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Advances in moduli of continuity through generalized Chebyshev wavelet approximation: Applications to mathematical models

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Abstract. In this work, we introduce the generalized Chebyshev wavelet over $[0, l)$. We estimate the generalized Chebyshev wavelet approximation of solution functions belonging to the generalized Hölder's class, along with their modulus of continuity. The modulus of continuity is a mathematical tool for measuring a function's smoothness and describing its properties on local and global levels. We aim to integrate moduli of continuity into wavelet approaches that significantly enhances the precision and effectiveness of solutions in diverse mathematical and engineering domains. Additionally, we propose a numerical algorithm based on collocation techniques using the generalized Chebyshev wavelet. We compare our outcomes with the results obtained by other methods that are documented in the literature. The analysis shows that the proposed method is remarkably efficient and precise depicted by the facts and figures of this research.

1. Introduction

Recently, several scientists from many different fields have given wavelet analysis a lot of attention. Electrical engineers, physicists, and mathematicians can collaborate more easily with the applications of wavelet analysis, see, for instance, [2]. Wavelets are mathematical tools that may be used to extract information from a wide variety of data types, audio signals, and images. Today, advances in seismic data processing have made the wavelet approximation technique and moduli of continuity useful tools for identifying and analyzing rapid changes for further study [11]. It enables precise representation of a wide range of operators and functions in the form of wavelet series [18].

Nonlinear singular boundary value problems serve as models for several phenomena, ranging from astrophysics and aerodynamics to stellar structure, chemistry, and biochemistry [7, 10]. Accurate and effective numerical approaches are generally necessary for the solution of nonlinear singular boundary value problems (SBVPs) [13]. Here, we explore the following class of strongly nonlinear SBVPs governed by Neumann and Robin boundary conditions:

$$y''(t) + \frac{\xi}{t} y'(t) = h(t, y(t)),\ 0 < t \leq 1,\ \xi \geq 1 \tag{1}$$

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constrained by the boundary conditions: $y'(0) = 0,\ \alpha y(1) + \beta y'(1) = \gamma.$

where $\alpha > 0, \beta \geq 0$ and γ are any finite real constants. For every $(x, y) \in (0, 1] \times (0, \infty)$, the nonlinear function $h(x, y)$ is continuous.

Wavelets have many of its applications in biomechanics that employ mechanics laws and computations to investigate human body behavior. The biological effects that arise from the interaction between microwave electromagnetic (EM) radiation and biological tissues rely on the exposure parameters, including intensity, frequency, waveform, and duration. The transfer of energy to the biological medium causes ion acceleration, molecule collisions, and a surge in tissue temperature. To mitigate the risks of temperature rise in biological tissues, it's crucial to investigate how microwave radiation alters delicate organs like the brain. The current study aims to assess how relaxation durations affect temperature changes in the human skull when exposed to microwaves. This article leverages wavelet approaches to investigate and resolve mathematical models of chemical and biological events [15].

In recent times, the literature has also utilized the variational iteration technique (VIM) and its modified forms. Only linear and nonlinear problems may be accurately approximated by these methods. When the nonlinear function has the form siny, sinhy, these approaches, however, are unable to solve the equation successfully. However, there are very few applications of VIM available for handling nonlinear situations. Therefore, many researchers have introduced the wavelet techniques to solve them effectively.

The main objectives of this article are to define the generalized Chebyshev wavelet in the interval $[0, l)$ and to evaluate the approximations of functions belonging to classes $H_{\omega}^{\alpha,2}[0, l), H_{\omega}^{\alpha,2,\phi}[0, l)$ and their moduli of continuity. The method of generalized Chebyshev wavelet and moduli of continuity offers the most accurate approximation of functions belonging to class of approximations. A collocation based algorithm has been developed to solve singular differential equations that models molecular and biological phenomenas by generalized Chebyshev wavelet on the interval $[0, l)$. The collocation approach improves accuracy and convergence as it enables adaptive refining by adding more collocation points. The method is expected to provide precise approximations of solution functions with complex features, such as rapid variations, discontinuities, and localized behavior. The generalized Chebyshev wavelet solutions are compared with the solutions of ADM, VIM, and other wavelets [3, 16].

Following is the representations of this research article: Section 2, presents some necessary definitions and essential results required for this article. In Section 3, estimators for the generalized Chebyshev wavelet approximations and moduli of continuity of functions in $H_{\omega}^{\alpha,2}[0, l)$ and $H_{\omega}^{\alpha,2,\phi}[0, l)$ have been developed along with their corresponding proofs. In Section 4, an algorithm for solving mathematical models by the generalized Chebyshev wavelets has been suggested and is illustrated by examples in Section 5. The results of this investigation have been presented in Section 6.

2. Preliminaries

In this section, some necessary definitions and mathematical preliminaries are given which will be used further in this paper.

Definition 2.1. ***Function of*** $H_{\omega}^{\alpha}[0, 1)$ ***class***
A continuous function $g \in H_{\omega}^{\alpha}[0, 1)$ *if* g *satisfies*

$$|g(x + t) - g(x)|\omega(x) = O(|t|^{\alpha})\ \ \forall\, x, t \in [0, 1),\ \omega(x) = \frac{1}{\sqrt{1 - x^2}},\ 0 < \alpha \leq 1.$$

For more details, see [17].

Definition 2.2. ***Function of*** $H_{\omega}^{\alpha,2}[0, l)$ ***class***
A continuous function $g \in H_{\omega}^{\alpha,2}[0, l)$ *if* g *satisfies*

$$\Big(\int_0^l |g(x + t) - g(x)|^2\omega(x)dx\Big)^{\frac{1}{2}} = O\Big(\Big(\frac{|t|}{m + 1}\Big)^{\alpha}\Big),\ 0 < \alpha \leq 1.$$

Definition 2.3. ***Function of*** $H_{\omega}^{\alpha,\phi,2}[0,l)$ ***class***
A continuous function $g \in H_{\omega}^{\alpha,\phi,2}[0,l)$ *if* g *satisfies*

$$\Big(\int_0^l |g(x+t)-g(x)|^2\omega(x)dx\Big)^{\frac{1}{2}} = O\Big(\Big(\frac{|t|}{m+1}\Big)^{\alpha}\phi\Big(\frac{|t|}{m+1}\Big)\Big),\ 0<\alpha\leq 1,\ m\geq 0,$$

where ϕ *is a non-negative, monotonically increasing function of* t *such that* $\frac{|t|^{\alpha}}{m+1}\phi\Big(\frac{|t|}{m+1}\Big)$ *approaches to* 0 *as* t *approaches to* 0^{+}.

Definition 2.4. ***Modulus of Continuity***
Modulus of continuity of $g \in L^2_{\omega}[0,l)$ *is given by*

$$W(g,\delta) = \sup_{0<h\leq\delta}\|g(.+h)-g(.)\|_2 = \sup_{0<h\leq\delta}\Big(\int_0^l |g(x+h)-g(x)|^2\omega(x)dx\Big)^{\frac{1}{2}}.$$

Notably, $W(g,\delta)$ *acts as a non-decreasing function of* δ *and* $W(g,\delta)$ *approaches to* 0 *as* δ *approaches to 0 [2] .*

2.1. *Generalized Chebyshev wavelet*

Generalized Chebyshev wavelet, denoted by ${}^{c}T_{n,m}^{(\mu)}$ on the interval $[0,l)$, is defined by

$${}^{c}T_{n,m}^{(\mu)}(t) = \begin{cases} \mu^{\frac{k}{2}}\sqrt{2}\,\tilde{T}_m\big(\frac{2\mu^k t}{l}-2n+1\big), & \text{if t} \in \big[\frac{(n-1)l}{\mu^k},\frac{nl}{\mu^k}\big],\\ 0\,, & \text{elsewhere.}\end{cases} \tag{2}$$

where

$$\tilde{T}_m(t) = \begin{cases} \frac{1}{\sqrt{\pi l}} &, \text{if } m=0\\ \sqrt{\frac{2}{\pi l}}\,\mathcal{T}_m(t), & \text{if } m\neq 0\end{cases}$$

$m = 0,1,2,\ldots,M-1$, $n = 1,2\ldots,\mu^k$, and k,μ are positive integer ($\mu > 1$). The first kind Chebyshev polynomials of degree m, denoted by $\mathcal{T}_m(t)$, are orthogonal with respect to the weight function $\omega(t) = \frac{1}{\sqrt{1-t^2}}$ on the interval $[-1,1]$ and satisfy:

$$\mathcal{T}_0(t) = 1,\ \mathcal{T}_1(t) = t,\ \mathcal{T}_{m+1}(t) = 2t\mathcal{T}_m(t)-\mathcal{T}_{m-1}(t),\ m=1,2,\ldots$$

Remark 2.5. *If* $\mu = 2$, *then the generalized Chebyshev wavelet reduces to classical Chebyshev wavelet. For more details, one can refer to [14].*

2.2. *Generalized Chebyshev wavelets series approximation*

Any square integrable function $g \in L^2_{\omega}[0,l)$ is expanded in terms of the generalized Chebyshev wavelets series as

$$g(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\ {}^{c}T_{n,m}^{(\mu)}(t) \tag{3}$$

where $c_{n,m} =< g(t), {}^{c}T_{n,m}^{(\mu)}(t) > \omega_{n,k}(t)$, $\omega_{n,k}(t) = \frac{1}{\sqrt{1-\big(\frac{2\mu^k t}{l}-2n+1\big)^2}}$.

The $(\mu^k,M)^{th}$ partial sums $S_{\mu^k,M}(g)$ of the series 3 is given by,

$$S_{\mu^k,M}(g)(t) = \sum_{n=1}^{\mu^k}\sum_{m=0}^{M-1} c_{n,m}\ {}^{c}T_{n,m}^{(\mu)}(t) \triangleq C^T T(t) \tag{4}$$

where C and $T(t)$ are column vectors given by,

$$C \triangleq [c_{1,0}, c_{1,1}, \ldots, c_{1,M-1}, c_{2,0}, \ldots, c_{2,M-1}, \ldots, c_{\mu^k,0}, \ldots, c_{\mu^k,M-1}]^T;$$

$$T(t) \triangleq [T^{(\mu)}_{1,0}(t), T^{(\mu)}_{1,1}(t), \ldots, T^{(\mu)}_{1,M-1}(t), T^{(\mu)}_{2,0}(t), \ldots, T^{(\mu)}_{2,M-1}(t), \ldots, T^{(\mu)}_{\mu^k,0}(t), \ldots, T^{(\mu)}_{\mu^k,M-1}(t)]^T.$$

The generalized Chebyshev wavelet approximation ${}^cE_{\mu^k,M}(g)$ of a function $g \in L^2_\omega[0,l)$ by $(\mu^k, M)^{th}$ partial sums $S_{\mu^k,M}(g)$ of its generalized Chebyshev wavelet series is given by

$${}^cE_{\mu^k,M}(g) = \min_{S_{\mu^k,M}(g)} \|g - S_{\mu^k,M}(g)\|_2 \;,\; \|g\|_2 = \Big(\int_0^1 |g(t)|^2 \omega(t) dt\Big)^{\frac{1}{2}}.$$

If ${}^cE_{\mu^k,M}(g)$ approaches to 0 as k, M approaches to ∞, then ${}^cE_{\mu^k,M}(g)$ is recognized as the best approximation of g, see [9].

2.3. ***Examples***

This includes the examples of a function belonging to $H^{\alpha,2}_\omega[0,l)$ but does not belongs to $H^\alpha_\omega[0,l)$.

Example 2.6. *Let*

$$f(x) = \begin{cases} |x|^\alpha \;,\; 0 \le x < \frac{l}{2} \\ 1 \;,\quad x = \frac{l}{2} \\ |x|^\alpha \;,\; \frac{l}{2} < x \le l \end{cases}$$

For $0 < \epsilon < \dfrac{l}{4}$, $x_1 = \dfrac{l}{2} - \dfrac{\epsilon}{2}$, $x_2 = \dfrac{l}{2} + \dfrac{\epsilon}{2}$ *then* $|x_1 - x_2| = \epsilon$.

$$\begin{aligned} \frac{(|f(x_1) - f(x_2)|)\; \omega(|x_1 - x_2|)}{|x_1 - x_2|} &= \frac{(|\frac{l}{2} - \frac{\epsilon}{2}|^\alpha - |\frac{l}{2} + \frac{\epsilon}{2}|^\alpha)\; \omega(|x_1 - x_2|)}{\epsilon} \\ &\quad (\because x^\alpha - y^\alpha \le |x-y|^\alpha,\; 0 < \alpha \le 1) \\ &\le \frac{\epsilon^\alpha\; \omega(\epsilon)}{\epsilon} \\ &\to \infty \text{ as } \epsilon \to 0^+. \end{aligned}$$

Therefore, $f(x) \notin H^\alpha_\omega[0,l)$.

$$\begin{aligned} \int_0^l |f(x+t) - f(x)|^2\; \omega(x)\; dx &= \int_0^{\frac{l}{2}-\frac{\epsilon}{2}} |f(x+t) - f(x)|^2\; \omega(x)\; dx + \int_{\frac{l}{2}+\frac{\epsilon}{2}}^l |f(x+t) - f(x)|^2\; \omega(x)\; dx \\ &\le \int_0^{\frac{l}{2}-\frac{\epsilon}{2}} |t|^{2\alpha}\; \omega(x)\; dx + \int_{\frac{l}{2}+\frac{\epsilon}{2}}^l |t|^{2\alpha}\; \omega(x)\; dx \\ &\le \sin^{-1}(l)\; |t|^{2\alpha} \text{ as } \epsilon \to 0^+. \end{aligned}$$

$$\Big(\int_0^l |f(x+t) - f(x)|^2 dx\Big)^{\frac{1}{2}} = \sqrt{\sin^{-1}(l)}|t|^\alpha = O(|t|^\alpha).$$

Hence, $f \in H^{\alpha,2}_\omega[0,l)$.

Example 2.7. *Let*

$$f(x) = \begin{cases} x \;, 0 \le x < \frac{l}{2} \\ 1 \;,\; \frac{l}{2} \le x < l \end{cases}$$

For $0 < \epsilon < \frac{l}{4}$, $x_1 = \frac{l}{2} - \frac{\epsilon}{2}$, $x_2 = \frac{l}{2} + \frac{\epsilon}{2}$ *then* $|x_1 - x_2| = \epsilon$.

$$\frac{|f(x_1) - f(x_2)|\ \omega(|x_1 - x_2|)}{|x_1 - x_2|} = \frac{|\frac{l}{2} - \frac{\epsilon}{2} - 1|\ \omega(\epsilon)}{|\epsilon|}$$
$$\to \infty \ \ as\ \epsilon \to 0^+$$

Therefore, $f(x) \notin H^\alpha[0, l)$.

$$\int_0^l |f(x+t) - f(x)|^2\ \omega(x)\ dx = \int_0^{\frac{l}{2}-\frac{\epsilon}{2}} |f(x+t) - f(x)|^2\ \omega(x)\ dx + \int_{\frac{l}{2}+\frac{\epsilon}{2}}^l |f(x+t) - f(x)|^2\ \omega(x)\ dx$$
$$\leq \int_0^{\frac{l}{2}-\frac{\epsilon}{2}} |t|^2 \frac{1}{\sqrt{1-x^2}}\ dx$$
$$= |t|^2\ \sin^{-1}\Big(\frac{l}{2} - \frac{\epsilon}{2}\Big)$$
$$= t^2 \sin^{-1}\Big(\frac{l}{2}\Big)\ as\ \epsilon \to 0^+.$$

$$\Big(\int_0^l |f(x+t) - f(x)|^2 \omega(x) dx\Big)^{\frac{1}{2}} = |t|\ \sqrt{\sin^{-1}\Big(\frac{l}{2}\Big)} = O(|t|).$$

Hence, $f \in H^{\alpha,2}_\omega[0, l)$.

3. Results of Approximations

In this section, the estimators and moduli of continuity for solution functions of mathematical models have been demonstrated.

Theorem 3.1. *If the solution function* $f \in H^{\alpha,2}_\omega[0, l)$ *and its generalized Chebyshev wavelet series* (3) *whose* $(\mu^k, M)^{th}$ *partial sums* (4) *then the approximation* ${}^cE^{(1)}_{\mu^k,M}(f)$ *satisfies:*

(i) *For* $M = 0$, ${}^cE^{(1)}_{\mu^k,0}(f) = min_{S_{\mu^k,0}(f)}\|f - S_{\mu^k,0}(f)\|_2$

$$= O\Big(\frac{l^\alpha}{\mu^{k\alpha}} + \frac{\sqrt{l}}{\mu^{\frac{k}{2}}}\Big);\ \ 0 < \alpha < 1.$$

(ii) *For* $M \geq 1$, ${}^cE^{(1)}_{\mu^k,M}(f) = min_{S_{\mu^k,M}(f)}\|f - S_{\mu^k,M}(f)\|_2$

$$= O\Big(\Big(\frac{l}{\mu^k}\Big)^\alpha \frac{1}{(M+1)^{\alpha - \frac{1}{2}}}\Big);\ \alpha > \frac{1}{2},\ k \geq 1.$$

Proof. (i) For $m = 0$, using the transformations $\frac{2\mu^k t}{l} - 2n + 1 = x$

$$f(t) = \sum_{n=1}^{\mu^k} c_{n,0}\ {}^cT^{(\mu)}_{n,0}(t),\ \text{where}$$

$$c_{n,0} = < f, {}^cT^{(\mu)}_{n,0} > \omega_n(t) = \int_{\frac{(n-1)l}{\mu^k}}^{\frac{nl}{\mu^k}} f(t)\ {}^cT^{(\mu)}_{n,0}(t)\ \omega_n(t)\ dt$$
$$= \int_{\frac{(n-1)l}{\mu^k}}^{\frac{nl}{\mu^k}} \Big(f(t) - f\Big(\frac{(n-1)l}{\mu^k}\Big)\Big)\ {}^cT^{(\mu)}_{n,0}(t)\ \omega_n(t)\ dt$$
$$+ f\Big(\frac{(n-1)l}{\mu^k}\Big) \int_{\frac{(n-1)l}{\mu^k}}^{\frac{nl}{\mu^k}} {}^cT^{(\mu)}_{n,0}(t)\ \omega_n(t)\ dt$$

$$
\begin{aligned}
|c_{n,0}| &\leq \int_{\frac{(n-1)l}{\mu^k}}^{\frac{nl}{\mu^k}} \Big|f(t) - f\Big(\frac{(n-1)l}{\mu^k}\Big)\Big| \times \Big(\frac{\sqrt{2}\mu^{\frac{k}{2}}}{\sqrt{\pi l}}\Big)\, \omega_n(t)\, dt \\
&\quad + \Big|f\Big(\frac{(n-1)l}{\mu^k}\Big)\Big| \times \Big(\frac{\sqrt{2}\mu^{\frac{k}{2}}}{\sqrt{\pi l}}\Big) \times \int_{\frac{(n-1)l}{\mu^k}}^{\frac{nl}{\mu^k}} \frac{1}{\sqrt{1-(\frac{2\mu^k t}{l}-2n+1)^2}} dt \\
&\leq \frac{\sqrt{2}\mu^{\frac{k}{2}}}{\sqrt{\pi l}} \Big(\int_{\frac{(n-1)l}{\mu^k}}^{\frac{nl}{\mu^k}} \Big|f(t) - f\Big(\frac{(n-1)l}{\mu^k}\Big)\Big|^2 \omega_n(t)\, dt\Big)^{\frac{1}{2}} \Big(\int_{\frac{(n-1)l}{\mu^k}}^{\frac{nl}{\mu^k}} |\omega_n(t)\, dt\Big)^{\frac{1}{2}} \\
&\quad + \Big|f\Big(\frac{(n-1)l}{\mu^k}\Big)\Big| \times \Big(\frac{\sqrt{2}\mu^{\frac{k}{2}}}{\sqrt{\pi l}}\Big) \times \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \frac{l\, dx}{2\mu^k} \\
&= \frac{l^\alpha}{\mu^{k(\alpha+\frac{1}{2})}} + f\Big(\frac{(n-1)l}{\mu^k}\Big) \frac{\sqrt{\pi l}}{\sqrt{2\mu^k}}.
\end{aligned}
$$

$$
\begin{aligned}
\|f - S_{\mu^k,0}(f)\|_2^2 &\leq \sum_{n=1}^{\mu^k} |c_{n,0}|^2 \int_{\frac{(n-1)l}{\mu^k}}^{\frac{nl}{\mu^k}} \psi_{n,0}^2 dx + \sum_{\substack{1\leq n,n'\leq \mu^k \\ n\neq n}} |c_{n,0}\, c_{n',0}| < \psi_{n,0}, \psi_{n',0} > \\
&\leq \sum_{n=1}^{\mu^k} |c_{n,0}|^2, \Big(\because \Big[\frac{(n-1)l}{\mu^k}, \frac{nl}{\mu^k}\Big) \cap \Big[\frac{(n'-1)l}{\mu^k}, \frac{n'l}{\mu^k}\Big) = \phi\Big) \\
&= \sum_{n=1}^{\mu^k} \Big(\frac{l^\alpha}{\mu^{k(\alpha+\frac{1}{2})}}\Big)^2 + \sum_{n=1}^{\mu^k} \Big|f\Big(\frac{(n-1)l}{\mu^k}\Big)\Big|^2 \times \Big(\frac{\pi l}{\mu^k}\Big) \\
&\quad + 2\sum_{n=1}^{\mu^k} \frac{l^\alpha}{\mu^{k(\alpha+\frac{1}{2})}} \times f\Big(\frac{(n-1)l}{\mu^k}\Big) \times \Big(\frac{\sqrt{\pi l}}{\mu^{\frac{k}{2}}}\Big) \\
&= \frac{l^{2\alpha} \times \mu^k}{\mu^{k(2\alpha+1)}} + \frac{A\pi l}{\mu^k} + \frac{2l^\alpha}{\mu^{k(\alpha+\frac{1}{2})}} \times \frac{B\sqrt{\pi l}}{\mu^{\frac{k}{2}}} \\
&\leq \Big(1 + A\pi + 2B\sqrt{\pi}\Big)\Big(\frac{l^\alpha}{\mu^{k\alpha}} + \frac{\sqrt{l}}{\mu^{\frac{k}{2}}}\Big)^2 \\
&= O\Big(\frac{l^\alpha}{\mu^{k\alpha}} + \frac{\sqrt{l}}{\mu^{\frac{k}{2}}}\Big)^2, \text{where } \Big(A = \sum_{n=1}^{\mu^k} \Big|f\Big(\frac{(n-1)l}{\mu^k}\Big)\Big|^2,\ B = \sum_{n=1}^{\mu^k} \Big|f\Big(\frac{(n-1)l}{\mu^k}\Big)\Big|\Big)
\end{aligned}
$$

$$
\begin{aligned}
\|f - S_{\mu^k,0}(f)\|_2 &= O\Big(\frac{l^\alpha}{\mu^{k\alpha}} + \frac{\sqrt{l}}{\mu^{\frac{k}{2}}}\Big), \\
\text{Hence, } {}^cE^{(1)}_{\mu^k,0}(f) &= \min_{S_{\mu^k,0}(f)} \|f - S_{\mu^k,0}(f)\|_2 \\
&= O\Big(\frac{l^\alpha}{\mu^{k\alpha}} + \frac{\sqrt{l}}{\mu^{\frac{k}{2}}}\Big);\ \ 0 < \alpha < 1.
\end{aligned}
$$

(ii) For $m \geq 1$, using the transformations $\frac{2\mu^k t}{l} - 2n + 1 = \cos(\theta)$

$$
c_{n,m} =< f, {}^c T^{(\mu)}_{n,m} > \omega_n(t) = \int_{\frac{(n-1)l}{\mu^k}}^{\frac{nl}{\mu^k}} f(t)\, {}^cT^{(\mu)}_{n,m}(t)\, \omega_n(t)\, dt
$$

$$= \int_{\frac{(n-1)l}{\mu^k}}^{\frac{nl}{\mu^k}} \Big(f(t) - f\Big(\frac{(n-1)l}{\mu^k}\Big)\Big)\, {}^cT^{(\mu)}_{n,m}(t)\, \omega_n(t)\, dt$$

$$+ f\Big(\frac{(n-1)l}{\mu^k}\Big) \times \int_{\frac{(n-1)l}{\mu^k}}^{\frac{nl}{\mu^k}} {}^cT^{(\mu)}_{n,m}(t)\, \omega_n(t)\, dt$$

$$|c_{n,m}| \leq \int_{\frac{(n-1)l}{\mu^k}}^{\frac{nl}{\mu^k}} \Big|f(t) - f\Big(\frac{(n-1)l}{\mu^k}\Big)\Big|\, {}^cT^{(\mu)}_{n,m}(t)\, \omega_n(t)\, dt$$

$$\leq \Big(\int_{\frac{(n-1)l}{\mu^k}}^{\frac{nl}{\mu^k}} \Big|f(t) - f\Big(\frac{(n-1)l}{\mu^k}\Big)\Big|^2\, \omega_n(t)\, dt\Big)^{\frac{1}{2}} \times$$

$$\Big(\int_{\frac{(n-1)l}{\mu^k}}^{\frac{nl}{\mu^k}} |\, {}^cT^{(\mu)}_{n,m}(t)|^2\, \omega_n(t)\, dt\Big)^{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{\mu^k}}\Big(\frac{l}{(m+1)\mu^k}\Big)^{\alpha} \Big(\int_{\frac{(n-1)l}{\mu^k}}^{\frac{nl}{\mu^k}} \frac{4\mu^k}{\pi l}\, T^2_m\Big(\frac{2\mu^k t}{l} - 2n + 1\Big) \times$$

$$\frac{1}{\sqrt{1 - (\frac{2\mu^k t}{l} - 2n + 1)^2}} dt\Big)^{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{\mu^k}}\Big(\frac{l}{(m+1)\mu^k}\Big)^{\alpha} \Big(\int_0^{\pi} \frac{2}{\pi}\, {}^cT^2_m(\cos\theta)\, d\theta\Big)^{\frac{1}{2}}$$

$$= \frac{\sqrt{2}}{\mu^{\frac{k}{2}}\sqrt{\pi}}\Big(\frac{l}{(m+1)\mu^k}\Big)^{\alpha} \Big(\int_0^{\pi} (\cos(m\theta))^2\, d\theta\Big)^{\frac{1}{2}}$$

$$\leq \frac{2\sqrt{2}}{\mu^{\frac{k}{2}}\sqrt{\pi}}\Big(\frac{l}{(m+1)\mu^k}\Big)^{\alpha}.$$

By Cauchy integral test,

$$\|f - S_{\mu^k,M}(f)\|_2^2 = \sum_{n=1}^{\mu^k}\sum_{m=M}^{\infty} |c_{n,m}|^2$$

$$\leq \sum_{n=1}^{\mu^k}\sum_{m=M}^{\infty} \frac{8l^{2\alpha}}{\mu^{k(2\alpha+1)}} \frac{1}{(m+1)^{2\alpha}}$$

$$= \frac{8l^{2\alpha}}{\mu^{2k\alpha}} \sum_{m=M}^{\infty} \frac{1}{(m+1)^{2\alpha}}$$

$$= \frac{8l^{2\alpha}}{\mu^{2k\alpha}}\Big(\frac{1}{(M+1)^{2\alpha}} + \int_M^{\infty} \frac{1}{(m+1)^{2\alpha}}\, dm\Big)$$

$$\leq \frac{8l^{2\alpha}}{\mu^{2k\alpha}}\Big(1 + \frac{1}{2\alpha - 1}\Big)\Big(\frac{1}{(M+1)^{2\alpha-1}} + \frac{1}{(M+1)^{2\alpha-1}}\Big)$$

$$\|f - S_{\mu^k,M}(f)\|_2 \leq \frac{4l^{\alpha}}{\mu^{k\alpha}} \sqrt{\Big(1 + \frac{1}{2\alpha-1}\Big)} \times \frac{1}{(M+1)^{\alpha - \frac{1}{2}}}$$

$$= O\Big(\Big(\frac{l}{\mu^k}\Big)^{\alpha} \frac{1}{(M+1)^{\alpha-\frac{1}{2}}}\Big).$$

Hence, ${}^cE^{(1)}_{\mu^k,M}(f) = \min_{S_{\mu^k,M}(f)} \|f - S_{\mu^k,M}(f)\|_2$

$$= O\Big(\Big(\frac{l}{\mu^k}\Big)^{\alpha} \frac{1}{(M+1)^{\alpha-\frac{1}{2}}}\Big);\ \alpha > \frac{1}{2},$$

completes the proof. □

Theorem 3.2. *If the solution function $f \in H_{\omega}^{\alpha,\phi,2}[0,l)$ then the approximation ${}^{c}E^{(2)}_{\mu^k,M}(f)$ satisfies:*

$$(i)\ E^{(2)}_{\mu^k,0}(f) = min_{S_{\mu^k,0}(f)} \|f - S_{\mu^k,0}(f)\|_2 = O\Big(\frac{1}{\mu^{k(\alpha-\frac{1}{2})}}\phi\Big(\frac{l}{\mu_k}\Big) + 1\Big);\ \ \alpha > \frac{1}{2}.$$

$$(ii)\ E^{(2)}_{\mu^k,M}(f) = min_{S_{\mu^k,M}(f)} \|f - S_{\mu^k,M}(f)\|_2 = O\Big(\Big(\frac{l}{\mu^k}\Big)^{\alpha}\phi\Big(\frac{l}{M\mu^k}\Big)\Big(1 + \frac{1}{2\alpha-1}\Big)^{\frac{1}{2}} \frac{1}{(M+1)^{\alpha-\frac{1}{2}}}\Big);\ \alpha > \frac{1}{2}, k \geq 1.$$

Proof. (i) For $m = 0$, following the proof of Theorem 3.1,

$$|c_{n,0}| \leq \frac{\sqrt{2}\mu^{\frac{k}{2}}}{\sqrt{\pi l}}\Big(\int_{\frac{(n-1)l}{\mu^k}}^{\frac{nl}{\mu^k}} \Big|f(t) - f\Big(\frac{(n-1)l}{\mu^k}\Big)\Big|^2 \omega_n(t)dt\Big)^{\frac{1}{2}} \times \Big(\int_{\frac{(n-1)l}{\mu^k}}^{\frac{nl}{\mu^k}} |\omega_n(t)|\, dt\Big)^{\frac{1}{2}}$$
$$+ \Big|f\Big(\frac{(n-1)l}{\mu^k}\Big)\Big| \times \Big(\frac{\sqrt{2}\mu^{\frac{k}{2}}}{\sqrt{\pi l}}\Big) \times \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \times \Big(\frac{l}{2\mu^k}\Big)dx$$
$$= \frac{\sqrt{2}\mu^{\frac{k}{2}}}{\sqrt{\pi l}} \times \frac{1}{\sqrt{\mu^k}}\Big(\frac{l}{\mu^k}\Big)^{\alpha}\phi\Big(\frac{l}{\mu^k}\Big) \times \Big(\frac{l\pi}{2\mu^k}\Big)^{\frac{1}{2}} + f\Big(\frac{(n-1)l}{\mu^k}\Big) \times \frac{\sqrt{\pi l}}{\sqrt{2\mu^k}}$$
$$= \frac{l^{\alpha}}{\mu^{k(\alpha+\frac{1}{2})}}\ \phi\Big(\frac{l}{\mu^k}\Big) + f\Big(\frac{(n-1)l}{\mu^k}\Big) \times \frac{\sqrt{\pi l}}{\sqrt{2\mu^k}}.$$

$$\|f - S_{\mu^k,0}(f)\|_2^2 = \sum_{n=1}^{\mu^k} |c_{n,0}|^2$$
$$= \sum_{n=1}^{\mu^k}\Big(\frac{l^{\alpha}}{\mu^{k(\alpha+\frac{1}{2})}}\Big)^2\ \phi^2\Big(\frac{l}{\mu^k}\Big) + \sum_{n=1}^{\mu^k}\Big|f\Big(\frac{(n-1)l}{\mu^k}\Big)\Big|^2 \times \frac{\pi l}{\mu^k}$$
$$+ 2\sum_{n=1}^{\mu^k} \frac{l^{\alpha}}{\mu^{k(\alpha+\frac{1}{2})}}\ \phi\Big(\frac{l}{\mu^k}\Big) \times f\Big(\frac{(n-1)l}{\mu^k}\Big) \times \frac{\sqrt{\pi l}}{\mu^{\frac{k}{2}}}$$
$$= \frac{l^{2\alpha}}{\mu^{k(2\alpha+1)}}\ \phi^2\Big(\frac{l}{\mu^k}\Big) \times \mu^k + \frac{A\pi l}{\mu^k} + \frac{2l^{\alpha}}{\mu^{k(\alpha+\frac{1}{2})}} \times \frac{B\sqrt{\pi l}}{\mu^{\frac{k}{2}}}\ \phi\Big(\frac{l}{\mu^k}\Big)$$
$$\leq \Big(1 + A\pi + 2B\sqrt{\pi}\Big)\Big(\frac{l^{\alpha}}{\mu^{k\alpha}}\ \phi\Big(\frac{l}{\mu^k}\Big) + \frac{\sqrt{l}}{\mu^{\frac{k}{2}}}\Big)^2$$
$$= O\Big(\frac{l^{\alpha}}{\mu^{k\alpha}}\ \phi\Big(\frac{l}{\mu^k}\Big) + \frac{\sqrt{l}}{\mu^{\frac{k}{2}}}\Big);\ \ \alpha > \frac{1}{2}.$$

Hence,
$${}^{c}E^{(2)}_{\mu^k,0}(f) = \min_{S_{\mu^k,0}(f)} \|f - S_{\mu^k,0}(f)\|_2 = O\Big(\frac{l^{\alpha}}{\mu^{k\alpha}}\ \phi\Big(\frac{l}{\mu^k}\Big) + \frac{\sqrt{l}}{\mu^{\frac{k}{2}}}\Big);\ \ \alpha > \frac{1}{2}.$$

(ii) For $m \geq 1$, following the proof of Theorem 3.1,

$$|c_{n,m}| \leq \Big(\int_{\frac{(n-1)l}{\mu^k}}^{\frac{nl}{\mu^k}} \Big|f(t) - f\Big(\frac{(n-1)l}{\mu^k}\Big)\Big|^2 \omega_n(t)\,dt\Big)^{\frac{1}{2}} \Big(\int_{\frac{(n-1)l}{\mu^k}}^{\frac{nl}{\mu^k}} |T_{n,m}^{(\mu)}(t)|^2\,\omega_n(t)\,dt\Big)^{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{\mu^k}}\Big(\frac{l}{(m+1)\mu^k}\Big)^{\alpha} \phi\Big(\frac{l}{(m+1)\mu^k}\Big) \times \Big(\int_{\frac{(n-1)l}{\mu^k}}^{\frac{nl}{\mu^k}} \Big(\frac{4\mu^k}{\pi l}\Big) \times$$

$$T_m^2\Big(\frac{2\mu^k t}{l} - 2n + 1\Big) \frac{1}{\sqrt{1 - (\frac{2\mu^k t}{l} - 2n+1)^2}}\,dt\Big)^{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{\mu^k}}\Big(\frac{l}{(m+1)\mu^k}\Big)^{\alpha} \phi\Big(\frac{l}{(m+1)\mu^k}\Big)\Big(\int_0^{\pi} \frac{2}{\pi}\, T_m^2(\cos\theta)d\theta\Big)^{\frac{1}{2}}$$

$$= \frac{\sqrt{2}}{\mu^{\frac{k}{2}}\sqrt{\pi}}\Big(\frac{l}{(m+1)\mu^k}\Big)^{\alpha} \phi\Big(\frac{l}{(m+1)\mu^k}\Big)\Big(\int_0^{\pi} (\cos(m\theta))^2 d\theta\Big)^{\frac{1}{2}}$$

$$\leq \frac{2\sqrt{2}}{\mu^{\frac{k}{2}}\sqrt{\pi}}\Big(\frac{l}{(m+1)\mu^k}\Big)^{\alpha} \phi\Big(\frac{l}{(m+1)\mu^k}\Big).$$

By Cauchy integral test,

$$\|f - S_{\mu^k,M}(f)\|_2^2 = \sum_{n=1}^{\mu^k}\sum_{m=M}^{\infty} |c_{n,m}|^2$$

$$\leq \sum_{n=1}^{\mu^k}\sum_{m=M}^{\infty} \frac{8l^{2\alpha}}{\pi\mu^{k(2\alpha+1)}(m+1)^{2\alpha}}\, \phi^2\Big(\frac{l}{(m+1)\mu^k}\Big)$$

$$\leq \frac{8l^{2\alpha}}{\pi\mu^{2k\alpha}}\, \phi^2\Big(\frac{l}{M\mu^k}\Big) \sum_{m=M}^{\infty} \frac{1}{(m+1)^{2\alpha}}$$

$$\leq \frac{8l^{2\alpha}}{\pi\mu^{2k\alpha}}\, \phi^2\Big(\frac{l}{M\mu^k}\Big)\Big(\frac{1}{(M+1)^{2\alpha}} + \int_M^{\infty} \frac{1}{(m+1)^{2\alpha}} dm\Big)$$

$$\leq \frac{8l^{2\alpha}}{\pi\mu^{2k\alpha}}\, \phi^2\Big(\frac{l}{M\mu^k}\Big)\Big(1 + \frac{1}{2\alpha - 1}\Big)\Big(\frac{1}{(M+1)^{2\alpha-1}} + \frac{1}{(M+1)^{2\alpha-1}}\Big)$$

$$= \frac{16l^{2\alpha}}{\pi\mu^{2k\alpha}}\, \phi^2\Big(\frac{l}{M\mu^k}\Big)\Big(1 + \frac{1}{2\alpha - 1}\Big) \times \frac{1}{(M+1)^{2\alpha-1}}$$

$$\|f - S_{\mu^k,M}(f)\|_2 \leq \frac{4l^{\alpha}}{\sqrt{\pi}\mu^{k\alpha}} \phi\Big(\frac{l}{M\mu^k}\Big)\sqrt{\Big(1 + \frac{1}{2\alpha - 1}\Big)} \times \frac{1}{(M+1)^{\alpha - \frac{1}{2}}}$$

$$= O\Big(\Big(\frac{l}{\mu^k}\Big)^{\alpha} \phi\Big(\frac{l}{M\mu^k}\Big)\frac{1}{(M+1)^{\alpha-\frac{1}{2}}}\Big);\ \alpha > \frac{1}{2}$$

Hence, ${}^{c}E_{\mu^k,M}^{(2)}(f) = \min_{S_{\mu^k,M}(f)} \|f - S_{\mu^k,M}(f)\|_2$

$$= O\Big(\Big(\frac{l}{\mu^k}\Big)^{\alpha} \phi\Big(\frac{l}{M\mu^k}\Big)\frac{1}{(M+1)^{\alpha-\frac{1}{2}}}\Big);\ \alpha > \frac{1}{2}.$$

□

Theorem 3.3. *If a function $f \in H_{\omega}^{\alpha,2}[0,l)$ then the moduli of continuity of $f - S_{\mu^k,M}(f)$ satisfies:*

$$(i)\ W^{(1)}\Big(f - S_{\mu^k,0}(f), \frac{1}{\mu^k}\Big) = \sup_{0<h\leq\frac{1}{\mu^k}} \|(f - S_{\mu^k,0}(f))(\cdot + h) - (f - S_{\mu^k,0}(f))(\cdot)\|_2$$

$$= O\Big(\frac{l^\alpha}{\mu^{k\alpha}} + \frac{\sqrt{l}}{\mu^{\frac{k}{2}}}\Big);\ 0 < \alpha < 1.$$

$$(ii)\ W^{(1)}\Big(f - S_{\mu^k,M}(f), \frac{1}{\mu^k}\Big) = \sup_{0<h\leq\frac{1}{\mu^k}} \|(f - S_{\mu^k,M}(f))(.+h) - (f - S_{\mu^k,M}(f))(.)\|_2$$
$$= O\Big(\Big(\frac{l}{\mu^k}\Big)^\alpha \frac{1}{(M+1)^{\alpha-\frac{1}{2}}}\Big);\ \alpha > \frac{1}{2},\ k \geq 1.$$

Proof. Following the proof of Theorem 3.1,

$$(i)\ \text{For M=0},\ \|f - S_{\mu^k,0}\|_2 = O\Big(\frac{l^\alpha}{\mu^{k\alpha}} + \frac{\sqrt{l}}{\mu^{\frac{k}{2}}}\Big);\ \ 0 < \alpha < 1$$
$$W^{(1)}\Big(f - S_{\mu^k,0}(f), \frac{l}{\mu^k}\Big) = \sup_{0<h\leq\frac{1}{\mu^k}} \|(f - S_{\mu^k,0}(f))(.+h) - (f - S_{\mu^k,0}(f)(.))\|_2$$
$$\leq \|f - S_{\mu^k,0}(f)\|_2 + \|f - S_{\mu^k,0}(f)\|_2$$
$$= 2.\|f - S_{\mu^k,0}\|_2$$
$$= 2.O\Big(\frac{l^\alpha}{\mu^{k\alpha}} + \frac{\sqrt{l}}{\mu^{\frac{k}{2}}}\Big);\ 0 < \alpha < 1.$$

$$(ii)\ \text{For } M \geq 1,\ \|f - S_{\mu^k,M}\|_2 = O\Big(\Big(\frac{l}{\mu^k}\Big)^\alpha \frac{1}{(M+1)^{\alpha-\frac{1}{2}}}\Big);\ \alpha > \frac{1}{2},\ k \geq 1.$$
$$W^{(1)}\Big(f - S_{\mu^k,M}(f), \frac{l}{\mu^k}\Big) = \sup_{0<h\leq\frac{l}{\mu^k}} \|(f - S_{\mu^k,M}(f))(.+h) - (f - S_{\mu^k,M}(f))(.)\|_2$$
$$\leq \|f - S_{\mu^k,M}(f)\|_2 + \|f - S_{\mu^k,M}(f)\|_2$$
$$= 2.\|f - S_{\mu^k,M}\|_2$$
$$= O\Big(\Big(\frac{l}{\mu^k}\Big)^\alpha \frac{1}{(M+1)^{\alpha-\frac{1}{2}}}\Big);\ \alpha > \frac{1}{2},\ k \geq 1.$$

□

Theorem 3.4. *If a function $f \in H_\omega^{\alpha,2,\phi}[0,l)$ then the moduli of continuity of $f - S_{\mu^k,M}(f)$ satisfies:*

$$(i)\ W^{(2)}\Big(f - S_{\mu^k,0}(f), \frac{1}{\mu^k}\Big) = O\Big(\frac{l^\alpha}{\mu^{k\alpha}}\phi\Big(\frac{l}{\mu^k}\Big) + \frac{\sqrt{l}}{\mu^{\frac{k}{2}}}\Big);\ \alpha > \frac{1}{2}.$$

$$(ii)\ W^{(2)}\Big(f - S_{\mu^k,M}(f), \frac{1}{\mu^k}\Big) = \sup_{0<h\leq\frac{1}{\mu^k}} \|(f - S_{\mu^k,M}(f))(.+h) - (f - S_{\mu^k,M}(f))(.)\|_2$$
$$= O\Big(\Big(\frac{l}{\mu^k}\Big)^\alpha \phi\Big(\frac{l}{M\mu^k}\Big)\frac{1}{(M+1)^{\alpha-\frac{1}{2}}}\Big);\ \alpha > \frac{1}{2}, k \geq 1.$$

Proof. Following the proof of Theorem 3.2 and Theorem 3.3,

$$(i)\ \text{For M=0},\ \|f - S_{\mu^k,0}\|_2 = O\Big(\frac{l^\alpha}{\mu^{k\alpha}}\phi\Big(\frac{l}{\mu^k}\Big) + \frac{\sqrt{l}}{\mu^{\frac{k}{2}}}\Big);\ \ \alpha > \frac{1}{2}$$
$$W^{(2)}\Big(f - S_{\mu^k,0}(f), \frac{1}{\mu^k}\Big) = O\Big(\frac{1}{\mu^{k(\alpha-\frac{1}{2})}}\phi\Big(\frac{l}{\mu^k}\Big) + 1\Big);\ \ \alpha > \frac{1}{2}.$$

(ii) For M ≥ 1, $\|f - S_{\mu^k,M}\|_2 = O\Big(\big(\frac{l}{\mu^k}\big)^{\alpha}\phi\big(\frac{l}{M\mu^k}\big)\frac{1}{(M+1)^{\alpha-\frac{1}{2}}}\Big);\ \alpha > \frac{1}{2}, k \geq 1.$

$$W^{(2)}\Big(f - S_{\mu^k,M}(f), \frac{1}{\mu^k}\Big) = O\Big(\big(\frac{l}{\mu^k}\big)^{\alpha}\phi\big(\frac{l}{M\mu^k}\big)\frac{1}{(M+1)^{\alpha-\frac{1}{2}}}\Big);\ \alpha > \frac{1}{2}, k \geq 1.$$

□

Remark 3.5. *From the above theorems, it is observed that:*

$$\begin{aligned}
W^{(1)}(f - S_{\mu^k,0}(f), \frac{1}{\mu^k}) &= O\big(\frac{l^{\alpha}}{\mu^{k\alpha}} + \frac{\sqrt{l}}{\mu^{\frac{k}{2}}}\big) \to 0 \text{ as } k \to \infty, M \to \infty.\\
W^{(1)}(f - S_{\mu^k,M}(f), \frac{1}{\mu^k}) &= O\Big(\big(\frac{l}{\mu^k}\big)^{\alpha}\frac{1}{(M+1)^{\alpha-\frac{1}{2}}}\Big) \to 0 \text{ as } k \to \infty, M \to \infty.\\
W^{(2)}(f - S_{\mu^k,0}(f), \frac{1}{\mu^k}) &= O\big(\frac{l^{\alpha}}{\mu^{k\alpha}}\phi\big(\frac{l}{\mu^k}\big) + \frac{\sqrt{l}}{\mu^{\frac{k}{2}}}\big) \to 0 \text{ as } k \to \infty, M \to \infty.\\
W^{(2)}(f - S_{\mu^k,M}(f), \frac{1}{\mu^k}) &= O\Big(\big(\frac{l}{\mu^k}\big)^{\alpha}\phi\big(\frac{l}{M\mu^k}\big)\frac{1}{(M+1)^{\alpha-\frac{1}{2}}}\Big) \to 0 \text{ as } k \to \infty, M \to \infty.
\end{aligned}$$

It is notable that

$$W^{(1)}\Big(f - S_{\mu^k,M}(f), \frac{1}{\mu^k}\Big) \leq 2.E^{(1)}_{\mu^k,M}(f)$$

and

$$W^{(2)}\Big(f - S_{\mu^k,M}(f), \frac{1}{\mu^k}\Big) \leq 2.E^{(2)}_{\mu^k,M}(f).$$

Hence, the moduli of continuity $W^{(1)}\big(f - S_{\mu^k,M}(f)\big)$ and $W^{(2)}\big(f - S_{\mu^k,M}(f)\big)$ are better and finer than $E^{(1)}_{\mu^k,M}(f)$ and $E^{(2)}_{\mu^k,M}(f)$, respectively.

4. Numerical Simulations

The following section presents the algorithm for the solution of the nonlinear heat conduction models in the human head, the spherical catalyst equations, and the radial stress models are demonstrated by examples.

4.1. *Model of heat conduction in human head:*

The following Emden-type equation is used to model the distribution of heat source in the human head:

$$\gamma t y''(t) + 2\gamma y'(t) + t a(y) = 0,\ 0 < t < 1, \tag{5}$$
subject to the boundary conditions: $y'(0) = 0,\ -\xi y'(1) = \tau(y - y_k)$,

where $a(y)$ is the heat production rate per unit volume, y is the absolute temperature; t is the radial distance from centre; γ is the average thermal conductivity inside the head; τ is a heat exchange coefficient; and y_k is the ambient temperature [5]. Nonlinear conduction modeling is employed in cooling techniques for traumatic brain injuries (TBI) to minimize further damage. The creation of thermoregulated helmets and wearable headgear depends on analyzing and applying nonlinear heat conduction principles. These are employed to forecast thermal effects from devices such as transcranial magnetic stimulation (TMS) or deep brain stimulation (DBS). The nonlinear heat conduction model not only enhances our comprehension of thermal behavior in biological tissues but also opens avenues for applications in wearable technology and thermal protection systems [6].

4.2. *The spherical catalyst equation:*

The following Lane-Emden equation is used to model the dimensionless concentration of the chemical spices which occur in a spherical catalyst:

$$y''(t) + \frac{2}{t}y'(t) - \kappa^2 y(t)\exp\Big(\frac{\sigma\upsilon(1-y(t))}{1+\upsilon(1-y(t))}\Big) = 0,\ 0 < t < 1, \tag{6}$$
subject to the boundary conditions: $y'(0) = 0,\ y(1) = 1,$

where κ^2, σ, and υ represents the Thiele modulus, dimensionless activation energy, and dimensionless heat of reaction as evaluated at the surface of the spherical catalytic pellet, respectively [12]. The Haber-Bosch process uses spherical catalyst equations to simulate the diffusion of hydrogen and nitrogen inside catalyst particles and optimize the reaction rate for producing ammonia [8]. These are used in direct air capture technologies to boost the processes that extract CO_2 from the air and transform it into fuel or other compounds.

Let $y(t)$ be the solution of the above nonlinear singular boundary value problems. Then,

$$y(t) \approx \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\ {}^{c}T^{(\mu)}_{n,m}(t),$$

having $(\mu^k, M)^{th}$ partial sum as

$$y(t) = (S_{\mu^k,M}f)(t) \approx \sum_{n=1}^{\mu^k}\sum_{m=0}^{M-1} c_{n,m}\ {}^{c}T^{(\mu)}_{n,m}(t) = C^T\ T^{(\mu)}(t). \tag{7}$$

By initial conditions of given nonlinear singular boundary value problems, Eq. (7) reduces to

$$\mathrm{y}(0) = \sum_{n=1}^{\mu^k}\sum_{m=0}^{M-1} c_{n,m}\ {}^{c}T^{(\mu)}_{n,m}(0) = a, \quad \mathrm{y}'(1) = \frac{d}{dt}\Big(\sum_{n=1}^{\mu^k}\sum_{m=0}^{M-1} c_{n,m}\ {}^{c}T^{(\mu)}_{n,m}(t)\Big)_{t=1} = b.$$

In Eq. (7), C^T contains $\mu^k M$ unknown coefficients. Hence, excluding initial conditions, $\mu^k M - 2$ extra conditions are needed for the solution of the differential equation. For determining the values of $\mu^k M$ unknown coefficients $c_{n,m}$, collocation points $t_i = \frac{i-1}{\mu^k M}$, i $= 1, 2, \ldots, \mu^k M$ are substituted in Eq. (7) to obtain $\mu^k M - 2$ equations. Hence, the values of unknown coefficients $c_{n,m}$ are obtained by these $\mu^k M$ equations. For more details, see [1].

5. Applications of GCWM

This study presents the approximate solutions of nonlinear model that better represents temperature fluctuations caused by the metabolic process and perfusion dynamics.

Example 5.1. ***Numerical simulations of models of the nonlinear heat conduction in human head***
Consider the nonlinear singular boundary value problem (5) *in the study of heat distribution in the human head for* $a(y) = e^{-y}$, $\gamma = 1$, $\tau = 2$, *and* $y_k = 0$,

$$ty''(t) + 2y'(t) + te^{-y(t)} = 0,\ 0 < t < 1, \tag{8}$$
subject to the boundary conditions: $y'(0) = 0,\ y'(1) + 2y(1) = 0.$

Now, we outline the algorithm for the numerical method discussed above:

Algorithm: To calculate the numerical solution of Eq. 8.

Input: Give values to numbers k=0, M=7, $\mu = 2$, and l=1.

Output: The approximate solution $y(t)$.

Steps for the numerical solution

Step 1: Construct the wavelet basis function ${}^{c}T^{(\mu)}_{n,m}(t)$ in [0,l), $n = 1, 2, \ldots, \mu^{k-1}$, $m = 0, 1, 2, \ldots, M-1$, where k and μ are positive integers.

Step 2: Approximate the solution function $y(t)$ as a linear combination of wavelet basis functions:
$y(t) = \sum_{n=1}^{\mu^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\, {}^{c}T^{(\mu)}_{n,m}(t) = \sum_{m=0}^{6} c_{1,m}\, {}^{c}T^{(\mu)}_{1,m}(t) = c_{1,0}\, {}^{c}T^{(\mu)}_{1,0}(t) + c_{1,1}\, {}^{c}T^{(\mu)}_{1,1}(t) + \cdots + c_{1,6}\, {}^{c}T^{(\mu)}_{1,6}(t)$.

Step 3: Substitute the wavelet approximation of $y(t)$ into the boundary conditions of the given equation.

Step 4: Divide the interval into equally spaced collocation points $t_i = \frac{i-1}{\mu^k M}$, i $= 1, 2, \ldots, \mu^{k-1}M$ and apply the differential equation at each of these points.

Step 5: The above process will results in a system of linear equations to find the values of the unknown coefficients $c_{n,m}$.

Step 6: This system of equations can be solved by variable precision arithmetic technique in MATLAB R2018a programming platform.

Step 7: Then substitute the values of coefficients $c_{n,m}$ back into the wavelet approximation and then, evaluate $y(t)$ at the desired points in the domain to obtain the approximate solution.

Step 8: One can increase the resolution of the approximate solution by refining the collocation points.

Table 1: The numerical solutions for different values of γ for Example 5.1.

x	GCWM k=0,M=7,γ=1	GCWM k=0,M=16,γ=1	GCWM k=0,M=7,γ=2	GCWM k=0,M=7,γ=3	ADM γ=1	JWSM M=7,γ=1
0	0.270027816	0.270029647	0.148488411	0.102598219	0.26990	0.2700
0.1	0.268755019	0.268756900	0.147769910	0.102096759	0.26862	0.2688
0.2	0.264930875	0.264932817	0.145612555	0.100591477	0.26480	0.2649
0.3	0.258537810	0.258539789	0.142010754	0.098079649	0.25841	0.2585
0.4	0.249546174	0.249548180	0.136955127	0.094556721	0.24943	0.2495
0.5	0.237913852	0.237915887	0.130432436	0.090016278	0.23781	0.2379
0.6	0.223585638	0.223587707	0.122425472	0.084450010	0.22349	0.2236
0.7	0.206492377	0.206494483	0.122425472	0.077847660	0.20641	0.2065
0.8	0.186549861	0.186552014	0.101869124	0.070196969	0.18648	0.1866
0.9	0.163657495	0.163659681	0.089263967	0.061483597	0.16359	0.1637
1	0.137696720	0.137698746	0.075062510	0.0516910419	0.13762	0.1377

Table 1, represents the comparisons of numerical solutions for M=7 for different values of γ. Figure 1, plots the approximate solution for $\gamma = 1, 2, 3$. Figure 2, depicts the comparisons of solutions of Eq. (8) *obtained by the generalized Chebyshev wavelet, Adomian decomposition method and by the Jacobi wavelet series method.*

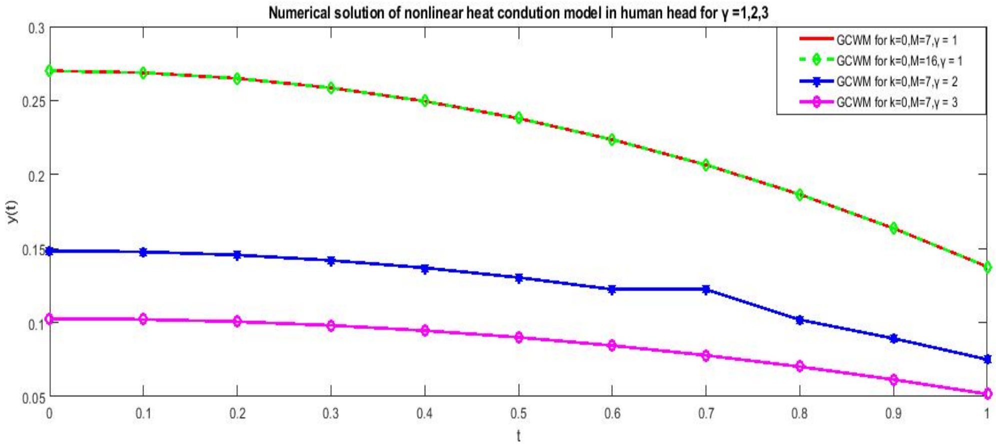

Figure 1: The graph represents the approximate solution of Eq. (8), for M=7, γ=1,2,3.

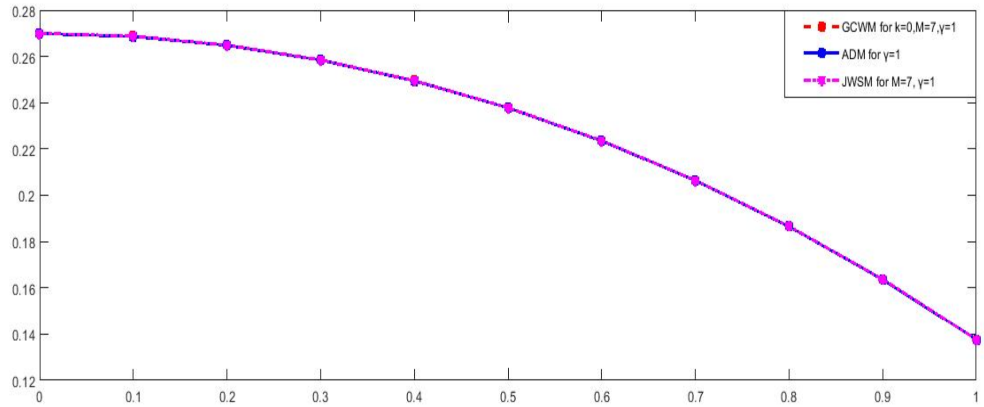

Figure 2: The graph represents the approximate solution of Eq. (8) by GCWM, ADM, JWSM for γ=1.

Example 5.2. ***Numerical results of the spherical catalyst equation***

Consider the nonlinear singular boundary value problem (6) *of the spherical catalyst equation for* $\upsilon = 1$ *and* $\kappa = 1$*:*

$$ty''(t) + 2y'(t) - ty(t)\exp\Big(\frac{\sigma(1-y(t))}{(2-y(t))}\Big) = 0,\ 0 < t < 1, \tag{10}$$

subject to the boundary conditions: $y'(0) = 0,\ y(1) = 1.$

Following the procedure adopted in Example 5.1, the required solution can be obtained.

Table 2: The numerical solutions for different values of σ for Example 5.2.

x	GCWM k=0,M=8,σ=0.5	GCWM k=0,M=8,σ=1	GCWM k=0,M=8,σ=1.5	Bernstein sol k=0,M=8,σ=1	VIM sol σ=1
0	0.844259943	0.836759799	0.828224752	0.836759840	0.828749891
0.1	0.845765694	0.838364881	0.829944702	0.838364897	0.830409262
0.2	0.850289481	0.843184259	0.835105737	0.843184251	0.835393813
0.3	0.857850980	0.851230223	0.843710888	0.851230207	0.843722865
0.4	0.868482709	0.862522431	0.855763731	0.862522411	0.855428619
0.5	0.8822295142	0.877086550	0.871266203	0.877086527	0.870556156
0.6	0.899147830	0.894952325	0.890215469	0.894952300	0.889163435
0.7	0.919304639	0.916150964	0.912599822	0.916150938	0.911321297
0.8	0.942776021	0.940711726	0.938393623	0.940711700	0.9371132129
0.9	0.969645200	0.968657603	0.967551240	0.968657588	0.966636539
1	1.0000000	1.000000	1.000000	1.00000000	1.000000000108

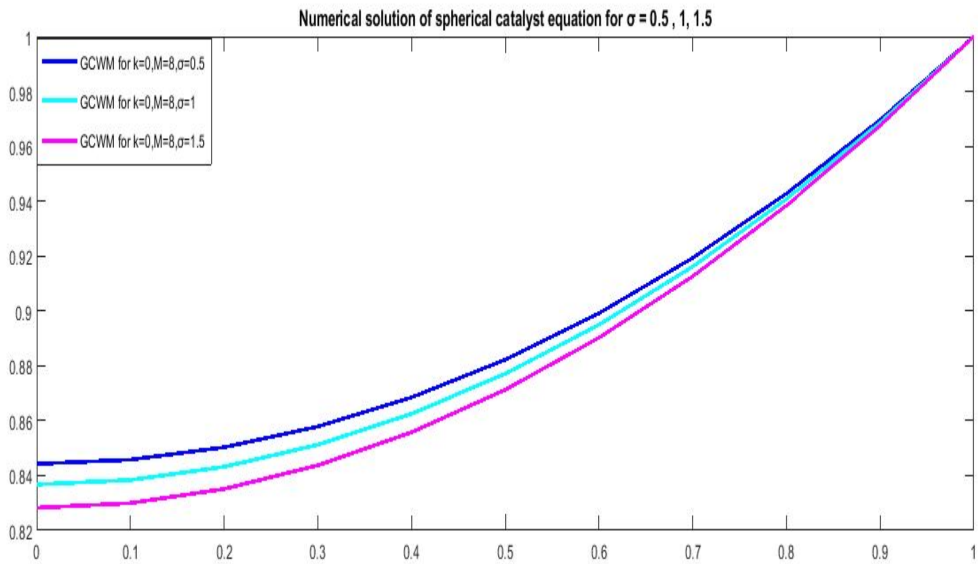

Figure 3: The graph represents the approximate solution of Eq. (10) by GCWM for M=8, and σ=0.5,1,1.5.

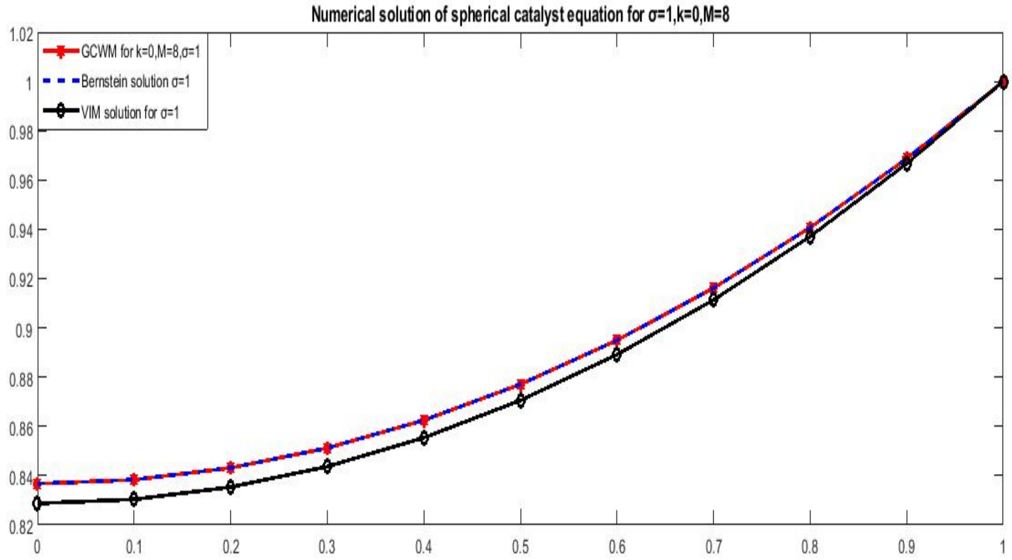

Figure 4: The graph represents the approximate solution of Eq. (10) by GCWM, Bernstein, and VIM for σ=1.

Table 2, displays the comparison of numerical solutions of Eq. (10) *for different values of σ obtained by GCWM, Bernstein wavelet and VIM method. Figure 3, 4, plots the approximate solutions of Eq.* (10) *for M=8.*

Example 5.3. *The following nonlinear singular boundary value problem models the radial stress on a rotationally symmetric shallow membrane cap[4]:*

$$y''(t) + \frac{3}{t}y'(t) = \frac{1}{2} - \frac{1}{8y^2(t)} \text{ for } t \in (0,1) \tag{11}$$

subject to the boundary conditions: $y'(0) = 0,\ y(1) = 1.$

Table 3, represents the numerical solutions of Eq. (11) *by GCWM, Adomian decomposition method and Varitional iteration method. Figure 5, plots the solutions of Eq.* (11).

Table 3: The numerical solutions for GCWM, ADM, VIM for Example 5.3.

x	GCWM k=0, M=8	ADM sol	VIM sol
0	0.954135307697	0.95413	0.95223
0.1	0.954588729206	0.95458	0.95263
0.2	0.955949645465	0.95594	0.95408
0.3	0.958220005108	0.95822	0.95649
0.4	0.961403036543	0.96140	0.95986
0.5	0.965503219590	0.96550	0.96420
0.6	0.970526246089	0.97052	0.96948
0.7	0.976478970230	0.97647	0.97571
0.8	0.983369349291	0.98336	0.98288
0.9	0.991206375540	0.99120	0.99098
1	1.000000000000	1.00000	1.000000000108

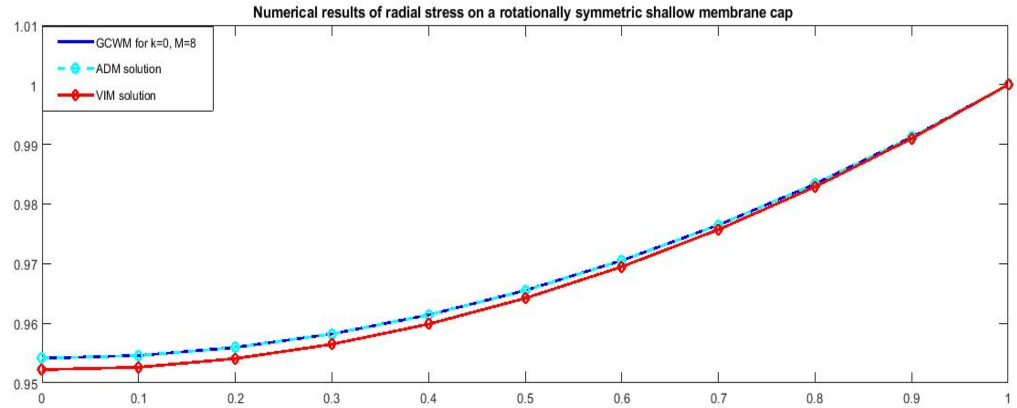

Figure 5: The graph represents the approximate solution of Eq. (11) by GCWM, ADM, and VIM for M=8.

6. Conclusions

In this paper, we have proposed a collocation based numerical method to solve the nonlinear mathematical models. The proposed method, which utilizes the generalized Chebyshev wavelet method to solve nonlinear singular boundary value problems, yields almost identical answers. The solution obtained by the generalized Chebyshev wavelet are compared with the solutions of the ADM, VIM, Bernstein, and Jacobi wavelets. It was discovered that the solutions found with these wavletes had a significant degree of resemblance. However, for high M values, the generalized Chebyshev wavelet solution gradually improved. This demonstrates the usefulness and precision of the suggested approach for resolving the nonlinear heat conduction models in the human head, the spherical catalyst equations, and the radial stress models on a shallow membrane cap that rotates symmetrically. From the tables, it is clear that the results obtained by GCWM are more accurate, and they converge toward an accurate approximation more competently with less time consumption. All computations are carried out using MATLAB R2018a. The method can be extended to solve higher-dimensional models of fractional partial differential equations easily, appearing in mathematical modeling and engineering applications.

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References

[1] I. Çelik. Chebyshev wavelet collocation method for solving generalized burgers–huxley equation. Mathematical methods in the applied sciences, 39(3):366–377, 2016.
[2] C. CHUI. Wavelet analysis and its applications. Wavelets, page 725, 1992.
[3] M. Danish, S. Kumar, and S. Kumar. A note on the solution of singular boundary value problems arising in engineering and applied sciences: use of oham. Computers and chemical engineering, 36:57–67, 2012.
[4] R. Dickey. Rotationally symmetric solutions for shallow membrane caps. Quarterly of Applied Mathematics, 47(3):571–581, 1989.
[5] R. Duggan and A. Goodman. Pointwise bounds for a nonlinear heat conduction model of the human head. Bulletin of mathematical biology, 48:229–236, 1986.
[6] R. A. El-Nabulsi and W. Anukool. Nonlocal thermal effects on biological tissues and tumors. Thermal Science and Engineering Progress, 34:101424, 2022.
[7] E. D. Fylladitakis, M. P. Theodoridis, and A. X. Moronis. Review on the history, research, and applications of electrohydrodynamics. IEEE Transactions on Plasma Science, 42(2):358–375, 2014.
[8] J. Humphreys, R. Lan, and S. Tao. Development and recent progress on ammonia synthesis catalysts for haber–bosch process. Advanced Energy and Sustainability Research, 2(1):2000043, 2021.

[9] J. Hyslop. A. zygmund, trigonometric series (cambridge university press, 1959) two volumes. volume i, xii+ 383 pp., volume ii, vii+ 354 pp., 84s. per volume.. Proceedings of the Edinburgh Mathematical Society, 12(1):57–58, 1960.
[10] H. Kaur, R. Mittal, and V. Mishra. Haar wavelet approximate solutions for the generalized lane–emden equations arising in astrophysics. Computer physics communications, 184(9):2169– 2177, 2013.
[11] S. Lal and U. Vats. Multi-resolution analysis and moduli of continuity of solutions for nonlinear li´enard equations by vieta-lucas wavelet. 2024.
[12] R. Rach, J.-S. Duan, and A.-M. Wazwaz. On the solution of non-isothermal reaction-diffusion model equations in a spherical catalyst by the modified adomian method. Chemical Engineering Communications, 202(8):1081–1088, 2015.
[13] P. Roul, V. P. Goura, and R. Agarwal. A compact finite difference method for a general class of nonlinear singular boundary value problems with neumann and robin boundary conditions. Applied Mathematics and Computation, 350:283–304, 2019.
[14] B. Sripathy, P. Vijayaraju, and G. Hariharan. Chebyshev wavelet-based approximation method to some non-linear differential equations arising in engineering. Int. J. Math. Anal, 9(20):993– 1010, 2015.
[15] R. A. Van Gorder. Exact first integrals for a lane–emden equation of the second kind modeling a thermal explosion in a rectangular slab. New Astronomy, 16(8):492–497, 2011.
[16] A.-M. Wazwaz and R. Rach. Comparison of the adomian decomposition method and the variational iteration method for solving the lane-emden equations of the first and second kinds. Kybernetes, 40(9/10):1305–1318, 2011.
[17] X. Zheng, Z. Wei, et al. Estimates of approximation error by legendre wavelet. Applied Mathematics, 7(7):694–700, 2016.
[18] L. Zhu and Q. Fan. Numerical solution of nonlinear fractional-order volterra integrodifferential equations by scw. Communications in Nonlinear Science and Numerical Simulation, 18(5):1203–1213, 2013.