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Characterization of Banach-space-valued functions involving the Weinstein transform

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Abstract. In this paper, the space $H(A)$ is defined by exploiting the theory of the Weinstein transform, and proved that Weinstein transform $\mathcal{F}_w(\phi)$ is an automorphism on the space $H(A)$. The Banach space-valued test functions of Beurling type ultradistribution $H_\omega(A)$ is defined by taking the weight function ω. It is shown that the subspace $D_{\mathbb{R}_+^{n+1}}(A)$ is dense in $H_\omega(A)$ and the Weinstein transform $\mathcal{F}_w(\phi)$ is an automorphism on the space $H_\omega(A)$. Further showed that the linear space $\omega D_{\mathbb{R}_+^{n+1}}(A) \oplus (A)$ is dense in $H_\omega(A)$.

1. Introduction

The Weinstein transform has rich calculus and nice mathematical backgrounds, which play an important role in solving problems in partial differential equations, waveforms, signal processing, fluid mechanics, and other areas of mathematics. Using the aforesaid transform theory, many research works have been made in [7, 10, 20, 32] by exploiting the theory of the Weinstein transform. In this connection, Mejjaoli et al. [15, 16], Salhi [17], Nahia and Salem [26, 27], Mehraj [14], Salem and Nasr [25] found many important observations by utilizing the aforesaid transform theory.
The theory of Banach-space-valued testing functions to the distributions was found by Zemanian [37]. This space is more general than scalar distributions. He introduced the inductive limit space $D^m(A)$ which is given by $D^m(A) = D^m_{\mathbb{R}^n}(A) = \cup_{i=1}^{\infty} D^m_{K_i}(A)$. where $D^m_{K_i}(A)$ is the linear space of all smooth function f from $\mathbb{R}^n$ into a Banach space A such that $supp\ \phi \subset K_i$ are the compact subsets of $\mathbb{R}^n$ and $K_i \subset K_{i+1}, \cup_{i=1} K_i = \mathbb{R}^n$. The topology generated of the semi-norm of $D^m_{K_i}(A)$ is defined by $\gamma_k(\phi) = \sup_{t\in K_i} \| D_t^k \phi(t) \|_A, \quad 0 \le k \le m$. Motivated by the result of Zemanian [37], exploiting the theory of Mellin transform, Tiwari [35] defined Banach space-valued distributions and proved several properties, including a Mellin-type convolution theorem. From the result of [37], Koh and Lie [13] proved the subspace ${}_\mu D_I(A)$ is dense in $H_\mu(A)$. They further showed that there is a bijection from $[H_\mu(A); B]$ onto $[H_\mu; [A; B]]$. After that, it is shown that the Hankel transformation of an arbitrary order on $H_\mu(A)$ is an automorphism on $H_\mu(A)$. Upadhyay [34] investigated the $H_\mu^\omega(A)$ type space, which was defined by the set of all those smooth, complex valued functions $\phi(x)$ on $I = (0, \infty)$ such that

$$\gamma^\mu_{\lambda,k}(\phi) = \sup_{x\in I} \left\| exp[\lambda\omega(x)]\left(x^{-1}\frac{d}{dx}\right)^k x^{-\mu-\frac{1}{2}}\phi(x)\right\|_A < \infty, \ \forall\ \lambda, k \in \mathbb{N}_0,$$

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and studied their algebraic and topological properties by using the theory of the Hankel transform.
The theory of ultradistribution, which is the generalization of Schwartz distributions, was introduced by Beurling [1], Bjo¨rck [4] and Roumieu [22]. In 1961, Beurling found many important observations, partly based on ideas published already in [1] and [2]. From the development of [1] and [2], Bjo¨rck [4] introduced the generalized distribution theory, including the spaces $\mathcal{B}_{p,k}$ of [8]. He also studied the questions of existence and approximation and interior regularity of solutions of equations with constant coefficients, and also consider equations that have no solutions. A unification of these results were given by Komtsu [12]. The Hankel transformation of ultradistribution was first introduced by Pathak and Pandey [18] and examined many properties. From the aforesaid concepts, Pathak and Shrestha [21] defined the Beurling type ultradistribution space H^{μ}_{ω} and its important properties were studied. Our main objective of the present paper is to investigate the Banach-space-valued ultradistributions associated with the Weinstein transform. The organization of the present paper is given below:
Section 1 is introduction, in which we provides a brief description of the Weinstein transform, the Hankel transform, various background of ultradistributions, and also the ideas of Banach-space-valued testing function to the distributions. In section 2, some important definitions, properties and results are given which are useful for our present manuscript. In section 3, the space $H_{\omega}(A)$ is defined and its various properties are discussed by utilizing the theory of the Weinstein transform. In section 4, we present several remarks and observations which concern the subject-matter of this paper.

2. Preliminaries

In this section, we discuss some properties related to the Weinstein transform that are useful for our research. From [26, 27] and [31], standard notations, definitions, and properties are given below:
- $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,\infty)$.
- $x = (x', x_{n+1}) = (x_1, x_2, x_3 ... x_n, x_{n+1}) \in \mathbb{R}^{n+1}_+$.
- $-x = (-x', x_{n+1}) = (-x_1, -x_2, -x_3 ... -x_n, x_{n+1}) \in \mathbb{R}^{n+1}_+$.
- $\langle x, \xi\rangle = \sum_{k=1}^{n+1} x_k \xi_k$.
- $\xi^m = \xi_1^{m_1}.\xi_2^{m_2}.......\xi_n^{m_n}.\xi_{n+1}^{m_{n+1}} \ \forall\ \xi \in \mathbb{R}^{n+1}_+$
- $\|\xi\|^2 = \sum_{k=1}^{n+1} \xi_k^2$.
- $D^{\alpha} = D_1^{\alpha_1} D_2^{\alpha_2} ....D_n^{\alpha_n} D_{n+1}^{\alpha_{n+1}}$ and $\alpha \in \mathbb{N}_0^{n+1}$.
- $C_*\left(\mathbb{R}^{n+1}_+\right)$, represents the space of continuous functions over $\mathbb{R}^{n+1}_+$, even with respect to the last variable.
- $C^{\infty}_*\left(\mathbb{R}^{n+1}_+\right)$, represents the space of infinitely differentiable function over $\mathbb{R}^{n+1}_+$, even with respect to the last variable.
- $S_*\left(\mathbb{R}^{n+1}_+\right)$, represents Schwartz space of rapidly decreasing functions over $\mathbb{R}^{n+1}_+$, even with respect to the last variable.
- L^p_{β} represents the space of measurable functions ϕ over $\mathbb{R}^{n+1}_+$ such that

$$\|\phi\|_{\beta,p} = \left(\int_{\mathbb{R}^{n+1}_+} |\phi(y)|^p \, d\mu_{\beta}(y)\right)^{\frac{1}{p}} < \infty,\ \ 1 \le p < \infty \text{ and } \|\phi\|_{\beta,\infty} = ess \sup_{y\in\mathbb{R}^{n+1}_+} |\phi(y)| < \infty,$$

where $d\mu_{\beta}(y)$ is the measure over $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,\infty)$ and defined as $d\mu_{\beta}(y) = d\mu_{\beta}(y', y_{n+1}) = \frac{y_{n+1}^{2\beta+1}}{(2\pi)^{\frac{n}{2}} 2^{\beta}\Gamma(\beta+1)} dy$.
The Weinstein operator $\Delta^n_{W,\beta}$ over $\mathbb{R}^{n+1}_+$ is given by $\Delta^n_{W,\beta} = \sum_{j=1}^{n+1} \frac{\partial^2}{\partial x_j^2} + \frac{2\beta+1}{x_{n+1}} \frac{\partial}{\partial x_{n+1}},\ \ \beta > -1/2$.

Definition 2.1. *Let $\phi \in L^1_{\beta}(\mathbb{R}^{n+1}_+)$. The Weinstein transform of ϕ is defined by*

$$(\mathcal{F}_w\phi)(\xi) = \int_{\mathbb{R}^{n+1}_+} e^{-i\langle \xi', y'\rangle} j_{\beta}(\xi_{n+1} y_{n+1}) \phi(y) \, d\mu_{\beta}(y),\ \ \xi \in \mathbb{R}^{n+1}_+, \tag{1}$$

where j_β is the normlized Bessel function of index β, which is given by $j_\beta(y) = \Gamma(\beta+1)\sum_{m=0}^{\infty}\frac{(-1)^m(y)^{2m}}{2^m m!\Gamma(\beta+m+1)}$ and $d\mu_\beta(y)$ is the measure over $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times (0,\infty)$ given by $d\mu_\beta(y) = d\mu_\beta(y', y_{n+1}) = \frac{y_{n+1}^{2\beta+1}}{(2\pi)^{\frac{n}{2}}2^\beta\Gamma(\beta+1)}dy$. The function $(\xi, y) \longrightarrow W_{\beta,n}(\xi, y) = e^{-i\langle\xi', y'\rangle} j_\beta(\xi_{n+1}y_{n+1})$ is called the Weinstein kernel, and it satisfies the following properties: For all $(\xi, y) \in \mathbb{C}^{n+1}\times\mathbb{C}^{n+1}$, from [33], we have

$$\mid W_{\beta,n}(\xi, y) \mid \leq 1 \tag{2}$$

Definition 2.2. *The translation operator τ_y^β, $y \in \mathbb{R}_+^{n+1}$ associated with the Weinstein operator $\Delta_{W,\beta}^n$ is given by*

$$\phi(x,y) = \tau_y^\beta\phi(x) = \int_{\mathbb{R}_+^{n+1}}\phi(z)D_\beta(x,y,z)\,d\mu_\beta(z), \quad x, z \in \mathbb{R}_+^{n+1}. \tag{3}$$

where $D_\beta(x,y,z)$ [33], is basic function and defined by

$$D_\beta(x,y,z) = \int_{\mathbb{R}_+^{n+1}} e^{-i\langle x',\xi'\rangle} j_\beta(x_{n+1}\xi_{n+1})e^{i\langle y',\xi'\rangle} j_\beta(y_{n+1}\xi_{n+1})e^{-i\langle z',\xi'\rangle} j_\beta(z_{n+1}\xi_{n+1})\,d\mu_\beta(\xi). \tag{4}$$

By using the generalized translation, we define the generalized convolution product $\phi\#_\beta\psi$ of the functions $\phi, \psi \in L^1(\mathbb{R}_+^{n+1})$ as follows

$$(\phi\#_\beta\psi)(y) = \int_{\mathbb{R}_+^{n+1}}\phi(x)(\tau_y^\beta\psi)(x)\,d\mu_\beta(x) = \int_{\mathbb{R}_+^{n+1}}\phi(x)\psi(x,y)\,d\mu_\beta(x)\,. \tag{5}$$

Inversion formula: Let $\phi \in L_\beta^1(\mathbb{R}_+^{n+1})$. If $F_w\phi \in L_\beta^1(\mathbb{R}_+^{n+1})$ then,

$$\phi(y) = \int_{\mathbb{R}_+^{n+1}} e^{i\langle y',\xi'\rangle} j_\beta(y_{n+1}\xi_{n+1})(\mathcal{F}_w\phi)(\xi)\,d\mu_\beta(\xi), \ a.e. \tag{6}$$

- For $f \in S_*(\mathbb{R}_+^{n+1})$, we have

$$\mathcal{F}_w((\Delta_{W,\beta}^n)^\alpha f)(\xi) = (-\|\xi\|^2)^\alpha(\mathcal{F}_w f)(\xi) \text{ and } (\Delta_{W,\beta}^n)^\alpha(\mathcal{F}_w f)(\xi) = \mathcal{F}_w[(-\|x\|^2)^\alpha f](\xi) \quad (\forall\ \xi\ \in\ \mathbb{R}_+^{n+1}). \tag{7}$$

- Let $\phi \in S_*\Big(\mathbb{R}_+^{n+1}\Big)$, then the inverse Weinstein transform is defined by

$$\mathcal{F}_w^{-1}\phi(y) = \mathcal{F}_w\phi(-y) \ \ \forall y \in \mathbb{R}_+^{n+1}. \tag{8}$$

Theorem 2.3. *(See [33]) Let $\phi, \psi \in \mathbb{R}_+^{n+1}$ and α be any multi-index then*

$$\begin{aligned}(\Delta_{n,\beta})_x^\alpha[\phi(x)\psi(x)] \quad &= \sum_{j=0}^{\alpha}\sum_{m=1}^{2j}\sum_{q=0}^{m}\sum_{|\rho'|\leq 2(\alpha-j)}\sum_{\delta_1,\delta_2,\ldots,\delta_n\geq 0}\binom{\alpha}{j}\binom{m}{q}\binom{\alpha-j}{\delta_1,\delta_2,....,\delta_n}\frac{1}{\rho'!}E'_{\beta,m}x_{n+1}^{m-\alpha}\\ &\times\Big(D_x^{\rho'+q}\phi(x)\Big)\Big(D_x^{\rho'+2\delta'+m-q}\psi(x)\Big).54321'54\end{aligned} \tag{9}$$

3. The Space $H_\omega(A)$ and its Properties

In this section the space $H_\omega(A)$ is defined and discuss its various properties by utilizing the theory of the Weinstein transform.

Definition 3.1. *The space $H(A)$ is the set of all those smooth, complex-valued function $\phi(x)$ from $\mathbb{R}_+^{n+1}$ into A which satisfies the following norm,*

$$\gamma_{m,k}(\phi) = \sup_{x\in\mathbb{R}_+^{n+1}} \| x^m (\Delta^n_{W,\beta})^k_x \phi(x) \|_A < \infty, \ \forall\, m,k \in \mathbb{N}_0. \tag{10}$$

Theorem 3.2. *Let $\phi \in H(A)$. Then $\mathcal{F}_w(\phi) \in H(A)$. Moreover, the Weinstein transform $\mathcal{F}_w$ defines an automorphism on $H(A)$.*

Proof. From (10), we have

$$\gamma_{\alpha,m}[(\mathcal{F}_w\phi)(\xi)] = \sup_{\xi\in\mathbb{R}_+^{n+1}} \left\| \xi^m (\Delta^n_{W,\beta})^\alpha_\xi (\mathcal{F}_w\phi)(\xi) \right\|_A. \tag{11}$$

We first compute

$$(\Delta^n_{W,\beta})^\alpha_\xi (\mathcal{F}_w\phi)(\xi) = \int_{\mathbb{R}_+^{n+1}} (\Delta^n_{W,\beta})^\alpha_\xi \Big(e^{-i\langle \xi', y'\rangle} j_\beta(\xi_{n+1}y_{n+1}) \Big) \phi(y)\, d\mu_\beta(y).$$

In view of (7), this reduces to

$$(\Delta^n_{W,\beta})^\alpha_\xi (\mathcal{F}_w\phi)(\xi) = \int_{\mathbb{R}_+^{n+1}} (-\|y\|^2)^\alpha e^{-i\langle \xi', y'\rangle} j_\beta(\xi_{n+1}y_{n+1}) \phi(y)\, d\mu_\beta(y).$$

Let $g(y) := (-\|y\|^2)^\alpha \phi(y)$. Then the above expression can be rewritten as

$$(\Delta^n_{W,\beta})^\alpha_\xi (\mathcal{F}_w\phi)(\xi) = \int_{\mathbb{R}_+^{n+1}} e^{-i\langle \xi', y'\rangle} j_\beta(\xi_{n+1}y_{n+1}) g(y)\, d\mu_\beta(y).$$

Substituting into (11), we obtain

$$\gamma_{\alpha,m}[(\mathcal{F}_w\phi)(\xi)] \le \sup_{\xi\in\mathbb{R}_+^{n+1}} \left\| \int_{\mathbb{R}_+^{n+1}} (-1)^m (\Delta^n_{W,\beta})^m_y \Big[e^{-i\langle \xi', y'\rangle} j_\beta(\xi_{n+1}y_{n+1}) \Big] g(y)\, d\mu_\beta(y) \right\|_A.$$

By integrating by parts and applying (2), it follows that

$$\gamma_{\alpha,m}[(\mathcal{F}_w\phi)(\xi)] \le \sup_{\xi\in\mathbb{R}_+^{n+1}} \left\| \sum_{r=0}^{k} \binom{k}{r} (y^2)^r (\Delta^n_{W,\beta})^m_y g(y) \right\|_A \int_{\mathbb{R}_+^{n+1}} (1+y^2)^{-k}\, d\mu_\beta(y) < \infty, \quad \text{for any integer } k \ge 1.$$

Thus, $\mathcal{F}_w(\phi) \in H(A)$ whenever $\phi \in H(A)$. Moreover, by the inversion formula for the Weinstein transform, $\mathcal{F}_w^{-1}\phi(y) = \mathcal{F}_w\phi(-y)$. This shows that $\mathcal{F}_w$ is one-to-one. Hence, $\mathcal{F}_w$ is an automorphism on $H(A)$. □

Definition 3.3. *Let ω be a continuous real-valued function defined on $\mathbb{R}_+^{n+1}$ possessing the following condition:*

(i) $0 = \omega(0) = \lim_{x\to 0} \omega(x) \le \omega(x+y) \le \omega(x) + \omega(y), \forall\, x,\ y \in \mathbb{R}_+^{n+1}$.

(ii) $J_n(\omega) = \displaystyle\int_{|y|\ge 1} \frac{\omega(y)}{1+\parallel y \parallel^2}\, d\mu_\beta(y) < \infty$.

(iii) $a + b\log(1+y) \le \omega(y), \forall\, y \in \mathbb{R}_+^{n+1}, a \in \mathbb{R},$ *and* $b > 0$.

We denote by M the set of all continuous functions satisfying (i), (ii) and (iii).

Definition 3.4. *Let $\omega \in M$. The space $H_\omega(A)$ is the set of all complex valued infinitely differentiable functions ϕ from $\mathbb{R}_+^{n+1}$ into a Banach space A, satisfying*

$$p_{\lambda,k}^{\omega}(\phi) = \sup_{x\in\mathbb{R}_+^{n+1}} \| e^{\lambda\omega(x)} D^k \phi(x) \|_A < \infty, \ \forall \ \lambda, k \ \in \mathbb{N}_0. \tag{12}$$

Theorem 3.5. *let $\alpha \in \mathbb{N}_0$, and ϕ be any C^∞- function then,*

$$e^{\lambda\omega(x)}(\Delta_{W,\beta}^n)_x^\alpha \phi(x) = \sum_{j=0}^{\alpha}\sum_{r=1}^{2j}\sum_{\delta_1,\delta_2,\dots,\delta_n\geq 0}\binom{\alpha}{j}\binom{\alpha-j}{\delta_1,\delta_2,\dots.\delta_n}E'_{\beta,r}x_{n+1}^{r-\alpha}e^{\lambda\omega(x)}D_x^{2\delta'+r}\phi(x). \tag{13}$$

Proof. The proof of this theorem follows directly from [5, Pg. 14] and [36]. □

Lemma 3.6. *Let $\phi \in H_\omega(A)$. Then, for any non-negative real number λ, any multi-index α, and positive constants $E'_{\beta,\alpha}$, the following inequality holds:*

$$\left|e^{\lambda\omega(x)}(\Delta_{W,\beta}^n)_x^\alpha \phi(x)\right| \leq E'_{\beta,\alpha}\left|x_{n+1}^\alpha\, e^{\lambda\omega(x)}D_x^{2\alpha}\phi(x)\right|.$$

Proof. Let $\phi \in H_\omega(A)$. From (13), we obtain

$$e^{\lambda\omega(x)}(\Delta_{W,\beta}^n)_x^\alpha \phi(x) = \sum_{j=0}^{\alpha}\sum_{r=1}^{2j}\binom{\alpha}{j}\binom{\alpha-j}{\delta_1,\delta_2,\dots,\delta_n}E'_{\beta,r}\, e^{\lambda\omega(x)}x_{n+1}^{r-\alpha}D_x^{2|\delta'|+r}\phi(x).$$

Therefore,

$$\begin{aligned}\left|e^{\lambda\omega(x)}(\Delta_{W,\beta}^n)_x^\alpha \phi(x)\right| &= \left|\sum_{j=0}^{\alpha}\sum_{r=1}^{2j}\binom{\alpha}{j}\binom{\alpha-j}{\delta_1,\delta_2,\dots,\delta_n}E'_{\beta,r}\, e^{\lambda\omega(x)}x_{n+1}^{r-\alpha}D_x^{2|\delta'|+r}\phi(x)\right|\\ &\leq \sum_{j=0}^{\alpha}\sum_{r=1}^{2j}\binom{\alpha}{j}E'_{\beta,r}\left|e^{\lambda\omega(x)}x_{n+1}^{r-\alpha}D_x^{r}\phi(x)\right|\\ &\leq \sum_{j=0}^{\alpha}\binom{\alpha}{j}E'_{\beta,j}\left|x_{n+1}^{2j-\alpha}e^{\lambda\omega(x)}D_x^{2j}\phi(x)\right| \leq E'_{\beta,\alpha}\left|x_{n+1}^{\alpha}e^{\lambda\omega(x)}D_x^{2\alpha}\phi(x)\right|.\end{aligned}$$

Hence,

$$\left|e^{\lambda\omega(x)}(\Delta_{W,\beta}^n)_x^\alpha \phi(x)\right| \leq E'_{\beta,\alpha}\left|x_{n+1}^\alpha e^{\lambda\omega(x)}D_x^{2\alpha}\phi(x)\right|.$$

From the above estimate, we define the function space $H_\omega^\beta(A)$ as follows: □

Definition 3.7. *The space $H_\omega^\beta(A)$ is defined as the set of all functions $\phi \in H_\omega(A)$ such that*

$$\gamma_{\lambda,k}^{\omega}(\phi) = \sup_{x\in\mathbb{R}_+^{n+1}} \| e^{\lambda\omega(x)}(\Delta_{W,\beta}^n)_x^k \phi(x) \|_A < \infty, \ \forall \ \lambda, k \ \in \mathbb{N}_0. \tag{14}$$

Theorem 3.8. *For $\omega \in M$, the space $H_\omega^\beta(A)$ is a subspace of $H(A)$.*

Proof. Let $\phi \in H_\omega^\beta(A)$. We need to show that $\gamma_{m,k}(\phi) < \infty$. From (10), we have

$$\sup_{x\in\mathbb{R}_+^{n+1}}\left\|x^m(\Delta_{W,\beta}^n)_x^k\phi(x)\right\|_A = \sup_{x\in\mathbb{R}_+^{n+1}}\left\|x^m e^{\lambda\omega(x)}e^{-\lambda\omega(x)}(\Delta_{W,\beta}^n)_x^k\phi(x)\right\|_A.$$

Since $x^m e^{-\lambda\omega(x)} \leq 1$ for some $\lambda > 0$, it follows that

$$\sup_{x\in\mathbb{R}_+^{n+1}}\left\|x^m(\Delta_{W,\beta}^n)_x^k\phi(x)\right\|_A \leq \sup_{x\in\mathbb{R}_+^{n+1}}\left\|e^{\lambda\omega(x)}(\Delta_{W,\beta}^n)_x^k\phi(x)\right\|_A < \infty.$$

Therefore, we conclude that $\gamma_{m,k}(\phi) < \infty$. This shows that $H_\omega^\beta(A) \subset H(A)$. □

Definition 3.9. *A function $\phi(x)$ belongs to $D_{\mathbb{R}_+^{n+1}}(A)$ if and only if ϕ is defined on $\mathbb{R}_+^{n+1}$, is smooth, and there exists $b \in \mathbb{R}_+^{n+1}$ such that $\phi(x) = 0$ whenever $|x| \geq b$.*

Furthermore, we define

$$\omega D_{\mathbb{R}_+^{n+1}}(A) = D_{\mathbb{R}_+^{n+1}}(A) \cap H_\omega(A). \tag{15}$$

Theorem 3.10. *The subspace $\omega D_{\mathbb{R}_+^{n+1}}(A)$ is dense in $H_\omega(A)$.*

Proof. Let $\theta(x) \in D_{\mathbb{R}_+^{n+1}}(A)$ be a cutoff function defined as

$$\theta(x) = \begin{cases} 1, & 0 < |x| \leq 1, \\ 0, & |x| \geq 2. \end{cases}$$

Now, let $\phi(x) \in H_\omega(A)$ and $\lambda, k \in \mathbb{N}_0^{n+1}$. Then, we have

$$e^{\lambda\omega(x)}(\Delta^n_{W,\beta})^k_x\phi(x)\left[\theta\Big(\frac{x}{\eta}\Big)\phi(x) - \phi(x)\right] = e^{\lambda\omega(x)}(\Delta^n_{W,\beta})^k_x\phi(x)\left[\phi(x)\Big(\theta\Big(\frac{x}{\eta}\Big) - 1\Big)\right].$$

From (9), it follows that

$$\begin{aligned} e^{\lambda\omega(x)}(\Delta^n_{W,\beta})^k_x\phi(x)\left[\theta\Big(\frac{x}{\eta}\Big)\phi(x) - \phi(x)\right] &= e^{\lambda\omega(x)}\sum_{j=1}^{k}\sum_{m=1}^{2j}\sum_{q=0}^{m}\sum_{|\rho'|\leq 2(k-j)}\sum_{\delta_1,\delta_2,\ldots,\delta_n\geq 0}\binom{k}{j}\binom{m}{q}\binom{k-j}{\delta_1,\delta_2,\ldots,\delta_n} \\ &\times\frac{1}{\rho'!}E'_{\beta,m}x_{n+1}^{m-k}\,D_x^{\rho+q}\phi(x)\cdot D_x^{\rho'+2\delta'+m-q}\left[\theta\Big(\frac{x}{\eta}\Big) - 1\right]. \end{aligned}$$

Therefore,

$$\begin{aligned} \sup_{x\in\mathbb{R}_+^{n+1}}\left\|e^{\lambda\omega(x)}(\Delta^n_{W,\beta})^k_x\phi(x)\left[\theta\Big(\frac{x}{\eta}\Big)\phi(x) - \phi(x)\right]\right\|_A &= \sup_{x\in\mathbb{R}_+^{n+1}}\Bigg\|e^{\lambda\omega(x)}\sum_{j=1}^{k}\sum_{m=1}^{2j}\sum_{q=0}^{m}\sum_{|\rho'|\leq 2(k-j)}^{\max}\sum_{\delta_1,\delta_2,\ldots,\delta_n\geq 0}\binom{k}{j}\binom{m}{q} \\ &\times\binom{k-j}{\delta_1,\delta_2,\ldots,\delta_n}\frac{1}{\rho'!}E'_{\beta,m}x_{n+1}^{m-k}\,D_x^{\rho+q}\phi(x)\cdot D_x^{\rho'+2\delta'+m-q}\left[\theta\Big(\frac{x}{\eta}\Big) - 1\right]\Bigg\|_A \\ &\leq \sum_{j=1}^{k}\sum_{m=1}^{2j}\sum_{q=0}^{m}\sum_{|\rho'|\leq 2(k-j)}^{\max}\binom{k}{j}\binom{m}{q}\frac{1}{\rho'!}E'_{\beta,m}\cdot\sup_{x\in\mathbb{R}_+^{n+1}}\left\|e^{\lambda\omega(x)}D_x^{\rho+q}\phi(x)\right\|_A\cdot\sup_{x\geq\eta}\left|\frac{D_x^{\rho'+2\delta'+m-q}\left[\theta\big(\frac{x}{\eta}\big) - 1\right]}{x_{n+1}^{k-m}}\right|. \end{aligned}$$

From (12), we know that $\sup_{x\in\mathbb{R}_+^{n+1}}\left\|e^{\lambda\omega(x)}D_x^{\rho+q}\phi(x)\right\|_A < \infty$. Moreover, since $\theta(x)$ and all its derivatives are bounded, we obtain $\sup_{x\geq\eta}\left|\frac{D_x^{\rho'+2\delta'+m-q}\left[\theta\left(\frac{x}{\eta}\right)-1\right]}{x_{n+1}^{k-m}}\right| \longrightarrow 0$ as $|\eta| \to \infty$, for fixed k and $0 \leq \rho' + 2\delta' + m - q \leq k$.

Hence, we conclude that $\theta\left(\frac{x}{\eta}\right)\phi(x) \longrightarrow \phi(x)$ in $H_\omega(A)$. This shows that the subspace $\omega D_{\mathbb{R}_+^{n+1}}(A)$ is dense in $H_\omega^\beta(A)$. □

Theorem 3.11. *If $\phi \in H_\omega^\beta(A)$ and $\omega \in M$ then the Weinstein transform $\mathcal{F}_w\phi$ is an automorphism on $H_\omega^\beta(A)$.*

Proof. Let $\phi \in H_\omega^\beta(A)$. We aim to show that $\mathcal{F}_w(\phi) \in H_\omega^\beta(A)$. From (14), we obtain

$$\gamma^\omega_{\alpha,\lambda}[\mathcal{F}_w\phi(\xi)] = \sup_{\xi\in\mathbb{R}_+^{n+1}}\left\|e^{\lambda\omega(\xi)}(\Delta^n_{W,\beta})^\alpha_\xi\int_{\mathbb{R}_+^{n+1}}e^{-i\langle\xi',y'\rangle}j_\beta(\xi_{n+1}y_{n+1})\phi(y)\,d\mu_\beta(y)\right\|_A.$$

Using (7), this can be written as

$$\gamma^{\omega}_{\alpha,\lambda}[\mathcal{F}_w\phi(\xi)] = \sup_{\xi\in\mathbb{R}^{n+1}_+}\left\| e^{\lambda\omega(\xi)}\int_{\mathbb{R}^{n+1}_+} e^{-i\langle\xi',y'\rangle} j_\beta(\xi_{n+1}y_{n+1})(-\|y\|^2)^\alpha\phi(y)\,d\mu_\beta(y)\right\|_A.$$

Now, setting $f(y) = (-\|y\|^2)^\alpha\phi(y)$, the above expression yields

$$\gamma^{\omega}_{\alpha,\lambda}[\mathcal{F}_w\phi(\xi)] \leq \sup_{\xi\in\mathbb{R}^{n+1}_+}\left\| e^{\lambda|\omega(\xi)|}\int_{\mathbb{R}^{n+1}_+} e^{-i\langle\xi',y'\rangle} j_\beta(\xi_{n+1}y_{n+1})f(y)\,d\mu_\beta(y)\right\|_A.$$

From [21], for every $\epsilon > 0$ there exists a constant $C(\epsilon)$ such that

$$|\omega(\xi)| \leq \epsilon\|\xi\|^2 + C(\epsilon).$$

Therefore, we obtain

$$\begin{aligned}\gamma^{\omega}_{\alpha,\lambda}[\mathcal{F}_w\phi(\xi)] &\leq \sup_{\xi\in\mathbb{R}^{n+1}_+}\left\| e^{\lambda(\epsilon\|\xi\|^2+C(\epsilon))}\int_{\mathbb{R}^{n+1}_+} e^{-i\langle\xi',y'\rangle} j_\beta(\xi_{n+1}y_{n+1})f(y)\,d\mu_\beta(y)\right\|_A\\ &\leq e^{\lambda C(\epsilon)}\sup_{\xi\in\mathbb{R}^{n+1}_+}\left\|\sum_{m=0}^{\infty}\frac{(\lambda\epsilon)^m}{m!}\|\xi\|^{2m}\int_{\mathbb{R}^{n+1}_+} e^{-i\langle\xi',y'\rangle} j_\beta(\xi_{n+1}y_{n+1})f(y)\,d\mu_\beta(y)\right\|_A\\ &= e^{\lambda C(\epsilon)}\sum_{m=0}^{\infty}\frac{(\lambda\epsilon)^m}{m!}\sup_{\xi\in\mathbb{R}^{n+1}_+}\left\|\int_{\mathbb{R}^{n+1}_+}(\Delta^n_{W,\beta})^m_y\left(e^{-i\langle\xi',y'\rangle} j_\beta(\xi_{n+1}y_{n+1})\right)f(y)\,d\mu_\beta(y)\right\|_A.\end{aligned}$$

By applying integration by parts, we further deduce

$$\begin{aligned}\gamma^{\omega}_{\alpha,\lambda}[\mathcal{F}_w\phi(\xi)] &\leq e^{\lambda C(\epsilon)}\sum_{m=0}^{\infty}\frac{(\lambda\epsilon)^m}{m!}\sup_{\xi\in\mathbb{R}^{n+1}_+}\left\|(-1)^m\int_{\mathbb{R}^{n+1}_+} e^{-i\langle\xi',y'\rangle} j_\beta(\xi_{n+1}y_{n+1})(\Delta^n_{W,\beta})^m_y f(y)\,d\mu_\beta(y)\right\|_A\\ &\leq e^{\lambda C(\epsilon)}\sum_{m=0}^{\infty}\frac{(\lambda\epsilon)^m}{m!}p_{m,\lambda}(f(y))\int_{\mathbb{R}^{n+1}_+} e^{-\lambda\omega(y)}\,d\mu_\beta(y) < \infty.\end{aligned}$$

Thus, $\mathcal{F}_w(\phi) \in H_\omega(A)$. Moreover, by the inversion formula for the Weinstein transform, we have $\mathcal{F}_w^{-1}\phi(y) = \mathcal{F}_w\phi(-y)$. Hence $\mathcal{F}_w$ is injective, and consequently an automorphism. □

Definition 3.12. *We denote by $\omega D_{\mathbb{R}^{n+1}_+}(A)\circledast(A)$ the linear space consisting of all functions $\phi \in \omega D_{\mathbb{R}^{n+1}_+}(A)$ that can be represented in the form $\phi = \sum_k a_k h_k$, where $a_k \in \omega D_{\mathbb{R}^{n+1}_+}(A)$, $h_k \in A$, and the summation is taken over a finite number of terms.*

Theorem 3.13. *The space $\omega D_{\mathbb{R}^{n+1}_+}(A)\circledast(A)$ is dense in $H_\omega(A)$.*

Proof. Let $\theta(x)$ denote the function defined in Theorem 3.10. For $\phi \in \omega D_{\mathbb{R}^{n+1}_+}(A)$, it suffices to show that

$$\theta\left(\tfrac{x}{\eta}\right)\mathcal{F}_w(\phi) \longrightarrow \mathcal{F}_w(\phi) \quad \text{as } \eta\to\infty \text{ in } H^\beta_\omega(A).$$

Consider $\phi(x) \in H_\omega(A)$ and $\lambda, k \in \mathbb{N}^{n+1}_0$. By (9), we obtain

$$\begin{aligned}e^{\lambda\omega(x)}(\Delta^n_{W,\beta})^k_x\left[\theta\left(\tfrac{x}{\eta}\right)(\mathcal{F}_w\phi)(x) - (\mathcal{F}_w\phi)(x)\right] &= e^{\lambda\omega(x)}\sum_{j=1}^{k}\sum_{m=1}^{2j}\sum_{q=0}^{m}\sum_{|\rho'|\leq 2(k-j)}\sum_{\delta_1,\dots,\delta_n\geq 0}\binom{k}{j}\binom{m}{q}\binom{k-j}{\delta_1,\dots,\delta_n}\frac{1}{\rho'!}\\ &\quad\times E'_{\beta,m}\,x^{m-k}_{n+1}D^{\rho+q}_x(\mathcal{F}_w\phi)(x)\cdot D^{\rho'+2\delta'+m-q}_x\left[\theta\left(\tfrac{x}{\eta}\right)-1\right].\end{aligned}$$

Therefore,

$$\sup_{x\in\mathbb{R}_+^{n+1}}\left\|e^{\lambda\omega(x)}(\Delta_{W,\beta}^n)_x^k\Big[\theta\Big(\tfrac{x}{\eta}\Big)(\mathcal{F}_w\phi)(x)-(\mathcal{F}_w\phi)(x)\Big]\right\|_A \le \sum_{j=1}^{k}\sum_{m=1}^{2j}\sum_{q=0}^{m}\sum_{|\rho'|\le 2(k-j)}\binom{k}{j}\binom{m}{q}\frac{1}{\rho'!}E'_{\beta,m}$$
$$\times \sup_{x\in\mathbb{R}_+^{n+1}}\left\|e^{\lambda\omega(x)}D_x^{\rho+q}(\mathcal{F}_w\phi)(x)\right\|_A \sup_{x\ge\eta}\left|\frac{D_x^{\rho'+2\delta'+m-q}\Big[\theta\Big(\frac{x}{\eta}\Big)-1\Big]}{x_{n+1}^{k-m}}\right|.$$

Since $\sup_{x\ge\eta}\left|\frac{D_x^{\rho'+2\delta'+m-q}\left[\theta\left(\frac{x}{\eta}\right)-1\right]}{x_{n+1}^{k-m}}\right| \longrightarrow 0$ as $\eta\to\infty$, for fixed k and $0\le\rho'+2\delta'+m-q\le k$, we conclude that $\theta\Big(\frac{x}{\eta}\Big)\mathcal{F}_w(\phi)\longrightarrow\mathcal{F}_w(\phi)$ in $H_\omega(A)$ as $\eta\to\infty$. This completes the proof. □

Theorem 3.14. *Let $\omega\in M$. Then $H_\omega^\beta(A)$ is a topological algebra with respect to pointwise multiplication.*

Proof.

Proof. Let $\phi,\psi\in H_\omega(A)$. From (14), we obtain

$$\gamma_{\lambda,k}^{\omega}(\phi\psi)=\sup_{x\in\mathbb{R}_+^{n+1}}\left\|e^{\lambda\omega(x)}(\Delta_{W,\beta}^n)_x^k(\phi\psi)(x)\right\|_A. \tag{16}$$

Applying (9) to the expression in (16), we deduce

$$\gamma_{\lambda,k}^{\omega}(\phi\psi)=\sup_{x\in\mathbb{R}_+^{n+1}}e^{\lambda\omega(x)}\Bigg\|\sum_{j=0}^{k}\sum_{r=1}^{2j}\sum_{q=0}^{r}\sum_{|\rho'|\le2(k-j)}\sum_{\delta_1,\delta_2,\dots,\delta_n\ge0}\binom{k}{j}\binom{r}{q}\binom{k-j}{\delta_1,\delta_2,\dots,\delta_n}\frac{1}{\rho'!}E'_{\beta,r}\,x_{n+1}^{r-k}$$
$$\times\Big(D_x^{\rho'+q}\phi(x)\Big)\cdot\Big(D_x^{\rho'+2\delta'+r-q}\psi(x)\Big)\Bigg\|_A$$
$$\le\sum_{j=0}^{k}\sum_{r=1}^{2j}\sum_{q=0}^{r}\sum_{|\rho'|\le2(k-j)}\sum_{\delta_1,\delta_2,\dots,\delta_n\ge0}\binom{k}{j}\binom{r}{q}\binom{k-j}{\delta_1,\delta_2,\dots,\delta_n}\frac{1}{\rho'!}E'_{\beta,r}$$
$$\times\sup_{x\in\mathbb{R}_+^{n+1}}e^{\frac{\lambda\omega(x)}{2}}\left|D_x^{\rho'+q}\phi(x)\right|\cdot\sup_{x\in\mathbb{R}_+^{n+1}}e^{\frac{\lambda+2k}{2}\omega(x)}\left|D_x^{\rho'+2\delta'+r-q}\psi(x)\right|.$$

In view of (10) and (13), it follows that

$$\gamma_{\lambda,k}^{\omega}(\phi\psi)\le\sum_{j=0}^{k}\sum_{r=1}^{2j}\sum_{q=0}^{r}\sum_{|\rho'|\le2(k-j)}\sum_{\delta_1,\delta_2,\dots,\delta_n\ge0}\binom{k}{j}\binom{r}{q}\binom{k-j}{\delta_1,\delta_2,\dots,\delta_n}p_{\rho'+q,\frac{\lambda}{2}}(\phi)\,p_{\rho'+2\delta'+r-q,\frac{\lambda+2k}{2}}(\psi)<\infty.$$

Thus, we conclude that $\gamma_{\lambda,k}^{\omega}(\phi\psi)<\infty$. This completes the proof. □

Theorem 3.15. *Let $\omega\in M$. Then $H_\omega^\beta(A)$ is a topological algebra with respect to convolution.*

Proof. Let $\phi,\psi\in H_\omega(A)$. By definition, we have

$$\gamma_{\lambda,k}^{\omega}(\phi\#_\beta\psi)=\sup_{x\in\mathbb{R}_+^{n+1}}e^{\lambda\omega(x)}\left\|(\Delta_{W,\beta}^n)_x^k\Big(\phi\#_\beta\psi\Big)(x)\right\|_A. \tag{17}$$

From [29, p. 21] and by applying (5), it follows that

$$\left\|(\phi\#_\beta(\Delta_{W,\beta}^n)_x^k\psi)(x)\right\|_A=\left\|\int_{\mathbb{R}_+^{n+1}}\phi(y)\,(\Delta_{W,\beta}^n)_x^k\psi(x,y)\,d\mu_\beta(y)\right\|_A. \tag{18}$$

In view of (3), we obtain

$$\left\|(\phi\#_\beta(\Delta^n_{W,\beta})^k_x\psi)(x)\right\|_A = \left\|\int_{\mathbb{R}^{n+1}_+}\phi(y)\left(\int_{\mathbb{R}^{n+1}_+}\psi(z)\,(\Delta^n_{W,\beta})^k_x D_\beta(x,y,z)\,d\mu_\beta(z)\right)d\mu_\beta(y)\right\|_A.$$

Using (4), this expression becomes

$$\left\|(\phi\#_\beta(\Delta^n_{W,\beta})^k_x\psi)(x)\right\|_A = \left\|\int_{\mathbb{R}^{n+1}_+}\phi(y)\left(\int_{\mathbb{R}^{n+1}_+}\psi(z)\left(\int_{\mathbb{R}^{n+1}_+}e^{-i\langle x',\xi'\rangle}J_\beta(x_{n+1}\xi_{n+1})e^{i\langle y',\xi'\rangle}J_\beta(y_{n+1}\xi_{n+1})\right.\right.\right.$$
$$\left.\left.\left.\times\, e^{-i\langle z',\xi'\rangle}J_\beta(z_{n+1}\xi_{n+1})\,d\mu_\beta(\xi)\right)d\mu_\beta(z)\right)d\mu_\beta(y)\right\|_A.$$

Applying the supremum bound, we obtain

$$\left\|(\phi\#_\beta(\Delta^n_{W,\beta})^k_x\psi)(x)\right\|_A \le \sup_{z\in\mathbb{R}^{n+1}_+}\left\|(\Delta^n_{W,\beta})^k_z\psi(z)\right\|_A\cdot\left\|\int_{\mathbb{R}^{n+1}_+}\phi(y)\left(\int_{\mathbb{R}^{n+1}_+}D_\beta(x,y,z)\,d\mu_\beta(z)\right)d\mu_\beta(y)\right\|_A.$$

Substituting this estimate into (18), we get

$$\begin{aligned}\gamma^\omega_{\lambda,k}(\phi\#_\beta\psi) &\le \sup_{z\in\mathbb{R}^{n+1}_+}e^{\lambda\omega(z)}\left\|(\Delta^n_{W,\beta})^k_z\psi(z)\right\|_A\cdot\left\|\int_{\mathbb{R}^{n+1}_+}\phi(y)\,d\mu_\beta(y)\right\|_A\\ &\le \gamma^\omega_{\lambda,k}(\psi)\cdot\sup_{y\in\mathbb{R}^{n+1}_+}e^{l\omega(y)}\left\|\phi(y)\right\|_A\int_{\mathbb{R}^{n+1}_+}e^{-l\omega(y)}\,d\mu_\beta(y).\end{aligned}$$

This yields

$$\gamma^\omega_{\lambda,k}(\phi\#_\beta\psi) \le \gamma^\omega_{\lambda,k}(\psi)\,\gamma^\omega_{l,0}(\phi)\int_{\mathbb{R}^{n+1}_+}e^{-l\omega(y)}\,d\mu_\beta(y) < \infty.$$

Hence, the desired result follows. □

4. Conclusion

The Weinstein transform has useful tool and strong calculus which is applicable for the various problems image processing, signal processing, wavelets, quantum calculus, and other areas of mathematical sciences. Utilizing the aforesaid theory, many important research work have been done by many authors [3, 5, 6, 14, 16] and [17, 25, 26, 29, 30]. The theory of ultradistributions is the origination of many research works. Exploiting the theory of ultradistributions, the characterizations of functional space are investigated by many authors [12, 19]. Later on, the problems of pseudo-differential operators and wavelet transform have been done by the authors see [20, 23].
Motivated from the above result, the authors introduced the space $H_\omega(A)$ and by utilizing the theory of the Weinstein transform studied its various properties.

For applications point of view Beurling type ultradistributions associated with the Weinstein transform provide a framework for defining and studying various properties of pseudo-differential operators of Banach space-valued functions associated with certain symbol. Exploiting the theory of pseudo-differential operators Sobolev spaces with Banach space-valued functions can be produced and studied many properties. The analysis of wavelet transforms with Banach space-valued functions as input or output can be found by using afoersaid theory.

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