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# On the higher power moments of Dirichlet coefficients of symmetric power $L$-functions and its applications

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**Abstract.** Let $f$ be a normalized primitive holomorphic cusp form of even integral weight for the full modular group $\Gamma = SL(2,\mathbb{Z})$. Let $\lambda_{\mathrm{sym}^j f}(n)$ denote the $n$-th normalized coefficients of the Dirichlet expansion of the $j$-th symmetric power $L$-function $L(\mathrm{sym}^j f, s)$. In this paper, we are interested in the average behaviour of the higher moments of $\lambda_{\mathrm{sym}^j f}(n)$ for $j \geqslant 2$, which refines the previous results in this direction. As an application, we also consider the number of sign changes of the sequence $\{\lambda_{\mathrm{sym}^j f}(n)\}$ for $j \geqslant 3$ in the interval $(x, 2x]$.

## 1. Introduction

The Fourier coefficients of modular forms are important and interesting objects in number theory. Let $H_k^*$ be the set of all normalized primitive holomorphic cusp forms of even integral weight $k \geqslant 2$ for the full modular group $\Gamma = SL(2,\mathbb{Z})$. Then, the Hecke eigenform $f \in H_k^*$ has the following Fourier expansion at the cusp $\infty$:

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e^{2\pi i n z}, \qquad \Im(z) > 0,$$

where $\lambda_f(n)$ is the $n$-th normalized Fourier coefficient (Hecke eigenvalue) such that $\lambda_f(1) = 1$. Then, $\lambda_f(n)$ is real and satisfy the multiplicative property

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f\Big(\frac{mn}{d^2}\Big), \tag{1}$$

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where $m, n \geqslant 1$ are integers. In 1974, Deligne [5] proved the Ramanujan-Petersson conjecture

$$|\lambda_f(n)| \leqslant d(n), \tag{2}$$

where $d(n)$ is the divisor function. By (2), Deligne's bound is equivalent to the fact that there exist $\alpha_f(p), \beta_f(p) \in \mathbb{C}$ satisfying

$$\alpha_f(p) + \beta_f(p) = \lambda_f(n), \qquad \alpha_f(p)\beta_f(p) = |\alpha_f(p)| = |\beta_f(p)| = 1. \tag{3}$$

More generally, for all integers $\ell \geqslant 1$, one has

$$\lambda_f(p^\ell) = \alpha_f(p)^\ell + \alpha_f(p)^{\ell-1}\beta_f(p) + \cdots + +\alpha_f(p)\beta_f(p)^{\ell-1} + \beta_f(p)^\ell.$$

It is generally conjectured that the $j$-th symmetric power $L$-function $L(\mathrm{sym}^j f, s)$ for all $j \geqslant 1$ is an entire function and satisfies certain Riemann type functional equation, which is a special case of the Langlands functoriality and this conjecture has recently been settled by the breakthrough works of Newton and Thorne [27, 28]. Let $\lambda_{\mathrm{sym}^j f}(n)$ denote the $n$-th normalized coefficient of the Dirichlet expansion of the $j$-th symmetric power $L$-function. Fomenko [6] proved that

$$\sum_{n \leqslant x} \lambda_{\mathrm{sym}^2 f}(n) \ll x^{\frac{1}{2}} (\log x)^2.$$

Later, this sum has been studied by many authors (see, e.g., [15, 20, 36]). The analogous cases for symmetric power lifting $\mathrm{sym}^j \pi_f$ for large $j$ were considered by Lau and Lü [21], and Tang and Wu [39].

On the other hand, Fomenko [7] studied the sum of $\lambda^2_{\mathrm{sym}^2 f}(n)$. Later, this result has been improved and generalized by a number of authors (see, e.g., [9, 22, 37, 38]). Recently, Sankaranarayanan, Singh and Srinivas [37] proved that

$$\begin{aligned}
\sum_{n \leqslant x} \lambda^2_{\mathrm{sym}^3 f}(n) &= c_1 x + O\Big(x^{\frac{15}{17}+\varepsilon}\Big), \\
\sum_{n \leqslant x} \lambda^2_{\mathrm{sym}^4 f}(n) &= c_2 x + O\Big(x^{\frac{12}{13}+\varepsilon}\Big),
\end{aligned}$$

where $c_1, c_2 > 0$ are some suitable constants. More recently, Luo et al. [22, Theorem 1.1] established the following

$$\sum_{n \leqslant x} \lambda^j_{\mathrm{sym}^2 f}(n) = x P_j(\log x) + O\Big(x^{\theta_j+\varepsilon}\Big),$$

where $P_j(t)$ is a polynomial in $t$ with $\deg P_3 = 0, \deg P_4 = 2, \deg P_5 = 5, \deg P_6 = 14, \deg P_7 = 35$ and $\deg P_8 = 90$, and the exponents are given by

$$\begin{aligned}
\theta_3 &= \frac{971}{1055}, \qquad \theta_4 = \frac{262}{269}, \qquad \theta_5 = \frac{3237}{3265}, \\
\theta_6 &= \frac{4923}{4937}, \qquad \theta_7 = \frac{7442}{7449}, \qquad \theta_8 = \frac{89771}{89799}.
\end{aligned}$$

Furthermore, in the same paper, they [22, Theorem 1.2] also established the following asymptotic formulae

$$\begin{aligned}
\sum_{n \leqslant x} \lambda^2_{\mathrm{sym}^j f}(n) &= \widetilde{c_j} x + O\Big(x^{\widetilde{\theta_j}+\varepsilon}\Big), \qquad 3 \leqslant j \leqslant 6, \\
\sum_{n \leqslant x} \lambda^2_{\mathrm{sym}^j f}(n) &= \widetilde{c_j} x + O\Big(x^{\widetilde{\theta_j}}\Big), \qquad j = 7, 8,
\end{aligned} \tag{4}$$

where $\widetilde{c_j}, (3 \leqslant j \leqslant 8)$ is a suitable constant, and

$$\begin{aligned}
\widetilde{\theta}_3 &= \frac{551}{635}, \qquad \widetilde{\theta}_4 = \frac{929}{1013}, \qquad \widetilde{\theta}_5 = \frac{1391}{1475}, \\
\widetilde{\theta}_6 &= \frac{979}{1021}, \qquad \widetilde{\theta}_7 = \frac{63}{65}, \qquad \widetilde{\theta}_8 = \frac{40}{41}.
\end{aligned}$$

Define

$$S_j(f;x) := \sum_{n\leqslant x} \lambda^j_{\mathrm{sym}^2 f}(n), \qquad j \geqslant 1.$$

In light of the recent progress of Newton and Thorne [27, 28] that $\mathrm{sym}^j f$ corresponds to a cuspidal automorphic representation of $GL_{j+1}(\mathbb{A}_\mathbb{Q})$ for all $j \geqslant 1$ and the nice analytic properties of these $L$-functions therein, the first purpose in this paper is to refine the results given by [22, Theorem 1.1]. More precisely, we are able to establish the following results.

**Theorem 1.1.** *Let $5 \leqslant j \leqslant 8$ be any fixed integer, and let $f \in H_k^*$ be a normalized Hecke eigenform. Then,*

$$S_j(f;x) = xP_j(\log x) + O\Big(x^{\vartheta_j+\varepsilon}\Big),$$

*where $P_j(t)$ is a polynomial in $t$ of degree $\deg P_5 = 5, \deg P_6 = 14, \deg P_7 = 35$ and $\deg P_8 = 90$, and*

$$\vartheta_5 = \frac{8061}{8131}, \qquad \vartheta_6 = \frac{2451}{2458}, \qquad \vartheta_7 = \frac{14821}{14835}, \qquad \vartheta_8 = \frac{47893}{47908}.$$

**Remark 1.2.** For comparison, we have $\frac{8061}{8131} = 0.99139\cdots < \frac{3237}{3265} = 0.99142\cdots$, $\frac{2451}{2458} = 0.99715\cdots < \frac{4923}{4937} = 0.99716\cdots$, $\frac{14821}{14835} = 0.99905\cdots < \frac{7442}{7449} = 0.99906\cdots$, and $\frac{47893}{47908} = 0.999686\cdots < \frac{89771}{89799} = 0.999688\cdots$, so the results in Theorem 1.1 indeed improves the previous results of Luo et al. [22, Theorem 1.1].

Let

$$S_j^*(f;x) := \sum_{n\leqslant x} \lambda^2_{\mathrm{sym}^j f}(n), \qquad j \geqslant 1.$$

Then, we can prove the following theorem which improves and generalizes the results of Luo et al. [22, Theorem 1.2].

**Theorem 1.3.** *Let $j \leqslant 3$ be any fixed integer, and let $f \in H_k^*$ be a normalized Hecke eigenform. Then,*

$$S_j^*(f;x) = c_{f,j}x + O\Big(x^{1-\frac{210}{105(j+1)^2-103}+\varepsilon}\Big),$$

*where $c_{f,j} > 0$ is some suitable constant depending on $f$ and $j$.*

**Remark 1.4.** In fact, for $j \geqslant 3$, by similar argument as that of Lao and Luo [23, Proposition 3.5], one can also establish the result

$$S_j^*(f;x) = c_{f,j}x + O\Big(x^{1-\frac{84}{42(j+1)^2-37}+\varepsilon}\Big),$$

where $c_{f,j} > 0$ is some suitable constant depending on $f$ and $j$. Obviously, our results improves the results of Luo et al. [22, Theorem 1.2] and Lao-Luo [23, Proposition 3.5], respectively.

In recent times, the sign changes of Fourier coefficients attached to the cusp forms has becomes an important and prominent topic, and has also attracted the attentions of a large number of scholars. Let $f \in H_k^*$ be a normalized Hecke eigenform, using the classical theorem of Landau and certain analytic properties of the associated $L$-functions, one can prove that the sequence $\{\lambda_f(n)\}_{n\in\mathbb{N}}$ has infinitely many sign changes (cf. [19]). The sign changes of subsequence of Fourier coefficients at prime arguments was firstly studied by Ram Murty [25]. Afterwards, Meher et al. [26] provided a quantitative version of sign changes of the sequences $\{\lambda_f(n^j)\}_{n\in\mathbb{N}}, (j = 2, 3, 4)$ in short intervals. More recently, Lao and Luo [23, Theorem 1.1] considered the number of sign changes of the sequences $\{\lambda_f(n^j)\}_{n\in\mathbb{N}}, (j \geqslant 3)$ in the interval $(x, 2x]$, which improved and generalized the results in [26]. In fact, for $j \geqslant 3$, Lao and Luo proved that the sequence $\{\lambda_f(n^j)\}_{n\in\mathbb{N}}$ has at least $\gg x^{1-r}$ sign changes for $1 - \frac{84}{42(j+1)^2-37} < r < 1$.

As an application of Theorem 1.3, in this paper, by combining Theorem 1.3 and a general result of Lao and Luo [23, Lemma 2.3], we are able to prove the following result concerning the sign changes of the sequence $\{\lambda_{\mathrm{sym}^j f}(n)\}_{n\in\mathbb{N}}$ for $j \geqslant 3$ in short intervals.

**Theorem 1.5.** *Let $f \in H_k^*$ be a normalized Hecke eigenform. Then, for $j \geqslant 3$ and $1 - \frac{210}{105(j+1)^2-103} < \eta^* < 1$, the sequence $\{\lambda_{sym^j f}(n)\}_{n\in\mathbb{N}}$ has at least one sign change in the interval $(x, x + x^{\eta^*}]$ for sufficiently large $x$. In particular, the number of sign changes of the same sequence is $\gg x^{1-\eta^*}$ for $n \in (x, 2x]$.*

**Remark 1.6.** In fact, for $j \geqslant 3$, using the similar argument of Lao and Luo [23, Theorem 1.1], one can also prove that the sequence $\{\lambda_{\mathrm{sym}^j f}(n)\}_{n\in\mathbb{N}}$ has at least $\gg x^{1-\eta^*}$ sign changes for $1 - \frac{84}{42(j+1)^2-37} < r < 1$. Clearly, our result in Theorem 1.5 improves that of Lao and Luo.

Throughout the paper, we always assume that $f \in H_k^*$ a normalized Hecke eigenform. And denote by $\varepsilon > 0$ an arbitrarily small positive constant that may vary in different occurrences.

## 2. Preliminaries

In this section, we introduce some background on the analytic properties of automorphic $L$-functions, and give some useful lemmas which play important roles in the course of the proof of the main results in this paper.

Let $f \in H_k^*$ be a normalized Hecke eigenform, and let $\lambda_f(n)$ denote its $n$-th normalized Fourier coefficient. The Hecke $L$-function $L(f,s)$ associated to $f$ is defined as

$$\begin{aligned} L(f,s) &= \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \lambda_f(p)p^{-s} + p^{-2s}\right)^{-1} \\ &= \prod_p \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1}\left(1 - \frac{\beta_f(p)}{p^s}\right)^{-1}, \qquad \Re(s) > 1, \end{aligned}$$

where $\alpha_f(p), \beta_f(p)$ are the local parameters satisfying (3). The $j$-th symmetric power $L$-function $L(\mathrm{sym}^j f, s)$ associated to $f$ is defined as

$$L(\mathrm{sym}^j f, s) = \prod_p \prod_{m=0}^{j} \left(1 - \alpha_f(p)^{j-m}\beta_f(p)^m p^{-s}\right)^{-1}, \qquad \Re(s) > 1.$$

We may expand it into a Dirichlet series and also a Euler product

$$L(\mathrm{sym}^j f, s) = \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n)}{n^s} = \prod_p \left(1 + \sum_{r\geqslant 1} \frac{\lambda_{\mathrm{sym}^j f}(p^r)}{p^{rs}}\right), \qquad \Re(s) > 1.$$

Obviously, $\lambda_{\mathrm{sym}^j f}(n)$ is a real multiplicative function. For $j = 1$, we have $L(\mathrm{sym}^1 f, s) = L(f,s)$. Similarly, the Rankin-Selberg $L$-function $L(\mathrm{sym}^i f \times \mathrm{sym}^j f, s)$ attached to $\mathrm{sym}^i f$ and $\mathrm{sym}^j f$ can defined as

$$\begin{aligned} L(\mathrm{sym}^i f \times \mathrm{sym}^j f, s) &= \prod_p \prod_{m=0}^{i} \prod_{m'=0}^{j} \left(1 - \frac{\alpha_f(p)^{i-m}\beta_f(p)^m \alpha_f(p)^{j-m'}\beta_f(p)^{m'}}{p^s}\right)^{-1} \\ &= \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j f}(n)}{n^s} \qquad \Re(s) > 1. \end{aligned} \tag{5}$$

It is standard to find that

$$\lambda_f(p^j) = \lambda_{\mathrm{sym}^j f}(p) = \frac{\alpha_f(p)^{j+1} - \beta_f(p)^{j+1}}{\alpha_f(p) - \beta_f(p)} = \sum_{m=0}^{j} \alpha_f(p)^{j-m}\beta_f(p)^m,$$

which can be rewritten as

$$\lambda_f(p^j) = \lambda_{\mathrm{sym}^j f}(p) = \widetilde{U}_j(\lambda_f(p)/2), \tag{6}$$

where $\widetilde{U}_j(x)$ is the $j$-th Chebyshev polynomial of second kind. For any prime number $p$, we also have

$$\lambda_{\mathrm{sym}^i f\times \mathrm{sym}^j f}(n) = \lambda_{\mathrm{sym}^i f}(n)\lambda_{\mathrm{sym}^j f}(n) = \lambda_f(p^i)\lambda_f(p^j). \tag{7}$$

As is well-known, to a primitive form $f$ one can associated to an automorphic cuspidal representation $\pi_f$ of $GL_2(\mathbb{A}_{\mathbb{Q}})$, and hence an automorphic $L$-function $L(\pi_f, s)$ which coincides with $L(f, s)$. It is predicted that $\pi_f$ gives rise to a symmetric power lift–an automorphic representation whose $L$-function is the symmetric power $L$-function attached to $f$.

For $1 \leqslant j \leqslant 8$, the Langlands functoriality conjecture which states that $\mathrm{sym}^j f$ is a cuspidal automorphic representation has been established in a series of important works of Gelbart and Jacquet [8], Kim [18], Kim and Shahidi [16, 17], Shahidi [35], Clozel and Thorne [2–4]. Very recently, Nowton and Thorne [27, 28] proved that $\mathrm{sym}^j f$ corresponds to a cuspidal automorphic representation of $GL_{j+1}(\mathbb{A}_{\mathbb{Q}})$ for all $j \geqslant 1$ (with $f$ being a holomorphic cusp form). From the works of the Rankin-Selberg convolution $L$-functions developed by Jacquet, Piatetski-Shapiro and Shalika [14], Jacquet and Shalika [12, 13], Shahidi [31–34], and the reformulation of Rudnick and Sarnak [30], we know that $L(\mathrm{sym}^j f, s)$ and $L(\mathrm{sym}^i f \times \mathrm{sym}^j f, s)$, $(1 \leqslant i \leqslant j)$ has the analytic continuations to the whole $\mathbb{C}$-plane (except possibly for simple poles at $s = 0, 1$ whenever $i = j$) and satisfy certain Riemann-type functional equations. We refer the interested reader to [11, Chapter 5] for a more comprehensive exposition.

We define the generating function $\mathfrak{L}_j(f, s)$ via

$$\mathfrak{L}_j(f, s) = \sum_{n=1}^{\infty} \frac{\lambda^j_{\mathrm{sym}^2 f}(n)}{n^s},$$

for $\Re(s) > 1$ and any given integer $j \geqslant 1$.

**Lemma 2.1.** *Let $f \in H_k^*$ be a normalized Hecke eigenform. For $5 \leqslant j \leqslant 8$ being a fixed integer, we have*

$$\mathfrak{L}_j(f, s) = F_j(s)U_j(s),$$

*where*

$$\begin{aligned}
F_5(s) &= \zeta(s)^6 L(\mathrm{sym}^2 f, s)^{15} L(\mathrm{sym}^4 f, s)^{15} L(\mathrm{sym}^6 f, s)^{10} L(\mathrm{sym}^8 f, s)^4 L(\mathrm{sym}^{10} f, s),\\
F_6(s) &= \zeta(s)^{15} L(\mathrm{sym}^2 f, s)^{36} L(\mathrm{sym}^4 f, s)^{40} L(\mathrm{sym}^6 f, s)^{29} L(\mathrm{sym}^8 f, s)^{15} L(\mathrm{sym}^{10} f, s)^5 L(\mathrm{sym}^{12} f, s),\\
F_7(s) &= \zeta(s)^{36} L(\mathrm{sym}^2 f, s)^{91} L(\mathrm{sym}^4 f, s)^{105} L(\mathrm{sym}^6 f, s)^{84} L(\mathrm{sym}^8 f, s)^{39} L(\mathrm{sym}^{10} f, s)^{21} L(\mathrm{sym}^{12} f, s)^6 L(\mathrm{sym}^{14} f, s),\\
F_8(s) &= \zeta(s)^{91} L(\mathrm{sym}^2 f, s)^{232} L(\mathrm{sym}^4 f, s)^{280} L(\mathrm{sym}^6 f, s)^{238} L(\mathrm{sym}^8 f, s)^{154} L(\mathrm{sym}^{10} f, s)^{76} L(\mathrm{sym}^{12} f, s)^{28}\\
&\quad \cdot L(\mathrm{sym}^{14} f, s)^7 L(\mathrm{sym}^{16} f, s).
\end{aligned}$$

*The L-series $\mathfrak{L}_j(f, s)$ is of degree $3^j$, and the function $U_j(s)$ admits a Dirichlet series which converges uniformly and absolutely in the half-plane $\Re(s) \geqslant \frac{1}{2} + \varepsilon$ and $U_j(s) \neq 0$ on $\Re(s) = 1$.*

**Proof.** The results follows from the similar argument of [22, Lemma 2.1], by noting the Hecke relation (1) and identities (6), (7).

For simplicity, we only consider the case for $j = 8$, since other cases can be handled by the similar approach. From [21, (13)], we know that

$$L(\mathrm{sym}^a f \times \mathrm{sym}^b f, s) = \prod_{r=0}^{b} L(\mathrm{sym}^{a+b-2r} f, s) \tag{8}$$

for any $a \geqslant b \geqslant 1$. Then, the result is the combination of the identity (8) and [22, Lemma 2.1]. □

Define

$$\mathfrak{L}^*_j(f, s) = \sum_{n=1}^{\infty} \frac{\lambda^2_{\mathrm{sym}^j f}(n)}{n^s}, \qquad \Re(s) > 1.$$

We have the following lemma concerning the decomposition of $\mathfrak{L}_j^*(f,s)$.

**Lemma 2.2.** *Let $f \in H_k^*$ be a normalized Hecke eigenform. For any given integer $j \geqslant 8$, we have*

$$\mathfrak{L}_j^*(f,s) = G_j(s)U_j^*(s),$$

*where*

$$G_j(s) = \zeta(s)\prod_{i=1}^{j} L(\mathrm{sym}^{2r}f, s),$$

*where the function $U_j^*(s)$ admits a Dirichlet series which converges uniformly and absolutely in the half-plane $\Re(s) \geqslant \frac{1}{2} + \varepsilon$ and $U_j^*(s) \neq 0$ on $\Re(s) = 1$.*

**Proof.** This result follows from the same argument as in the proof of Lemma 2.1 by noting the relation

$$\lambda^2_{\mathrm{sym}^j f}(p) = \lambda_f^2(p^j) = 1 + \sum_{r=1}^{j} \lambda_{\mathrm{sym}^{2r} f}(p). \qquad \square$$

In the following, we record a number of results of the subconvexity and convexity bounds for the associated $L$-functions available in the literature, which contributes the strengths of our results.

**Lemma 2.3.** *For any $\varepsilon > 0$, one has*

$$\int_1^T \left|\zeta\Big(\frac{5}{7} + it\Big)\right|^{12} dt \ll T^{1+\varepsilon},$$

*uniformly for $T \geqslant 1$.*

**Proof.** The result can be found in [10, Theorem 8.4]. $\square$

**Lemma 2.4.** *For any $\varepsilon > 0$, we have*

$$\begin{aligned} \zeta(\sigma + it) &\ll \Big(1 + |t|\Big)^{\max\{\frac{13}{42}(1-\sigma),0\}+\varepsilon}, \\ L(\mathrm{sym}^2 f, \sigma + it) &\ll \Big(1 + |t|\Big)^{\max\{\frac{6}{5}(1-\sigma),0\}+\varepsilon}, \end{aligned}$$

*uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 2$ and $|t| \geqslant 1$. .*

**Proof.** The first result is the recent breakthrough of Bourgain [1, Theorem 5], and the second result follows from the impressive work of Lin, Nunes and Qi [24, Corollary 1.2] and the Phragmén-Lindelöf convexity principle for a strip [11, Theorem 5.53]. $\square$

We state some basic definitions and analytic properties about general $L$-functions. A general $L$-function $L(\phi, s)$ is a Dirichlet series (associated to the object $\phi$) that admits an Euler product of degree $m \geqslant 1$, namely

$$L(\phi, s) = \sum_{n=1}^{\infty} \frac{\lambda_\phi(n)}{n^s} = \prod_{p<\infty}\prod_{j=1}^{m}\Big(1 - \frac{\alpha_\phi(p,j)}{p^s}\Big)^{-1},$$

where $\alpha_\phi(p,j), (j = 1, 2, \cdots, m)$ are the local parameters of $L(\phi, s)$ at finite prime $p$. Suppose that this series and its Euler product are absolutely convergent for $\Re(s) > 1$. We denote the gamma factor by

$$L_\infty(\phi, s) = \prod_{j=1}^{m} \pi^{-\frac{s+\mu_\phi(j)}{2}}\Gamma\Big(\frac{s + \mu_\phi(j)}{2}\Big),$$

with local parameters $\mu_\phi(j), (j = 1, 2, \cdots, m)$ of $L(\phi, s)$ at $\infty$. The completed $L$-function $\Lambda(\phi, s)$ is defined as

$$\Lambda(\phi, s) = q(\phi)^{\frac{s}{2}} L_\infty(\phi, s)L(\phi, s),$$

where $q(\phi)$ is the arithmetic conductor of $L(\phi, s)$. We assume that $\Lambda(\phi, s)$ admits an analytic continuation to the whole $\mathbb{C}$-plane and is holomorphic everywhere except for possible poles of finite order at $s = 0, 1$. Furthermore, it satisfies a functional equation of the Riemann zeta-type

$$\Lambda(\phi, s) = \epsilon_\phi \Lambda(\widetilde{\phi}, 1 - s),$$

where $\epsilon_\phi$ is the root number with $|\epsilon_\phi| = 1$, and $\widetilde{\phi}$ is the dual of $\phi$ such that $\lambda_{\widetilde{\phi}}(n) = \overline{\lambda_\phi(n)}, L_\infty(\widetilde{\phi}, s) = L_\infty(\phi, s)$ and $q(\widetilde{\phi}) = q(\phi)$. We call the $L$-function satisfy the Ramanujan conjecture if $\lambda_\phi(n) \ll n^\varepsilon$ for any $\varepsilon > 0$.

Form above, we observe that $L(\mathrm{sym}^j f, s), (j \geqslant 1)$ is a general $L$-function in the sense of Perelli [29]. For general $L$-functions, we have the following average and individual convexity bounds.

**Lemma 2.5.** *Assume that $\mathfrak{L}(s)$ is a general L-function of degree m. Then,*

$$\int_T^{2T} \left|\mathfrak{L}(\sigma + it)\right|^2 dt \ll T^{m(1-\sigma)+\varepsilon} \tag{9}$$

*uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 1 + \varepsilon$ and $T \geqslant 1$, and .*

$$\mathfrak{L}(\sigma + it) \ll \left(1 + |t|\right)^{\max\{\frac{m}{2}(1-\sigma), 0\}+\varepsilon} \tag{10}$$

*uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 1 + \varepsilon$ and $|t| \geqslant 1$.*

**Proof.** The results follows from Perelli's mean value theorem and convexity bounds for general $L$-functions as given by [29]. □

In what follows, we give a general criterion for the sign changes of any real sequence $\{a_n b_n c_n\}_{n\in\mathbb{N}}$ due to Lao and Luo [23], here $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are three real sequences.

**Lemma 2.6.** *Let $\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}$ and $\{c_n\}_{n\in\mathbb{N}}$ be three real sequences satisfying*
(i) $a_n = O(n^{\alpha_1}), b_n = O(n^{\alpha_2}), c_n = O(n^{\alpha_3})$,
(ii) $\sum_{n\leqslant x} a_n b_n c_n = O(n^\beta)$,
(iii) $\sum_{n\leqslant x} a_n^2 b_n^2 c_n^2 = cx + O(x^\gamma)$,
*where $\alpha_1, \alpha_2, \alpha_3, \beta, \gamma, c \geqslant 0$. If $\alpha_1 + \alpha_2 + \alpha_3 + \beta < 1$, then for any r satisfying*

$$\max\{\alpha_1 + \alpha_2 + \alpha_2 + \beta, \gamma\} < r < 1,$$

*the sequence $\{a_n b_n c_n\}_{n\in\mathbb{N}}$ has at least one sign change for $n \in (x, x + x^r]$. In particular, the number of sign changes of the sequence $\{a_n b_n c_n\}_{n\in\mathbb{N}}$ for $n \in (x, 2x]$ is at least $\gg x^{1-r}$ for sufficiently large x.*

**Proof.** This result follows from Lao and Luo [23, Lemma 2.3]. □

## 3. Proof of Theorem 1.1

In this section, we only prove the case for $j = 8$, since other cases can be handled in a similar manner. Applying Perron's formula (see [11, Proposition 5.54]) to the generating function $\mathfrak{L}_j(f, s)$, we have

$$\sum_{n\leqslant x} \lambda^8_{\mathrm{sym}^2 f}(n) = \frac{1}{2\pi i} \int_{1+\varepsilon-iT}^{1+\varepsilon+iT} \mathfrak{L}_8(f, s) \frac{x^s}{s} ds + O\Big(\frac{x^{1+\varepsilon}}{T}\Big),$$

where $s = \sigma + it$, and $1 \leqslant T \leqslant x$ is some parameter to be chosen later. It is clear that $\mathfrak{L}_8(f, s)$ can be analytically continued to the half-plane $\Re(s) > \frac{1}{2}$, and it only has a pole at $s = 1$ of order 91. Therefore, by shifting the line of integration to the parallel line with $\Re(s) = \frac{1}{2} + \varepsilon$ and invoking Cauchy's residue theorem, we get

$$\begin{aligned} \sum_{n\leqslant x} \lambda^8_{\mathrm{sym}^2 f}(n) &= \operatorname*{Res}_{s=1}\Big\{\mathfrak{L}_8(f, s)\frac{x^s}{s}\Big\} + \frac{1}{2\pi i}\Big\{\int_{\kappa-iT}^{\kappa+iT} + \int_{\kappa+iT}^{\eta+iT} + \int_{\eta-iT}^{\kappa-iT}\Big\}\mathfrak{L}_8(f, s)\frac{x^s}{s} ds + O\Big(\frac{x^{1+\varepsilon}}{T}\Big) \\ &:= xP_8(\log x) + I_1 + I_2 + I_3 + O\Big(\frac{x^{1+\varepsilon}}{T}\Big), \end{aligned} \tag{11}$$

where $\kappa = \frac{1}{2} + \varepsilon, \eta = 1 + \varepsilon$, and $P_8(t)$ is a polynomial in $t$ of degree 90.

Now, we need to evaluate the integrals $I_1, I_2$ and $I_3$, respectively. For the sake of simplicity, by Lemma 2.1, we write

$$\mathfrak{L}_8(f,s) = F_8(s)U_8(s) = \zeta(s)^{91}L(\mathrm{sym}^2 f,s)^{232}L_8^*(f,s)U_8(s),$$

where

$$L_8^*(f,s) := L(\mathrm{sym}^4 f,s)^{280}L(\mathrm{sym}^6 f,s)^{238}L(\mathrm{sym}^8 f,s)^{154}L(\mathrm{sym}^{10} f,s)^{76}L(\mathrm{sym}^{12} f,s)^{28}L(\mathrm{sym}^{14} f,s)^{7}L(\mathrm{sym}^{16} f,s).$$

It is not hard to find that the $L$-function $L_8^*(f,s)$ is of degree $3^8 - 787 = 5774$.

For $I_1$, by lemma 2.4 and (9), we get

$$\begin{aligned}
I_1 &\ll x^{\frac{1}{2}+\varepsilon}\int_1^T \left|\mathfrak{L}_8(f,\kappa+it)\right|t^{-1}dt + x^{\frac{1}{2}+\varepsilon} \\
&\ll x^{\frac{1}{2}+\varepsilon}\log T \max_{1\leqslant T_1\leqslant T/2}\left(\max_{T_1\leqslant t\leqslant 2T_1} T_1^{-1}\left|\zeta(\kappa+it)\right|^{91}\left(\left|L(\mathrm{sym}^2 f,\kappa+iT_1)\right|^{462}\int_{T_1}^{2T_1}\left|L(\mathrm{sym}^2 f,\kappa+it)\right|^2 dt\right)^{\frac{1}{2}}\right. \\
&\quad \left.\cdot\left(\int_{T_1}^{2T_1}\left|L_8^*(f,\kappa+it)\right|^2 dt\right)^{\frac{1}{2}}\right) + x^{\frac{1}{2}+\varepsilon} \\
&\ll x^{\frac{1}{2}+\varepsilon}T^{\frac{13}{42}\cdot\frac{1}{2}\cdot 91+(\frac{6}{5}\cdot\frac{1}{2}\cdot 462+3\cdot\frac{1}{2})\cdot\frac{1}{2}+5774\cdot\frac{1}{2}\cdot\frac{1}{2}-1+\varepsilon} + x^{\frac{1}{2}+\varepsilon} \\
&\ll x^{\frac{1}{2}+\varepsilon}T^{\frac{23939}{15}+\varepsilon}. \qquad (12)
\end{aligned}$$

The integrals $I_2$ and $I_3$ over the horizontal segments can be treated in a similar manner. By Lemma 2.4 and (10), we have

$$\begin{aligned}
I_2 + I_3 &\ll \int_\kappa^\eta x^\sigma\left|\mathfrak{L}_8(f,\sigma+it)\right|T^{-1}d\sigma \\
&\ll \max_{\kappa\leqslant\sigma\leqslant\eta} x^\sigma T^{(\frac{13}{42}\cdot 91+\frac{6}{5}\cdot 232+5774\cdot\frac{1}{2})(1-\sigma)+\varepsilon}T^{-1} \\
&\ll \frac{x^{1+\varepsilon}}{T} + x^{\frac{1}{2}+\varepsilon}T^{\frac{95747}{60}+\varepsilon}. \qquad (13)
\end{aligned}$$

Combining the estimates (18)-(13), we obtain

$$\sum_{n\leqslant x}\lambda^8_{\mathrm{sym}^2 f}(n) = xP_8(\log x) + O\left(x^{\frac{1}{2}+\varepsilon}T^{\frac{23939}{15}+\varepsilon}\right) + O\left(\frac{x^{1+\varepsilon}}{T}\right). \qquad (14)$$

On taking $T = x^{\frac{15}{47908}}$ in (14), we have

$$\sum_{n\leqslant x}\lambda^8_{\mathrm{sym}^2 f}(n) = xP_8(\log x) + O\left(x^{\frac{47893}{47908}+\varepsilon}\right).$$

## 4. Proof of Theorem 1.3

Let $j \geqslant 3$ be any given integer. Applying Perron's formula, and shifting the line of integration to the parallel line with $\Re(s) = \kappa' := \frac{5}{7}$, along with Cauchy's residue theorem, we obtain

$$\begin{aligned}
\sum_{n\leqslant x}\lambda^2_{\mathrm{sym}^j f}(n) &= \operatorname*{Res}_{s=1}\left\{\mathfrak{L}_j(f,s)\frac{x^s}{s}\right\} + \frac{1}{2\pi i}\left\{\int_{\kappa'-iT}^{\kappa'+iT} + \int_{\kappa'+iT}^{\eta+iT} + \int_{\eta-iT}^{\kappa'-iT}\right\}\mathfrak{L}_j^*(f,s)\frac{x^s}{s}ds + O\left(\frac{x^{1+\varepsilon}}{T}\right) \\
&:= c_{f,j}x + J_1 + J_2 + J_3 + O\left(\frac{x^{1+\varepsilon}}{T}\right), \qquad (15)
\end{aligned}$$

where $c_{f,j} > 0$ is some suitable constant depending on $f$ and $j$, and $s = \sigma + it, \eta = 1 + \varepsilon$, and $1 \leqslant T \leqslant x$ is some parameter to be specified later.

Now, it remains to estimate the integrals $J_1, J_2$ and $J_3$, respectively. For simplicity, by Lemma 2.2, we write

$$\mathfrak{L}_j^*(f,s) = G_j(s)U_j^*(s) := \zeta(s)L(\mathrm{sym}^2 f,s)L(\mathrm{sym}^4 f,s)G_j^*(s)U_j^*(s),$$

where

$$G_j^*(s) := \prod_{r=3}^{j} L(\mathrm{sym}^{2r} f,s).$$

For $J_1$, by Lemmas 2.3-2.4 and (9), together with Hölder's inequality, we have

$$\begin{aligned} J_1 &\ll x^{\frac{5}{7}+\varepsilon}\int_1^T \left|\mathfrak{L}_j^*(f,\kappa'+it)\right|t^{-1}dt + x^{\frac{5}{7}+\varepsilon} \\ &\ll x^{\frac{5}{7}+\varepsilon}\log T \max_{1\leqslant T_1\leqslant T/2}\Big(T_1^{-1}\left|L(\mathrm{sym}^2 f,\kappa'+iT_1)\right|\Big(\int_{T_1}^{2T_1}\left|\zeta(\kappa'+it)\right|^{12}dt\Big)^{\frac{1}{12}}\Big(\left|L(\mathrm{sym}^4 f,\kappa'+it)\right|^{\frac{2}{5}} \\ &\quad\cdot\int_{T_1}^{2T_1}\left|L(\mathrm{sym}^4 f,\kappa'+it)\right|^2 dt\Big)^{\frac{5}{12}}\Big(\int_{T_1}^{2T_1}\left|G_j^*(f,\kappa'+it)\right|^2 dt\Big)^{\frac{1}{2}}\Big) + x^{\frac{5}{7}+\varepsilon} \\ &\ll x^{\frac{5}{7}+\varepsilon}T^{\frac{6}{5}\cdot\frac{2}{7}+\frac{1}{12}+(\frac{2}{5}\cdot\frac{2}{7}\cdot\frac{5}{2}+5\cdot\frac{2}{7})\cdot\frac{5}{12}+((j+1)^2-9)\cdot\frac{2}{7}\cdot\frac{1}{2}-1} + x^{\frac{5}{7}+\varepsilon} \\ &\ll x^{\frac{5}{7}+\varepsilon}T^{\frac{(j+1)^2}{7}-\frac{481}{420}+\varepsilon}. \end{aligned} \tag{16}$$

For $J_2$ and $J_3$, by Lemma 2.4 and (10), we get

$$\begin{aligned} J_2 + J_3 &\ll \int_{\kappa'}^{\eta} x^{\sigma}\left|\mathfrak{L}_j^*(f,\sigma+it)\right|T^{-1}d\sigma \\ &\ll \max_{\kappa'\leqslant\sigma\leqslant\eta} x^{\sigma}T^{(\frac{13}{42}+\frac{6}{5}+((j+1)^2-4)\cdot\frac{1}{2})(1-\sigma)}T^{-1} \\ &\ll \frac{x^{1+\varepsilon}}{T} + x^{\frac{5}{7}+\varepsilon}T^{\frac{(j+1)^2}{7}-\frac{838}{735}+\varepsilon}. \end{aligned} \tag{17}$$

Putting together (15)-(17), we obtain

$$\sum_{n\leqslant x}\lambda^2_{\mathrm{sym}^j f}(n) = c_{f,j}x + O\Big(x^{\frac{5}{7}+\varepsilon}T^{\frac{(j+1)^2}{7}-\frac{838}{735}+\varepsilon}\Big) + O\Big(\frac{x^{1+\varepsilon}}{T}\Big). \tag{18}$$

On taking $T = x^{\frac{210}{105(j+1)^2-103}}$ in (18), we have

$$\sum_{n\leqslant x}\lambda^2_{\mathrm{sym}^j f}(n) = c_{f,j}x + O\Big(x^{1-\frac{210}{105(j+1)^2-103}+\varepsilon}\Big).$$

## 5. Proof of Theorem 1.5

Let $j \geqslant 1$ be any fixed integer. From [21, Theorem 1.1(b)], we know that

$$\sum_{n\leqslant x}\lambda_{\mathrm{sym}^j f}(n) \ll x^{1-\frac{2}{j+2}}. \tag{19}$$

By taking $a_n = \lambda_{\mathrm{sym}^j f}(n)$ and $b_n = c_n \equiv 1$, and noting Theorem 1.3 and (19), along with Deligne's bound (2), we can directly get the conclusion.

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