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Submanifolds of an extended complex manifold

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Abstract. The present paper is study to investigate some properties of various lifts on an extended complex structure defined over extended complex manifold ${}^kM$. We define the submanifold ${}^kV$ of an extended complex manifold ${}^kM$ and obtain fundamental results of it. Further, we elaborated the conditions of existing distributions of real dimensions of an extended complex manifold ${}^kM$ in the term of partial differential equations. In the last, we define Haantje's tensor on ${}^kM$ and discuss various properties of it.

1. Introduction

The lifting theory has an important place in differentiable geometry because it may be extended through the lift function to geometric structures on any manifold. Tekkoyun et. al. [23, 24], investigated the geometrical properties of an extended complex manifold ${}^kM$ defined over the k-th order of the complex manifold M and developed lifts of functions, vector fields, and 1-forms on M to ${}^kM$.

Khan [11–13, 17, 18] investigated some properties of different lifts on extendeed complex structure on an extended complex manifold ${}^kM$. Nivas et.al. [8, 26, 27] had studied some special structures and defined their properties. Saxena et. al [21, 22] elaborate the properties of hypersuface and decomposition of curvature tensor field. Several geometric structure like Metallic, Super, Hermitian, etc. on the manifold to the tangent bundle are studied by earlier investigators [1–4, 7, 9, 10, 19, 29, 30].

In this paper, we define the submanifold ${}^kV$ of an extended complex manifold ${}^kM$ and obtain fundamental results of it. The conditions of existing distributions of real dimensions of an extended complex manifold ${}^kM$ are elaborated. Finally, Haantje's tensor on ${}^kM$ is defined and also discusses various properties of it.

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2. Preliminaries

Assume that M is a 2m-dimensional real manifold while ^kM is an extended manifold of k orders. If a tensor field J_k of type (1, 1) on ^kM is an endomorphism of the tangent space $Tp(^kM)$, then $(J_k)^2 = -I$ for each point p of ^kM, then J_k is referred to as an extended almost complex structure on ^kM. An extended manifold ^kM with J_k is said to be an extended almost complex manifold [14–16, 24]. If $k = 0$, J_0 is said to be an almost complex structure, hence the manifold $^0M = M$ on which such structure J_0 is defined is called the complex manifold.

Assume that M is any complex manifold and that ^kM is its k^{th} order extension. Let f be a function on M, then the functions f^{v^k} and f^{c^k} represent the vertical and complete lifts of f on ^kM, respectively and defined by [20, 25]

$$f^{v^k} = f \circ \tau M \circ \tau^2 M \circ ... \circ \tau^{k-1} M, \tag{1}$$

and

$$f^{c^k} = \dot{z}^{ri}(\frac{\partial f^{c^k}}{\partial z^{ri}})^v + \dot{\bar{z}}^{ri}(\frac{\partial f^{c^k}}{\partial \bar{z}^{ri}})^v, \tag{2}$$

where $\tau^{k-1}M \rightarrow^k M^{k-1}M$ is a canonical projection.

Properties for both lifts of the complex functions are given as follows:

$$\begin{aligned} i)\ (f+g)^{v^r} &= f^{v^r} + g^{v^r}, (f.g)^{v^r} = f^{v^r}.g^{v^r} \\ ii)\ (f+g)^{c^r} &= f^{c^r} + g^{c^r}, (f.g)^{c^r} = \sum_{j=0}^{r} C_j^r f^{c^{r-j}v^j}.g^{c^j v^{r-j}}, \end{aligned} \tag{3}$$

where f and g are the complex functions and C_j^r is the combination also f^v represent vertical lift and f^c represent complete lift.

Let $\aleph$ is defind as a complex vector field with property

$$\aleph = Z^{0i}\frac{\partial}{\partial z^{ki}} + \bar{Z}^{0i}\frac{\partial}{\partial \bar{z}^{ki}}.$$

Then the local expression of the vertical and complete lifts of $\aleph$ to ^kM in the term of partial differential equations are

$$\aleph^{v^k} = (Z^{0i})^{v^k}\frac{\partial}{\partial z^{ki}} + (\bar{Z}^{0i})^{v^k}\frac{\partial}{\partial \bar{z}^{ki}}. \tag{4}$$

and

$$\aleph^{c^k} = C_j^r (Z^{0i})^{v^{k-r}c^r}\frac{\partial}{\partial z^{ri}} + C_j^r (\bar{Z}^{0i})^{v^{k-r}c^r}\frac{\partial}{\partial \bar{z}^{ri}}. \tag{5}$$

The following are the extended properties for the complete lifts and vertical lifts of complex vector fields:

$$\begin{aligned} i)\ (\aleph + \Re)^{v^r} &= \aleph^{v^r} + \Re^{v^r}, (\aleph + \Re)^{c^r} = \aleph^{c^r} + \Re^{c^r}, \\ ii)\ \aleph^{v^k}(f^{c^k}) &= (\aleph f)^{v^k},\ \ \aleph^{c^k}(f^{c^k}) = (\aleph f)^{c^k}, \\ iii)\ (f\aleph)^{v^r} &= f^{v^r}.\aleph^{v^r}, (f\aleph)^{c^r} = \sum_{j=0}^{r} C_j^r f^{c^{r-j}v^j}.\aleph^{c^j v^{r-j}}, \\ iv)\ \aleph^{v^r}(f^{v^r}) &= 0,\ \aleph^{c^r}(f^{c^r}) = (\aleph f)^{c^r},\ \aleph^{c^r}(f^{v^r}) = \aleph^{v^r}(f^{c^r}) = (\aleph f)^{v^r}, \\ v)\ [\aleph^{v^r}, \Re^{v^r}] &= 0, [\aleph^{c^r}, \Re^{c^r}] = [\aleph, \Re]^{c^r}, [\aleph^{v^r}, \Re^{c^r}] = [\aleph^{c^r}, \Re^{v^r}] = [\aleph, \Re]^{v^r}, \end{aligned} \tag{6}$$

where $\aleph, \Re$ are the complex vector fields.

Let α be a complex 1-form which holds the following condition

$$\alpha = \alpha_{0i} dz^{0i} + \bar{\alpha}_{0i} d\bar{z}^{0i},$$

then the local expression of the complete and vertical lifts of α to ${}^kM$ are, respectively, given by

$$\alpha^{c^k} = (\alpha_{0i})^{c^{k-r}v^r} dz^{ri} + (\bar{\alpha}_{0i})^{c^{k-r}v^r} d\bar{z}^{ri}. \tag{7}$$

$$\alpha^{v^k} = (\alpha_{0i})^{v^k} dz^{0i} + (\bar{\alpha}_{0i})^{v^k} d\bar{z}^{0i}, \tag{8}$$

The following are the extended properties for the complete lifts and vertical lifts of complex 1-forms [5, 28, 32]:

$$\begin{aligned} i)(\alpha+\lambda)^{v^r} &= \alpha^{v^r} + \lambda^{v^r}, (\alpha+\lambda)^{c^r} = \alpha^{c^r} + \lambda^{c^r}, \\ ii) \ \ (f\alpha)^{v^r} &= f^{v^r}\alpha^{v^r}, (f\alpha)^{c^r} = \sum_{j=0}^{r} C^r_j f^{c^{r-j}v^j} \alpha^{c^r v^{r-j}}. \end{aligned} \tag{9}$$

Assume that M is any complex manifold and that ${}^kM$ is its k^{th} order extension. Let F be a (1, 1) tensor field. Then

$$\begin{aligned} \alpha^{c^k}(F^{c^k}) &= (\alpha F)^{c^k}, \ \ \alpha^{v^k}(F^{v^k}) = (\alpha F)^{v^k}, \\ F^{c^k}(\aleph^{c^k}) &= (F\aleph)^{c^k}, \ \ F^{v^k}(\aleph^{c^k}) = (F\aleph)^{v^k}, \end{aligned} \tag{10}$$

where $\aleph$ and α are a vector field and a 1-form respectively.

3. Submanifold of an extended complex manifold

Let ${}^kM$ be the k^{th} order extension of M a complex manifold. If the tensor field F of type (1,1) on ${}^kM$ which satisfies the equation

$$F^2 + I = 0, \tag{11}$$

in the above equation (11) I is an identity tensor field. F is an almost complex structure on extended manifold ${}^kM$ taking lift and conditions on rank are as follows:

$$Rank(F) = \frac{1}{2}(RankF^2 + dim^kM) \tag{12}$$

$$Rank(F) = \frac{1}{3}(RankF^2 + 2 \ times \ dim^kM)$$

Let ${}^kV$ be a C^∞ k-dimensional extended complex manifold is a sub manifold in ${}^kM$ with structure tensor field F satisfying (11) let ϕ be the embedding as follow

$$\phi : \ {}^kV \to {}^kM \tag{13}$$

and Φ be the mapping defined by ϕ, which is,

$$\Phi = d\phi$$

such that,

$$\Phi : \ T({}^kV) \ \to \ T({}^kM), \tag{14}$$

where T^kV and T^kM are respective tangent spaces of ${}^kV$ and ${}^kM$ at $p \in^k V$ and $q \in^k M$.

$T({}^kV, {}^kM)$ be define as tangent vectors to the sub manifold $\phi({}^kV)$ and mapping Φ is defined as

$$\Phi : T({}^kV) \rightarrow T({}^kV, {}^kM), \tag{15}$$

Mapping defined in (15) is isomorphic. All defined vectors over $\phi({}^kV)$ form a vector bundle in the extended manifold ${}^kM$ and if the defined vectors are normal, then the vector bundle will be normal and represented as $N({}^kV, {}^kM)$. The vector bundle induced by the ϕ form $N({}^kV,{}^k M)$ is $N({}^kV)$.

If isomorphism is normal, then

$$\Gamma : N({}^kV) \rightarrow N({}^kV, {}^kM), \tag{16}$$

and the space of all C^∞ tensor fields of type (r, s) associated with $N({}^kV)$ by $\eta^r_s({}^kV)$. Thus,

$$\eta^0_0({}^kV) = \zeta^0_0({}^kV) \tag{17}$$

is the space of all C^∞ function defined on ${}^kV$ while an element $\eta^1_0({}^kV)$ is a C^∞ vector field normal to ${}^kV$ and the element $\zeta^1_0({}^kV)$ is a C^∞ vector field tangential to ${}^kV$.

Vector fields $\overline{\aleph}$ and $\overline{\mathfrak{R}}$ along $\phi({}^kV)$ defined over extended manifold $({}^kM)$. Let $\widetilde{\aleph}$ and $\widetilde{\mathfrak{R}}$ are local extension on $\overline{\aleph}$ and $\overline{\mathfrak{R}}$. Then $[\widetilde{\aleph}, \widetilde{\mathfrak{R}}]$ is a tangential vector field on ${}^kM$ and its restriction $[\widetilde{\aleph}, \widetilde{\mathfrak{R}}]/\phi({}^kV)$ to $\phi({}^kV)$ is determined independently respect to $\widetilde{\aleph}$ and $\widetilde{\mathfrak{R}}$. Therefore we define $[\widetilde{\aleph}, \widetilde{\mathfrak{R}}]$ by

$$[\overline{\aleph}, \overline{\mathfrak{R}}] = [\widetilde{\aleph}, \widetilde{\mathfrak{R}}/\phi({}^kV)]$$

Since Φ is an isomorphism, we have

$$[\Phi\aleph, \Phi\mathfrak{R}] = \Phi[\aleph, \mathfrak{R}]$$

for all $\aleph \in \zeta^1_0({}^kV)$ and $\mathfrak{R} \in \zeta^1_0({}^kV)$, where $\zeta^r_s({}^kV)$ is defined over C^∞ tensor field of type (r, s) connected with $T({}^kV)$.

Definition 3.1. *An k-dimensional C^∞ manifold ${}^kV$ is defined to be an invariant submanifold of ${}^kM$, if the tangent spaces $T_p(\phi({}^kV))$ of $\phi({}^kV)$ is invariant by the linear mapping F at each point P' of $\phi({}^kV)$.*

Thus we have for vector field $\aleph, \mathfrak{R}$ in ${}^kV$, we have

$$\widetilde{F}(\Phi\aleph) = \Phi(F(\aleph)), \tag{18}$$

where F is an almost complex structure defined over extended complex manifold.

Theorem 3.2. *In order that an extended complex manifold ${}^{k+l}M$, it is necessary that its existing distributions D_m, D_n and D_l are of real dimensions m, n, l respectively with the condition $1 \leq m < k$, $m + n = k$ and such that they have no common direction with each other and span simultaneously the linear extended complex manifold of dimension $k + l$.*

Proof: Let eigen values of F be η and corresponding eigen vector Λ, so we have

$$\bar{\Lambda} = \eta\Lambda. \tag{19}$$

Barring above equation and use (3.1) we get as follows

$$\Lambda^2 = \eta^2\Lambda \tag{20}$$

From (20), following cases arise.

Case 1 Let $\Lambda = f$, where f is some scalar. Thus in view of above equation it follows that $\eta = 0$. Hence their are ξ eigen value of f each equal to zero reference to the distribution D_ξ.

Case 2 Λ is linearly independent eigen vector and from (20) we obtained $\eta = \pm\sqrt{-1}$. Let m eigen values be $\sqrt{-1}$ and $(k-m)$ eigen values be $-\sqrt{-1}$.

Let $\mathfrak{U}^1, \mathfrak{U}^2, ..., \mathfrak{U}^m$ be the eigen vectors corresponding to the eigen value $\sqrt{-1}$ and $\mathfrak{V}^m+1, \mathfrak{V}^m+2, ..., \mathfrak{V}^k$ be the eigen vectors corresponding to the eigen value $-\sqrt{-1}$. the eigen vectors defined by $\mathfrak{U}$ will leads the distribution D_m of real dimension m and vectors defined by $\mathfrak{V}$ leads the distribution D_n of real dimension n such that $m+n=k$.

Let us now consider

$$\begin{aligned} 2\mathfrak{L}(\aleph) &= \bar{\bar{\aleph}} + \sqrt{-1}\bar{\aleph} \\ 2\mathfrak{M}(\aleph) &= \bar{\bar{\aleph}} - \sqrt{-1}\bar{\aleph} \\ \mathfrak{G}(\aleph) &= -\bar{\bar{\aleph}} + \aleph. \end{aligned} \tag{21}$$

Thus we have

$$\begin{aligned} \mathfrak{L}(\aleph) + \mathfrak{M}(\aleph) + \mathfrak{G}(\aleph) &= \aleph \\ \mathfrak{L}(\aleph) - \mathfrak{M}(\aleph) &= \sqrt{-1}\bar{\aleph} \\ \mathfrak{L}(\aleph) + \mathfrak{M}(\aleph) &= \bar{\bar{\aleph}}. \end{aligned} \tag{22}$$

Let $[\mathfrak{B}_\aleph, \mathfrak{C}_\mathfrak{R}, \mathfrak{A}_\mathfrak{S}]$ be the inverse of $[\mathfrak{U}^\aleph, \mathfrak{V}^\mathfrak{R}, \mathfrak{T}^\mathfrak{S}]$, where $\aleph = 1,2,3,..,m$, $\mathfrak{R} = m+1, m+2, m+3, ...k$ and $\mathfrak{S} = 1,2,3,...\xi$. Then obviously

$$\begin{aligned} \mathfrak{B}_\aleph(\mathfrak{U}_\aleph^{\aleph^m}) &= \delta_\aleph^{\aleph^m}, \;\; 1 \le \aleph, \aleph^m \le m \\ \mathfrak{C}_\mathfrak{R}(\mathfrak{V}^{\mathfrak{R}^m}) &= \delta_\mathfrak{R}^{\mathfrak{R}^m}, \;\; m+1 \le \mathfrak{R}, \mathfrak{R}^m \le k \\ \mathfrak{A}_n(\mathfrak{T}^\mathfrak{S}) + \delta_n^\mathfrak{S} &= 0 \\ \mathfrak{B}_\aleph(\mathfrak{V}^\mathfrak{R}) = \mathfrak{C}_\mathfrak{R}(\mathfrak{U}^\aleph) = \mathfrak{B}_\aleph(\mathfrak{T}^\mathfrak{S}) = \mathfrak{C}_\mathfrak{R}(\mathfrak{T}^\mathfrak{S}) = \mathfrak{A}_\mathfrak{S}(\mathfrak{U}^\aleph) = \mathfrak{A}_\mathfrak{S}(\mathfrak{V}^\mathfrak{R}) &= 0 \\ \mathfrak{B}_\aleph(\aleph)\mathfrak{U}^\aleph + \mathfrak{C}_\mathfrak{R}(\aleph)\mathfrak{V}^\mathfrak{R} - \mathfrak{A}_\mathfrak{S}(\aleph)\mathfrak{T}^\mathfrak{S} &= \aleph. \end{aligned} \tag{23}$$

Now using last equation of (23) and conditions of extended complex manifold along with the fact that $\mathfrak{T}^\mathfrak{S}$ are eigen vector corresponding the zero eigen value we obtain the following equation

$$(\sqrt{-1})\mathfrak{B}_\aleph(\aleph)\mathfrak{U}^\aleph - (\sqrt{-1})\mathfrak{C}_\mathfrak{R}(\aleph)\mathfrak{V}^\mathfrak{R} = \bar{\bar{\aleph}}. \tag{24}$$

Using equation(24), also considering the fact that $\mathfrak{U}^\aleph$ and $\mathfrak{V}^\mathfrak{R}$ are eigen vectors and their corresponding eigen values are $\sqrt{-1}$ and $-\sqrt{-1}$ respectively therefore we get the following result

$$\sqrt{-1}\mathfrak{B}_\aleph(\aleph)\mathfrak{U}^\aleph + \sqrt{-1}\mathfrak{C}_\mathfrak{R}(\aleph)\mathfrak{V}^\mathfrak{R} = \bar{\bar{\aleph}}. \tag{25}$$

Considering (22), (23), (24), (25)we have following set of equations,

$$\begin{aligned} \mathfrak{L}(\aleph) &= \mathfrak{B}_\aleph(\aleph)\mathfrak{U}^\aleph \\ \mathfrak{M}(\aleph) &= \mathfrak{C}_\mathfrak{R}(\aleph)\mathfrak{V}^\mathfrak{R} \\ \mathfrak{G}(\aleph) &= -\mathfrak{A}_\mathfrak{S}(\aleph)\mathfrak{T}^\mathfrak{S}. \end{aligned} \tag{26}$$

Considering equation (23) and (26)

$$\begin{aligned}\mathfrak{L}(\mathfrak{V}^{\aleph}) &= \mathfrak{U}^{\aleph}\\ \mathfrak{M}(\mathfrak{V}^{\Re}) &= \mathfrak{V}^{\Re}\\ \mathfrak{G}(\mathfrak{T}^{\mathfrak{S}}) &= \mathfrak{T}^{\mathfrak{S}}.\end{aligned} \tag{27}$$

Further more we get the following condition

$$\mathfrak{L}(\mathfrak{V}^{\Re}) = \mathfrak{M}(\mathfrak{U}^{\aleph}) = \mathfrak{L}(\mathfrak{T}^{\mathfrak{S}}) = \mathfrak{M}(\mathfrak{T}^{\mathfrak{S}}) = \mathfrak{G}(\mathfrak{U}^{\aleph}) = 0. \tag{28}$$

Also we have,

$$\begin{aligned}\mathfrak{L}^2(\aleph) &= \mathfrak{L}(\aleph)\\ \mathfrak{M}^2(\aleph) &= \mathfrak{M}(\aleph)\\ \mathfrak{G}^2(\aleph) &= \mathfrak{G}(\aleph).\end{aligned} \tag{29}$$

Further,

$$\begin{aligned}\mathfrak{L}(\mathfrak{M}(\aleph)) = \mathfrak{M}(\mathfrak{L}(\aleph)) &= \mathfrak{L}(\mathfrak{G}(\aleph)) = \mathfrak{M}(\mathfrak{G}(\aleph))\\ &= \mathfrak{G}(\mathfrak{L}(\aleph)) = \mathfrak{G}(\mathfrak{M}(\aleph)) = 0\end{aligned} \tag{30}$$

4. Some calculations of Haantje's tensor on an extended complex manifold

Tensor field $\widetilde{H}$ of type (1,2) defined over ${}^kM$ is Haantje's tensor field [31] if it follow the condition defined as

$$\widetilde{H}(\widetilde{\aleph}, \widetilde{\Re}) = \widetilde{N}(\widetilde{\aleph}, \widetilde{\Re}) - \widetilde{N}(\widetilde{m}\widetilde{\aleph}, \widetilde{\Re}) - \widetilde{N}(\widetilde{\aleph}, \widetilde{m}\widetilde{\Re}) + \widetilde{N}(\widetilde{m}\widetilde{\aleph}, \widetilde{m}\widetilde{\Re}), \tag{31}$$

where $\widetilde{N}$ is Nijenhuise tensor.

Theorem 4.1. *Tensor field $\widetilde{H}$ defined over extended manifold ${}^kM$ satisfies*

$$\widetilde{H}(\Phi\aleph, \Phi\Re) = \widetilde{N}(\Phi\aleph, \Phi\Re) = \Phi(N(\aleph, \Re)), \tag{32}$$

Proof: Any vector field which is tangential to extended complex submanifold ${}^kV$ is not connected in the distribution ${}^kD$, we have

$$\widetilde{m}(\Phi, \aleph) = 0 \tag{33}$$

where $\aleph$ is any vector field in ${}^kV$. Considering the property of Nijenhuise tensor we have

$$\widetilde{H}(\Phi\aleph, \Phi\Re) = \Phi(N(\aleph, \Re)).$$

Definition 4.2. *A Haantje's operator is a (1,1) tensor whose Haantje's torsion vanishes [6].*

Let us define a tensor Γ which is Haaantje's operator if Γ takes a diagonal form on the defined chart $\aleph = (\aleph^1, \aleph^2, ..., \aleph^n)$ in the term of partial differential equations

$$\Gamma(\aleph) = \sum_{i=1}^{n} \beta_i(\aleph) \frac{\partial}{\partial \aleph_i} \otimes d\aleph^i, \tag{34}$$

where $\beta_i(\aleph) = \beta_i^i(\aleph)$ are the eigen values of Γ and partial differential equations $(\frac{\partial}{\partial \aleph^1}, ... \frac{\partial}{\partial \aleph^n})$ are the fields forming natural frame associated with the local chart $(\aleph^1, \aleph^2, ..., \aleph^n)$.

Combining the above results leads to the theorem.

Theorem 4.3. *An extended complex submanifold ^kV embedded in a extended complex manifold ^kM such that the distribution ^kD is never tangential to ϕ^kV is the manifold with induced extended structure the the Haantje's tensor vanishes in ^kM, if and only if it vanishes in ^kV.*

Theorem 4.4. *Let M be any complex manifold and ^kM be an extended complex manifold. Then*

$$(H_F)^{C^k} = H_{F^{C^k}}, \tag{35}$$

where $\aleph, \mathfrak{R}$ are arbitrary extended complex vector fields.

Proof: Taking lifts on the both side of the equation (4.1) and using equation (2.9), then

$$\begin{aligned}(H_F)^{C^k} &= (N[\aleph,\mathfrak{R}] - N[N\aleph,\mathfrak{R}] - N[\aleph,N\mathfrak{R}] + N[N\aleph,N\mathfrak{R}])^{C^k},\\ &= ([F\aleph,F\mathfrak{R}] + F^2[\aleph,\mathfrak{R}] - F[\aleph,F\mathfrak{R}] - F[F\aleph,\mathfrak{R}])^{C^k},\\ &= [F^{C^k}\aleph^{C^k},F^{C^k}\mathfrak{R}^{C^k}] + (F^{C^k})^2[\aleph,\mathfrak{R}] - F^{C^k}[\aleph^{C^k},F^{C^k}\mathfrak{R}^{C^k}]\\ &\quad -F^{C^k}[F^{C^k}\aleph^{C^k},\mathfrak{R}^{C^k}],\\ &= H_{F^{C^k}}.\end{aligned}$$

Hence prove the theorem.

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