = [0,100]$, such as
$U_1 = \{Y_1 = [87.5,100], Y_2 = [81.5,87.5), Y_3 = [75.5,81.5), Y_4 = [65.5,75.5), Y_5 = [59.5,65.5),$
$Y_6 = [54.5,59.5), Y_7 = [44.5,54.5), Y_8 = [29.5,44.5), Y_9 = [0,29.5)\}$ and
$U_2 = \{V_1 = [89.5,100], V_2 = [84.5,89.5), V_3 = [79.5,84.5), V_4 = [74.5,79.5), V_5 = [69.5,74.5),$
$V_6 = [64.5,69.5), V_7 = [59.5,64.5), V_8 = [49.5,59.5), V_9 = [0,49.5)\}$ are used in the grading systems. In this case, when the exam grade of a student taking a Linear Algebra course from the (KAEU) Mathematics department is to be compared with the exam grade of a student taking a Linear Algebra course from the (METU) Mathematics department, arbitrary strait soft sets are used. At the same time, (KAEU) and (METU) grading systems can also be expressed with strait soft sets. If F_{KAEU} and F_{METU} are constructed in the following form, they are the (KAEU) and (METU) grading systems, respectively, can be expressed in terms of strait soft sets using the partition sets U_1 and U_2, as following:
$F_{KAEU} = \{(AA,Y_1),(BA,Y_2),(BB,Y_3),(CB,Y_4),(CC,Y_5),(DC,Y_6),(DD,Y_7),(FD,Y_8),(FF,Y_9)\}$ and
$F_{METU} = \{(AA,V_1),(BA,V_2),(BB,V_3),(CB,V_4),(CC,V_5),(DC,V_6),(DD,V_7),(FD,V_8),(FF,V_9)\}$.

The rest of this study is arranged as follows. Section 2 has brief information on the preliminaries related to soft sets, soft matrices, and strait soft sets. In Section 3, common image strait soft sets are introduced, the Strait AND-product of common image strait soft sets and of their reduced soft matrices is defined and their properties are investigated. In Section 4, a decision-making algorithm called *CISSDM* is presented, which is obtained by using the operation Strait AND product on reduced soft matrices of common image strait soft sets, and an application is given that provides performance analysis of a company's personnel. In Section 5, the granular intersection of two partitions of a set is defined, and thus the Strait AND product defined on common image strait soft sets and on their reduced soft matrices is generalized to arbitrary strait soft sets and their reduced soft matrices. In Section 6, the *CISSDM* method is modified for arbitrary strait soft sets, and the Strait AND decision-making method (*SSDM*) is proposed, which is suitable for both group decision-making and *MADM* problems. In addition, a group decision-making application is given regarding the change in sales rates of electric cars depending on battery quality, and the *MADM* problem, whose solution was given in [6] with the help of strait soft sets and strait rough sets, is solved with *SSDM*, and the effectiveness, advantages, and disadvantages of the given method are discussed comparatively. Section 7 presents some conclusions.

2. Preliminaries

Definition 2.1. ([12]) Let U be an initial universe set, E be a set of parameters, $P(U)$ be the power set of U and $A \subseteq E$. A soft set (F,A) or simply F_A on the universe U is defined by the ordered pairs

$$(F,A) = \{(x, F(x)) | x \in E, F(x) \in P(U)\},$$

where $F : E \to P(U)$ such that $F(x) = \emptyset$ if $x \notin A$.

In other words, a soft set over U is a parameterized family of subsets of the universe U.

Definition 2.2. ([36]) Let (F,A) and (G,B) be soft sets over U.

a) If $A \subseteq B$ and $F(x) \subseteq G(x)$ for all $x \in A$, then (F,A) is a soft subset of (G,B), denoted by $(F,A)\widetilde{\subseteq}(G,B)$.

b) If $(F,A)\widetilde{\subseteq}(G,B)$ and $(G,B)\widetilde{\subseteq}(F,A)$, then (F,A) and (G,B) is said to be soft equal and denoted by $(F,A) = (G,B)$.

Definition 2.3. ([16]) Let (F, A) be soft set over U. Then the set

$$supp(F, A) = \{x \in A | F(x) \neq \emptyset\}$$

is called the *support* of the soft set (F, A). The *null soft set* is a soft set with empty support and we denote it by $\emptyset_E$. A soft set (F, A) is called *non-null* if $supp(F, A) \neq \emptyset$.

Definition 2.4. ([17]) Let (F, A) be soft set over U.

a) If $\bigcup_{x \in A} F(x) = U$, then (F, A) is called a *full soft set* over U.

b) A full soft set (F, A) is called a *covering soft set* if $F(x) \neq \emptyset$ for all $x \in A$.

c) (F, A) is said to be a *partition soft set*, if the set $\{F(x) : x \in A\}$ forms a partition of U.

Definition 2.5. ([18]) Let (F, A) be soft set over U such that A is a nonempty parameter set. We say that (F, A) is a bijective soft set, if (F, A) such that

a) $\bigcup_{x \in A} F(x) = U$,

b) $F(e_i) \cap F(e_j) = \emptyset$ for all $e_i, e_j \in A$ such that $e_i \neq e_j$.

Let $U = \{u_1, u_2, ..., u_n\}$ be a universal set and let $E = \{e_1, e_2, ..., e_m\}$ be the set of all parameters.

Definition 2.6. ([11]) For the soft set F_A, if

$$a_{ij} = \begin{cases} 1, & u_i \in F(e_j) \\ 0, & u_i \notin F(e_j) \end{cases}$$

then the matrix $[a_{ij}]_{n \times m}$ is called a *soft matrix* of the soft set F_A over U.

The set of all $n \times m$ soft matrices is denoted by $SM_{n \times m}$.

Definition 2.7. ([11]) If $[a_{ij}], [b_{ik}] \in SM_{n \times m}$, then *And-product* of $[a_{ij}]$ and $[b_{ik}]$ is defined by

$$\begin{aligned} \wedge : \quad & SM_{n \times m} \times SM_{n \times m} \to SM_{n \times m^2} \\ & [a_{ij}], [b_{ik}] \to [a_{ij}] \wedge [b_{ik}] = [c_{ip}] \end{aligned}$$

where $c_{ip} = min\{a_{ij}, b_{ik}\}$ such that $p = m(j - 1) + k$.

Definition 2.8. ([5]) Let $U = \{u_1, u_2, ..., u_n\}$, $E = \{e_1, e_2, ..., e_m\}$, $A \subseteq E$ and let cardinality of the set A be s. For the soft set F_A, if

$$a_{it} = \begin{cases} 1, & \text{if } e_t \in A \text{ and } u_i \in F(e_t) \\ 0, & \text{if } e_t \in A \text{ and } u_i \notin F(e_t) \end{cases}$$

then the matrix $[a_{it}]_{n \times s}$ is called a *reduced soft matrix* of the soft set F_A over U. Here $1 \leq s \leq m$.

Since $x \notin A$ implies $F(x) = \emptyset$, by eliminating the parameters in $E \backslash A$ from the set E, the soft matrix corresponding to soft set F_A can be constructed, and then the type of this soft matrix will be $n \times s$.

Definition 2.9. ([5]) If $[a_{ij}] \in SM_{n \times m}$, $[b_{ik}] \in SM_{n \times t}$, then *generalized And-product* of $[a_{ij}]$ and $[b_{ik}]$ is defined by

$$\begin{aligned} \wedge : \quad & SM_{n \times m} \times SM_{n \times t} \to SM_{n \times mt} \\ & [a_{ij}], [b_{ik}] \to [a_{ij}] \wedge [b_{ik}] = [c_{ip}] \end{aligned}$$

where $c_{ip} = min\{a_{ij}, b_{ik}\}$ such that $j = \beta$, $p = (\beta - 1)t + k$ and β is the smallest positive integer such that $p \leq t\beta$.

Definition 2.10. ([11]) Let $[a_{ij}], [b_{ij}] \in SM_{n \times m}$.

a) If $a_{ij} = 0$ for all i, j, then the soft matrix $[a_{ij}]$ is called a zero soft matrix and denoted by $[0]$.

b) $a_{ij} = 1$ for all i, j, then the soft matrix $[a_{ij}]$ is called a universal soft matrix and denoted by $[1]$.

c) $[a_{ij}]$ is a soft submatrix of $[b_{ij}]$ if $a_{ij} \leq b_{ij}$ for all i, j, denoted by $[a_{ij}] \subseteq [b_{ij}]$.

d) The soft matrix $[c_{ij}]$ is said to be a union of $[a_{ij}]$ and $[b_{ij}]$ if $c_{ij} = max\{a_{ij}, b_{ij}\}$ for all i, j, denoted by $[c_{ij}] = [a_{ij}] \cup [b_{ij}]$.

e) The soft matrix $[c_{ij}]$ is said to be a intersection of $[a_{ij}]$ and $[b_{ij}]$ if $c_{ij} = min\{a_{ij}, b_{ij}\}$ for all i, j, denoted by $[c_{ij}] = [a_{ij}] \cap [b_{ij}]$.

$PA(U) = \bigcup_{i \in I} U_i$ is the set of all partitions of U, $|U_k|$ is the number of elements in U_k and $supp(F, A)$ is abbreviated as SA if it does not cause any confusion.

Definition 2.11. ([6]) Let (F, A) be a soft set over U. For the soft set (F, A), if

$$F : SA \to PA(U)$$

is a set valued function such that $F(SA) = U_k$ for a fixed $k \in I$, then the soft set (F, SA) is called a strait soft set of (F, A).
A strait soft set over U can be represented by the set of ordered pairs

$$(F, SA) = \{(x, Y) | x \in SA, Y \in U_k \text{ for a fixed } k \in I\}.$$

From now on, $SS(U)$ denotes the set of all strait soft sets over U.

3. Common image strait soft sets with new operations on their soft matrices

Considering that the set containing all partitions of U is expressed as $PA(U) = \bigcup_{i \in I} U_i$, common image strait soft sets, which are a special case of strait soft sets, are defined as follows.

Definition 3.1. Let (F, SA) and (G, SB) be strait soft sets of the soft sets (F, A) and (G, B), respectively. If $F(SA) = G(SB) = U_k$ for a fixed $k \in I$, then (F, SA) and (G, SB) are called ***common image strait soft sets*** over U.

The set of all common image strait soft sets on U is denoted by $CISS(U)$.
The relationships diagram given in Figure 3 can be seen from the Definitions 2.1, 2.4, 2.5 and 2.11.

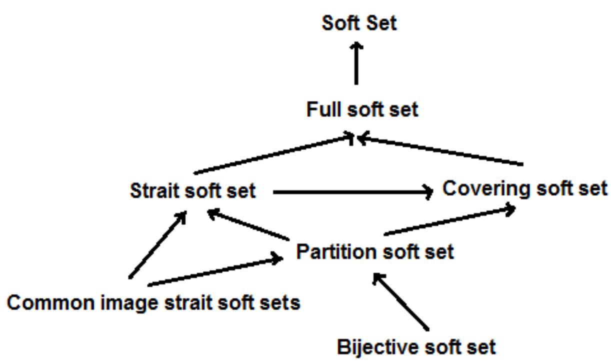


Figure 3: Connections of common image strait soft sets with other soft set types

Definition 3.2. Let $(F, SA), (G, SB) \in CISS(U)$. The operation Strait AND product of (F, SA) and (G, SB), denoted by $\wedge_s^c$, is defined by the following

$$\wedge_s^c : CISS(U) \times CISS(U) \to CISS(U), F_{SA} \wedge_s^c G_{SB} = H_{S(SA\times SB)}$$

where, $H(e_i, e_j) = F(e_i)$ if $F(e_i) = G(e_j)$ for all $e_i \in SA$ and $e_j \in SB$.
Here, the soft set $(H, SA \times SB)$ is constructed as,

$$H(e_i, e_j) = \begin{cases} F(e_i), & if \quad F(e_i) = G(e_j) \\ \emptyset, & if \quad F(e_i) \neq G(e_j) \end{cases},$$

then $H_{S(SA\times SB)}$ is the strait soft set of $(H, SA \times SB)$.

Theorem 3.1. *The operation Strait AND product $\wedge_s^c$ is associative on $CISS(U)$.*

Proof. Let $(F, SA), (G, SB), (K, SC) \in CISS(U)$. $F_{SA} \wedge_s^c G_{SB} = H_{S(SA\times SB)}$, where $H_{S(SA\times SB)}$ is the strait soft set of $(H, SA \times SB)$ and $(H, SA \times SB)$ is obtained by

$$H(e_i, e_j) = \begin{cases} F(e_i), & if \quad F(e_i) = G(e_j) \\ \emptyset, & if \quad F(e_i) \neq G(e_j) \end{cases}.$$

Now, $H_{S(SA\times SB)} \wedge_s^c K_{SC} = M_{S((SA\times SB)\times SC)}$, where $M_{S((SA\times SB)\times SC)}$ is the strait soft set of the soft set $M_{(SA\times SB)\times SC}$ is obtained by

$$M((e_i, e_j), e_k) = \begin{cases} H(e_i, e_j) = F(e_i), & if \quad F(e_i) = G(e_j) = K(e_k) \\ \emptyset, & if \quad H(e_i, e_j) = F(e_i) \neq K(e_k) \\ \emptyset, & if \quad F(e_i) \neq G(e_j) = K(e_k) \\ \emptyset, & if \quad F(e_i) \neq G(e_j) \neq K(e_k) \end{cases}.$$

Now, assume that $G_{SB} \wedge_s^c K_{SC} = N_{S(SB\times SC)}$, where $N_{S(SB\times SC)}$ is the strait soft set of $(N, SB\times SC)$ and $(N, SB\times SC)$ is obtained by

$$N(e_i, e_j) = \begin{cases} G(e_i), & if \quad G(e_i) = K(e_j) \\ \emptyset, & if \quad G(e_i) \neq K(e_j) \end{cases}.$$

$F_{SA} \wedge_s^c N_{S(SB\times SC)} = P_{S(SA\times(SB\times SC))}$, where $P_{S(SA\times(SB\times SC))}$ is the strait soft set of the soft set $P_{SA\times(SB\times SC)}$ is obtained by

$$P(e_i, (e_j, e_k)) = \begin{cases} F(e_i), & if \quad F(e_i) = G(e_j) = K(e_k) \\ \emptyset, & if \quad F(e_i) \neq K(e_k) \\ \emptyset, & if \quad F(e_i) \neq G(e_j) = K(e_k) \\ \emptyset, & if \quad F(e_i) \neq G(e_j) \neq K(e_k) \end{cases}.$$

Since $M_{S((SA\times SB)\times SC)} = P_{S(SA\times(SB\times SC))}$, then $(F_{SA} \wedge_s^c G_{SB}) \wedge_s^c K_{SC} = F_{SA} \wedge_s^c (G_{SB} \wedge_s^c K_{SC})$. Therefore Strait AND product is an associative operation on $CISS(U)$.

Since strait soft sets are already defined on the support set, when constructing matrices of strait soft sets, it is more convenient to use the concept of reduced soft matrix by Definition 2.8.

Let $CISSM(U)$ denotes the set of reduced soft matrices of all common image strait soft sets. The reduced soft matrix of $(F, SA) \in CISS(U)$ is denoted by $[F_{SA}]$ and $|F_{SA}|_t$ denotes the t-th column of the matrix $[F_{SA}]$. Now, we are ready to define a new operation on $CISSM(U)$.

Definition 3.3. Let $[F_{SA}], [G_{SB}] \in CISSM(U)$. The operation Strait And product of $[F_{SA}]$ and $[G_{SB}]$, denoted by $\wedge_s^c$, is defined by the following

$$\wedge_s^c : CISSM(U) \times CISSM(U) \to CISSM(U), [F_{SA}] \wedge_s^c [G_{SB}] = [C_P]$$

where, $|C_P|_t = |F_{SA}|_k$, $t \in I_k$ and I_k is given by

$$I_1 = \{1, 2, ..., |S_1|\} \text{ and for } k > 1, I_k = \{1, 2, ..., \sum_{i=1}^{k} |S_i|\} \setminus \bigcup_{i=1}^{k-1} I_i,$$

where $S_k = \{j : |F_{SA}|_k = |G_{SB}|_j\}$.

Example 3.1. Let the universe $U = \{u_1, u_2, u_3, u_4\}$, the parameter set $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ and two subsets of E be $A = \{e_1, e_2, e_3, e_4, e_5, e_7\}$ and $B = \{e_1, e_2, e_3, e_4, e_5, e_6\}$. Suppose that corresponding soft sets of A and B are:

$$F_A = \{(e_1, \{u_1, u_2\}), (e_2, \{u_1, u_2\}), (e_3, \{u_3\}), (e_4, \{u_1, u_2\}), (e_5, \emptyset), (e_7, \{u_4\})\},$$

$$G_B = \{(e_1, \{u_3\}), (e_2, \{u_1, u_2\}), (e_3, \{u_4\}), (e_4, \{u_1, u_2\}), (e_5, \emptyset), (e_6, \emptyset)\}.$$

Then, the support sets of F_A and G_B are:

$$SA = supp(F, A) = \{e_1, e_2, e_3, e_4, e_7\},$$

$$SB = supp(G, B) = \{e_1, e_2, e_3, e_4\}.$$

Since $F(SA) = G(SB) = U_2$, where $U_2 = \{\{u_1, u_2\}, \{u_3\}, \{u_4\}\}$ is a partition of U, then the strait soft sets of F_A and G_B, respectively are

$$F_{SA} = \{(e_1, \{u_1, u_2\}), (e_2, \{u_1, u_2\}), (e_3, \{u_3\}), (e_4, \{u_1, u_2\}), (e_7, \{u_4\})\},$$

$$G_{SB} = \{(e_1, \{u_3\}), (e_2, \{u_1, u_2\}), (e_3, \{u_4\}), (e_4, \{u_1, u_2\})\}.$$

By Definition 3.2, the soft set $(H, SA \times SB)$ is constructed as,

$$H(e_i, e_j) = \begin{cases} F(e_i), & if \quad F(e_i) = G(e_j) \\ \emptyset, & if \quad F(e_i) \neq G(e_j) \end{cases},$$

then $(H, SA \times SB) = \{((e_1, e_1), \emptyset), ((e_1, e_2), \{u_1, u_2\}), ((e_1, e_3), \emptyset), ((e_1, e_4), \{u_1, u_2\}),$
$((e_2, e_1), \emptyset), ((e_2, e_2), \{u_1, u_2\}), ((e_2, e_3), \emptyset), ((e_2, e_4), \{u_1, u_2\}),$
$((e_3, e_1), \{u_3\}), ((e_3, e_2), \emptyset), ((e_3, e_3), \emptyset), ((e_3, e_4), \emptyset)$
$((e_4, e_1), \emptyset), ((e_4, e_2), \{u_1, u_2\}), ((e_4, e_3), \emptyset), ((e_4, e_4), \{u_1, u_2\}),$
$((e_7, e_1), \emptyset), ((e_7, e_2), \emptyset), ((e_7, e_3), \{u_4\}), ((e_7, e_4), \emptyset)\}.$

Then the strait soft set of $(H, SA \times SB)$ is $F_{SA} \wedge_s^c G_{SB} = H_{S(SA \times SB)}$, where
$(H, S(SA \times SB)) = \{((e_1, e_2), \{u_1, u_2\}), ((e_1, e_4), \{u_1, u_2\}),$
$((e_2, e_2), \{u_1, u_2\}), ((e_2, e_4), \{u_1, u_2\}),$
$((e_3, e_1), \{u_3\}),$
$((e_4, e_2), \{u_1, u_2\}), ((e_4, e_4), \{u_1, u_2\}),$
$((e_7, e_3), \{u_4\})\}.$

Similarly, $G_{SB} \wedge_s^c F_{SA} = T_{S(SB \times SA)}$, where
$(T, S(SB \times SA)) = \{((e_1, e_3), \{u_3\}),$
$((e_2, e_1), \{u_1, u_2\}), ((e_2, e_2), \{u_1, u_2\}), ((e_2, e_4), \{u_1, u_2\}),$
$((e_3, e_7), \{u_4\}),$
$((e_4, e_1), \{u_1, u_2\}), ((e_4, e_2), \{u_1, u_2\}), ((e_4, e_4), \{u_1, u_2\})\}.$

Since $(H, S(SA \times SB)) \neq (T, S(SB \times SA))$, the operation Strait AND product of strait soft sets is not commutative, in general.

Now, corresponding reduced soft matrices of the strait soft sets F_{SA} and G_{SB} are

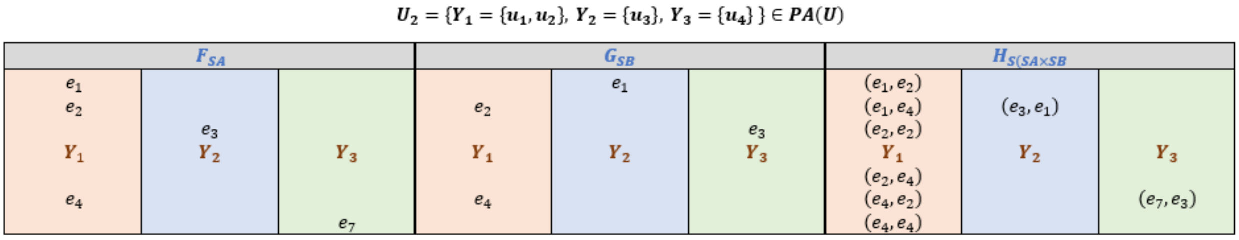


Figure 4: Distributions of parameters on the partition sets in strait soft sets

$$[F_{SA}] = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad [G_{SB}] = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Since $S_1 = \{j : |F_{SA}|_1 = |G_{SB}|_j\} = \{2, 4\}$,
$S_2 = \{j : |F_{SA}|_2 = |G_{SB}|_j\} = \{2, 4\}$,
$S_3 = \{j : |F_{SA}|_3 = |G_{SB}|_j\} = \{1\}$,
$S_4 = \{j : |F_{SA}|_4 = |G_{SB}|_j\} = \{2, 4\}$ and
$S_5 = \{j : |F_{SA}|_5 = |G_{SB}|_j\} = \{3\}$, then, $I_1 = \{1, 2, ..., |S_1|\} = \{1, 2\}$,
$I_2 = \{1, 2, ..., |S_1| + |S_2|\} \setminus I_1 = \{3, 4\}$,
$I_3 = \{1, 2, ..., |S_1| + |S_2| + |S_3|\} \setminus (I_1 \cup I_2) = \{5\}$,
$I_4 = \{1, 2, ..., |S_1| + |S_2| + |S_3| + |S_4|\} \setminus (I_1 \cup I_2 \cup I_3) = \{6, 7\}$ and
$I_5 = \{1, 2, ..., |S_1| + |S_2| + |S_3| + |S_4| + |S_5|\} \setminus (I_1 \cup I_2 \cup I_3 \cup I_4) = \{8\}$.
Therefore,

$$[F_{SA}] \wedge_s^c [G_{SB}] = [C_P] = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Similarly,

$$[G_{SB}] \wedge_s^c [F_{SA}] = [D_Q] = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, the operation Strait And-product of reduced soft matrices of strait soft sets is not commutative, in general.

Theorem 3.2. *There is a strait soft set corresponding to each element of $CISSM(U)$.*

Proof. Let $[a_{ij}]_{m\times n} \in CISSM(U)$. Then, $|a_{ij}|_j \neq 0$ for all $j \in \{1, 2, ..., n\}$, $|a_{ij}|_{j_k} = |a_{ij}|_{j_t}$ or $|a_{ij}|_{j_k} \cap |a_{ij}|_{j_t} = [0]_{m\times 1}$ for all $j_k, j_t \in \{1, 2, ..., n\}$ and $\bigcup_{j=1}^{n} |a_{ij}|_j = [1]_{m\times 1}$. By Definition 2.8, the soft set (F, A) obtained by

$$a_{ij} = \begin{cases} 1, & \text{if } e_j \in A \text{ and } u_i \in F(e_j) \\ 0, & \text{if } e_j \in A \text{ and } u_i \notin F(e_j) \end{cases}$$

is a strait soft set $(F, A) = (F, SA) = \{(x, Y) | x \in SA, Y \in U_k \text{ for a fixed } k\}$, where U_k is a partition of U, by Definition 2.11.

Theorem 3.3. *The operation Strait AND product $\wedge_s^c$ is associative on $CISSM(U)$.*

Proof. The proof is omitted since it can be proven using similar logic to Theorem 3.1.

4. Decision making application on performance analysis

In this section, a decision-making algorithm, called *CISSDM*, obtained using the operation Strait AND product on reduced soft matrices of common image strait soft sets will be presented, and an application that provides performance analysis of a company's personnel will be given.

Definition 4.1. Let $[F^i_{A_i}], [F^j_{A_j}] \in CISSM(U)$. The binary decision soft matrix of $[F^i_{A_i}]$ and $[F^j_{A_j}]$, denoted by $[B_{P_{ij}}]$,is defined as

$$[B_{P_{ij}}] = ([F^i_{A_i}] \wedge^c_s [F^j_{A_j}]) \cap ([F^j_{A_j}] \wedge^c_s [F^i_{A_i}])$$

Definition 4.2. Let $[B_{P_{ij}}] = [b_{ij}]$ be the binary decision soft matrix of $[F^i_{A_i}], [F^j_{A_j}] \in CISSM(U)$. Then, the row-sum matrix of $[B_{P_{ij}}]$, denoted by $R[B_{P_{ij}}]$, is the one column matrix defined as $R[B_{P_{ij}}] = [r_{i1}]$, where $r_{i1} = \sum_j b_{ij}$.

Definition 4.3. Let $F^1_{A_1}, F^2_{A_2}, ..., F^k_{A_k} \in CISSM(U)$ and $[B_{P_{ij}}] = [b_{ij}]$ be the binary decision soft matrices for all $i, j \in \{1, 2, ..., k\}$. The one-column matrix obtained by summing matrices $R[B_{P_{ij}}]$ for all $i, j \in \{1, 2, ..., k\}$, denoted by $\sum R[B_{P_{ij}}]$ is called the decision matrix of U.

Definition 4.4. Let $F^1_{A_1}, F^2_{A_2}, ..., F^k_{A_k} \in CISSM(U)$ and $\sum R[B_{P_{ij}}] = [d_{i1}]$ is the decision matrix of U. Then the optimum set of U is the set $Opt_{[d_{i1}]}(U) = \{\frac{u_i}{di1} : u_i \in U\}$ and the ranking is given by $d_{k1} \leq d_{t1}$ if and only if $u_k \leq u_t$.

Decision Making Algorithm for *CISSDM*

Step 1. k decision makers obtain a suitable partition of U.

Step 2. k decision makers form their strait soft sets: $F^1_{A_1}, F^2_{A_2}, ..., F^k_{A_k}$.

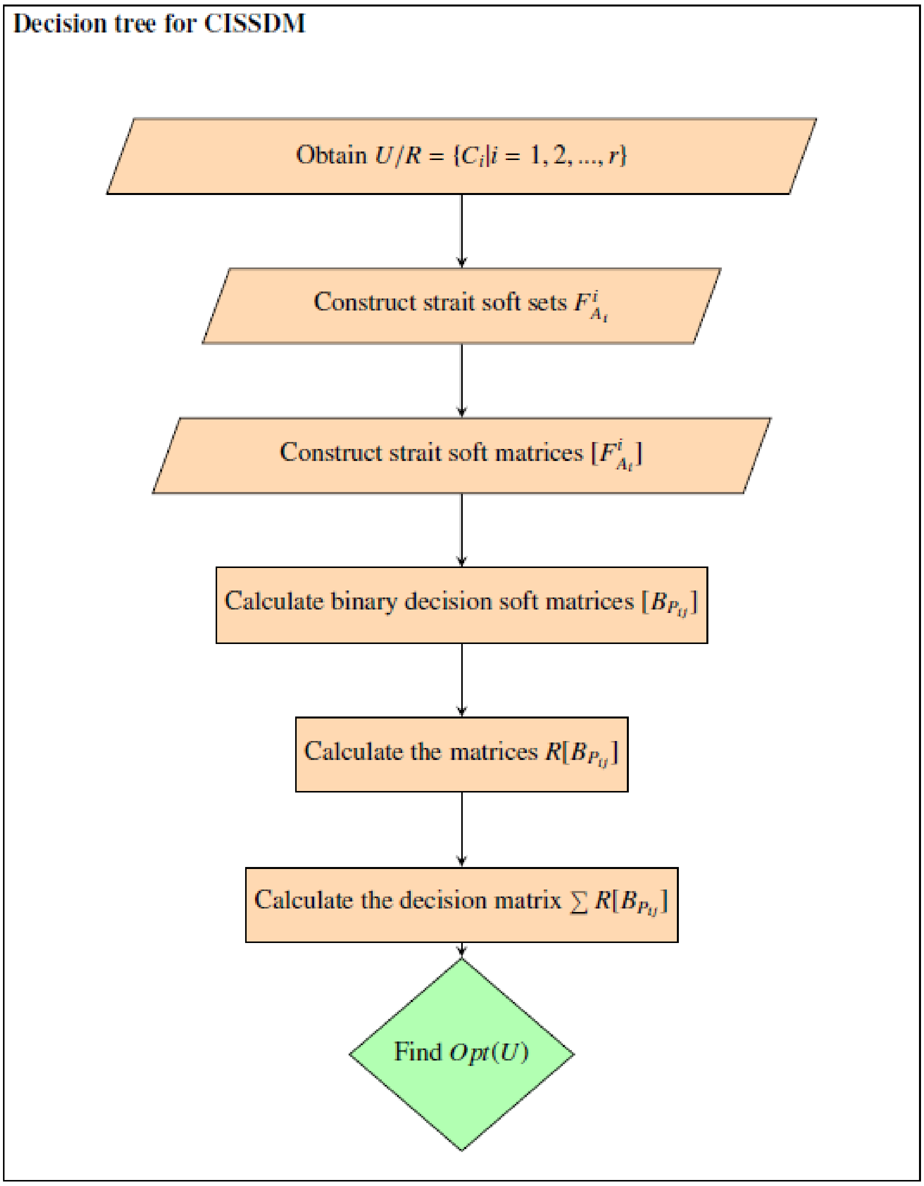
Step 3. The soft matrices $[F^1_{A_1}], [F^2_{A_2}], ..., [F^k_{A_k}]$ are constructed.

Step 4. The binary decision soft matrix $[B_{P_{ij}}] = ([F^i_{A_i}] \wedge^c_s [F^j_{A_j}]) \cap ([F^j_{A_j}] \wedge^c_s [F^i_{A_i}])$ is calculated for all $i, j \in \{1, 2, ..., k\}$.

Step 5. The one-column row-sum matrices $R[B_{P_{ij}}]$ is calculated for all $i, j \in \{1, 2, ..., k\}$.

Step 6. The decision matrix $\sum R[B_{P_{ij}}]$ is calculated.

Step 7. The common optimum set $Opt(U)$ of U and the ranking of $u_i \in U$ are obtained.



Example 4.1. Company X wants to conduct a performance analysis for the employees of its three branches located in the same city. Let these branches and their personnel be determined by the sets $Y_1 = \{u_1, u_2, u_3, u_4\}$, $Y_2 = \{u_5, u_6, u_7, u_8\}$ and $Y_3 = \{u_9, u_{10}, u_{11}, u_{12}\}$. Then (**Step 1.**) $U_1 = \{Y_1, Y_2, Y_3\}$ is a partition of $U = \{u_1, u_2, ..., u_{12}\}$. For this purpose, the company appoints 3 decision-makers, and these decision-makers determine the following common parameters as $E = \{e_1, e_2, ..., e_5\}$, where e_1 : Low number of staff complaints; e_2 : The abundance of staff satisfaction feedback; e_3 : Compatibility of sales volume with workload; e_4 : Efficient work; e_5 : Harmony of personnel with each other. As a result of the examinations, the decision makers create the following strait soft sets(**Step 2.**):

$$F^1_E = \{(e_1, Y_1), (e_2, Y_1), (e_3, Y_2), (e_4, Y_2), (e_5, Y_3)\},$$

$$F^2_E = \{(e_1, Y_2), (e_2, Y_3), (e_3, Y_2), (e_4, Y_1), (e_5, Y_3)\},$$

$$F^3_E = \{(e_1, Y_1), (e_2, Y_2), (e_3, Y_2), (e_4, Y_3), (e_5, Y_2)\}.$$

Then, the corresponding strait soft matrices are (**Step 3.**)

$$[F_E^1]=\begin{bmatrix}1&1&0&0&0\\1&1&0&0&0\\1&1&0&0&0\\1&1&0&0&0\\0&0&1&1&0\\0&0&1&1&0\\0&0&1&1&0\\0&0&1&1&0\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\end{bmatrix},[F_E^2]=\begin{bmatrix}0&0&0&1&0\\0&0&0&1&0\\0&0&0&1&0\\0&0&0&1&0\\1&0&1&0&0\\1&0&1&0&0\\1&0&1&0&0\\1&0&1&0&0\\0&1&0&0&1\\0&1&0&0&1\\0&1&0&0&1\\0&1&0&0&1\end{bmatrix},[F_E^3]=\begin{bmatrix}1&0&0&0&0\\1&0&0&0&0\\1&0&0&0&0\\1&0&0&0&0\\0&1&1&0&1\\0&1&1&0&1\\0&1&1&0&1\\0&1&1&0&1\\0&0&0&1&0\\0&0&0&1&0\\0&0&0&1&0\\0&0&0&1&0\end{bmatrix}.$$

To calculate the binary decision soft matrices $[B_{P_{12}}]$, $[B_{P_{13}}]$ and $[B_{P_{23}}]$, the following products are necessary:

$$[F_E^1]\wedge_s^c[F_E^2]=\begin{bmatrix}1&1&0&0&0&0&0&0\\1&1&0&0&0&0&0&0\\1&1&0&0&0&0&0&0\\1&1&0&0&0&0&0&0\\0&0&1&1&1&1&0&0\\0&0&1&1&1&1&0&0\\0&0&1&1&1&1&0&0\\0&0&1&1&1&1&0&0\\0&0&0&0&0&0&1&1\\0&0&0&0&0&0&1&1\\0&0&0&0&0&0&1&1\\0&0&0&0&0&0&1&1\end{bmatrix},[F_E^2]\wedge_s^c[F_E^1]=\begin{bmatrix}0&0&0&0&0&1&1&0\\0&0&0&0&0&1&1&0\\0&0&0&0&0&1&1&0\\0&0&0&0&0&1&1&0\\1&1&0&1&1&0&0&0\\1&1&0&1&1&0&0&0\\1&1&0&1&1&0&0&0\\1&1&0&1&1&0&0&0\\0&0&1&0&0&0&0&1\\0&0&1&0&0&0&0&1\\0&0&1&0&0&0&0&1\\0&0&1&0&0&0&0&1\end{bmatrix}.$$

Since $[B_{P_{ij}}]=([F^i_{A_i}]\wedge_s^c[F^j_{A_j}])\cap([F^j_{A_j}]\wedge_s^c[F^i_{A_i}])$, then

$$[B_{P_{12}}]=\begin{bmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&1&1&0&0&0\\0&0&0&1&1&0&0&0\\0&0&0&1&1&0&0&0\\0&0&0&1&1&0&0&0\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\end{bmatrix}.$$

Since,

$$[F_E^1] \wedge_s^c [F_E^3] = \begin{bmatrix} 1&1&0&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0&0\\ 0&0&1&1&1&1&1&1&0\\ 0&0&1&1&1&1&1&1&0\\ 0&0&1&1&1&1&1&1&0\\ 0&0&1&1&1&1&1&1&0\\ 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&1 \end{bmatrix}, [F_E^3] \wedge_s^c [F_E^1] = \begin{bmatrix} 1&1&0&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0&0\\ 0&0&1&1&1&1&0&1&1\\ 0&0&1&1&1&1&0&1&1\\ 0&0&1&1&1&1&0&1&1\\ 0&0&1&1&1&1&0&1&1\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0&0 \end{bmatrix},$$

then,

$$[B_{P_{13}}] = \begin{bmatrix} 1&1&0&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0&0\\ 0&0&1&1&1&1&0&1&0\\ 0&0&1&1&1&1&0&1&0\\ 0&0&1&1&1&1&0&1&0\\ 0&0&1&1&1&1&0&1&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{bmatrix}.$$

Similarly,

$$[F_E^2] \wedge_s^c [F_E^3] = \begin{bmatrix} 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1&0\\ 1&1&1&0&1&1&1&0&0\\ 1&1&1&0&1&1&1&0&0\\ 1&1&1&0&1&1&1&0&0\\ 1&1&1&0&1&1&1&0&0\\ 0&0&0&1&0&0&0&0&1\\ 0&0&0&1&0&0&0&0&1\\ 0&0&0&1&0&0&0&0&1\\ 0&0&0&1&0&0&0&0&1 \end{bmatrix}, [F_E^3] \wedge_s^c [F_E^2] = \begin{bmatrix} 1&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0\\ 0&1&1&1&1&0&0&1&1\\ 0&1&1&1&1&0&0&1&1\\ 0&1&1&1&1&0&0&1&1\\ 0&1&1&1&1&0&0&1&1\\ 0&0&0&0&0&1&1&0&0\\ 0&0&0&0&0&1&1&0&0\\ 0&0&0&0&0&1&1&0&0\\ 0&0&0&0&0&1&1&0&0 \end{bmatrix}.$$

Then,

$$[B_{P_{23}}]=\begin{bmatrix}0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0\\0&1&1&0&1&0&0&0&0\\0&1&1&0&1&0&0&0&0\\0&1&1&0&1&0&0&0&0\\0&1&1&0&1&0&0&0&0\\0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0\end{bmatrix}.$$

(**Step 5.**) For all $i, j \in \{1,2,3\}$, the one-column row-sum matrices $R[B_{P_{ij}}]$ are

$$R[B_{P_{12}}]=\begin{bmatrix}0\\0\\0\\0\\2\\2\\2\\2\\1\\1\\1\\1\end{bmatrix}, R[B_{P_{13}}]=\begin{bmatrix}2\\2\\2\\2\\5\\5\\5\\5\\0\\0\\0\\0\end{bmatrix}, R[B_{P_{23}}]=\begin{bmatrix}0\\0\\0\\0\\3\\3\\3\\3\\0\\0\\0\\0\end{bmatrix}.$$

Then, (**Step 6.**) the decision matrix is

$$\sum R[B_{P_{ij}}]=[d_{i1}]=\begin{bmatrix}2\\2\\2\\2\\10\\10\\10\\10\\1\\1\\1\\1\end{bmatrix}.$$

Finally, $Opt_{[d_{i1}]}(U)=\{\frac{u_i}{di1}: u_i \in U\}=\{\frac{u_1}{2},\frac{u_2}{2},\frac{u_3}{2},\frac{u_4}{2},\frac{u_5}{10},\frac{u_6}{10},\frac{u_7}{10},\frac{u_8}{10},\frac{u_9}{1},\frac{u_{10}}{1},\frac{u_{11}}{1},\frac{u_{12}}{1}\}=\{\frac{Y_1}{2},\frac{Y_2}{10},\frac{Y_3}{1}\}$ and the ranking is obtained as $Y_3 < Y_1 < Y_2$. Here, the interaction of the parameters with each other is obtained by using the Strait AND product operation. Additionally, by using binary decision soft matrix, the alternatives that have the most parameters of the decision-makers are determined.

5. New operations on arbitrary strait soft sets

In this section, we first define the granular intersection of any two partitions of a universal set U and show that the granular intersection of the partitions is also a partition of U. Using this result, we define the operation Strait AND product of any strait soft sets on U.

Definition 5.1. Let $\emptyset \neq U$ be an universal set and $PA(U) = \bigcup_{i\in I} U_i$ be the set of all partitions of U. If $U_k = \{Y_i | i = 1, 2, ..., r_1\}, U_t = \{V_i | i = 1, 2, ..., r_2\} \in PA(U)$, then the **granular intersection** of the partitions U_k and U_t, denoted $U_k \cap_g U_t$, is the set $U_k \cap_g U_t = \{\emptyset \neq Y_i \cap V_j | i = 1, 2, ..., r_1, j = 1, 2, ..., r_2\}$.

Proposition 5.1. *If $U_k, U_t \in PA(U)$, then $U_k \cap_g U_t \in PA(U)$.*

Proof. Let $U_k = \{Y_i | i = 1, 2, ..., r_1\}, U_t = \{V_i | i = 1, 2, ..., r_2\} \in PA(U)$. By Definition 5.1,
$U_k \cap_g U_t = \{\emptyset \neq Y_i \cap V_j | i = 1, 2, ..., r_1, j = 1, 2, ..., r_2\} = \{S_i | i = 1, 2, ..., r_3\}$.
Then, we immediately have,
$S_i \neq \emptyset$ for all $i \in \{1, 2, ..., r_3\}$.
Let $S_i, S_j \in U_k \cap_g U_t$ and $i \neq j$. Then, there exist $a, c \in \{1, 2, ..., r_1\}, b, d \in \{1, 2, ..., r_2\}$ such that $S_i = Y_a \cap V_b$ and $S_j = Y_c \cap V_d$. Since $Y_a \cap Y_c = \emptyset$ and $V_b \cap V_d = \emptyset$, then,
$S_i \cap S_j = (Y_a \cap V_b) \cap (Y_c \cap V_d) = \emptyset$, for all $i, j \in \{1, 2, ..., r_3\}$ and $i \neq j$.
Also,
$\bigcup_{i=1}^{r_3} S_i = \bigcup_{a=1}^{r_1} \bigcup_{b=1}^{r_2} (Y_a \cap V_b) = \bigcup_{a=1}^{r_1} Y_a \cap \bigcup_{b=1}^{r_2} V_b = U \cap U = U$.
Therefore, $U_k \cap_g U_t = \{S_i | i = 1, 2, ..., r_3\}$ is a partition of U.

Now, we are ready to introduce a novel operation on $SS(U)$.

Definition 5.2. Let $PA(U) = \bigcup_{i\in I} U_i$ and $(F, SA), (G, SB) \in SS(U)$. Then, $F(SA) = U_k$ and $G(SB) = U_t$ for fixed $k, t \in I$. The operation Strait AND product of (F, SA) and (G, SB), denoted by $\wedge_s$, is defined by the following

$$\wedge_s : SS(U) \times SS(U) \to SS(U), F_{SA} \wedge_s G_{SB} = H_{S(SA\times SB)}$$

where, the soft set $(H, SA \times SB)$ is constructed as $H(e_i, e_j) = F(e_i) \cap G(e_j)$ for all $e_i \in SA$ and $e_j \in SB$, then $H_{S(SA\times SB)}$ is the strait soft set of $(H, SA \times SB)$. Here, $H(S(SA \times SB)) = U_k \cap_g U_t$.

Example 5.1. Let $U = \{u_1, u_2, ..., u_5\}$ and
$U_k = \{Y_1 = \{u_1, u_2, u_3\}, Y_2 = \{u_4, u_5\}\}, U_t = \{V_1 = \{u_1, u_2\}, V_2 = \{u_3, u_4, u_5\}\} \in PA(U)$. Then,
$U_k \cap_g U_t = \{Y_1 \cap V_1, Y_1 \cap V_2, Y_2 \cap V_2\} = \{\{u_1, u_2\}, \{u_3\}, \{u_4, u_5\}\} \in PA(U)$.
Now, let the strait soft sets $F(SA) = U_k$ and $G(SB) = U_t$ as

$$F_{SA} = \{(e_1, \{u_1, u_2, u_3\}), (e_2, \{u_4, u_5\})\},$$

$$G_{SB} = \{(e_2, \{u_1, u_2\}), (e_3, \{u_3, u_4, u_5\})\}.$$

Then, $F_{SA} \wedge_s G_{SB} = H_{S(SA\times SB)}$, where$(H, SA \times SB) = \{((e_1, e_2), \{u_1, u_2\}), ((e_1, e_3), \{u_3\}), ((e_2, e_2), \emptyset), ((e_2, e_3), \{u_4, u_5\})\}$ and the strait soft set of $(H, SA \times SB)$ is$(H, S(SA \times SB)) = \{((e_1, e_2), \{u_1, u_2\}), ((e_1, e_3), \{u_3\}), ((e_2, e_3), \{u_4, u_5\})\}$.
It is seen that $H(S(SA \times SB)) = U_k \cap_g U_t$.

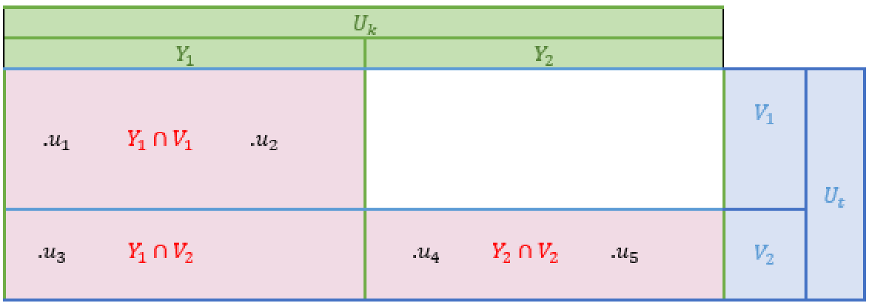


Figure 5: Granular intersection of the partitions U_k and U_t

Similarly,
$G_{SB} \wedge_s F_{SA} = \{((e_2, e_1), \{u_1, u_2\}), ((e_3, e_1), \{u_3\}), ((e_3, e_2), \{u_4, u_5\})\}$ is obtained. Since $F_{SA} \wedge_s G_{SB} \neq G_{SB} \wedge_s F_{SA}$, then the operation Strait AND product $\wedge_s$ on $SS(U)$ is not commutative, in general.

Theorem 5.1. *The operation Strait AND product $\wedge_s$ is associative on $SS(U)$.*

Proof. Let $(F, SA), (G, SB), (K, SC) \in SS(U)$. $F_{SA} \wedge_s G_{SB} = H_{S(SA\times SB)}$, where $H_{S(SA\times SB)}$ is the strait soft set of $(H, SA \times SB)$ and $(H, SA \times SB)$ is obtained by $H(e_i, e_j) = F(e_i) \cap G(e_j)$. Now, $H_{S(SA\times SB)} \wedge_s K_{SC} = M_{S((SA\times SB)\times SC)}$, where $M_{S((SA\times SB)\times SC)}$ is the strait soft set of the soft set $M_{(SA\times SB)\times SC}$ is obtained by $M((e_i, e_j), e_k) = (F(e_i) \cap G(e_j)) \cap K(e_k)$.
Now, assume that $G_{SB} \wedge_s K_{SC} = N_{S(SB\times SC)}$, where $N_{S(SB\times SC)}$ is the strait soft set of $(N, SB \times SC)$ and $(N, SB\times SC)$ is obtained by $N(e_j, e_k) = G(e_j) \cap K(e_k)$. Now, $F_{SA} \wedge_s N_{S(SB\times SC)} = P_{S(SA\times(SB\times SC))}$, where $P_{S(SA\times(SB\times SC))}$ is the strait soft set of the soft set $P_{SA\times(SB\times SC)}$ is obtained by $P(e_i, (e_j, e_k)) = F(e_i) \cap (G(e_j) \cap K(e_k))$. Since $M_{S((SA\times SB)\times SC)} = P_{S(SA\times(SB\times SC))}$, then $(F_{SA} \wedge_s G_{SB}) \wedge_s K_{SC} = F_{SA} \wedge_s (G_{SB} \wedge_s K_{SC})$. Therefore Strait AND product is an associative operation on $SS(U)$.

The operation Strait AND product $\wedge_s^c$ defined on $CISS(U)$ makes it easier to perform operations on common image strait soft sets. But the operation Strait AND product given by Definition 5.2, valid on arbitrary strait soft sets, can also be used on $CISS(U)$. Therefore, in this study, all of these processes are referred to as Strait AND product. This is proven by the following theorem:

Theorem 5.2. *The operation Strait AND product $\wedge_s$ is a generalization of the operation Strait AND product $\wedge_s^c$ on $CISS(U)$.*

Proof. Since $PA(U) = \bigcup_{i\in I} U_i$ is the set of all partitions of U and $(F, SA), (G, SB) \in CISS(U)$, then $F(SA) = G(SB) = U_k$ for a fixed $k \in I$, by Definition 3.1. $F_{SA} \wedge_s^c G_{SB} = H_{S(SA\times SB)}$, where $H(e_i, e_j) = F(e_i)$ if $F(e_i) = G(e_j)$ for all $e_i \in SA$ and $e_j \in SB$, By Definition 3.2.
Here, the soft set $(H, SA \times SB)$ is constructed as,

$$H(e_i, e_j) = \begin{cases} F(e_i), & if \quad F(e_i) = G(e_j) \\ \emptyset, & if \quad F(e_i) \neq G(e_j) \end{cases},$$

then $H_{S(SA\times SB)}$ is the strait soft set of $(H, SA \times SB)$. For all $e_i \in SA$ and $e_j \in SB$, $F(e_i) = G(e_j)$ or $F(e_i) \neq G(e_j)$, implies that $F(e_i) \cap G(e_j) = F(e_i)$ or $F(e_i) \cap G(e_j) = \emptyset$. Then $F_{SA} \wedge_s G_{SB} = H_{S(SA\times SB)}$, by Definition 5.2. Therefore, $F_{SA} \wedge_s^c G_{SB} = F_{SA} \wedge_s G_{SB}$.

We mentioned in the previous section that it is necessary to use the reduced soft matrix structure to generate matrices of strait soft sets. Let $SSM(U)$ denotes the set of reduced soft matrices of all strait soft sets over U. The reduced soft matrix of $(F, SA) \in SS(U)$ is denoted by $[F_{SA}]$ and $|F_{SA}|_t$ denotes the t-th column of the matrix $[F_{SA}]$. The following definition presents the operation Strait And product of any two reduced matrices corresponding to strait soft sets.

Definition 5.3. Let $U = \{u_1, u_2, ..., u_m\}$ be the universe and $[F_{SA}] = [a_{ij}]_{m\times n}, [G_{SB}] = [b_{ik}]_{m\times t} \in SSM(U)$. The operation Strait And product of $[F_{SA}]$ and $[G_{SB}]$, denoted by $\wedge_s$, is defined by the following

$$\wedge_s : SSM(U) \times SSM(U) \to SSM(U), [F_{SA}] \wedge_s [G_{SB}] = [C_P] = [c_{ip}]$$

where, $c_{ip} = min\{a_{iv}, b_{iq_{S_v}}\}$ for $p \in I_v$ and $p = q_{I_v}$, here
q_{I_v} denotes the q-th element of the set I_v and q_{S_v} denotes the q-th element of the set S_v, where
$I_1 = \{1, 2, ..., |S_1|\}$ and for $v > 1$, $I_v = \{1, 2, ..., \sum_{i=1}^{v} |S_i|\} \setminus \bigcup_{i=1}^{v-1} I_i$ and S_v is given by
$S_v = \{k : min\{|F_{SA}|_v, |G_{SB}|_k\} \neq 0\}$.

Example 5.2. Let the strait soft sets

$$F_{SA} = \{(e_1, \{u_1, u_2, u_3\}), (e_2, \{u_4, u_5\})\},$$

$$G_{SB} = \{(e_2, \{u_5\}), (e_3, \{u_3, u_4\}), (e_4, \{u_1, u_2\})\}.$$

over $U = \{u_1, u_2, ..., u_5\}$. The corresponding reduced soft matrices of the strait soft sets F_{SA} and G_{SB} are

$$[F_{SA}] = [a_{ij}]_{5\times 2} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad [G_{SB}] = [b_{ik}]_{5\times 3} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

By Definition 5.3, since
$S_1 = \{k | min\{|F_{SA}|_1, |G_{SB}|_k\} \neq 0\} = \{2, 3\}$,
$S_2 = \{k | min\{|F_{SA}|_2, |G_{SB}|_k\} \neq 0\} = \{1, 2\}$, then,
$I_1 = \{1, 2, ..., |S_1|\} = \{1, 2\}$ and $I_2 = \{1, 2, ..., \sum_{i=1}^{2} |S_i|\} \setminus \bigcup_{i=1}^{2-1} I_i = \{1, 2, 3, 4\} \setminus \{1, 2\} = \{3, 4\}$.
For instance, to obtain c_{43}, since $p = 3 \in I_2$, $p = 3 = 1_{I_2}$ and $1_{S_2} = 1$, then $c_{43} = min\{a_{42}, b_{41}\} = 0$. Similarly, to obtain c_{31}, since $p = 1 \in I_1$, $p = 1 = 1_{I_1}$ and $1_{S_1} = 2$, then, $c_{31} = min\{a_{31}, b_{32}\} = 1$. Then, $[F_{SA}] \wedge_s [G_{SB}] = [C_P] = [c_{ip}]$ is the reduced soft matrix

$$[c_{ip}]_{5\times 4} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Similarly, $[G_{SB}] \wedge_s [F_{SA}] = [D_P] = [d_{ip}]$ is the reduced soft matrix

$$[d_{ip}]_{5\times 4} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Since $[F_{SA}] \wedge_s [G_{SB}] \neq [G_{SB}] \wedge_s [F_{SA}]$, then the operation Strait AND product on $SSM(U)$ are not commutative, in general.

Theorem 5.3. *The operation Strait AND product* $\wedge_s$ *is associative on* $SSM(U)$.

Proof. The proof is omitted since it can be proven using similar logic to Theorem 5.1.

The result that the operation Strait AND product on arbitrary strait soft sets is a generalization of the Strait AND product operation on $CISS(U)$ also holds for reduced soft matrices of strait soft sets. This is given by the following theorem.

Theorem 5.4. *The operation Strait AND product* $\wedge_s$*, given by Definition 5.3, is a generalization of the operation Strait AND product* $\wedge_s^c$ *on* $CISSM(U)$.

Proof. The proof is omitted since it can be proven using similar logic to Theorem 5.2.

6. A decision making application on sales rates of electric cars according to battery quality

In this section, *CISSDM* will be reconstructed for reduced matrices of arbitrary strait soft sets, and an application on the sales rates of electric cars depending on battery quality will be presented.

Depending on the type of problem or the narrowness and width of the desired solution range for the problem, decision-makers continue the solution process with one of the following binary decision soft matrix definitions.

Definition 6.1. Let $[F^i_{A_i}], [F^j_{A_j}] \in SSM(U)$.

a. The crisp binary decision soft matrix of $[F^i_{A_i}]$ and $[F^j_{A_j}]$, denoted by $[B^c_{P_{ij}}]$,is defined as

$$[B^c_{P_{ij}}] = ([F^i_{A_i}] \wedge_s [F^j_{A_j}]) \cap ([F^j_{A_j}] \wedge_s [F^i_{A_i}]).$$

b. The relaxed binary decision soft matrix of $[F^i_{A_i}]$ and $[F^j_{A_j}]$, denoted by $[B^r_{P_{ij}}]$,is defined as

$$[B^r_{P_{ij}}] = ([F^i_{A_i}] \wedge_s [F^j_{A_j}]) \cup ([F^j_{A_j}] \wedge_s [F^i_{A_i}]).$$

Definition 6.2. Let $[B_{P_{ij}}] = [b_{ij}]$ be the one of $[B^c_{P_{ij}}]$ or $[B^r_{P_{ij}}]$ of $[F^i_{A_i}], [F^j_{A_j}] \in SSM(U)$. Then, the row-sum matrix of $[B_{P_{ij}}]$, denoted by $R[B_{P_{ij}}]$, is the one column matrix defined as $R[B_{P_{ij}}] = [r_{i1}]$, where $r_{i1} = \sum_j b_{ij}$.

Definition 6.3. Let $F^1_{A_1}, F^2_{A_2}, ..., F^k_{A_k} \in SSM(U)$ and $[B_{P_{ij}}] = [b_{ij}]$ be the binary decision soft matrices for all $i, j \in \{1, 2, ..., k\}$. The one-column matrix obtained by summing matrices $R[B_{P_{ij}}]$ for all $i, j \in \{1, 2, ..., k\}$, denoted by $\sum R[B_{P_{ij}}]$ is called the decision matrix of U.

Definition 6.4. Let $F^1_{A_1}, F^2_{A_2}, ..., F^k_{A_k} \in SSM(U)$ and $\sum R[B_{P_{ij}}] = [d_{i1}]$ is the decision matrix of U. Then, **the optimum set** of U is the set

$$Opt_{[d_{i1}]}(U) = \{\tfrac{u_i}{di1} : u_i \in U\}$$

and the ranking is given by $d_{k1} \leq d_{t1}$ if and only if $u_k \leq u_t$.
Let $PA(U) = \bigcup_{i\in I} U_i$ is the set of all partitions of U and $U_k = \{Y_1, Y_2, ..., Y_t\} \in PA(U)$. For the partition U_k, **the partition optimum set** $Popt(U_k)$ is the set,

$$Popt(U_k) = \{\tfrac{Y_s}{\sum di1} : u_i \in Y_s\}$$

and the ranking is given by $\sum d_{k1} \leq \sum d_{t1}$ if and only if $Y_k \leq Y_t$, using the set $Opt_{[d_{i1}]}(U)$.

The following algorithm gives the Strait AND decision making method (*SSDM*) obtained by using the Strait AND product on reduced matrices of arbitrary strait soft sets.

Decision Making Algorithm for *SSDM*

Step 1. k decision makers obtain suitable partitions of U.

Step 2. k decision makers form their strait soft sets: $F^1_{A_1}, F^2_{A_2}, ..., F^k_{A_k}$.

Step 3. The soft matrices $[F^1_{A_1}], [F^2_{A_2}], ..., [F^k_{A_k}]$ are constructed.

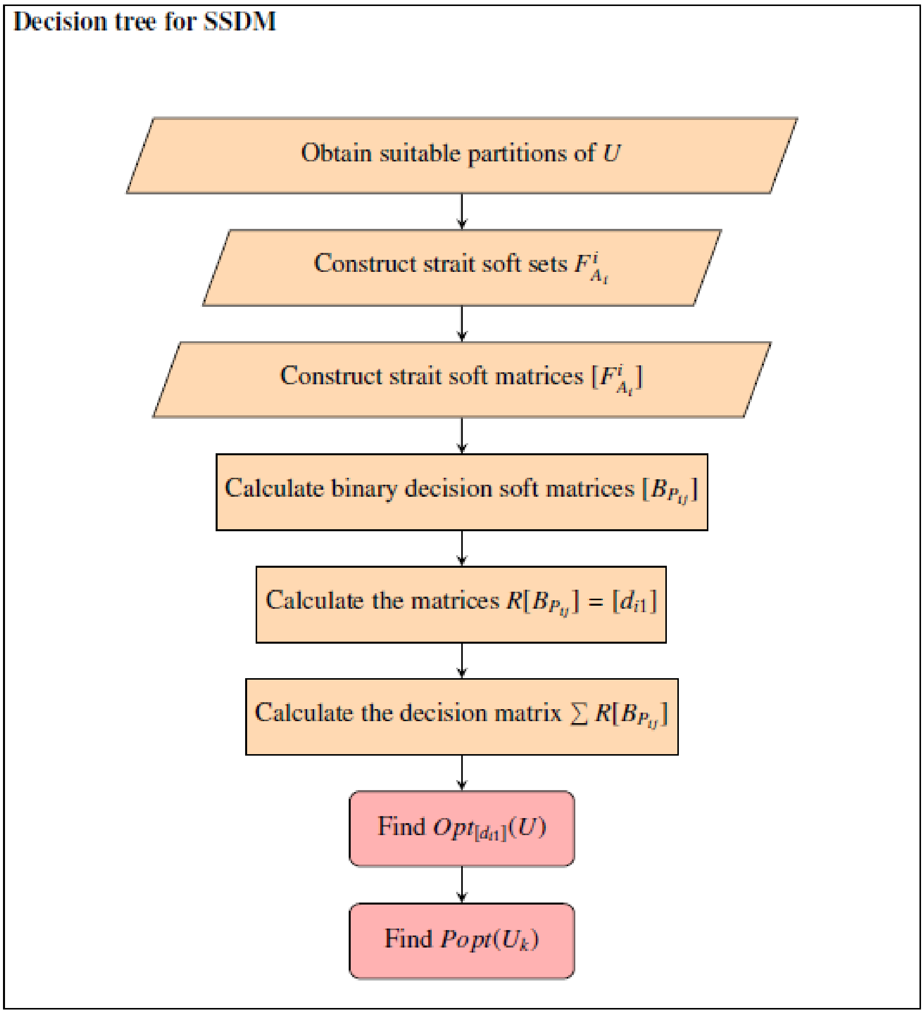
Step 4. The binary decision soft matrix is chosen as the one of $[B_{P_{ij}}] = [B^c_{P_{ij}}]$ or $[B_{P_{ij}}] = [B^r_{P_{ij}}]$, $i, j \in \{1, 2, ..., k\}$.

Step 5. The one-column row-sum matrices $R[B_{P_{ij}}]$ is calculated for all $i, j \in \{1, 2, ..., k\}$.

Step 6. The decision matrix $\sum R[B_{P_{ij}}] = [d_{i1}]$ is calculated.

Step 7. The common optimum set $Opt_{[d_{i1}]}(U)$ of U and the ranking of $u_i \in U$ are obtained.

Step 8. The partition optimum set $Popt(U_k)$ of the partition $U_k \in PA(U)$ and the ranking of $Y_i \in U_k$ are obtained.



Example 6.1. A company that produces electric cars is examining 8 models, expressed by the set $U = \{u_1, u_2, ..., u_8\}$, produced by rival companies to investigate the effect of battery quality on sales rates. When determining battery quality, partitions are also obtained (**Step 1.**).
According to battery cost: Costs less than 5000 dollars are $Y_1 = \{u_1, u_4, u_7\}$,
Costs between 5000 and 7000 dollars are $Y_2 = \{u_2, u_3, u_5\}$ and
Those costing 7,000 dollars and above are $Y_3 = \{u_6, u_8\}$. Then, the first partition of U is $U_1 = \{Y_1, Y_2, Y_3\}$.
According to the range provided by the battery: Those with a range of less than 300 km are $V_1 = \{u_1, u_3\}$,
Those with a range between 300 km and 450 km are $V_2 = \{u_4, u_5, u_7\}$ and
Those with a range of more than 450 km are $V_3 = \{u_2, u_6, u_8\}$. Then, the other partition of U is $U_2 = \{V_1, V_2, V_3\}$.
According to the maximum usage time of the battery: Battery life less than 6 years are $T_1 = \{u_1, u_5\}$,
Those with a battery life of 6 to 8 years are $T_2 = \{u_2, u_3, u_4\}$ and
Those with a battery life of more than 8 years are $T_3 = \{u_6, u_7, u_8\}$. Then the last partition of U is

$U_3 = \{T_1, T_2, T_3\}$.

Now a decision maker creates the following strait soft sets regarding the sales rates of these cars in the countries C_1, C_2, and C_3 and whether buyers have an idea about batteries. Then, the parameters are obtained as e_1 : Cars purchased by battery-conscious consumers with high sales in C_1, e_2 : Cars purchased by battery-conscious consumers with high sales in C_2, and e_3 : Cars purchased by battery-conscious consumers with high sales in C_3. As a result of the examinations, the decision maker determines the following strait soft sets according to parameter set $E = \{e_1, e_2, e_3\}$ (**Step 2.**):

$$F_E^1 = \{(e_1, Y_2), (e_2, Y_3), (e_3, Y_1)\},$$

$$F_E^2 = \{(e_1, V_1), (e_2, V_2), (e_3, V_3)\},$$

$$F_E^3 = \{(e_1, T_2), (e_2, T_1), (e_3, T_3)\}.$$

Then, the corresponding reduced soft matrices of these strait soft sets are (**Step 3.**)

$$[F_E^1] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, [F_E^2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, [F_E^3] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

For a more precise solution, if the solution is continued by choosing the crisp binary decision soft matrix $[B_{P_{ij}}] = [B^c_{P_{ij}}]$, then to compute the binary decision soft matrices $[B_{P_{12}}], [B_{P_{13}}]$ and $[B_{P_{23}}]$, the following products are required (**Step 4.**):

$$[F_E^1] \wedge_s [F_E^2] = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, [F_E^2] \wedge_s [F_E^1] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since $[B_{P_{ij}}] = ([F^i_{A_i}] \wedge_s [F^j_{A_j}]) \cap ([F^j_{A_j}] \wedge_s [F^i_{A_i}])$, then

$$[B_{P_{12}}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since,

$$[F_E^1] \wedge_s [F_E^3] = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, [F_E^3] \wedge_s [F_E^1] = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then,

$$[B_{P_{13}}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Similarly,

$$[F_E^2] \wedge_s [F_E^3] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, [F_E^3] \wedge_s [F_E^2] = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then,

$$[B_{P_{23}}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(**Step 5.**) For all $i, j \in \{1, 2, 3\}$, the one-column row-sum matrices $R[B_{P_{ij}}]$ are

$$R[B_{P_{12}}] = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, R[B_{P_{13}}] = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, R[B_{P_{23}}] = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Then, (**Step 6.**) the decision matrix is

$$\sum R[B_{P_{ij}}] = [d_{i1}] = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Finally, (**Step 7.**) $Opt_{[d_{i1}]}(U) = \{\frac{u_i}{di1} : u_i \in U\} = \{\frac{u_1}{0}, \frac{u_2}{1}, \frac{u_3}{3}, \frac{u_4}{0}, \frac{u_5}{0}, \frac{u_6}{1}, \frac{u_7}{1}, \frac{u_8}{1}\}$ and the ranking is obtained as $u_1 = u_4 = u_5 < u_2 = u_6 = u_7 = u_8 < u_3$. Accordingly, in production planning, the decision maker may foresee giving priority to cars that have the features of the car u_3.
(**Step 8.**)
$Popt(U_1) = \{\frac{Y_s}{\sum di1} : u_i \in Y_s\} = \{\frac{Y_1}{1}, \frac{Y_2}{4}, \frac{Y_3}{2}\}$ and the ranking is $Y_1 < Y_3 < Y_2$.
$Popt(U_2) = \{\frac{V_s}{\sum di1} : u_i \in V_s\} = \{\frac{V_1}{3}, \frac{V_2}{1}, \frac{V_3}{3}\}$ and the ranking is $V_2 < V_1 = V_3$.
$Popt(U_3) = \{\frac{T_s}{\sum di1} : u_i \in T_s\} = \{\frac{T_1}{0}, \frac{T_2}{4}, \frac{T_3}{3}\}$ and the ranking is $T_1 < T_3 < T_2$.

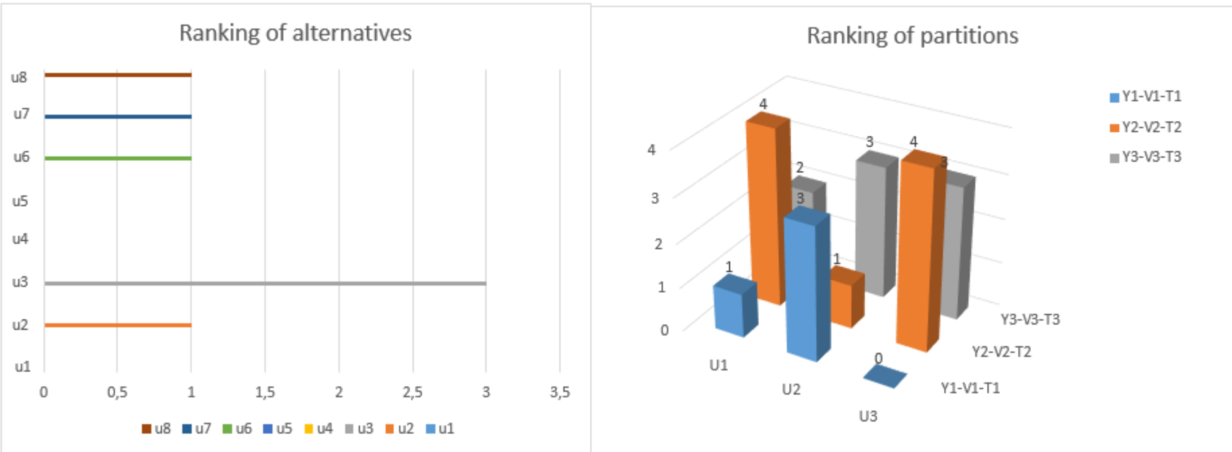


Figure 6: The optimum set and the partition optimum sets

According to these results, the decision maker can predict that potential customers prefer cars with a battery cost between 5000 and 7000 dollars, a battery range of less than 300 km or more than 450 km, and a battery life of 6 to 8 years.

A comparative application.

To discuss the validity, advantages and disadvantages of *SSDM*, the following example is obtained by adapting Example 6 and Example 7 from the paper [6].

Example 6.2. (adapted from [6], Examples 6. and 7.) Let the sets of the best-selling smartphones in the countries C_1, C_2 and C_3 of the 8 different types of smartphones indicated with $u_1, u_2, ..., u_8$ in 2021 be given as $X_1 = \{u_1, u_3, u_7, u_8\}, X_2 = \{u_1, u_5, u_6, u_8, \}, X_3 = \{u_2, u_5, u_7\}$, respectively.
An expert group will decide which attributes can be recommended for future smartphones, considering the best-selling smartphones in the set $U = \{u_1, u_2, ..., u_8\}$ in countries C_1, C_2 and C_3. To do this, they first determine some attributes or parameters.
The sets of attributes for different colors, screen technologies, materials and screen types are, respectively,
$E_1 = \{e^1_1$: light color, e^1_2 : dark color, e^1_3 : golden silver, e^1_4 : mixed color$\}$,
$E_2 = \{e^2_1$: OLED, e^2_2 : AMOLED, e^2_3 : Super AMOLED, e^2_4 : TFT LCD, e^2_5 : IPS LCD, e^2_6 : PLS LCD$\}$,

$E_3 = \{e^3_1 : \text{p-g}, e^3_2 : \text{p-a}, e^3_3 : \text{pc-g}, e^3_4 : \text{al-g}, e^3_5 : \text{al-pc}\}$,
abbreviations here refer to: p:plastic, g:glass, al:aluminum and pc:polycarbonate.
$E_4 = \{e^4_1 : \text{d sc}, e^4_2 : \text{cl-single}, e^4_3 : \text{exp-single}, e^4_4 : \text{r/s sc}, e^4_5 : \text{fd sc}\}$,
abbreviations here refer to: sc:screen, d:dual, cl:classic, exp:expandable, r/s:rollable/sliding and fd:foldable.
Let the sets $\beta_1 = E_1$, $\beta_2 = E_2 \setminus \{e6^2\}$, $\beta_3 = E_3$ and $\beta_4 = E_4$ be subsets of the disjoint feature sets determined by the expert group.
The partition sets of U are **(Step 1)**:
$U_1 = \{Y_1 = \{u_1\}, Y_2 = \{u_2, u_6\}, Y_3 = \{u_3, u_4, u_5\}, Y_4 = \{u_7, u_8\}\}$,
$U_2 = \{V_1 = \{u_1, u_6\}, V_2 = \{u_2\}, V_3 = \{u_3, u_8\}, V_4 = \{u_4\}, V_5 = \{u_5, u_7\}\}$ and,
$U_3 = \{T_1 = \{u_1, u_6\}, T_2 = \{u_2, u_5, u_7\}, T_3 = \{u_3, u_4\}, T_4 = \{u_8\}\}$.
$U_4 = \{W_1 = \{u_1, u_5\}, W_2 = \{u_2, u_6, u_7\}, W_3 = \{u_3, u_4, u_8\}\}\}$. After evaluating the smartphones, the expert group creates strait soft sets as follows **(Step 2)**:
$F^1_{\beta_1} = \{(e^1_1, Y_4), (e^1_2, Y_1), (e^1_3, Y_2), (e^1_4, Y_3)\}$,
$F^2_{\beta_2} = \{(e^2_1, V_2), (e^2_2, V_3), (e^2_3, V_5), (e^2_4, V_1), (e^2_5, V_4)\}$,
$F^3_{\beta_3} = \{(e^3_1, T_3), (e^3_2, T_1), (e^3_3, T_2), (e^3_4, T_4)\}$ and
$F^4_{\beta_4} = \{(e^4_1, W_1), (e^4_2, W_2), (e^4_3, W_3), (e^4_4, W_3), (e^4_5, W_1)\}$.

Then, the corresponding reduced soft matrices of these strait soft sets are (**Step 3.**)

$$[F^1_{\beta_1}] = \begin{bmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 0&0&0&1\\ 0&0&0&1\\ 0&0&1&0\\ 1&0&0&0\\ 1&0&0&0 \end{bmatrix}, [F^2_{\beta_2}] = \begin{bmatrix} 0&0&0&1&0\\ 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&0&1\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&1&0&0\\ 0&1&0&0&0 \end{bmatrix}, [F^3_{\beta_3}] = \begin{bmatrix} 0&1&0&0\\ 0&0&1&0\\ 1&0&0&0\\ 1&0&0&0\\ 0&0&1&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{bmatrix}, [F^4_{\beta_4}] = \begin{bmatrix} 1&0&0&0&1\\ 0&1&0&0&0\\ 0&0&1&1&0\\ 0&0&1&1&0\\ 1&0&0&0&1\\ 0&1&0&0&0\\ 0&1&0&0&0\\ 0&0&1&1&0 \end{bmatrix}.$$

For a wider range of solutions, if the solution is continued by choosing the relaxed binary decision soft matrix $[B_{P_{ij}}] = [B^r_{P_{ij}}]$, then to compute the binary decision soft matrices $[B_{P_{12}}]$, $[B_{P_{13}}]$, $[B_{P_{14}}]$, $[B_{P_{23}}]$, $[B_{P_{24}}]$ and $[B_{P_{34}}]$, the following products are required (**Step 4.**):

$$[F^1_{\beta_1}] \wedge_s [F^2_{\beta_2}] = \begin{bmatrix} 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0&0\\ 0&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0 \end{bmatrix}, [F^2_{\beta_2}] \wedge_s [F^1_{\beta_1}] = \begin{bmatrix} 0&0&0&0&0&1&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&1&0&0&0&0\\ 0&1&0&0&0&0&0&0 \end{bmatrix}.$$

Since $[B_{P_{ij}}] = ([F^i_{A_i}] \wedge_s [F^j_{A_j}]) \cup ([F^j_{A_j}] \wedge_s [F^i_{A_i}])$, then

$$[B_{P_{12}}] = \begin{bmatrix} 0&0&1&0&0&1&0&0\\ 1&0&0&1&0&0&0&0\\ 0&0&1&0&0&1&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&1&0&1&0\\ 0&0&0&0&1&0&1&0\\ 0&1&0&1&0&0&0&0\\ 1&1&0&0&0&0&0&0 \end{bmatrix}.$$

For $[B_{P_{13}}]$,

$$[F^1_{\beta_1}] \wedge_s [F^3_{\beta_3}] = \begin{bmatrix} 0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1\\ 0&0&0&1&0&0&0\\ 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0 \end{bmatrix}, [F^3_{\beta_3}] \wedge_s [F^1_{\beta_1}] = \begin{bmatrix} 0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0\\ 1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1 \end{bmatrix}.$$

Then,

$$[B_{P_{13}}] = \begin{bmatrix} 0&1&1&0&0&0&0\\ 0&0&0&0&1&0&0\\ 1&0&0&0&0&1&0\\ 1&0&0&0&0&1&0\\ 0&0&0&0&0&1&1\\ 0&0&1&1&0&0&0\\ 1&0&0&1&0&0&0\\ 0&1&0&0&0&0&1 \end{bmatrix}.$$

For $[B_{P_{14}}]$,

$$[F^1_{\beta_1}] \wedge_s [F^4_{\beta_4}] = \begin{bmatrix} 0&0&0&1&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&1&1&0\\ 0&0&0&0&0&0&0&1&1&0\\ 0&0&0&0&0&0&1&0&0&1\\ 0&0&0&0&0&1&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0\\ 0&1&1&0&0&0&0&0&0&0 \end{bmatrix}, [F^4_{\beta_4}] \wedge_s [F^1_{\beta_1}] = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&1&0&0\\ 0&0&0&0&0&1&0&1&0&0\\ 0&1&0&0&0&0&0&0&0&1\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&1&0&0&0 \end{bmatrix}.$$

Then,

$$[B_{P_{14}}] = \begin{bmatrix} 1&0&0&1&1&0&0&0&0&0\\ 0&0&0&1&0&1&0&0&1&0\\ 0&0&0&0&0&1&0&1&1&0\\ 0&0&0&0&0&1&0&1&1&0\\ 0&1&0&0&0&0&1&0&0&1\\ 0&0&0&1&0&1&0&0&0&0\\ 1&0&1&0&0&0&0&0&0&0\\ 0&1&1&0&1&0&1&0&0&0 \end{bmatrix}.$$

For $[B_{P_{23}}]$,

$$[F^2_{\beta_2}] \wedge_s [F^3_{\beta_3}] = \begin{bmatrix} 0&0&0&0&1&0\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0 \end{bmatrix}, [F^3_{\beta_3}] \wedge_s [F^2_{\beta_2}] = \begin{bmatrix} 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&1&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix}.$$

Then,

$$[B_{P_{23}}] = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

For $[B_{P_{24}}]$,

$$[F^2_{\beta_2}] \wedge_s [F^4_{\beta_4}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$[F^4_{\beta_4}] \wedge_s [F^2_{\beta_2}] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then,

$$[B_{P_{24}}] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Finally, for $[B_{P_{34}}]$,

$$[F^3_{\beta_3}] \wedge_s [F^4_{\beta_4}] = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, [F^4_{\beta_4}] \wedge_s [F^3_{\beta_3}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Then,

$$[B_{P_{34}}] = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

(**Step 5.**) For all $i, j \in \{1, 2, 3, 4\}$, the one-column row-sum matrices $R[B_{P_{ij}}]$ are

$$R[B_{P_{12}}] = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}, R[B_{P_{13}}] = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}, R[B_{P_{14}}] = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 2 \\ 2 \\ 4 \end{bmatrix}.$$

$$R[B_{P_{23}}] = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}, R[B_{P_{24}}] = \begin{bmatrix} 4 \\ 2 \\ 4 \\ 4 \\ 4 \\ 2 \\ 2 \\ 4 \end{bmatrix}, R[B_{P_{34}}] = \begin{bmatrix} 5 \\ 2 \\ 4 \\ 4 \\ 4 \\ 2 \\ 2 \\ 4 \end{bmatrix}.$$

Then, (**Step 6.**) the decision matrix is

$$\sum R[B_{P_{ij}}] = [d_{i1}] = \begin{bmatrix} 18 \\ 12 \\ 17 \\ 16 \\ 17 \\ 12 \\ 12 \\ 18 \end{bmatrix}.$$

Finally, (**Step 7.**) $Opt_{[d_{i1}]}(U) = \{\frac{u_i}{di1} : u_i \in U\} = \{\frac{u_1}{18}, \frac{u_2}{12}, \frac{u_3}{17}, \frac{u_4}{16}, \frac{u_5}{17}, \frac{u_6}{12}, \frac{u_7}{12}, \frac{u_8}{18}\}$ and the ranking is obtained as $u_2 = u_6 = u_7 < u_4 < u_3 = u_5 < u_1 = u_8$. Accordingly, in production planning, the decision maker may foresee giving priority to smart phones that have the features of u_1 and u_8.

Since the partition sets of U are
$U_1 = \{Y_1 = \{u_1\}, Y_2 = \{u_2, u_6\}, Y_3 = \{u_3, u_4, u_5\}, Y_4 = \{u_7, u_8\}\}$,
$U_2 = \{V_1 = \{u_1, u_6\}, V_2 = \{u_2\}, V_3 = \{u_3, u_8\}, V_4 = \{u_4\}, V_5 = \{u_5, u_7\}\}$ and,
$U_3 = \{T_1 = \{u_1, u_6\}, T_2 = \{u_2, u_5, u_7\}, T_3 = \{u_3, u_4\}, T_4 = \{u_8\}\}$.
$U_4 = \{W_1 = \{u_1, u_5\}, W_2 = \{u_2, u_6, u_7\}, W_3 = \{u_3, u_4, u_8\}\}\}$,
then (**Step 8.**)

$Popt(U_1) = \{\frac{Y_s}{\sum di1} : u_i \in Y_s\} = \{\frac{Y_1}{18}, \frac{Y_2}{24}, \frac{Y_3}{50}, \frac{Y_4}{30}\}$ and the ranking is $Y_1 < Y_2 < Y_4 < Y_3$.
$Popt(U_2) = \{\frac{V_s}{\sum di1} : u_i \in V_s\} = \{\frac{V_1}{30}, \frac{V_2}{12}, \frac{V_3}{35}, \frac{V_4}{16}, \frac{V_5}{30}\}$ and the ranking is $V_2 < V_4 < V_1 = V_5 < V_3$.

$Popt(U_3) = \{\frac{T_s}{\sum di1} : u_i \in T_s\} = \{\frac{T_1}{30}, \frac{T_2}{41}, \frac{T_3}{33}, \frac{T_4}{18}\}$ and the ranking is $T_4 < T_1 < T_3 < T_2$.
$Popt(U_4) = \{\frac{W_s}{\sum di1} : u_i \in W_s\} = \{\frac{W_1}{35}, \frac{W_2}{36}, \frac{W_3}{51}\}$ and the ranking is $W_1 < W_2 < W_3$.

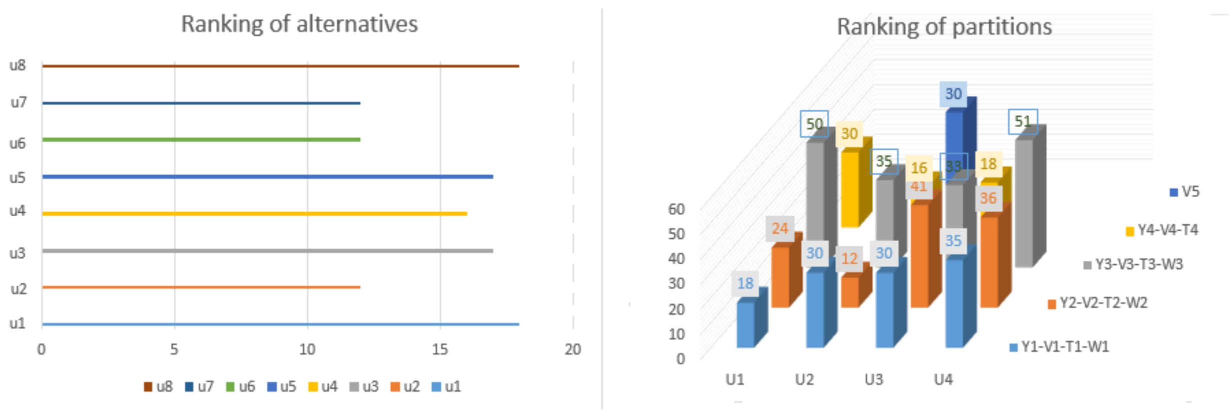


Figure 7: The optimum set and the partition optimum sets

Accordingly, considering the parameters of the strait soft sets $F^1_{\beta_1}, F^2_{\beta_2}, F^3_{\beta_3}$ and $F^4_{\beta_4}$, the following results are obtained:

Results according to smartphone features				
Bestsellers in C_1, C_2 and C_3	**Color**	**Screen technology**	**Material**	**Screen type**
Rank 1	mixed color	AMOLED	polycarbonate-glass	expandable single/rollable-sliding
Rank 2	light	TFT-LCD/Super AMOLED	plastic-glass	classic single
Rank 3	golden silver	IPS-LCD	plastic-aluminum	dual/foldable screen
Rank 4	dark	OLED	aluminum-glass	-

Results by smartphone models	
Bestsellers in C_1, C_2 and C_3	**Featured smartphone models**
Rank 1	u_1: dark, TFT-LCD, plastic-aluminum, dual/foldable screen u_8: light, AMOLED, aluminum-glass, expandable single/rollable-sliding screen
Rank 2	u_3: mixed color, AMOLED, plastic-glass, expandable single/rollable-sliding screen u_5: mixed color, Super AMOLED, polycarbonate-glass, dual/foldable screen
Rank 3	u_4: mixed color, TFT-LCD, plastic-glass, expandable single/rollable-sliding screen
Rank 4	u_2: golden silver, OLED, plastic-aluminum, classic single screen u_6: golden silver, TFT-LCD, plastic-aluminum, classic single screen u_7: light, Super AMOLED, polycarbonate-glass, classic single screen

The solution suggested in [6], Examples 6. and 7. is as follows:

The real numbers $\alpha_k \geq 1$	Bestsellers in C_1, C_2 and C_3
$\alpha_1 = \alpha_2 = \alpha_3 = 2$	−light, Super AMOLED, polycarbonate-glass, dual-foldable −light, Super AMOLED, aluminum-glass, dual-foldable −dark, Super AMOLED, polycarbonate-glass, dual-foldable −dark, Super AMOLED, aluminum-glass, dual-foldable
$\alpha_1 = \alpha_2 = \alpha_3 = 5$	dark, Super AMOLED, aluminum-glass, dual-foldable
$\alpha_1 = \alpha_2 = \alpha_3 = 1$	−light, Super AMOLED, polycarbonate-glass, dual-foldable −mixed, Super AMOLED, polycarbonate-glass, dual-foldable
$\alpha_1 = 4, \alpha_2 = 3, \alpha_3 = 4$	dark, Super AMOLED, aluminum-glass, dual-foldable

Comparisons, advantages and disadvantages:

Based on the results obtained above, the advantages of *SSDM* and the comparison of both methods are as follows:

a_1. *SSDM* facilitates analysis by decision-makers as it presents both preferred alternatives and their features as two separate results.

a_2. Since the Strait AND product is used in *SSDM*, the interactions of the parameters with each other are also included in the process. A potential customer researching a product such as a smartphone will, of course evaluate many features of the product together. In the method given in [6], the interaction of product features is not included in the solution process. For this reason, the screen technology feature is only featured as Super AMOLED, and the screen type feature is only featured as a dual/foldable screen.

a_3. With *SSDM*, it can be determined which smartphone models, with which features, will reach higher sales volumes.

a_4. *SSDM* can be applied both to *MCGDM*, as in Example 6.1, and to multi-attribute decision-making problems, as in this example.

a_5. *SSDM* also determines which features need to be updated to make an already offered product better.

A special situation that can be considered as a disadvantage is the following: When choosing $[B^c_{P_{ij}}]$ or $[B^r_{P_{ij}}]$, decision makers are expected to have information about the strait soft sets created. Because, as in this example, since $F^1_{\beta_1}, F^2_{\beta_2}$ and $F^3_{\beta_3}$ are bijective soft sets, $[B^c_{P_{ij}}]$ will produce very narrow results and therefore decision makers will have to use $[B^r_{P_{ij}}]$, which will lengthen the process.

7. Conclusion

In this study, the strait soft set structure, which was first introduced in [6], is examined in detail. This study can be considered in two parts. In the first part, the introduction of common image strait soft sets, which examine soft sets that parameterize the same partitioning of a universal set, the definition of a new operation Strait AND product on both soft sets and soft matrices, and an applied decision-making algorithm proposed using all these new concepts are seen. In the second part, almost all the concepts given in the first part are generalized to arbitrary strait soft sets, a new applied multi-criteria group decision-making algorithm is proposed, and a multi-attribute decision-making problem given in [6] is solved with comparative analysis.

Since the same partitioning of a set is parameterized in common image strait soft sets due to its structure, the definition of Strait AND product is first defined on common image strait soft sets and on their reduced soft matrices for ease of application. In order to prove that common image strait soft sets have many areas of application, a decision-making application that provides performance analysis of a company's personnel is given. The proposed *CISSDM* method can be briefly described as an effective method for solving all types of decision-making problems for which efficiency analysis is required under the same characteristics.

In order to expand the range of problems mentioned above and to provide solutions to different types of problems, the *SSDM* method has been proposed by using the Strait AND product operation defined on arbitrary strait soft sets and on their reduced matrices. Both the proposed methods *CISSDM* and *SSDM* are two-stage decision-making methods that sequentially yield both alternatives and partition sets.

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