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On bivariate Fibonacci-Fubini polynomials

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Abstract. Recently, Fibonacci extensions of several special polynomials, such as Fibonacci-Hermite, Fibonacci-Euler, Fibonacci-Genocchi, Fibonacci-Bernoulli, and Fibonacci-Bernstein polynomials, have been considered, and diverse of their properties and relations have been thoroughly analyzed using the content of Golden calculus. In this paper, we first define the generating function of the bivariate Fibonacci-Fubini polynomials and derive some valuable relations and properties. These involve summation formulas, addition formulas, golden derivative property, and golden integral representation for the bivariate Fibonacci-Fubini polynomials. We also provide implicit summation formulas and a symmetric identity for these polynomials. Moreover, we investigate multifarious correlations and formulas for the bivariate Fibonacci-Fubini polynomials associated with the Fibonacci-Euler polynomials, the Fibonacci-Bernoulli polynomials, and the Fibonacci-Stirling polynomials of the second kind. Lastly, we provide a Fibonacci differential operator formula for the bivariate Fibonacci-Fubini polynomials.

1. Introduction

The Fibonacci number sequence, taking its name from Leonardo Fibonacci (1170-1250), is defined, for $n \geq 2$, by the following recurrence relation

$$F_{n+1} = F_n + F_{n-1}$$

with primary conditions $F_0 = 0$ and $F_1 = 1$. The first few terms of this sequence are $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55,$ $89, \ldots$ (cf. [3–6, 13–17, 19–25, 33]). The Binet formula of the Fibonacci sequence is as follows:

$$F_n = \frac{\varphi^n - \varphi'^n}{\varphi - \varphi'}, \tag{1}$$

where $\varphi' = \frac{1-\sqrt{5}}{2} \approx -0,618033\ldots$ termed the Silver ratio and $\varphi = \frac{1+\sqrt{5}}{2} \approx 1,618033\ldots$ termed the Golden ratio or Golden proportion, which also satisfies $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi$. It appears in architecture, art, geometry, and varies from sunflowers to human body proportions. Due to numerous applications in mathematics, art, and science, Fibonacci numbers are also called "Nature's Perfect numbers".

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Golden calculus (or, say, F-calculus) can be described as properties and applications of Fibonacci numbers. The main component of this calculus is the Golden derivative in (9) as a finite difference derivative with the Golden and Silver ratios as bases, introduced by Pashaev and Nalci [22]. After defining the Golden derivative, the Golden binomials and Taylor expansion in terms of the Golden binomials have been constructed in [22]. Then, according to this expansion, the Golden exponential functions have been introduced, and trigonometric and hyperbolic Golden functions, as solutions of the Golden ordinary differential equation and partial differential equation, have been considered in [22, 23]. Recently [19–21], the Golden calculus was generalized to higher order Golden calculus by introducing higher order Golden Fibonacci derivatives, by which the generating functions for higher order Fibonacci numbers, higher Fibonomials, and higher Golden binomials have been derived. Also, it is shown that the higher Golden binomials are equivalent to Carlitz's characteristic polynomials for combinatorial matrices, cf. [19–21]. Moreover, utilizing the Golden exponential function in (5), the Fibonacci-Bernoulli polynomials were defined in (14), and several properties, relations, and applications were provided. Also, the roots of Fibonacci-Bernoulli polynomials were analyzed in [25]. After the definition and examinations of the Fibonacci-Bernoulli polynomials [17, 19–21, 25, 33], Fibonacci-type special polynomials, such as Fibonacci-Euler polynomials in (cf. [33]), Fibonacci-Apostol-Euler polynomials in (cf. [6, 14]), Fibonacci-Apostol-Bernoulli polynomials in (cf. [6, 14]), Fibonacci-Apostol-Genocchi polynomials in (cf. [6, 14]), Fibonacci-sigmoid polynomials in (cf. [13]), Fibonacci-Hermite polynomials in (cf. [3]), Fibonacci-Bell polynomials in (cf. [5]), Fibonacci-Bernstein polynomials in (cf. [3]), Fibonacci-harmonic polynomials in (cf. [33]), and Fibonacci-Stirling polynomials of the second kind in (cf. [5]), have recently been considered and some of their properties and applications have been investigated by many mathematicians, cf. [3–5, 13, 14, 17, 19–21, 25, 33] and also see the references cited therein.

For more information on Golden calculus, see [3–5, 13–17, 19–25, 33].

2. Preliminaries

Along the paper, let $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of all natural numbers, and $\mathbb{N} \cup \{0\}$, respectively. Here, we provide some content of F-calculus taken from [3–5, 13–17, 19–25, 33].

Fibonomials (Golden binomial coefficients) are defined, for $1 \leq k \leq n$, as follows

$$\binom{n}{k}_F = \frac{F_n!}{F_k! F_{n-k}!}, \tag{2}$$

where F-factorial (or, say, Golden factorial) is given as

$$F_n! := F_n F_{n-1} F_{n-2} ... F_2 F_1, \tag{3}$$

with $F_0! := 1$. Here $\binom{n}{0}_F = 1$ and $\binom{n}{k}_F = 0$ for $k > n$. Many properties of $\binom{n}{k}_F$ are listed in [21].

F-analog of $(x+y)^n$ (Golden binomial theorem) is provided, for $n \in \mathbb{N}$, by (cf. [20])

$$(x+y)_F^n = \sum_{k=0}^{n} \binom{n}{k}_F (-1)^{\binom{k}{2}} x^k y^{n-k} \tag{4}$$

with $(x+y)_F^0 := 1$. Also, it is noted that $(x+y)_F := (x+y)_F^1$ and $(x-y)_F^n := (x+(-y))_F^n$.

The F-analogues of exponential function (say, also the golden exponential functions) are defined by (cf. [3–5, 13–17, 19–22, 25, 33])

$$e_F^t = \sum_{n=0}^{\infty} \frac{t^n}{F_n!}, \tag{5}$$

and

$$E_F^t = \sum_{n=0}^{\infty} (-1)^{\binom{n}{2}} \frac{t^n}{F_n!}. \tag{6}$$

It is noted that

$$e_F^{xt}E_F^{yt} = e_F^{(x+y)_F t} \text{ and } e_F^{xt}e_F^{yt} = e_F^{(x+_F y)t}, \tag{7}$$

where (cf. [33])

$$(x +_F y)^n = \sum_{k=0}^{n} \binom{n}{k}_F x^{n-k}y^k = \sum_{k=0}^{n} \binom{n}{k}_F x^k y^{n-k}. \tag{8}$$

Also, note that $(x -_F y)^n := (x +_F (-y))^n$.

The golden F-derivative operator is defined, for $f(x)$ being any function, by (cf. [22])

$$D_F^x [f(x)] = \frac{f(\varphi x) - f(\varphi' x)}{x(\varphi - \varphi')}, \tag{9}$$

which is linear, namely, it satisfies for a, b being two scalars and $f(x), g(x)$ being any functions,

$$D_F^x [af(x) + bg(x)] = aD_F^x [f(x)] + bD_F^x [g(x)].$$

It can be readily seen from (5), (6), and (9) that

$$D_F^x [x^n] = F_n x^{n-1}, D_F^x \left[e_F^{xt}\right] = te_F^{xt} \text{ and } D_F^x \left[E_F^{xt}\right] = tE_F^{-xt}. \tag{10}$$

Also, the multiplication rule and quotient rule, for $f(x)$ and $g(x)$, of the golden derivative, are given as

$$D_F^x [f(x) g(x)] = g(\varphi x) D_F^x [f(x)] + f(\varphi' x) D_F^x [g(x)] \tag{11}$$

and

$$D_F^x \left[\frac{f(x)}{g(x)}\right] = \frac{g(\varphi x) D_F^x [f(x)] - f(\varphi' x) D_F^x [g(x)]}{g(\varphi x) g(\varphi' x)}. \tag{12}$$

The fibonomial convolution of two sequences is defined by Krot [16] as follows

$$c_n = a_n * b_n = \sum_{k=0}^{n} \binom{n}{k}_F a_k b_{n-k},$$

where a_n and b_n are two sequences provided by

$$A_F(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{F_n!} \text{ and } B_F(t) = \sum_{n=0}^{\infty} b_n \frac{t^n}{F_n!},$$

therefore, it can be written that

$$C_F(t) = A_F(t) B_F(t) = \sum_{n=0}^{\infty} c_n \frac{t^n}{F_n!}, \tag{13}$$

which resembles the golden form of the usual Cauchy product and will be commonly used in the proofs of some theorems in the following sections.

Using (5), the Fibonacci-Bernoulli polynomials and the Fibonacci-Euler polynomials are introduced as follows:

$$\sum_{n=0}^{\infty} B_{n,F}(x) \frac{t^n}{F_n!} = \frac{te_F^{xt}}{e_F^t - 1} \tag{14}$$

and

$$\sum_{n=0}^{\infty} E_{n,F}(x) \frac{t^n}{F_n!} = \frac{2e_F^{xt}}{e_F^t + 1}. \tag{15}$$

The numbers of $B_{n,F}(x)$ and $E_{n,F}(x)$ are determined as $B_{n,F}(0) : B_{n,F}$ and $E_{n,F}(0) := E_{n,F}$, respectively. Several properties and relations of the polynomials in (14) and (15) have been derived and examined in [14, 17, 19–21, 24, 25, 33].

The Fibonacci-Stirling numbers $S_{2,F}(n,k)$ and polynomials $S_{2,F}(n,k:x)$ of the second kind are introduced as follows:

$$\sum_{n=k}^{\infty} S_{2,F}(n,k) \frac{t^n}{F_n!} = \frac{\left(e_F^t - 1\right)^k}{F_k!} \text{ and } \sum_{n=k}^{\infty} S_{2,F}(n,k:x) \frac{t^n}{F_n!} = \frac{\left(e_F^t - 1\right)^k}{F_k!} e_F^{tx}. \tag{16}$$

It can be readily seen from (16) that $S_{2,F}(n,k:0) = S_{2,F}(n,k)$. Numerous formulas and relations for the numbers $S_{2,F}(n,k)$ and polynomials $S_{2,F}(n,k:x)$ have been investigated in [5].

The bivariate Fibonacci-Bell polynomials are introduced via the golden exponential function (cf. [5]):

$$\sum_{n=0}^{\infty} \phi_{n,F}(x,y) \frac{t^n}{F_n!} = e_F^{x\left(e_F^t - 1\right)} e_F^{yt}. \tag{17}$$

Choosing $y = 0$ in (17), the polynomials $\phi_{n,F}(x,y)$ become the Fibonacci-Bell polynomials $\phi_{n,F}(x)$ as follows:

$$\sum_{n=0}^{\infty} \phi_{n,F}(x) \frac{t^n}{F_n!} = e_F^{x\left(e_F^t - 1\right)}. \tag{18}$$

Changing $x = 1 = y+1$ in (17), the polynomials $\phi_{n,F}(x,y)$ become the Fibonacci-Bell numbers $\phi_{n,F}$ as follows:

$$\sum_{n=0}^{\infty} \phi_{n,F} \frac{t^n}{F_n!} = e_F^{\left(e_F^t - 1\right)}. \tag{19}$$

Multifarious summation formulas, addition formulas, golden derivative properties, golden integral representation, symmetric identity, and implicit summation formulas for bivariate Fibonacci-Bell polynomials have been analyzed and derived in [5].

Recently, Duran and Acikgoz [3] defined the Fibonacci-Hermite polynomials $H_{n,F}(x)$ using the golden exponential function as follows:

$$\sum_{n=0}^{\infty} H_{n,F}(x) \frac{t^n}{F_n!} = e_F^{2tx} e_F^{-t^2}. \tag{20}$$

Several properties of the Fibonacci-Hermite polynomials $H_{n,F}(x)$ are provided in [3]. In addition, after giving some properties of the Fibonacci-Bernstein polynomials, a correlation between the Fibonacci-Hermite polynomials and the Fibonacci-Bernstein polynomials has been derived in [3]. Furthermore, in [3], the content of the Golden integral is reviewed and diverse definitions and properties are analyzed. Some of them are given as follows:

The definite Golden integral of $f(x)$ is defined as (cf. [3, 33])

$$\int_0^b f(x)\, d_F(x) = (\varphi - \varphi')\, b \sum_{n=0}^{\infty} \frac{\varphi'^n}{\varphi^{n+1}} f\left(\frac{\varphi'^n}{\varphi^{n+1}} b\right) \tag{21}$$

with

$$\int_a^b f(x)\, d_F(x) = \int_0^b f(x)\, d_F(x) - \int_0^a f(x)\, d_F(x), \tag{22}$$

which is always convergent because of $\left|\frac{\varphi'}{\varphi}\right| < 1$ for $a,b \in \mathbb{R}$ with $a < b$.

Let $g'(y)$ denotes the ordinary derivative of $g(y)$. I is obtained from (21) that

$$\int_a^b D_F^x [g(y)]\, d_F(y) = g(b) - g(a), \tag{23}$$

if $g(y)$ exists in a neighborhood of $y = 0$ and is continuous at $y = 0$.

Special polynomials with several extensions or versions possess many applications in probability, combinatorics, quantum calculus, elementary and analytic number theory, physics, and so on, cf. [1-33]. In this study, we consider a novel family of certain special polynomials with a Fibonacci calculus viewpoint, namely bivariate Fibonacci-Fubini polynomials, and analyze many of their properties.

3. Bivariate Fibonacci-Fubini Polynomials

We begin by recalling the usual two-variable Fubini polynomials as follows (cf. [1, 2, 7–9, 11, 12, 18, 26, 27, 30–32])

$$\sum_{n=0}^{\infty} w_n(x,y)\frac{t^n}{n!} = \frac{e^{xt}}{1-y(e^t-1)}. \tag{24}$$

Upon setting $x = 0$ in (24), we get the usual Fubini polynomials as follows (cf. [1, 2, 7–9, 11, 12, 18, 26, 27, 30–32])

$$\sum_{n=0}^{\infty} w_n(y)\frac{t^n}{n!} = \frac{1}{1-y(e^t-1)}. \tag{25}$$

It is obvious that for $n \in \mathbb{N}_0$ (cf. [7]):

$$w_n\left(x, -\frac{1}{2}\right) = E_n(x), \quad w_n\left(-\frac{1}{2}\right) = E_n \tag{26}$$

and (cf. [2, 7–9, 18, 27])

$$w_n(y) = \sum_{\mu=0}^{n} S_2(n,\mu)\,\mu!\,y^{\mu}, \tag{27}$$

where $E_n(x)$ denotes the usual nth Euler polynomial, E_n denotes the usual nth Euler number, and $S_2(n,\mu)$ denote the classical Stirling numbers of the second kind, defined by, respectively, cf. [2, 7–10, 18, 26, 27, 30, 31]:

$$\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} = \frac{2e^{xt}}{e^t+1}, \quad \sum_{n=0}^{\infty} E_n\frac{t^n}{n!} = \frac{2}{e^t+1} \text{ and } \sum_{n=0}^{\infty} S_2(n,\mu)\frac{t^n}{n!} = \frac{(e^t-1)^{\mu}}{\mu!}.$$

The familiar Fubini numbers can be generated by choosing $y = 1$ in (25), that is, $w_n(1) := w_n$ given by (*cf.* [1, 2, 7–9, 11, 12, 18, 26, 27, 30–32])

$$\sum_{n=0}^{\infty} w_n\frac{t^n}{n!} = \frac{1}{2-e^t}. \tag{28}$$

See the references [1, 2, 7–9, 11, 12, 18, 26, 27, 30–32] for the applications and analysis of $w_n(x,y)$.

We here put our main definition.

Definition 3.1. *The bivariate Fibonacci-Fubini polynomials are introduced as follows:*

$$\sum_{n=0}^{\infty} w_{n,F}(x,y)\frac{t^n}{F_n!} = \frac{e_F^{xt}}{1-y\left(e_F^t-1\right)}. \tag{29}$$

Taking $x = 0$ in (29) gives a new type of the Fubini polynomials, which we call the Fibonacci-Fubini polynomials:

$$\sum_{n=0}^{\infty} w_{n,F}(y)\frac{t^n}{F_n!} = \frac{1}{1-y\left(e_F^t-1\right)}. \tag{30}$$

Choosing $y = 1$ in (29) yields another new type of the Fubini polynomials:

$$\sum_{n=0}^{\infty} w_{n,F}(x)\frac{t^n}{F_n!} = \frac{e_F^{xt}}{2+e_F^t}. \tag{31}$$

Letting $x = 0$ and $y = 1$ in (29) provides the Fibonacci-Fubini numbers $w_{n,F}$:

$$\sum_{n=0}^{\infty} w_{n,F}\frac{t^n}{F_n!} = \frac{1}{2+e_F^t}. \tag{32}$$

Here are some identities, formulas, and relations for the polynomials $w_n(x,y)$.

Theorem 3.2. *The following summation formula holds for $n \in \mathbb{N}_0$:*

$$w_{n,F}(x,y) = \sum_{k=0}^{n}\binom{n}{k}_F w_{k,F}(y)\,x^{n-k}. \tag{33}$$

Proof. By (29), using (13), we observe that

$$\begin{aligned}\sum_{n=0}^{\infty} w_{n,F}(x,y)\frac{t^n}{F_n!} &= \frac{e_F^{xt}}{1-y\left(e_F^t-1\right)}\\ &= \sum_{n=0}^{\infty} w_{n,F}(y)\frac{t^n}{F_n!}\sum_{n=0}^{\infty} x^n\frac{t^n}{F_n!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}_F w_{k,F}(y)\,x^{n-k}\frac{t^n}{F_n!},\end{aligned}$$

which gives the claimed formula (33). □

Theorem 3.3. *The following summation formulas are valid for $n \in \mathbb{N}_0$.*

$$w_{n,F}(x_1 +_F x_2, y) = \sum_{k=0}^{n}\binom{n}{k} w_{k,F}(x_1,y)\,x_2^{n-k} \tag{34}$$

and

$$w_{n,F}(x_1 +_F x_2, y) = \sum_{k=0}^{n}\binom{n}{k} w_{k,F}(y)\,(x_1 +_F x_2)^{n-k}. \tag{35}$$

Proof. From (29) and (30), we get

$$\begin{aligned}\sum_{n=0}^{\infty} w_{n,F}\left(x_1 +_F x_2, y\right) \frac{t^n}{F_n!} &= \frac{e_F^{(x_1 +_F x_2)t}}{1 - y\left(e_F^t - 1\right)} \\ &= \frac{e_F^{x_1 t}}{1 - y\left(e_F^t - 1\right)} e_F^{x_2 t} \\ &= \sum_{n=0}^{\infty} w_{n,F}\left(x_1, y\right) \frac{t^n}{F_n!} \sum_{n=0}^{\infty} {x_2}^n \frac{t^n}{F_n!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k}_F w_{k,F}\left(x_1, y\right) x_2^{n-k} \frac{t^n}{F_n!}\end{aligned}$$

and similarly

$$\begin{aligned}\sum_{n=0}^{\infty} w_{n,F}\left(x_1 +_F x_2, y\right) \frac{t^n}{F_n!} &= \frac{e_F^{(x_1 +_F x_2)t}}{1 - y\left(e_F^t - 1\right)} \\ &= \sum_{n=0}^{\infty} w_{n,F}\left(y\right) \frac{t^n}{F_n!} \sum_{n=0}^{\infty} \left(x_1 +_F x_2\right)^n \frac{t^n}{F_n!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k}_F w_{k,F}\left(y\right) \left(x_1 +_F x_2\right)^{n-k} \frac{t^n}{F_n!},\end{aligned}$$

which yield the desired results (34) and (35). □

Theorem 3.4. *(Golden derivative property) We have*

$$D_F^x \left[w_{n,F}\left(x, y\right)\right] = F_n w_{n-1,F}\left(x, y\right). \tag{36}$$

Proof. Applying D_F^x (9) to both sides of (29) with respect to x and utilizing (10), we attain

$$\sum_{n=0}^{\infty} D_F^x \left[w_{n,F}\left(x, y\right)\right] \frac{t^n}{F_n!} = \frac{D_F^x\left[e_F^{xt}\right]}{1 - y\left(e_F^t - 1\right)} = t \frac{e_F^{xt}}{1 - y\left(e_F^t - 1\right)}.$$

Thus, we get the alleged result in the theorem. □

The direct result of Theorem 3.4 is given for $m < n$ as follows:

$$\left(D_F^{x,}\right)^{(m)} \left[w_{n,F}\left(x, y\right)\right] = (n)_{F,m}\, w_{n-m,F}\left(x, y\right),$$

where we define the Fibonnaci falling factorial $(n)_{F,m}$ given by $(n)_{F,m} = F_n F_{n-1} F_{n-2} \cdots F_{n-m+1}$ with $(n)_{F,0} = 1$ and $D_F^{x,(m)}$ is mth derivative operator, namely $\left(D_F^x\right)^{(m)} = \left(D_F^x\right)^{(m-1)} D_F^x$.

Theorem 3.5. *(Golden integral representation) We have*

$$\int_a^b w_{n,F}\left(x, y\right) d_F\left(x\right) = \frac{w_{n+1,F}\left(b, y\right) - w_{n+1,F}\left(a, y\right)}{F_{n+1}}.$$

Proof. By (23) and (36), we acquire

$$\begin{aligned}\int_a^b w_{n,F}(x,y)\, d_F(y) &= \frac{1}{F_{n+1}} \int_a^b D_F^x \left[w_{n+1,F}(x,y)\right] d_F(y) \\ &= \frac{w_{n+1,F}(b,y) - w_{n+1,F}(a,y)}{F_{n+1}},\end{aligned}$$

which ends the proof. □

The following series of equalities are correct (cf. [5]):

$$\sum_{M=0}^{\infty} g(M) \frac{(x_1 + x_2)^M}{F_M!} = \sum_{l,k=0}^{\infty} g(l+k) \frac{x_1^l}{F_l!} \frac{x_2^k}{F_k!} \tag{37}$$

and

$$\sum_{n,m=0}^{\infty} B(m,n) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} B(m, n-m). \tag{38}$$

Theorem 3.6. *(Implicit summation formula) We have*

$$w_{k+l,F}(x,y) = \sum_{n,m=0}^{k,l} \binom{k}{n}_F \binom{l}{m}_F w_{k+l-n-m,F}(z,y) (x -_F z)^{n+m}. \tag{39}$$

Proof. Changing t by $t + u$ in (17), we get

$$\frac{1}{1 - y\left(e_F^t - 1\right)} = e_F^{-(t+u)z} \sum_{l,k=0}^{\infty} w_{k+l,F}(z,y) \frac{t^k}{F_k!} \frac{u^l}{F_l!}.$$

Again, changing z by x in the previous equality, and utilizing (37), it is obtained that

$$\frac{1}{1 - y\left(e_F^t - 1\right)} = e_F^{-(t+u)x} \sum_{l,k=0}^{\infty} w_{k+l,F}(x,y) \frac{t^k}{F_k!} \frac{u^l}{F_l!}.$$

It is examined from two previous equations that

$$\sum_{k,l=0}^{\infty} w_{k+l,F}(x,y) \frac{t^k}{F_k!} \frac{u^l}{F_l!} = e_F^{(t+u)(x-_F z)} \sum_{l,k=0}^{\infty} w_{k+l,F}(z,y) \frac{t^k}{F_k!} \frac{u^l}{F_l!},$$

which means

$$\sum_{k,l=0}^{\infty} w_{k+l,F}(x,y) \frac{t^k}{F_k!} \frac{u^l}{F_l!} = \sum_{n,m=0}^{\infty} (x -_F z)^{n+m} \frac{t^n}{F_n!} \frac{u^m}{F_m!} \sum_{k,l=0}^{\infty} w_{k+l,F}(z,y) \frac{t^k}{F_k!} \frac{u^l}{F_l!}.$$

Hence, it is derived from (38) that

$$\sum_{k,l=0}^{\infty} w_{k+l,F}(x,y) \frac{t^k}{F_k!} \frac{u^l}{F_l!} = \sum_{k,l=0}^{\infty} \sum_{n,m=0}^{k,l} \frac{(x -_F z)^{n+m}\ w_{k+l-n-m,F}(z,y)}{F_n! F_m! F_{k-n}! F_{l-m}!} t^k u^l,$$

which means the alleged consequence (39). □

Corollary 3.7. *It can be analyzed from (39) that*

$$w_{k,F}(x,y)=\sum_{n=0}^{k}\binom{k}{n}_F w_{k-n,F}(z,y)(x-_F z)^n .$$

Corollary 3.8. *It can be analyzed from (39) that*

$$w_{k,F}(x+_F z,y)=\sum_{n=0}^{k}\binom{k}{n}_F w_{k-n,F}(z,y)x^n .$$

Theorem 3.9. *(Symmetric identity) The following*

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}_F a^{n-k}b^k w_{n-k,F}(bx,ay)\ w_{k,F}(ax,by)\\ =&\sum_{k=0}^{n}\binom{n}{k}_F a^k b^{n-k}\ w_{k,F}(bx,ay)\,w_{n-k,F}(ax,by)\end{aligned}\tag{40}$$

holds for $0\leq n$ *and* $b,a\in\mathbb{R}$.

Proof. Let

$$\Upsilon=\frac{e_F^{(1+_F1)xabt}}{\left(1-ay\left(e_F^{at}-1\right)\right)\left(1-by\left(e_F^{bt}-1\right)\right)},$$

which is symmetric in a and b, and we examine that

$$\begin{aligned}\Upsilon&=\sum_{n=0}^{\infty}w_{n,F}(bx,ay)\frac{(at)^n}{F_n!}\sum_{n=0}^{\infty}w_{n,F}(ax,by)\frac{(bt)^n}{F_n!}\\ &=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}_F a^{n-k}b^k w_{n-k,F}(bx,ay)\ w_{k,F}(ax,by)\frac{t^n}{F_n!}\end{aligned}$$

and similarly

$$\Upsilon=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}_F a^k b^{n-k} w_{k,F}(bx,ay)\,w_{n-k,F}(ax,by)\ \frac{t^n}{F_n!},$$

which give the desired result (40). □

Theorem 3.10. *The following equality*

$$w_{n,F}(x,y)=\frac{y}{1+y}\sum_{k=0}^{n}\binom{n}{k}_F w_{k,F}(x,y)+\frac{x^n}{1+y}\tag{41}$$

holds for non-negative integer n.

Proof. Utilizing (29), it is computed that

$$\begin{aligned}e_F^{xt}&=\left(1-y\left(e_F^{t}-1\right)\right)\sum_{n=0}^{\infty}w_{n,F}(x,y)\frac{t^n}{F_n!}\\ &=\left(y+1-ye_F^{t}\right)\sum_{n=0}^{\infty}w_{n,F}(x,y)\frac{t^n}{F_n!}\\ &=(y+1)\sum_{n=0}^{\infty}w_{n,F}(x,y)\frac{t^n}{F_n!}-y\left[\sum_{n=0}^{\infty}\frac{t^n}{F_n!}\sum_{n=0}^{\infty}w_{n,F}(x,y)\frac{t^n}{F_n!}\right],\end{aligned}$$

then we obtain

$$\sum_{n=0}^{\infty}\left(x^{n}+y\sum_{j=0}^{n}\binom{n}{j}_{F}w_{j,F}(x,y)\right)\frac{t^{n}}{F_{n}!}=\sum_{n=0}^{\infty}(y+1)\,w_{n,F}(x,y)\,\frac{t^{n}}{F_{n}!},$$

which completes the proof. □

A linear combination of the polynomials $w_{n,F}(x,y)$ for different x and y values is provided in the next theorem.

Theorem 3.11. *Let $y_1 \neq y_2$ and $n \in \mathbb{N}_0$. We have*

$$\sum_{k=0}^{n}\binom{n}{k}_{F}w_{n-k,F}(x_1,y_1)\,w_{k,F}(x_2,y_2)=\frac{y_2w_{n,F}(x_1+_F x_2,y_2)-y_1w_{n,F}(x_1+_F x_2,y_1)}{y_2-y_1}. \tag{42}$$

Proof. Utilizing (29), we consider

$$\begin{aligned}&\frac{e_F^{x_1t}}{1-y_1\left(e_F^{t}-1\right)}\frac{e_F^{x_2t}}{1-y_2\left(e_F^{t}-1\right)}\\&=\frac{y_2}{y_2-y_1}\frac{e_F^{(x_1+_Fx_2)t}}{1-y_2\left(e_F^{t}-1\right)}-\frac{y_1}{y_2-y_1}\frac{e_F^{(x_1+_Fx_2)t}}{1-y_1\left(e_F^{t}-1\right)},\end{aligned}$$

which yields

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}_{F}w_{n-k,F}(x_1,y_1)\,w_{k,F}(x_2,y_2)\,\frac{t^{n}}{F_{n}!}\\=\;&\frac{y_2}{y_2-y_1}\sum_{n=0}^{\infty}w_{n,F}(x_1+_F x_2,y_2)\,\frac{t^{n}}{F_{n}!}-\frac{y_1}{y_2-y_1}\sum_{n=0}^{\infty}w_{n-k,F}(x_1+_F x_2,y_1)\,\frac{t^{n}}{F_{n}!}.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}_{F}w_{n-k,F}(x_1,y_1)\,w_{k,F}(x_2,y_2)\right)\frac{t^{n}}{F_{n}!}\\=\;&\sum_{n=0}^{\infty}\left(\frac{y_2}{y_2-y_1}w_{n,F}(x_1+_F x_2,y_2)-\frac{y_1}{y_2-y_1}w_{n,F}(x_1+_F x_2,y_1)\right)\frac{t^{n}}{F_{n}!},\end{aligned}$$

which gives the desired result (42). □

4. Some Connected Formulas

In this part, we derive multifarious correlations for the bivariate Fibonacci-Fubini polynomials related to the Fibonacci-Stirling polynomials and numbers of the second kind, the Fibonacci-Euler polynomials and numbers, and the Fibonacci-Bernoulli polynomials and numbers.

We provide a relation between the polynomials $E_{n,F}(x)$ in (15) and the polynomials $w_{n,F}(x,y)$ in (29).

Theorem 4.1. *We have*

$$E_{n,F}(x)=w_{n,F}\left(x,-\frac{1}{2}\right). \tag{43}$$

Proof. We compute from (15) and (33) that

$$\sum_{n=0}^{\infty} w_{n,F}\left(x,-\frac{1}{2}\right)\frac{t^n}{F_n!} = \frac{e_F^{xt}}{1+\frac{1}{2}\left(e_F^t-1\right)} = \frac{2}{e_F^t+1}e_F^{xt} = \sum_{n=0}^{\infty} E_{n,F}(x)\frac{t^n}{F_n!},$$

which means the claimed relation (43). □

Corollary 4.2. *When $x=0$ in (43), we attain*

$$w_{n,F}\left(-\frac{1}{2}\right) = E_{n,F}. \tag{44}$$

Remark 4.3. *The formulas (43) and (44) are Fibonacci extensions of the identity in (26).*

A relation includes $w_{n,F}(y)$ and $S_{2,F}(n,k:x)$ is given as follows.

Theorem 4.4. *The following relation*

$$w_{n,F}(x,y) = y^k F_k! S_{2,F}(n,k:x) \tag{45}$$

is valid for $\left|y\left(e_F^t-1\right)\right|<1$ and a non-negative integer n.

Proof. By (16) and (29), we see that

$$\begin{aligned}\sum_{n=0}^{\infty} w_{n,F}(x,y)\frac{t^n}{F_n!} &= \frac{e_F^{xt}}{1-y\left(e_F^t-1\right)}\\ &= \sum_{k=0}^{\infty} y^k\left(e_F^t-1\right)^k e_F^{xt}\\ &= \sum_{k=0}^{\infty} y^k F_k! \sum_{n=k}^{\infty} S_{2,F}(n,k:x)\frac{t^n}{F_n!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} y^k F_k! S_{2,F}(n,k:x)\frac{t^n}{n!},\end{aligned}$$

which implies the desired result (45). □

Corollary 4.5. *Some exceptional cases of the relation (45) are as follows:*

$$w_{n,F}(y) = y^k F_k! S_{2,F}(n,k), \tag{46}$$

$$w_{n,F}(x) = F_k! S_{2,F}(n,k:x),$$

and

$$w_{n,F} = F_k! S_{2,F}(n,k).$$

Remark 4.6. *The relation (46) is a Fibonacci analog of the well-known formula in (27).*

For $-n\in\mathbb{N}$ and $|x|<a$, the negative binomial expansion is provided as follows (cf. [2]):

$$(x+a)^{-n} = \sum_{k=0}^{\infty}(-1)^k\binom{n+k-1}{k}x^k a^{-n-k}. \tag{47}$$

Theorem 4.7. *For $n \in \mathbb{N}_0$, we have*

$$w_{n,F}(x,y) = \sum_{k=0}^{\infty} \sum_{l=k}^{n} \binom{x+k-1}{k} F_k! \binom{n}{l}_F w_{n-l,F}(-k,y) S_{2,F}(l,k). \tag{48}$$

Proof. We derive from (29) and (16), we attain

$$\begin{aligned}
\sum_{n=0}^{\infty} w_{n,F}(x,y) \frac{t^n}{F_n!} &= \frac{1}{1-y\left(e_F^t - 1\right)} \left(e_F^{-t} - 1 + 1\right)^{-x} \\
&= \frac{1}{1-y\left(e_F^t - 1\right)} \sum_{k=0}^{\infty} \binom{x+k-1}{k} \left(1 - e_F^{-t}\right)^{-k} \\
&= \frac{1}{1-y\left(e_F^t - 1\right)} \sum_{k=0}^{\infty} \binom{x+k-1}{k} F_k! \frac{\left(e_F^t - 1\right)^k}{F_k!} e_F^{-kt} \\
&= \sum_{k=0}^{\infty} \binom{x+k-1}{k} F_k! \sum_{n=0}^{\infty} w_{n,F}(-k,y) \frac{t^n}{F_n!} \sum_{n=0}^{\infty} S_{2,F}(n,k) \frac{t^n}{F_n!} \\
&= \sum_{k=0}^{\infty} \binom{x+k-1}{k} F_k! \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l}_F w_{n-l,F}(-k,y) S_{2,F}(l,k) \right) \frac{t^n}{F_n!},
\end{aligned}$$

which means the asserted relation (48). □

We now state the following theorem.

Theorem 4.8. *The following formula*

$$w_{n,F}(x,y) = \frac{1}{2} \sum_{l=0}^{n} \binom{n}{l}_F E_{l,F} \left[w_{n-l,F}(x +_F 1, y) + w_{n-l,F}(x,y) \right]$$

is valid for a non-negative integer n.

Proof. We acquire from (15) and (29) that

$$\begin{aligned}
\sum_{n=0}^{\infty} w_{n,F}(x,y) \frac{t^n}{F_n!} &= \frac{e_F^{xt}}{1-y\left(e_F^t - 1\right)} \frac{2}{e_F^t + 1} \frac{e_F^t + 1}{2} \\
&= \frac{1}{2} \frac{e_F^{(x+_F 1)t}}{1-y\left(e_F^t - 1\right)} \frac{2}{e_F^t + 1} + \frac{1}{2} \frac{e_F^{xt}}{1-y\left(e_F^t - 1\right)} \frac{2}{e_F^t + 1} \\
&= \frac{1}{2} \sum_{n=0}^{\infty} w_{n,F}(x +_F 1, y) \frac{t^n}{F_n!} \sum_{n=0}^{\infty} E_{n,F} \frac{t^n}{F_n!} \\
&\quad + \frac{1}{2} \sum_{n=0}^{\infty} w_{n,F}(x,y) \frac{t^n}{F_n!} \sum_{n=0}^{\infty} E_{n,F} \frac{t^n}{F_n!},
\end{aligned}$$

which ends the proof. □

We finally state a relation for the Fibonacci-Bernoulli and Fibonacci-Fubini polynomials as follows.

Theorem 4.9. *The following relation*

$$w_{n,F}(x,y)=\frac{1}{n+1}\sum_{k=0}^{n+1}\binom{n+1}{k}_F B_{n+1-l,F}\left[w_{l,F}(x+_F 1,y)-w_{l,F}(x,y)\right] \tag{49}$$

is valid for $n\in\mathbb{N}_0$.

Proof. We acquire from (5) and (29) that

$$\begin{aligned}\sum_{n=0}^{\infty}w_{n,F}(x,y)\frac{t^n}{F_n!}&=\frac{e_F^{xt}}{1-y\left(e_F^t-1\right)}\frac{t}{e_F^t-1}\frac{e_F^t-1}{t}\\&=t^{-1}\frac{e_F^{(x+_F1)t}}{1-y\left(e_F^t-1\right)}\frac{t}{e_F^t-1}-t^{-1}\frac{e_F^{xt}}{1-y\left(e_F^t-1\right)}\frac{t}{e_F^t-1}\\&=\sum_{n=0}^{\infty}w_{n,F}\left(x+_f 1,y\right)\frac{t^{n-1}}{F_n!}\sum_{n=0}^{\infty}B_{n,F}\frac{t^n}{F_n!}-\sum_{n=0}^{\infty}w_{n,F}(x,y)\frac{t^{n-1}}{F_n!}\sum_{n=0}^{\infty}B_{m,n}(x)\frac{t^n}{F_n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}_F w_{l,F}(x+_F1,y)B_{n-l,F}\right)\frac{t^{n-1}}{F_n!}-\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}_F w_{l,F}(x,y)B_{n-l,F}\right)\frac{t^{n-1}}{F_n!},\end{aligned}$$

which means the asserted result (49). □

5. Further Remarks

Let us now consider the Fibonacci differential operator for $k\in\mathbb{N}_0$ as follows (cf. [5]):

$$\left(xD_F^x\right)^{(k)}, \tag{50}$$

where $\left(xD_F^x\right)^{(k)}=\left(xD_F^x\right)^{(k-1)}xD_F^x$. We observe from (50) that for $1\leq n$:

$$\left(xD_F^x\right)^{(k)}x^n=\left(xD_F^x\right)\left(xD_F^x\right)\ldots\left(xD_F^x\right)x^n=\left(F_n\right)^k x^n. \tag{51}$$

Hence, we obtain from (51) that (cf. [5])

$$\left(xD_F^x\right)^{(k)}(1+x)^n=\sum_{l=0}^{n}\binom{n}{l}\left(xD_F^x\right)^{(k)}x^l=\sum_{l=0}^{n}\binom{n}{l}\left(F_n\right)^l x^l$$

and

$$\left(xD_F^x\right)^{(k)}f(x)=\sum_{n=0}^{\infty}a_n\left(xD_F^x\right)^{(k)}x^n=f(x)=\sum_{n=0}^{\infty}a_n\left(F_n\right)^k x^n,$$

where $f(x)=\sum_{n=0}^{\infty}a_nx^n$ is a formal power series.

Theorem 5.1. *The following Fibonacci operator formula for* $w_{n,F}(x,y)$

$$\left(xD_F^x\right)^{(k)}w_{n,F}(x,y)=\sum_{l=0}^{n}\binom{n}{l}_F w_{n-l,F}(y)\left(F_l\right)^k x^l \tag{52}$$

holds for $n\in\mathbb{N}$ *and* $k\in\mathbb{N}_0$.

Proof. It is readily seen from (17) and (51) that

$$\begin{aligned}\left(xD_F^x\right)^{(k)} w_{n,F}\left(x,y\right) &= \sum_{l=0}^{n}\binom{n}{l}_F w_{n-l,F}\left(x\right)\left(yD_F^y\right)^{(k)} x^l \\ &= \sum_{l=0}^{n}\binom{n}{l}_F w_{n-l,F}\left(y\right)\left(F_l\right)^k x^l,\end{aligned}$$

which means the desired consequence (52). □

6. Conclusions

Recently, using the content of golden calculus, Fibonacci extensions of diverse special polynomials, such as Fibonacci-Hermite, Fibonacci-Euler, Fibonacci-Bernoulli, and Fibonacci-sigmoid polynomials, have been considered, and several of their properties and relations have been investigated. In this paper, we first considered bivariate Fibonacci Fubini polynomials and numbers, and we then obtained some identities and properties for these polynomials and numbers, involving summation formulas, recurrence relations, golden derivative property, and golden integral property. We have also proved some implicit summation formulas and a symmetric property. Furthermore, we have given some correlations, including the bivariate Fibonacci Fubini polynomials and numbers, the Fibonacci Stirling polynomials and numbers of the second kind, the Fibonacci Euler polynomials and numbers, and the Fibonacci Bernoulli polynomials and numbers. Finally, we have provided a Fibonacci differential operator formula for the bivariate Fibonacci-Fubini polynomials. To the best of our knowledge, the results derived in the paper are novel and don't seem to have been reported in the literature. Generally, these consequences can be utilized in numerous fields of statistics, mathematical physics, probability, engineering, and mathematics.

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