Filomat 39:30 (2025), 10705–10713
https://doi.org/10.2298/FIL2530705F

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some notes on convex F-contraction in b-metric spaces

Ali Farajzadeh[a,*], Mahtab Delfani[a], Ljubiša D. R. Kočinac[b]

[a]*Department of Mathematics, Razi University, Kermanshah, Iran*
[b]*Faculty of Sciences and Mathematics, University of Niš, 18000 Niš, Serbia*

Abstract.
In this paper, a generalized version of convex F-contraction, where F is not necessarily strictly increasing, in b-metric spaces is introduced. A weaker form of the continuity, called nearly continuity, is presented and it is also shown, by providing an example, that it is a real generalization of the continuity. A new version of the convex F-contraction, by comparing the old one, in order to establish an existence result of a fixed point for a self-mapping in the setting of b-metric spaces, has been offered. By using this version of the convex F-contraction, being a Cauchy sequence of the Picard's iteration is stated which by applying this result an affirmative answer to the first question raised in [(MDPI) Axioms, DOI:10.3390/axioms10020071] is given. By suitable conditions and replacing the strictly increasing by the quasi-convexity of the function F, the existence and uniqueness of fixed points for self-mappings which are satisfied in F-Kannan type contraction is proved. The results of the article improve the main results in this area as well answer to the open problems raised in the paper [On Convex F-Contraction in b-Metric Spaces, (MDPI) Axioms, DOI:10.3390/axioms10020071]. Moreover, all the fixed point theorems are obtained for F-contraction mappings whose F are strictly increasing instead of being quasi-convex.

1. Introduction and preliminaries

In [12], Wardowski introduced the concept of F-contraction and proved a fixed point theorem which generalizes the classical Banach contraction mapping principle [1].

Definition 1.1. ([12]) Let (X, d) be a metric space and $T : X \to X$ be a mapping. Then T is called an *F-contraction*, if there exists a function $F : (0, \infty) \to \mathbb{R}$ such that:

(F_1) F is strictly increasing on $(0, \infty)$;

(F_2) for every sequence $\{\alpha_n\}$ in $(0, \infty)$ we have $\lim_{n\to\infty} F(\alpha_n) = -\infty$ if and only if $\lim_{n\to\infty} \alpha_n = 0$;

(F_3) there exists a number $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$;

2020 *Mathematics Subject Classification*. Primary 47H10; Secondary 54E35, 54H25.
Keywords. F-contraction; convex F-contraction; fixed point; nearly continuous; quasi-convex; weakly continuous
Received: 06 April 2025; Revised: 28 July 2025; Accepted: 24 September 2025
Communicated by Dejan Ilić
* Corresponding autor: Ali Farajzadeh
Email addresses: farajzadehali@gmail.com (Ali Farajzadeh), ma.delfani@gmail.com (Mahtab Delfani), lkocinac@gmail.com (Ljubiša D. R. Kočinac)
ORCID iDs: https://orcid.org/0000-0001-5221-6741 (Ali Farajzadeh), https://orcid.org/0000-0002-4870-7908 (Ljubiša D. R. Kočinac)

(F_4) there exists $\tau > 0$ such that

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y)), \tag{1}$$

for all $x, y \in X$ with $Tx \neq Ty$.

Theorem 1.2. ([12]) *Let (X, d) be a complete metric space and $T : X \to X$ be an F-contraction. Then T has a unique fixed point x^* in X. Also, for every $x \in X$, the sequence $\{T^n x\}$ converges to x^*.*

Recall that F-contractions in metric spaces were considered in [4, 5, 10], and in b-metric spaces in [6, 11]. The concept of b-metric space, which is a great generalization of usual metric space, was introduced by Czerwik [3].

Definition 1.3. ([3]) Let X be a nonempty set. A *b-metric* on X is a function $d : X \times X \to [0, +\infty)$, if there exists a real number $s \geq 1$, such that for all $x, y, z \in X$ the following conditions hold:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

(X, d, s) is called a *b-metric space*.

Example 1.4. Let $X = [0, 1]$ and $d : X \times X \to \mathbb{R}^+$ be defined by $d(x, y) = |x - y|^2$, for all $x, y \in X$. Clearly, $(X, d, 2)$ is a b-metric space that is not a metric space.

Remark 1.5. It is well known that if (X, d) is a b-metric space, then the function $\rho(x, y) = \inf\{\sum_{i=1}^{n} d(x_{i-1}, x_i)\}$, where the infimum is taken over all natural numbers n and all chains $x = x_0, x_1, ..., x_n = y$ of elements in X, defines a pseudometric on X and $d(x, y) \leq \rho(x, y)$, for all $x, y \in X$.

Recall that a sequence $\{x_n\}$ in a b-metric space (X, d, s) *converges* to a point $x \in X$, if $\lim_{n\to\infty} d(x_n, x) = 0$. A sequence $\{x_n\}$ in a b-metric space (X, d, s) is a *Cauchy sequence*, if for each $\varepsilon > 0$, there exists a natural number $n_0(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$, for each $m, n \geq n_0(\varepsilon)$. A b-metric space (X, d, s) is *complete*, if each Cauchy sequence in X converges to some point of X.

Remark 1.6. ([2]) Let (X, d) be a b-metric space. The b-metric d is called *continuous* if

$$(d(x_n, x) \to 0 \ \text{ and } \ d(y_n, y) \to 0) \Longrightarrow d(x_n, y_n) \to d(x, y).$$

The following example shows that the continuity of d may fail.

Example 1.7. ([8]) Let $X = \mathbb{N} \cup \{\infty\}$ and let $d : X \times X \to \mathbb{R}$ be defined by

$$d(m, n) = \begin{cases} 0, & \text{if } m = n, \\ |\frac{1}{m} - \frac{1}{n}|, & \text{if one of m, n is even and the other is even or } \infty, \\ 5, & \text{if one of } m, n \text{ is odd and the other is odd (and } m \neq n) \text{ or } \infty, \\ 2, & \text{otherwise.} \end{cases}$$

Then, considering all possible cases, it can be checked that for all $m, n, p \in X$, we have

$$d(m, p) \leq \frac{5}{2}\left(d(m, n) + d(n, p)\right).$$

Thus, (X, d) is a b-metric space (with $s = 5/2$). Let $x_n = 2n$ and $y_n = 1$, for each $n \in \mathbb{N}$. Then

$$d(2n, \infty) = \frac{1}{2n} \to 0 \ \text{ as } n \to \infty,$$

that is, $x_n \to \infty$, but $d(x_n, 1) = 2 \nrightarrow 5 = d(\infty, 1)$ as $n \to \infty$.

Let (X,d,s) be a b-metric space and T be a self-mapping on X. The *Picard sequence* of T is given by $\{x_n\}_{n\in\mathbb{N}\cup\{0\}} = \{T^n x\}_{n\in\mathbb{N}\cup\{0\}}$ for any $x \in X$, where $T^0x = x$.

In this paper, a generalized version of convex F-contraction, where F is not necessarily strictly increasing, in b-metric spaces is introduced. A weaker form of the continuity, called nearly continuity, is presented and it is also shown, by providing an example, that it is a real generalization of the continuity. A new version of the convex F-contraction, by comparing the old one, in order to establish an existence result of a fixed point for a self-mapping in the setting of b-metric spaces, has been offered. By using this version of the convex F-contraction, being a Cauchy sequence of the Picard's iteration is stated which by applying this result an affirmative answer to the first question raised in [(MDPI) Axioms, DOI:10.3390/axioms10020071] is given. By suitable conditions and replacing the strictly increasing by the quasi-convexity of the function F, the existence and uniqueness of fixed points for self-mappings which are satisfied in F-Kannan type contraction is proved. The results of the article improve the main results in this area as well answer to the open problems raised in the paper [On Convex F-Contraction in b-Metric Spaces, (MDPI) Axioms, DOI:10.3390/axioms10020071]. Moreover, all the fixed point theorems are obtained for F-contraction mappings whose F are strictly increasing instead of being quasi-convex.

2. Main results

In this section we are going to introduce weak forms of the continuity which are needed in the sequel. Also, a weak version of the convex F-contraction is given. By using these notions, the results presented in [7] will be improved by mild assumptions. Finally, the questions posed by Huang et. al. in [7] are answered.

Definition 2.1. ([7]) Let (X,d,s) be a b-metric space and T be a self-mapping on X. We say that T is a *convex F-contraction*, if there exists a function $F : (0,+\infty) \to \mathbb{R}$ such that condition (F_1) holds and

(F_2^s) for each sequence $\{\alpha_n\}$ of positive numbers, if $\lim_{n\to\infty} F(\alpha_n) = -\infty$, then $\lim_{n\to\infty} \alpha_n = 0$;

(F_3^s) there exists $k \in \left(0, \dfrac{1}{1+\log_2^s}\right)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$;

(F_4^s) there exist $\tau > 0$ and $\lambda \in [0,1)$ such that

$$\tau + F(d(x_{n+1},x_n)) \leq F(\lambda d(x_{n+1},x_n) + (1-\lambda)d(x_n,x_{n-1})), \tag{2}$$

for all $d(x_{n+1},x_n) > 0$, where $n \in \mathbb{N}$.

Condition (F_4^s) in Definition 2.1 is a bit confusing and uncertain, because it depends on a recursive sequence which depends on the variable x which changes on the set X. Therefore, the sequence in root depends on all elements of X. Thus we modify it as follows.

$(F_4^{s'})$ there exist $\tau > 0$ and $\lambda \in [0,1)$ such that

$$\tau + F(d_n) \leq F(\lambda d_n + (1-\lambda)d_{n-1}), \tag{3}$$

where $\{d_n = d(T^{n+1}(x), T^n(x))\}$ and x an arbitrary element of X.

Remark 2.2. If the self-mapping $T : X \to X$ satisfies in conditions (F_1) and (F_4) then it fulfils in $(F_4^{s'})$, because if x an arbitrary element of X, we get by (F_4)

$$\tau + F(d(T(T^n(x)), T(T^{n-1}(x)))) \leq F(d(T^n(x), T^{n-1}(x))),$$

Hence, since F is strictly increasing and $\tau > 0$ we obtain

$$d(T(T^n(x)), T(T^{n-1}(x))) \leq d(T^n(x), T^{n-1}(x))$$

which implies

$$\tau + F(d(T^{n+1}(x), T^n(x))) \leq F(\frac{1}{2}d(T^{n+1}(x), T^n(x)) + (1-\frac{1}{2})d(T^n(x), T^{n-1}(x))).$$

Consequently the definition of F-contraction is a special case of *convex F-contraction*.

We introduce the following definitions which play important role in this article.

Definition 2.3. Let X be a topological space. The mapping $T : X \to X$ is called *weakly continuous* at $x_0 \in X$, if $x_i \to x_0$, then there exists a subnet $\{x_{i_j}\}$ of $\{x_i\}$ such that $Tx_{i_j} \to Tx_0$.

Remark 2.4. Let X be a topological space. If $T : X \to X$ is a continuous mapping at x^*, then it is obvious that T is weakly continuous at x^*. Now if T is not continuous at x^*, then there exists $x_i \to x^*$ such that $Tx_i \nrightarrow Tx^*$. Hence for each open set U of Tx^* there exists x_{i_U} with $Tx_{i_U} \notin U$. Hence net $\{Tx_{i_U}\}_U$ does not have a convergent subnet where x_{i_U} converges to x^* which means T is not weakly continuous at x^*. Consequently, the concepts of continuity and weakly continuity are equivalent.

Definition 2.5. Let (X, d, s) be a b-metric space. We say that $T : X \to X$ is *nearly continuous* at $x_0 \in X$, if $x_n \to x_0$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{Tx_{n_k}\}$ is convergent.

Remark 2.6. (a) If $T : X \to X$ is continuous at x_0, then it is nearly continuous at x_0, but the converse may fail; for example, if we take $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ with the Euclidean metric and define $T : X \to X$ by

$$Tx = \begin{cases} 1, & x \in \{\frac{1}{2k} : k \in \mathbb{N}\}, \\ 0, & x \in \{\frac{1}{2k-1} : k \in \mathbb{N}\} \cup \{0\}, \end{cases}$$

then T is not continuous at $x_0 = 0$, while it is nearly continuous at 0.

(b) If T is nearly continuous at x_0 and T is closed (that is, its graph is closed), then for each sequence $\{x_n\}$ converging to x_0 there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that Tx_{n_k} is convergent. Hence, it follows from the closedness of T that $Tx_{n_k} \to Tx_0$ and so it is weakly continuous at x_0.

(c) The mapping T defined by $T(x) = \frac{1}{x}$ for $x \neq 0$, and $T(0) = 0$, has closed graph, although it is not nearly continuous at $x = 0$. This counterexample demonstrates that the closedness of T need not imply nearly continuity, in general.

The following lemma plays a crucial role in proving the main results, and it gives an affirmative answer to the first question raised in [7]. In fact, we use the condition (F_1) instead of (F_2^s), with a new proof.

Lemma 2.7. *Let (X, d, s) be a b-metric space and $T : X \to X$ a self-mapping. If T satisfies in the conditions (F_1), (F_3) and $(F_4^{s'})$, then for each $x \in X$, the sequence $\{T^n x\}_{n \in \mathbb{N} \cup \{0\}}$ is a Cauchy sequence.*

Proof. Choose $x \in X$ and construct a sequence $\{x_n\}$ by $x_n = T^n x$, for all $n \in \mathbb{N} \cup \{0\}$. If there exists $n_0 \in \mathbb{N} \cup \{0\}$ such that $x_{n_0+1} = x_{n_0}$, then

$$\{x_n\} = \{x, Tx, T^2x, \ldots, T^{n_0-1}x, x_{n_0}, x_{n_0}, \ldots\}.$$

In this case, the sequence $\{T^n x\}_{n \in \mathbb{N} \cup \{0\}}$ is clearly Cauchy, and the proof is completed. So, we can assume that $x_{n+1} \neq x_n$, for each $n \in \mathbb{N} \cup \{0\}$. Therefore $d(x_{n+1}, x_n) > 0$, for all $n \in \mathbb{N} \cup \{0\}$. Now, if we define $d_n = d(x_{n+1}, x_n)$, then by $(F_4^{s'})$ we have

$$F(d(x_{n+1}, x_n)) < \tau + F(d(x_{n+1}, x_n)) \leq F(\lambda d(x_{n+1}, x_n) + (1-\lambda)d(x_n, x_{n-1})).$$

Using condition (F_1), we obtain

$$d(x_{n+1},x_n) \leq \lambda d(x_{n+1},x_n) + (1-\lambda)d(x_n,x_{n-1}).$$

Then $d(x_{n+1},x_n) \leq d(x_n,x_{n-1})$, for all $n \in \mathbb{N}$. Hence, $\{d(x_{n+1},x_n)\}$ is a convergent sequence. We claim that $\lim\limits_{n\to\infty} d(x_{n+1},x_n) = 0$. Otherwise, let $\lim\limits_{n\to\infty} d(x_{n+1},x_n) = \varepsilon > 0$. Since F is strictly increasing, then F has right and left limit in each positive real number, especially $L = \lim\limits_{t\to\varepsilon^+} F(t)$. Thus, since $\{d(x_{n+1},x_n)\}$ is a decreasing sequence (converges to ε) and $\lim\limits_{n\to\infty} d(x_{n+1},x_n) = \inf\limits_{k\in\mathbb{N}} d(x_{k+1},x_k)$, we have $d(x_{n+1},x_n) \to \varepsilon$, $d(x_n,x_{n-1}) \to \varepsilon$ and $\lambda d(x_{n+1},x_n) + (1-\lambda)d(x_n,x_{n-1}) \to \varepsilon$ which imply

$$L = \lim_{n\to\infty} F(d(x_{n+1},x_n)), \quad L = \lim_{n\to\infty} F(d(x_n,x_{n-1})),$$

and

$$L = \lim_{n\to\infty} F(\lambda d(x_{n+1},x_n) + (1-\lambda)d(x_n,x_{n-1})).$$

By condition $(F_4^{s'})$, we have

$$F(d(x_{n+1},x_n)) < \tau + F(d(x_{n+1},x_n)) \leq F(\lambda d(x_{n+1},x_n) + (1-\lambda)d(x_n,x_{n-1})).$$

Now, by taking the limit from both sides of the above inequalities, when $n \to \infty$, we get $L < \tau + L \leq L$, which is a contradiction. Therefore $\lim\limits_{n\to\infty} d(x_{n+1},x_n) = 0$. Hence, by $\lim\limits_{n\to\infty} d(x_{n+1},x_n) = 0$, via (F_3), there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} (d(x_{n+1},x_n))^k F(d(x_{n+1},x_n)) = 0. \tag{4}$$

We now proceed to show that

$$\tau + F(d(x_{n+1},x_n)) \leq F(d(x_n,x_{n-1})), \tag{5}$$

for all $n \in \mathbb{N}$. Indeed, if (9) is not true, then

$$\tau + F(d(x_{n+1},x_n)) > F(d(x_n,x_{n-1})),$$

for some $n \in \mathbb{N}$. Thus,

$$F(d(x_n,x_{n-1})) < \tau + F(d(x_{n+1},x_n)) \leq F(\lambda d(x_{n+1},x_n) + (1-\lambda)d(x_n,x_{n-1})).$$

Thus, it follows from the strictly increasing of F that

$$d(x_n,x_{n-1}) < \lambda d(x_{n+1},x_n) + (1-\lambda)d(x_n,x_{n-1}).$$

Hence, $d(x_n,x_{n-1}) < d(x_{n+1},x_n)$, which is a contradiction. Consequently, the relation (9) holds and by repeating it we get

$$F(d(x_{n+1},x_n)) \leq F(d(x_1,x_0)) - n\tau, \tag{6}$$

for all $n \in \mathbb{N}$.

From (10) we obtain

$$(d_n = d(x_{n+1},x_n))^k n\tau \leq (d(x_{n+1},x_n))^k F(d(x_1,x_0)) - (d(x_{n+1},x_n))^k F(d(x_{n+1},x_n)). \tag{7}$$

Therefore, the equations (8) and $\lim_{n\to\infty} d(x_{n+1},x_n) = 0$ conclude

$$\lim_{n\to\infty} d_n^k n = 0.$$

Hence, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$d_n \leq \frac{1}{n^{\frac{1}{k}}}.$$

Hence the series $\sum_{n=1}^{\infty} d_n$ is convergent, which implies that the sequence $\{S_m = \sum_{n=1}^{m} d_n\}$ of partial sums is convergent. Now, the result follows from Remark 1.5. □

Inspired by the proof of the previous result, we state the following proposition by removing the strictly increasing condition of the mapping F.

Remember that $F : (0, \infty) \to \mathbb{R}$ is called quasi-convex if

$$F(tx + (1-t)y) \le \max\{F(x), F(y)\}, \forall x, y \in (0, \infty), t \in [0, 1].$$

It is obvious that a convex function is quasi-convex function while the converse may fail, for instance $F(x) = [x]$ (the step function).

Proposition 2.8. *Let (X, d, s) be a b-metric space and $T : X \to X$ a self-mapping. If T satisfies in the conditions F_2, F_3 and $(F_4^{s'})$, where F is quasi-convex, then for each $x \in X$, the sequence $\{T^n x\}_{n\in\mathbb{N}\cup\{0\}}$ is a Cauchy sequence.*

Proof. Put $x_n = T^n x$, for all $n \in \mathbb{N} \cup \{0\}$. If there exists $n_0 \in \mathbb{N} \cup \{0\}$ such that $x_{n_0+1} = x_{n_0}$, then $\{x_n\} = x_{n_0}$, for all $n \ge n_0$, which converges to x_{n_0}. Then $x_n = T^n x$ is a Cauchy sequence and the proof is completed. Hence, suppose that $x_{n+1} \ne x_n$, for each $n \in \mathbb{N} \cup \{0\}$. Therefore $d(x_{n+1}, x_n) > 0$, for all $n \in \mathbb{N} \cup \{0\}$. Now, if we define $d_n = d(x_{n+1}, x_n)$, then by $(F_4^{s'})$ we have

$$F(d(x_{n+1}, x_n)) < \tau + F(d(x_{n+1}, x_n)) \le F(\lambda d(x_{n+1}, x_n) + (1-\lambda) d(x_n, x_{n-1})),$$

which implies, by the quasi-convexity of F, that $F(d(x_{n+1}, x_n)) - F(d(x_n, x_{n-1})) \le -\tau$. Hence, for each $m \ge 1$,

$$\sum_{n=1}^{m} (F(d(x_{n+1}, x_n)) - F(d(x_n, x_{n-1}))) \le \sum_{n=1}^{m} -\tau,$$

which implies $\lim_{n\to\infty} F(d(x_{n+1}, x_n)) = -\infty$, and by F_2 we get $d(x_n, x_{n-1}) \to 0$. By (F_3), there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} (d(x_{n+1}, x_n))^k F(d(x_{n+1}, x_n)) = 0. \tag{8}$$

By $(F_4^{s'})$ and the quasi-convexity of F, we deduce that

$$\tau + F(d(x_{n+1}, x_n)) \le F(d(x_n, x_{n-1})), \tag{9}$$

for all $n \in \mathbb{N}$, which implies

$$F(d(x_{n+1}, x_n)) \le F(d(x_1, x_0)) - n\tau, \tag{10}$$

for all $n \in \mathbb{N}$. From (10) we obtain

$$(d_n = d(x_{n+1}, x_n))^k n\tau \le (d(x_{n+1}, x_n))^k F(d(x_1, x_0)) - (d(x_{n+1}, x_n))^k F(d(x_{n+1}, x_n)). \tag{11}$$

Hence, by (8), (11) and $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$ we conclude

$$\lim_{n\to\infty} d_n^k n = 0.$$

Then, there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$,

$$d_n \le \frac{1}{n^{\frac{1}{k}}}.$$

This means the series $\sum_{n=1}^{\infty} d_n$ is convergent, equivalently the sequence $\{S_m = \sum_{n=1}^{m} d_n\}$ of partial sums of the series $\sum_{n=1}^{\infty} d_n$ is convergent. Now, the result follows from Remark 1.5. $\square$

The following fixed point theorem, by using Proposition 2.8, is established in the setting of complete b-metric spaces which improves the corresponding results given in [4, 6, 7, 10, 12].

Theorem 2.9. *Let (X, d, s) be a complete b-metric space and $T : X \to X$ a closed self-mapping. If T satisfies in (F_2), (F_3) and $(F_4^{s'})$, for at least an element of X where F is quasi-convex then T has a fixed point.*

Proof. Let x be an element of X which satisfies in $(F_4^{s'})$. By Proposition 2.8 the sequence $\{T^n(x)\}$ is a Cauchy sequence. Then, the completeness of X implies that there exists $z \in X$ such that $T^n(x) \to z$. It is obvious that $(T^n(x), T(T^n(x))) \in G_r(T)$ and $(T^n(x), T(T^n(x))) \to (z, z)$. Thus, the closedness of T assures that $(z, z) \in G_r(T)$. This means $z = T(z)$. and the proof is completed. □

Example 2.10. *Let $X = L^1([0, 1])$ (The set of all lebesgue integrable functions on [0,1]) with*

$$d(f, g) = \int_0^1 |f(x) - g(x)|dx, \quad \forall f, g \in X,$$

and $T : X \to X$ be defined by $T(f) = f^2$. Now, if we take $f_0(x) = x^2$, for all $x \in [0, 1]$, then

$$d(T^{n+1}(f_0), T^n(f_0)) = \int_0^1 |x^{2(n+1)} - x^{2n}|dx = \frac{1}{2n+1} - \frac{1}{2n+3} = \frac{2}{(2n+1)(2n+3)},$$

and

$$d(T^n(f_0), T^{n-1}(f_0)) = \int_0^1 |x^{2(n)} - x^{2(n-1)}|dx = \frac{1}{2(n-1)+1} - \frac{1}{2(n-1)+3} = \frac{2}{(2n-1)(2n+1)}.$$

Now, by taking $F(x) = -\frac{1}{\sqrt{x}}$, for all $x > 0$, we have

$$\tau + F(d(T^{n+1}(f_0), T^n(f_0))) \leq F(d(T^n(f_0), T^{n-1}(f_0))).$$

Hence

$$\tau - \frac{\sqrt{(2n+1)(2n+3)}}{\sqrt{2}} \leq -\frac{\sqrt{(2n-1)(2n+1)}}{\sqrt{2}} \Rightarrow \tau \leq \frac{\sqrt{4n^2+8n+3} - \sqrt{4n^2-1}}{\sqrt{2}}.$$

Thus, the self-mapping T fulfills all the hypotheses of Theorem 2.9.

The next result was stated in [7] for $\lambda = 0$. We are going to prove it for each $\lambda \in [0, 1)$.

Theorem 2.11. *Let (X, d, s) be a complete b-metric space and $T : X$ a closed self-mapping on X. If conditions (F_1), (F_3) and $(F_4^{s'})$ are satisfied, then, T has a fixed point in X, and the fixed point is unique when the following condition holds*

$(F_4^{s''})$ *there exist $\tau > 0$ and $\lambda \in [0, 1)$ such that*

$$\tau + F(d(T(x), T(y))) \leq F(\lambda d(T(x), T(y)) + (1 - \lambda)d(x, y)), \forall x, y \in X. \tag{12}$$

Proof. If $x \in X$, then by Lemma 2.7 and the completeness of X we deduce that the sequence $\{T^n x\}$ is convergent to an element x^*. Hence, $x^* = \lim_{n\to\infty} T^n x$. Due to the closedness of T, we conclude that x^* is a fixed points of T. Now, if we suppose x^*, y^* are two different fixed points of T, then by $(F_4^{s''})$ we have

$$\begin{aligned} F(d(x^*, y^*)) &= F(d(Tx^*, Ty^*)) < \tau + F(d(Tx^*, Ty^*)) \\ &\leq F(\lambda d(Tx^*, Ty^*) + (1 - \lambda)d(x^*, y^*)) \\ &= F(\lambda d(x^*, y^*) + (1 - \lambda)d(x^*, y^*)) = F(d(x^*, y^*)), \end{aligned}$$

which is a contradiction. So, $x^* = y^*$ and the proof is completed. □

Remark 2.12. It is obvious that he condition $(F_4^{s''})$ and the strictly increasing of F imply $d(T(x), T(y)) < d(x, y)$ for $x \neq y$. Hence T is contractible mapping. Then it is Lipschitz continuous. Thus, in this case we can relax the closedness of T in the first part of Theorem 2.11. Also condition (F_4) is a special case of $(F_4^{s''})$ by taking $\lambda = 0$. Therefore, Theorem 2.11 improves Theorem 3 of [7].

By combining Theorem 2.11 and part (b) of Remark 2.6 we get the following result.

Theorem 2.13. *Let (X, d, s) be a complete b-metric space and T be closed self-mapping on X. If conditions $(F_1), (F_3)$ and $(F_4^{s'})$ are satisfied, then T has a fixed point in X and the fixed point is unique if the following condition holds*

$(F_4^{s''})$ *there exist $\tau > 0$ and $\lambda \in [0, 1)$ such that*

$$\tau + F(d(T(x), T(y))) \leq F(\lambda d(T(x), T(y)) + (1 - \lambda)d(x, y)), \forall x, y \in X. \tag{13}$$

Remark 2.14. The result of the first part of the previous Theorem is still valid if $(F_4^{s'})$ holds for some element of X.

In order to state the next result we need the following definition.

Definition 2.15. ([7]) Let (X, d, s) be a b-metric space and T be a self-mapping on X. We say that T is an *F-contraction of Kannan type* if there exists a function $F : (0, \infty) \to \mathbb{R}$ such that condition (F_1) holds and

(F_4^k) there exist $\tau > 0$ such that

$$\tau + F(d(T(x), T(y))) \leq F\left(\frac{1}{2}(d(x, T(x)) + d(y, T(y)))\right), \tag{14}$$

for all $x, y \in X$ with $x \neq y$.

Remark 2.16. It is obvious from (F_1) and the last inequality that

$$d(T(x), T(y)) \leq \frac{1}{2}\left(d(x, T(x)) + d(y, T(y))\right) \leq d(x, T(x)) + d(y, T(y)),$$

for all $x, y \in X$ with $x \neq y$ which is Kannan contraction.

By using Definition 2.15 we can introduce the following condition.

We say that a self-mapping T on a b-metric X satisfies condition (F_4^k) if the following statement holds: there exists $\tau > 0$ such that

$$\tau + F(d_n) \leq F\left(\frac{1}{2}(d_n + d_{n-1})\right), \tag{15}$$

where $\{d_n = d(T^{n+1}(x), T^n(x))\}$ and $x \in X$.

We note that condition (F_4^k) is exactly $(F_4^{s'})$ when $\lambda = \frac{1}{2}$. Hence we have the following theorems which answers to the second question posed in [7]. Moreover it is a new version of Theorem 4 of [7] and they are new versions of Kannan type theorem.

Theorem 2.17. *Let (X, d, s) be a complete b-metric space and let $T : X \to X$ satisfy $(F_1), (F_3)$ and (F_4^k), for some $x \in X$. If T has closed graph, then T has a fixed point x^* in X and the fixed point is unique if at least one of the conditions (F_4^k) or $(F_4^{s''})$ holds. Further, if (F_4^k), is true for all $x \in X$, then for each $x \in X$, the sequence $\{T^n x\}$ converges to x^*.*

Proof. Let x be an arbitrary element of X and (F_4^k) be satisfied. Define $x_n = T^n(x)$ for each $n \in \mathbb{N} \cup \{0\}$. By Lemma 2.7, $\{x_n = T^n(x)\}$ is a Cauchy sequence and by the completeness of X x_n converges to an element of X, say x^*. Now the closedness of x^* assures that it is a fixed point of T. Now, if we suppose x^* and y^* are two distinct fixed points of T, then by (F_4^k) we get

$$\begin{aligned}
F(d(x^*, y^*)) &= F(d(T(x^*), T(y^*))) < \tau + F(d(T(x^*), T(y^*))) \\
&\leq F\left(\frac{1}{2}(d(x^*, T(x^*)) + d(y^*, T(y^*)))\right) \\
&= F\left(\frac{1}{2}(d(x^*, x^*) + d(y^*, y^*))\right)
\end{aligned}$$

which is a contradiction. Hence, T has a unique fixed point. Each iteration $\{T^n x\}$ converges to the unique fixed point x^* is obtained by Lemma 2.7 and the validity of (F_4^k) or $(F_4^{s''})$ for each element of X. □

Remark 2.18. The result of Theorem 2.17 is still valid if the hypothesis (F_1) is replaced by the quasi-convexity of T.

3. Conclusions

A generalized version of convex F-contraction, where F is not necessarily strictly increasing, in b-metric spaces is introduced. A weaker form of the continuity (is called nearly continuity) is given. A new contraction which is named convex F-contraction in order to establish an existence result of a fixed point for a self-mapping in the setting of b-metric spaces is provided. By suitable conditions and relaxing the strictly increasing by the quasi-convexity of the function F existence and uniqueness of fixed point for a self-mapping which is satisfied in F-Kannan type contraction is investigated. The results of the article improve the main results in this area as well answer the open problems raised in the paper [On Convex F-Contraction in b-Metric Spaces, (MDPI) Axioms, DOI:10.3390/axioms10020071]. Moreover, all the fixed point theorems are obtained for F-contraction mappings whose F are strictly increasing are replaced by quasi-convexity.

References

[1] S. Banach, *Sur les operations dans les ensembles abstraits et leur application aux equations integrales*, Fund. Math. **3** (1922), 133–181.

[2] S. Cobazas, *b-metric spaces, fixed points and Lipschitz functions*, arXiv:1802.02722v2 (2018), 35 pp.

[3] S. Czerwik, S. *Contraction mappings in b-metric spaces*, Acta Math. Inform. Univ. Ostrav. **1** (1993), 5–11.

[4] M. Delfani, A. P. Farajzadeh, C. F. Wen, *On fixed points theorems for generalized F-contraction mappings via t-property*, J. Nonlinear Convex Anal. **23** (2022), 2511–2521.

[5] A. P. Farajzadeh, M. Delfani, Y. H. Wang, *Existence and uniqueness of fixed points of generalized F-contraction mappings* J. Math. **2021** (2021), Art. ID 687238, Doi.org/10.1155/2021/6687238.

[6] N. Goswami, N. Haokip, V. N. Mishra, *F-contractive type mappings in b-metric spaces andsome related fixed point results*, Fixed Point Theory Appl. **2019** 2019(1):13, DOI:10.1186/s13663-019-0663-6.

[7] H. Huang, Z. D. Mitrović, k. Zoto, S. Radenović, *On convex F-contraction in b-metric spaces*, Axioms. **2021**, 10(2), 71; https://doi.org/10.3390/axioms10020071.

[8] N. Hussain, V. Parvaneh, J. R. Roshan, Z. Kadelburg, *Fixed points of cyclic (ψ, φ, L, A, B)-contractive mappings in ordered b-metric spaces with applications*, Fixed Point Theory Appl. **2013** (2013), http://dx.doi.org/10.1155/2014/471827.

[9] R. Kannan, *Some results on fixed points* II, Amer. Math. Monthly **76** (1969), 405–408.

[10] A. Taheri, A. Farajzadeh, *A new generalization of α-type almost-F-contractions and α-type F-Suzuki contractions in metric spaces and their fixed point theorems*, Carpathian Math. Publ. **11** (2019), 475–492.

[11] S. Tomonari, *Fixed point theorems for single-valued and set-valued F-contractions in b-metric spaces*, J. Fixed Point Theory Appl. **2018** (2018), 20–35.

[12] D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl. **2012** (2012), Art. ID 94.