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On recurrent linear relations

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Abstract. In this paper we extend the notions of recurrent vectors and recurrent opertaors to linear relations.

1. Introduction

Let X and Y be two infinite dimensional separable complex Banach spaces. The set of all bounded linear operators from X into Y is denoted by $\mathcal{B}(X, Y)$ with $\mathcal{B}(X) = \mathcal{B}(X, X)$. The most significant concepts related to linear dynamical properties are hypercyclicity, topologically transitive and recurrent. We say that $T \in \mathcal{B}(X)$ is *hypercyclic* if there exists a non-zero vector x in X such that the set

$$Orb(T, x) := \{T^n x : n \in \mathbb{N} \cup \{0\}\}$$

is dense in X. In this case, the vector x is called a *hypercyclic vector* of T. T is *topologically transitive* if for all non-empty open subsets U and V of X, there exists $n \in \mathbb{N} \cup \{0\}$ such that $T^n(U) \cap V \neq \emptyset$. In 1929, Birkhoff proved the equivalence between hypercyclicity and topologically transitive. A bounded operator $T \in \mathcal{B}(X)$ is said to be *recurrent* if for any non-empty open subset U of X, there exists some n in $\mathbb{N}$ such that

$$T^{-n}(U) \cap U \neq \emptyset.$$

A vector x in X is said to be a *recurrent vector* for T or *T-recurrent* if there is a strictly increasing sequence of positive integers $\{k_n\}$ for which $\{T^{k_n} x\}$ converges to x. The set of all recurrent vectors for T is denoted by $Rec(T)$. Then $Rec(T)$ is a G_δ set dense if and only if T is recurrent [10, Proposition 2.1]. For a more comprehensive understanding of recurrent linear operators and its related properties within linear dynamics, we refer the reader to [3, 5, 6, 9, 10, 13] and the reference therein.

Chen et al. [8] and Abakumov et al. [1] extended some dynamic notions to linear relations. In the same direction, we extend the central concept in topological dynamics that is the recurrent concept to the context of linear relations. This work on recurrent linear relations is divided into three parts. In the first

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two parts, we present and we investigate the concept of recurrent vectors of linear relations and the concept of recurrent linear relations. These two concepts extend the notion of recurrent vectors of bounded linear operators and the notion of recurrent bounded linear operators. We study the notion of recurrent linear relations, we prove that the same properties known for a recurrent bounded operator also hold true for recurrent linear relations and we give more characterizations to recurrent linear relations. In the third part, we prove that the point spectrum of the adjoint of a recurrent linear relation is included in the unit circle.

2. Preliminary of linear relations

A *linear relation* or *multivalued linear operator* T on X is a mapping from the subspace $\mathcal{D}(T) := \{x \in X : Tx \text{ is non-empty subset of } X\}$ into $2^X \setminus \emptyset$ the set of all non-empty subset of X provided that

$$T(x+y) = Tx + Ty \quad \text{and} \quad T(\lambda x) = \lambda Tx,$$

for all $x, y \in \mathcal{D}(T)$ and for all non-zero scalar λ. The subspace $\mathcal{D}(T)$ is called the domain of T. By $\mathcal{LR}(X)$ we denote the set of all linear relations on X. A linear relation T on X can be uniquely determined by its graph $G(T)$ which is defined by

$$G(T) := \{(x, y) \in X \times X : x \in \mathcal{D}(T) \text{ and } y \in T(x)\}.$$

For $T \in \mathcal{LR}(X)$, the inverse of T is the linear relation T^{-1} defined by

$$G(T^{-1}) := \{(y, x) \in X \times X : (x, y) \in G(T)\}.$$

Let A and B be two non-empty subsets of X, let

$$T(A) := \bigcup_{x \in A \cap \mathcal{D}(T)} Tx \text{ and } T^{-1}(B) := \{x \in \mathcal{D}(T) : Tx \cap B \neq \emptyset\}.$$

In particular, $R(T) := T(\mathcal{D}(T))$ is the range of T. The subspace $T^{-1}(0)$, denoted by $\ker(T)$, is called the null space of the linear relation T. The subspace $T(0)$ is called the multivalued part of T. T is said to be a single valued operator or an operator if Tx is a singleton, which is also equivalent to $T(0) = \{0\}$.
For T and $S \in \mathcal{LR}(X)$, the linear relations $T + S$ and TS are defined respectively by

$$G(T + S) := \{(x, y + z) \in X \times X \text{ such that } (x, z) \in G(S) \text{ and } (x, y) \in G(T)\}$$

and

$$G(TS) := \{(x, y) \in X \times X : \exists z \in X \text{ such that } (x, z) \in G(S) \text{ and } (z, y) \in G(T)\}.$$

For $n \in \mathbb{N} \cup \{0\}$, T^n is defined by $T^0 = I$ (the identity operator in X) and if T^{n-1} is defined, then $T^n x := T(T^{n-1}x) = \bigcup_{y \in \mathcal{D}(T) \cap T^{n-1}x} Ty$, where $x \in \mathcal{D}(T^n) := \{x \in \mathcal{D}(T) : \mathcal{D}(T) \cap T^{n-1}x \neq \emptyset\}$ (see [8]). By induction, we can show that $(T^n)^{-1} = (T^{-1})^n$, for all $n \in \mathbb{N}$.

$T \in \mathcal{LR}(X)$ is continuous if for all neighbourhood U in $R(T)$, $T^{-1}(U)$ is an neighbourhood set in $\mathcal{D}(T)$. If T is continuous and $\mathcal{D}(T) = X$, then T is said to be bounded. If $G(T)$ is closed then T is said to be closed. $\mathcal{BCR}(X)$ denotes the set of all closed and bounded linear relations. We notice that if T is closed, then $T(0)$ is closed. A single valued linear operator S is called a *selection* of T if $\mathcal{D}(T) = \mathcal{D}(S)$ and

$$Tx = Sx + T(0) \text{ for all } x \in \mathcal{D}(T).$$

Moreover, if S is continuous, then T is continuous. Now, we recall some algebraic properties of linear relation.

Lemma 2.1. *[11, Proposition I.2.8 and I.3.1] Let $T \in \mathcal{LR}(X)$.*

1. *If $x \in \mathcal{D}(T)$, then $y \in Tx$ equivalently $Tx = y + T(0)$. In particular,*

$$0 \in Tx \Longleftrightarrow Tx = T(0).$$

2. *If M is subset of X, then*

$$TT^{-1}(M) = M \cap R(T) + T(0) \ \ \text{and} \ \ T^{-1}T(M) = M \cap \mathcal{D}(T) + T^{-1}(0).$$

Lemma 2.2. *[2, Lemma 2.5] Let A, B and $C \in \mathcal{LR}(X)$. Then*

1. *$G((A+B)C) \subset G(AC+BC)$. If $C(0) \subset \ker(A) \cup \ker(B)$, then*

$$(A+B)C = AC + BC.$$

2. *If $\mathcal{D}(A) = X$, then $A(B+C) = AB + AC$.*

3. Recurrent vectors of linear relations

In this section, we introduce and study the notion of recurrent vectors of linear relations. We start by the following definition

Definition 3.1. Let $T \in \mathcal{BCR}(X)$. A vector x of X is called a *recurrent vector for T or T-recurrent* if there exists a strictly increasing sequence of positive integers $\{k_n\}$ such that there exists $y_{k_n} \in T^{k_n}x$, for all $n \in \mathbb{N}$ and

$$y_{k_n} \longrightarrow x \text{ as } n \to \infty.$$

We denote by $Rec(T)$ the set of all recurrent vectors for T.

Example 3.2. Let A be the bounded shift operator defined on $\ell_2(\mathbb{N})$ by

$$A(x_1, x_2, ...) = 2(x_2, x_3, ...).$$

Let $k \in \mathbb{N}$, then the vector $y := (x_1, x_2, ..., x_k, \frac{1}{2^k}x_1, \frac{1}{2^k}x_2, ..., \frac{1}{2^k}x_k, \frac{1}{2^{2k}}x_1, ..., \frac{1}{2^{2k}}x_k, ...)$ is a recurrent vector for A (see [13, Example 2.32]). Moreover, $y = A^{nk}y$ for all $n \in \mathbb{N}$. Now, we consider the bounded linear relation T defined by

$$\begin{array}{rccc} T: & \ell_2(\mathbb{N}) & \longrightarrow & 2^{\ell_2(\mathbb{N})} \setminus \emptyset \\ & x = (x_i)_{i\in\mathbb{N}} & \longmapsto & Ax + \ker(A). \end{array}$$

Then $T(0) = \ker(A)$. Let $x = (x_i)_{i\in\mathbb{N}} \in \ell_2(\mathbb{N})$, then $Tx = Ax + T(0)$. From Lemma 2.1, it follows that

$$\begin{array}{rcl} T^2x & = & T(Ax + T(0)) \\ & = & A^2x + T(0) + T^2(0) \\ & = & A^2x + T^2(0). \end{array}$$

Therefore, by induction we can show that

$$T^n x = A^n x + T^n(0), \ \text{for all} \ n \in \mathbb{N}.$$

This implies that $y = A^{nk}y \in T^{nk}y$, for all $n \in \mathbb{N}$. We set $z_{k_n} := y$, for all $n \in \mathbb{N}$. Hence

$$z_{k_n} \in T^{nk}y \ \text{ and } \ z_{k_n} \longrightarrow y \text{ as } n \longrightarrow \infty.$$

Finally, we deduce that y is a recurrent vector for T.

More generally, we prove that every selection of linear relation T has a recurrent vector, then T has a recurrent vector.

Proposition 3.3. *Let $A \in \mathcal{B}(X)$ be a selection of $T \in \mathcal{BCR}(X)$. If x is a recurrent vector for A, then it is a recurrent vector for T.*

Proof. As x is a recurrent vector for A, then there exists a strictly increasing sequence of positive integers $\{k_n\}$ such that

$$A^{k_n}x \longrightarrow x \text{ as } n \to \infty.$$

Since A is a selection of T, then by prove of example 3.2 (see also [4, Theorem 2.5.6], we have A^n is a selection of T^n for all $n \in \mathbb{N}$. Hence

$$T^n y = A^n y + T^n(0), \text{ for all } n \in \mathbb{N} \text{ and } y \in X. \tag{1}$$

Which implies that $A^{k_n}x \in T^{k_n}x$, for all $n \in \mathbb{N}$. This gives x is a recurrent vector for T. □

Example 3.4. Let $A \in \mathcal{B}(X)$ be non-injective and let T be a linear relation defined by

$$\begin{array}{rccc} T: & X & \longrightarrow & 2^X \setminus \emptyset \\ & x & \longmapsto & A^{-1}A^2x. \end{array}$$

Then for all $x \in \mathcal{D}(A) = X$, we have

$$\begin{array}{rcl} Tx & = & A^{-1}A^2x \\ & = & Ax + A^{-1}(0) \\ & = & Ax + T(0). \end{array}$$

This gives A is a selection of T. Since $\mathcal{D}(A) = \mathcal{D}(T)$ and A is continuous, then $T \in \mathcal{BCR}(X)$. If x is a recurrent vector for A, then x is a recurrent vector for T by the previous proposition 3.3.

Proposition 3.5. *Let $A \in \mathcal{BCR}(X)$, $B \in \mathcal{BCR}(Y)$ and $S \in \mathcal{B}(Y, X)$ such that $AS = SB$. Then, we have*

$$S(Rec(B)) \subset Rec(A).$$

Proof. Let $x \in Rec(B)$, then there exists a strictly increasing sequence of positive integers $\{k_n\}$ such that there exists $y_{k_n} \in B^{k_n}x$, for all $n \in \mathbb{N}$ and

$$y_{k_n} \longrightarrow x \text{ as } n \longrightarrow \infty.$$

As $AS = SB$, then

$$\begin{array}{rcl} Sy_{k_n} & \in & SB^{k_n}x \\ & = & A^{k_n}Sx. \end{array}$$

Since $\{y_{k_n}\}$ converges to x and S is continuous, then $\{Sy_{k_n}\}$ converge to Sx. Therefore Sx is a recurrent vector for the linear relation A. □

The following corollary is an immediate consequence of the previous proposition.

Corollary 3.6. *Let $T \in \mathcal{BCR}(X)$ and $S \in \mathcal{B}(X)$ such that T commute with S. Then, we have the following assertions*

i) $S(Rec(T))$ *is a subset of* $Rec(T)$.
ii) $\lambda Rec(T) = Rec(T)$, *for every* $\lambda \in \mathbb{C} \setminus \{0\}$.

In the sequel, the open ball with center x and radius r is denoted by $B(x, r)$.

Theorem 3.7. *Let $T \in \mathcal{BCR}(X)$. Then*

$$Rec(T) = \bigcap_{p \geqslant 1} \bigcup_{n \geqslant 1} \{y \in X : \exists y_n \in T^n y \text{ such that } \|y_n - y\| < \frac{1}{p}\}.$$

Proof. Let $x \in Rec(T)$, then there exists a strictly increasing sequence of positive integers $\{k_n\}$ such that there exists $y_{k_n} \in T^{k_n}x$, for all $n \in \mathbb{N}$ and

$$y_{k_n} \longrightarrow x \text{ as } n \to \infty.$$

Hence, for all $p \geqslant 1$, there exists $m \geqslant 0$ such that

$$\|y_{k_m} - x\| < \frac{1}{p}.$$

Therefore,

$$x \in \bigcap_{p\geqslant 1} \bigcup_{n\geqslant 1} \{y \in X : \exists y_n \in T^n y \text{ such that } \|y_n - y\| < \frac{1}{p}\}.$$

Conversely, let $x \in \bigcap_{p\geqslant 1} \bigcup_{n\geqslant 1} \{y \in X : \exists y_n \in T^n y \text{ such that } \|y_n - y\| < \frac{1}{p}\}$. Then, for all $p \geqslant 1$, there exists $y_{n_p} \in T^{n_p}x$ such that $\|y_{n_p} - x\| < \frac{1}{p}$. We set $B_p := B(x, \frac{1}{p})$, for all $p \in \mathbb{N}$. Then for $p = 1$, there exists $n_1 \geqslant 1$ such that $y_{n_1} \in T^{n_1}x \cap B_1$. Also, for $p = 2$, there exists $n_2 \geqslant 1$ such that $y_{n_2} \in T^{n_2}x \cap B_2$. Now, we show that there exists $k_2 \in \mathbb{N}$ such that $k_2 > k_1 := n_1$ and $T^{k_2}x \cap B_2 \neq \emptyset$. Since $y_2 \in T^{n_2}x \cap B_2$, then $x \in T^{-n_2}B_2 \cap B_2$. The continuity of T^{n_2} implies that $T^{-n_2}B_2 \cap B_2$ is a neighbourhood of x. Which means that there exists $r > 0$ such that

$$B(x, r) \subset T^{-n_2}B_2 \cap B_2.$$

Then there exists $i \in \mathbb{N}$ such that

$$B_i := B(x, \frac{1}{i}) \subset B(x, r) \subset T^{-n_2}B_2 \cap B_2.$$

For $p = i$, there exists $n_i \geqslant 1$ such that $y_{n_i} \in T^{n_i}x \cap B_i$. We then have

$$\begin{aligned} T^{n_2}B_i &\subset T^{n_2}(T^{-n_2}B_2 \cap B_2) \\ &\subset T^{n_2}T^{-n_2}B_2 \cap T^{n_2}B_2 \\ &\subset (B_2 + T^{n_2+n_i}(0)) \cap T^{n_2}B_2. \end{aligned}$$

Hence

$$T^{n_2}B_i \subset (B_2 + T^m(0)) \cap T^{n_2}B_2, \tag{2}$$

where $m := n_2 + n_i$. Since $y_{n_i} \in T^{n_i}x \cap B_i$, then

$$T^{n_2}y_{n_i} \subset T^m x \cap T^{n_2}B_i. \tag{3}$$

Combining (2) and (3), we obtain

$$\begin{aligned} \emptyset \neq T^{n_2}y_{n_i} &= T^{n_2}y_{n_i} \cap T^{n_2}B_i \\ &\subset T^m x \cap T^{n_2}B_i \cap (B_2 + T^m(0)) \cap T^{n_2}B_2, \end{aligned}$$

which implies that $T^m x \cap (B_2 + T^m(0)) \neq \emptyset$. Now, let $z \in T^m x \cap (B_2 + T^m(0))$, then there exist $b \in B_2$ and $c \in T^m(0)$ such that $z = b + c$. Therefore

$$\begin{aligned} T^m x &= z + T^m(0) \\ &= b + c + T^m(0) \\ &= b + T^m(0). \end{aligned}$$

This gives $b \in T^m x \cap B_2$. Hence $m > n_2$ and $T^m x \cap B_2 \neq \emptyset$. Consequently, if $m > k_1$, then we take $k_2 := m$. Otherwise, we use the same argument, and there exists $m' \in \mathbb{N}$ such that $T^{m+m'}x \cap B_2 \neq \emptyset$ and $m + m' > k_1$. Then we take $k_2 := m + m'$. Finally, we take $z_{k_1} := y_{n_1} \in T^{k_1}x \cap B_1$, $n_1 = k_1$, $z_{k_2} \in T^{k_2}x \cap B_2$ and $k_2 > k_1$. Therefore there exists a strictly increasing sequence $\{k_p\}$ of positive integers such that $z_{k_p} \in T^{k_p}x$ and $z_{k_p} \longrightarrow x$ as $p \longrightarrow \infty$. Therefore, $x \in Rec(T)$. □

Let $T \in \mathcal{LR}(X)$ and $S \in \mathcal{LR}(Y)$. Then the linear relation $T \oplus S$ is defined by

$$\begin{array}{rccc} T \oplus S: & \mathcal{D}(X \oplus Y) & \longrightarrow & 2^{X \oplus Y} \setminus \emptyset \\ & x \oplus y & \longmapsto & Tx \oplus Sy \end{array}$$

where $X \oplus Y := \{(x_1, x_2) : x_1 \in X, \text{ and } x_2 \in Y\}$ and $\mathcal{D}(T \oplus S) := \mathcal{D}(T) \oplus \mathcal{D}(S)$ Now, let $k \in \mathbb{N}$, then

$$(T \oplus S)^k x \oplus y = T^k x \oplus S^k y$$

Proposition 3.8. *Let $T \in \mathcal{BCR}(X)$ and $S \in \mathcal{BCR}(Y)$. Then*

$$Rec(T \oplus S) \subset Rec(T) \oplus Rec(S).$$

Proof. Let $x = x_1 \oplus x_2 \in Rec(T \oplus S)$, then there exists a strictly increasing sequence of positive integers $\{k_p\}$ such that there exists

$$y_{k_p} \in (T \oplus S)^{k_p} x = T^{k_p} x \oplus S^{k_p} x, \text{ for all } p \in \mathbb{N}$$

and $y_{k_p} \longrightarrow x$ as $p \to \infty$. Let P be the bounded projection defined by

$$\begin{array}{rccc} P: & X \oplus Y & \longrightarrow & Y \\ & x_1 \oplus x_2 & \longmapsto & x_2. \end{array}$$

We have

$$P(y_{k_p}) \longrightarrow x_2 \text{ as } p \longrightarrow \infty \text{ and } P(y_{k_p}) \in S^{k_p} x_2.$$

We deduce that x_2 is a recurrent vector for S. Similarly, we then get x_1 is a recurrent vector for T. □

4. Recurrent linear relations

In this section, we define and study the notion of recurrent linear relations.

Definition 4.1. Let $T \in \mathcal{BCR}(X)$. We say that T is *recurrent*, if for any non-empty open subset of X, there exists some $n \in \mathbb{N}$ such that

$$T^n(U) \cap U \neq \emptyset.$$

By $Rec(X)$ we denote the set of all recurrent linear relations on X.

Proposition 4.2. *Let $A \in \mathcal{B}(X)$ be a selection of $T \in \mathcal{BCR}(X)$. If A is a recurrent operator, then T is a recurrent linear relation.*

Proof. Let U be a non-empty open subset of X. Since A is recurrent single valued operator, then there exists $n \in \mathbb{N}$ such that

$$A^{-n}(U) \cap U \neq \emptyset.$$

On other hand, since A is selection of T, then by equality 1, we get $A^m y \in T^m y$ for all $m \in \mathbb{N}$ and $y \in X$. Since $A^{-n}(U) \cap U \neq \emptyset$, then there exists $x \in A^{-n}(U) \cap U$. So, $A^n x \in U$ and $A^n x \in T^n x$. Hence $A^n x \in T^n x \cap U$ and $x \in U$. We deduce that $T^n(U) \cap U \neq \emptyset$. Finally, we conclude that T is a recurrent linear relation. □

We show here that for every non-injective recurrent selections, we can construct recurrent linear relations. Indeed, let A be a non-injective bounded operator on X and let T be a linear relation defined by

$$\begin{array}{rccc} T: & X & \longrightarrow & 2^X \setminus \emptyset \\ & x & \longmapsto & A^{-1}A^2 x. \end{array}$$

Then by Example 3.4 we obtain A is a selection of $T \in \mathcal{BCR}(X)$. If S is a recurrent linear operator, then according to Proposition 4.2, we deduce that T is a recurrent linear relation.

Example 4.3. Let B be a shift operator defined on $\ell_2(\mathbb{N})$ by

$$B(x_0, x_1, ...) = 3(x_1, x_2, ...).$$

Then, the linear relation T defined on $\ell_2(\mathbb{N})$ by

$$\begin{array}{rccc} T: & \ell_2(\mathbb{N}) & \longrightarrow & 2^{\ell_2(\mathbb{N})} \setminus \emptyset \\ & x := (x_i)_{i\in\mathbb{N}} & \longmapsto & Bx + B^{-1}(0) \end{array}$$

is recurrent. Indeed, since $Tx = Bx + T(0)$, for all $x = (x_1, x_2, ...) \in \ell_2(\mathbb{N})$, where $T(0) = \ker(B)$, then B is a selection of T. Furthermore, by Example 2.22 in [13], B is a hypercyclic single valued operator. This implies that B is recurrent. Therefore, by Proposition 4.2, it follows that T is a recurrent linear relation.

Proposition 4.4. *Let $T \in \mathcal{BCR}(X)$, $S \in \mathcal{BCR}(Y)$ and $V \in \mathcal{B}(X, Y)$ such that $VT = SV$ and the range of V is dense in Y. If T is recurrent, then S is recurrent.*

Proof. Suppose that T is a recurrent linear relation. Let U be a non-empty subset open of Y. Since V is continuous and the range of V is dense in Y, then $V^{-1}(U)$ is a non-empty open subset of X. Since T is a recurrent linear relation, then there exists $n \in \mathbb{N}$ such that

$$T^n(V^{-1}(U)) \cap V^{-1}(U) \neq \emptyset.$$

This implies that there exist two elements x and y in $V^{-1}(U)$ such that $y \in T^n x$. Hence $V(x), V(y) \in U$ and $T^n x = y + T^n(0)$, therefore we have

$$\begin{aligned} S^n V(x) &= VT^n x \qquad\qquad (VT = SV) \\ &= V(y + T^n(0)) \\ &= V(y) + VT^n(0) \\ &= V(y) + S^n V(0) \\ &= V(y) + S^n(0). \end{aligned}$$

Then $V(y) \in S^n V(x) \subset S^n(U)$. Consequently $S^n(U) \cap U \neq \emptyset$ and so S is a recurrent linear relation. □

The immediate consequences of the last proposition are the following.

Corollary 4.5. *Let $T \in \mathcal{BCR}(X)$, $S \in \mathcal{BCR}(Y)$ and $V \in \mathcal{B}(X, Y)$ such that $VT = SV$ and V is invertible. Then T is a recurrent linear relation if and only if S is a recurrent linear relation.*

Corollary 4.6. *Let $T \in \mathcal{BCR}(X)$ and $S \in \mathcal{B}(X)$. If S is invertible, then T is a recurrent linear relation if and only if STS^{-1} or $S^{-1}TS$ is a recurrent linear relation.*

In the following results, we characterize the recurrent linear relation.

Theorem 4.7. *Let $T \in \mathcal{BCR}(X)$. Then, the following assertions are equivalent*

i) T is a recurrent linear relation.

ii) For each non-empty open subset V of X, there exists $n \in \mathbb{N}$ such that

$$T^{-n}(V) \cap V \neq \emptyset.$$

iii) For all $x \in X$, there are sequences $\{x_k\}$ in X, $\{y_k\}$ in X and $\{n_k\}$ in $\mathbb{N}$ such that

$$x_k \longrightarrow x, \quad y_k \longrightarrow x \quad \text{and} \quad T^{n_k} x_k = y_k + T^{n_k}(0).$$

iv) For all $x \in X$ and for a neighbourhood W of zero, there exist two vectors z and t in X and $p \in \mathbb{N}$ such that

$$t - x \in W, \quad z - x \in W \quad \text{and} \quad T^p(z) = t + T^p(0).$$

Proof. $i) \Longleftrightarrow ii)$. Suppose that T is a recurrent linear relation, then for any non-empty open subset V of X, there exists $n \in \mathbb{N}$ such that $T^n(V) \cap V \neq \emptyset$. Which implies that

$$(V + T^{-n}(0)) \cap T^{-n}(V) \neq \emptyset.$$

Then there exist $v, w \in V$ and $y \in T^{-n}(0)$ such that $x := v + y \in (V + T^{-n}(0)) \cap T^{-n}(V)$ and $x \in T^{-n}(w)$. Hence,

$$\begin{array}{rcl} T^{-n}(w) & = & x + T^{-n}(0) \\ & = & v + y + T^{-n}(0) \\ & = & v + T^{-n}(0). \end{array}$$

Thus $v \in T^{-n}(w) \cap V$. Therefore, $T^{-n}(V) \cap V \neq \emptyset$.
The converse goes similarly.

$i) \Longrightarrow iii)$. Suppose that T is a recurrent linear relation. Let $x \in X$. For each $k \in \mathbb{N}$, let $U_k := B(x, \frac{1}{k})$. Since T is a recurrent linear relation, then for all $k \in \mathbb{N}$, there exists $\{n_k\}$ in $\mathbb{N}$ such that

$$T^{n_k}(U_k) \cap U_k \neq \emptyset.$$

Then, there exists a sequence $\{y_k\}$ such that

$$y_k \in T^{n_k}(U_k) \cap U_k, \text{ for all } k \in \mathbb{N}.$$

Thus, there exists $\{x_k\}$ in U_k such that $y_k \in T^{n_k}(x_k)$, for all $k \in \mathbb{N}$. Therefore

$$\|x_k - x\| < \frac{1}{k}, \quad \|y_k - x\| < \frac{1}{k} \quad \text{and} \quad y_k \in T^{n_k}(x_k)$$

for all $k \in \mathbb{N}$. Which implies that $\{x_k\}$ converges to x, $\{y_k\}$ converges to x and $T^{n_k}(x_k) = y_k + T^{n_k}(0)$.

$iii) \Longrightarrow iv)$. Let $x \in X$ and W a neighbourhood of zero, then there are sequences $\{x_k\}$ in X, $\{y_k\}$ in X and $\{n_k\}$ in $\mathbb{N}$ such that

$$x_k \longrightarrow x, \quad y_k \longrightarrow x \quad \textit{and} \quad y_k \in T^{n_k}x_k.$$

This means that there exists $m \in \mathbb{N}$ such that $x_k - x \in W$ and $y_k - x \in W$, for all $k \geqslant m$. Set $x_m = z$, $t = y_m$ and $p := n_m$, then we obtain

$$t - x \in W, \quad z - x \in W \quad \text{and} \quad t \in T^p(z).$$

$iv) \Longrightarrow i)$. Let U be a non-empty open subset of X. Hence, there exists $x \in U$. For all $k \in \mathbb{N}$, we set $W_k := B(x, \frac{1}{k})$. Then, W_k is a neighbourhood of zero. By assumption of $iii)$, there exist z_k in X and t_k in $\mathbb{N}$ such that

$$\|x_k - x\| < \tfrac{1}{k}, \quad \|t_k - x\| < \tfrac{1}{k} \quad \text{and} \quad t_k \in T^{p_k}(x_k).$$

Which implies that $\{z_k\}$ converges to x and $\{t_k\}$ converges to x. Using the fact that U is open and $x \in U$, then there exists $m \in \mathbb{N}$ such that

$$z_k \in U \quad \text{and} \quad t_k \in U,$$

for all $k \geqslant m$. Therefore

$$t_k \in T^{p_k}(z_k) \cap U \subset T^{p_k}(U) \cap U.$$

Finally, we can say that T is a recurrent linear relation. □

Corollary 4.8. *Let $T \in \mathcal{BCR}(X)$. If T is bijective, then T is a recurrent linear relation if and only if T^{-1} is a recurrent linear operator.*

Theorem 4.9. *Let $T \in \mathcal{BCR}(X)$. If for each non-empty open subset U and for each W a neighbourhood of zero, there exists $n \in \mathbb{N}$ such that*

$$T^n(U) \cap W \neq \emptyset \quad \textit{and} \quad T^n(W) \cap U \neq \emptyset,$$

then T is a recurrent linear relation.

Proof. Let $x \in X$. For each $k \in \mathbb{N}$, set $U_k := B(x, \frac{1}{k})$ and $W_k := B(0, \frac{1}{k})$. By assumption, for every $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ provided that

$$T^{n_k}(U_k) \cap W_k \neq \emptyset \quad \text{and} \quad T^{n_k}(W_k) \cap U_k \neq \emptyset.$$

This implies that there exist $a_k \in T^{n_k}(U_k) \cap W_k$ and $d_k \in T^{n_k}(W_k) \cap U_k$, for each $k \in \mathbb{N}$. Therefore, there exist two sequences $\{b_k\}$ in W_k and $\{c_k\}$ in U_k such that

$$a_k \in T^{n_k}(c_k) \cap W_k \quad \text{and} \quad d_k \in T^{n_k}(b_k) \cap U_k$$

for all $k \in \mathbb{N}$. We set $\{x_k\} := \{b_k\} + \{c_k\}$ and $\{y_k\} := \{a_k\} + \{d_k\}$, then we obtain

$$\begin{array}{rcl} \|x_k - x\| & = & \|c_k - x + b_k\| \\ & \leqslant & \|c_k - x\| + \|b_k\| \\ & \leqslant & \frac{2}{k} \end{array}$$

and

$$\begin{array}{rcl} \|y_k - x\| & = & \|d_k - x + a_k\| \\ & \leqslant & \|d_k - x\| + \|a_k\| \\ & \leqslant & \frac{2}{k} \end{array}$$

for all $k \in \mathbb{N}$. Consequently $x_k \longrightarrow x$ and $y_k \longrightarrow x$ as $k \longrightarrow \infty$. Moreover, we have

$$\begin{array}{rcl} T^{n_k}(x_k) & = & T^{n_k}(b_k + c_k) \\ & = & T^{n_k}(b_k) + T^{n_k}(c_k) \\ & = & d_k + T^{n_k}(0) + a_k + T^{n_k}(0) \\ & = & a_k + d_k + T^{n_k}(0) \\ & = & y_k + T^{n_k}(0). \end{array}$$

Finally, using *ii*) of Theorem 4.7, we conclude that T is a recurrent linear relation. □

Lemma 4.10. *[1, Lemma 2.1] Let A and B be two subsets of X with $int(\overline{B}) = \emptyset$. Then*

$$int(\overline{A}) = int(\overline{A} \cup \overline{B}).$$

In the following theorem we characterize a recurrent linear relation by its recurrent vectors.

Theorem 4.11. *Let $T \in \mathcal{BCR}(X)$. Then, the following assertions are equivalents*

i) *T is a recurrent linear relation.*
ii) *$Rec(T) \setminus \{0\}$ is dense in X.*
iii) *$Rec(T)$ is dense in X.*

Proof. *iii*) $\Longleftrightarrow$ *ii*) If $Rec(T) \setminus \{0\}$ is dense in X, then obviously $Rec(T)$ is dense in X. Conversely, suppose that $Rec(T)$ is dense in X. We set $A = Rec(T) \setminus \{0\}$ and $B = \{0\}$. Since $int(\overline{B}) = \emptyset$, then by Lemma 4.10 it follows that

$$int(\overline{A}) = int(\overline{A} \cup \overline{B}).$$

Therefore, we have

$$\begin{array}{rcl} X & = & int(X) \\ & = & int(\overline{Rec(T)}) \\ & = & int(\overline{A \cup B}) \\ & = & int(\overline{A} \cup \overline{B}) \\ & = & int(\overline{A}) \\ & \subset & \overline{A} \\ & \subset & X. \end{array}$$

Hence $Rec(T) \setminus \{0\}$ is dense in X.

iii) $\Longrightarrow$ *i*) Suppose that $Rec(T)$ is dense in X. Let U be a non-empty open subset of X, then there exists some x in $U \cap Rec(T)$. Since $x \in Rec(T)$, then there exists a sequence $\{n_k\}$ such that there exists $y_{n_k} \in T^{n_k}x$ and $y_{n_k} \longrightarrow x$ as $k \longrightarrow \infty$. Since U is an open subset of X and $x \in U$, then there exists $N \in \mathbb{N}$ such that $y_{n_k} \in U$, for all $k \geqslant N$. Hence

$$\begin{aligned} y_{n_k} \in T^{n_k}x &\Longleftrightarrow (x, y_{n_k}) \in G(T^{n_k}) \\ &\Longleftrightarrow (y_{n_k}, x) \in G((T^{n_k})^{-1}) \\ &\Longleftrightarrow (y_{n_k}, x) \in G(T^{-n_k}) \\ &\Longleftrightarrow x \in T^{-n_k}y_{n_k}. \end{aligned}$$

Then $x \in T^{-n_k}(y_{n_k}) \cap U \subset T^{-n_k}(U) \cap U$, for all $k \geqslant N$. Therefore T is recurrent by Theorem 4.7.

i) $\Longrightarrow$ *iii*) Assume that T is a recurrent linear relation. Let $x \in X$, set $U_0 := B(x, \varepsilon)$, for some $0 < \varepsilon < \frac{1}{2}$. We show that there exists a vector recurrent for T in U_0. Indeed, using the fact T is recurrent, then there exists $n_1 \in \mathbb{N}$ such that

$$T^{-n_1}(U_0) \cap U_0 \neq \emptyset.$$

Hence there exists $x_1 \in T^{-n_1}(U_0) \cap U_0$. As T^{n_1} is a continuous linear relation and $x_1 \in T^{-n_1}(U_0) \cap U_0$, then $T^{-n_1}(U_0) \cap U_0$ is a neighbourhood of x_1. Then, there exists positive real number $\varepsilon_1 < \dfrac{1}{2^2}$ such that $U_1 := B(x_1, \varepsilon_1) \subset T^{-n_1}(U_0) \cap U_0$. Since T is recurrent, then there exists $m \in \mathbb{N}$ such that

$$T^{-m}(U_1) \cap U_1 \neq \emptyset.$$

So, there exists $x_2 \in T^{-m}(U_1) \cap U_1$. As T^m is a continuous linear relation and $x_2 \in T^{-m}(U_1) \cap U_1$, then $T^{-m}(U_1) \cap U_1$ is a neighbourhood of x_2. This implies that there exists positive real number $\varepsilon_2 < \frac{1}{2^3}$ such that $U_2 := B(x_2, \varepsilon_2) \subset T^{-m}(U_1) \cap U_1$. Since T is recurrent, then there exists $p \in \mathbb{N}$ such that

$$T^{-p}(U_2) \cap U_2 \neq \emptyset. \tag{4}$$

Since $U_2 \subset T^{-m}(U_1) \cap U_1$, then

$$T^{-p}(U_2) \subset T^{-p}(T^{-m}(U_1) \cap U_1) \subset T^{-(m+p)}(U_1) \cap T^{-p}(U_1) \text{ and } U_2 \subset U_1.$$

Which implies that

$$T^{-p}(U_2) \cap U_2 \subset T^{-(m+p)}(U_1) \cap T^{-p}(U_1) \cap U_1.$$

By (4), we obtain $T^{-(m+p)}(U_1) \cap T^{-p}(U_1) \cap U_1 \neq \emptyset$. This means that

$$T^{-(m+p)}(U_1) \cap U_1 \neq \emptyset.$$

Therefore there exists $n_2 \in \mathbb{N}$ such that $n_2 > n_1$ and

$$T^{-n_2}(U_1) \cap U_1 \neq \emptyset.$$

So, there exists $x_2 \in T^{-n_2}(U_1) \cap U_1$. As T^{n_2} is a continuous linear relation and $x_2 \in T^{-n_2}(U_1) \cap U_1$, then $T^{-n_2}(U_1) \cap U_1$ is a neighbourhood of x_2. This implies that there exists positive real number $\varepsilon_2 < \frac{1}{2^3}$ such that $U_2 := B(x_2, \varepsilon_2) \subset T^{-n_2}(U_1) \cap U_1$. By continuing inductively, we construct a sequence $\{x_n\}$ in X, a strictly increasing sequence of positive integer $\{n_k\}$ and a sequence of positive real numbers $\varepsilon_n < \frac{1}{2^{n+1}}$ such that

$$B(x_k, \varepsilon_k) = U_k \subset U_{k-1} \text{ and } U_k \subset T^{-n_k}U_{k-1} \cap U_{k-1}.$$

Since $diam(U_k) \longrightarrow 0$ as $k \longrightarrow \infty$ and X is a Banach space then from Cantor's theorem, we deduce that there exists $y \in X$ such that

$$\bigcap_{k=0}^{\infty} U_k = \{y\}.$$

Since $y \in U_k$, for all $k \in \mathbb{N}$, then there exists y_k in U_{k-1} such that

$$y \in T^{-n_k}(y_k) \cap U_{k-1},$$

for all $k \in \mathbb{N}$. Then

$$\begin{aligned} y \in T^{-n_k}(y_k) &\iff (y_k, y) \in G(T^{-n_k}) \\ &\iff (y_k, y) \in G((T^{n_k})^{-1}) \\ &\iff (y, y_k) \in G(T^{n_k}) \\ &\iff y_k \in T^{n_k} y \end{aligned}$$

Hence $y_k \in T^{n_k} y \cap U_{k-1}$, for all $k \in \mathbb{N}$. Therefore, we obtain

$$\begin{aligned} \|y_k - y\| &= \|y_k - x_k + x_k - y\| \\ &\leqslant \|y_k - x_k\| + \|x_k - y\| \\ &\leqslant \varepsilon_{k-1} + \varepsilon_{k-1} \\ &\leqslant 2\varepsilon_{k-1} \\ &\leqslant \frac{1}{2^k} \end{aligned}$$

for all $k \in \mathbb{N}$. So, $y_k \longrightarrow y$ as $k \longrightarrow \infty$. Consequently, y is a recurrent vector for T and $y \in U_0$. Finally, we conclude that the set $Rec(T)$ is dense in X. □

Definition 4.12. *[8]* Let $T \in \mathcal{LR}(X)$. We say that $x \in X$ is a *periodic point* of T if $x \in \mathcal{D}_\infty(T) := \bigcap_{n \in \mathbb{N}} \mathcal{D}(T^n)$ and there exists $n \in \mathbb{N}$ such that $T^n x = x + T^n(0)$.
We denote by $Per(T)$ the set of all periodic points of T.

Theorem 4.13. *Let $T \in \mathcal{BCR}(X)$ such that $T(0) \subset \ker(T)$. If $Per(T)$ is dense in X, then we have the following assertions:*
i) $T^n \in Rec(X)$ for all $n \in \mathbb{N}$.
ii) $\lambda T \in Rec(X)$ for all $\lambda \in \{\exp(\frac{2\pi i}{n}) : n \in \mathbb{N}\}$.

Proof. i) Let U be a non-empty subset of X and let $n \in \mathbb{N}$. Since the set $Per(T)$ is dense in X, then $Per(T) \cap U \neq \emptyset$. Let $x \in X$ such that $x \in Per(T) \cap U$. Then there exists $m \in \mathbb{N}$ such that

$$T^m x = x + T^m(0).$$

Since $T(0) \subset \ker(T)$, then by [7, Proposition 3.1], $T^p(0) = T(0)$ for all $p \in \mathbb{N}$. Moreover, from Lemma 2.2, it follows that

$$\begin{aligned} T^{2m}(x) &= T^m(x + T^m(0)) \\ &= T^m x + T^{2m}(0) \\ &= x + T^m(0) + T^{2m}(0) \\ &= x + T^{2m}(0). \end{aligned}$$

By induction, we prove that $T^{nm}(x) = x + T^{nm}(0)$. As result $x \in T^{nm}(x) \subset (T^n)^m(U)$ and so $(T^n)^m(U) \cap U \neq \emptyset$. Therefore T^n is a recurrent linear relation.
ii) Let $\lambda \in \{\exp(\frac{2\pi i}{n}) : n \in \mathbb{N}\}$ and let $x \in Per(T)$. Then there exist $n, m \in \mathbb{N}$ such that $\lambda = \exp(\frac{2\pi i}{n})$ and $T^m x = x + T^m(0)$. Which implies that

$$\begin{aligned} (\lambda T)^{nm} x &= \lambda^{nm} T^{nm} x \\ &= x + T^{nm}(0) \\ &= x + (\lambda T)^{nm}(0). \end{aligned}$$

This means that $x \in Per(\lambda T)$. Consequently, $Per(\lambda T)$ is dense in X. Therefore by *i*), we deduce that λT is a recurrent linear relation. □

Proposition 4.14. *Let $T \in \mathcal{BCR}(X)$ and $S \in \mathcal{BCR}(Y)$. If $T \oplus S$ is recurrent linear relation, then T and S are recurrent linear relations.*

Proof. Let U and V be two non-empty subsets open of X and Y respectively. Since $T \oplus S$ is recurrent linear relation, then there exists $n \in \mathbb{N} \cup \{0\}$ such that

$$(T \oplus S)^n(U \oplus V) \cap (U \oplus V) \quad = \quad T^n(U) \oplus S^n(V) \cap U \oplus V \quad \neq \emptyset$$

this implies

$$T^n(U) \cap U \neq \emptyset \text{ and } S^n(V) \cap V \neq \emptyset.$$

This means that T and S are recurrent linear relations. □

5. Spectral property of recurrent linear relations

Let $T \in \mathcal{LR}(X)$. Then, the adjoint T^* of T ([11]), is defined by

$$G(T^*) := \{(y^*, x^*) \in X^* \oplus X^* : \; x^*(x) = y^*(y), \text{ for all } (x, y) \in G(T)\}$$

For $N \subset X$ and $M \subset X^*$. Let

$$N^{\perp} := \{x^* \in X^* : x^*(x) = 0, \text{ for all } x \in N\}$$

and

$$M^{\top} := \{x \in X : x^*(x) = 0, \text{ for every } x^* \in M\}.$$

If N is a linear space of X and M is a linear subspace of X^*, then $\overline{N} = N^{\perp^{\top}}$ and $\overline{M} \subset M^{\top^{\perp}}$. By Proposition III.1.3 in [11], we have

$$\mathcal{D}(T)^{\perp} = T^*(0),\; R(T)^{\perp} = \ker(T^*) \text{ and } (\lambda T)^* = \lambda T^* \text{ for } \lambda \in \mathbb{C}.$$

We say that $\lambda \in \mathbb{C}$ is an *eigenvalue* of T if there exists some non-zero vector $x \in X$ such that $Tx = \lambda x + T(0)$. In this case, the vector x is called an *eigenvector* of T corresponding to the eigenvalue λ. The set of all eigenvalues of T is called the point spectrum and denoted by $\sigma_p(T)$. In the sequel, the circle with center zero and radius 1 in $\mathbb{C}$ is denoted by $\mathbb{T}$.

Theorem 5.1. *Let $T \in Rec(X)$. Then, for every $\lambda \in \mathbb{C} \setminus \mathbb{T}$ the range of the linear relation $T - \lambda I$ is dense in X and*

$$\sigma_p(T^*) \subset \mathbb{T}.$$

Proof. Let T be a recurrent linear relation and $\lambda \in \mathbb{C} \setminus \mathbb{T}$. Suppose that $R(T - \lambda I)$ is not dense in X. Hence, $X \setminus \overline{R(T - \lambda I)}$ is a non-empty open subset of X. Since T is recurrent, then by Theorem 4.11, is follows that

$$(Rec(T) \setminus \{0\}) \cap (X \setminus \overline{R(T - \lambda I)}) \neq \emptyset.$$

Then, there exists a vector x in $(Rec(T) \setminus \{0\}) \cap (X \setminus \overline{R(T - \lambda I)})$. From the Hahn-Banach theorem, there exists an continuous linear functional x^* in X^* such that $x^*(x) \neq 0$ and $x^*(\overline{R(T - \lambda I)}) = \{0\}$. Then $x^*((T - \lambda I)(X)) = \{0\}$. Let $n \in \mathbb{N}$. From Lemma 4.2 in [14], it follows that

$$x^*(R(T^n - \lambda^n I)) \subset x^*(R(T - \lambda I)) = \{0\}.$$

Which implies that

$$x^* T^n a - \lambda^n x^*(a) = x^*(T^n - \lambda^n I)a = 0$$

for all $a \in X$. So, we get $x^* T^n x = \lambda^n x^*(x)$. Since x is a recurrent vector for T, then there exists a strictly increasing sequence of positive integers $\{k_n\}$ such that there exists $y_{k_n} \in T^{k_n}x$, for all $n \in \mathbb{N}$ and $y_{k_n} \longrightarrow x$ as $n \to \infty$. Then, we obtain

$$\begin{aligned} \lambda^{k_n} x^*(x) &= x^*(T^{k_n} x) \\ &= x^*(y_{k_n} + T^{k_n}(0)) \\ &= x^*(y_{k_n}). \end{aligned}$$

Since x^* is continuous and $\{y_{k_n}\}$ converges to x, then

$$\lambda^{k_n}x^*(x) = x^*(y_{k_n}) \longrightarrow x^*(x)$$

as $n \longrightarrow \infty$. This means that, $\{\lambda^{k_n}\}$ converges to 1, which is a contradiction with $\lambda \notin \mathbb{T}$. Therefore $\overline{R(T-\lambda I)} = X$. Moreover, we obtain

$$\begin{aligned}
\ker(T-\lambda I)^* &= \overline{\ker(T-\lambda I)^*} \\
&\subset ((\ker(T-\lambda I)^*)^{\top})^{\perp} \\
&= ((R(T-\lambda I)^{\perp})^{\top})^{\perp} \\
&= \overline{R(T-\lambda I)}^{\perp} \\
&= X^{\perp} \\
&= \{0\}.
\end{aligned}$$

Using [11, Proposition III.1.5], it follows that

$$\ker(T-\lambda I)^* = \ker(T^* - \lambda I) = \{0\}.$$

Thus $\lambda \notin \sigma(T^*)$. Finally, we get $\sigma(T^*)$ is a subset of $\mathbb{T}$. □

In particular, if T is a recurrent linear relation, then the range of T is dense in X.

Theorem 5.2. *Let $T \in \mathcal{BCR}(H)$. If the subspace $\mathcal{D} := span(\bigcup_{\lambda\in\mathbb{T}} \ker(T-\lambda I))$ is dense in X, then T is a recurrent linear relation.*

Proof. Let $x \in \mathcal{D}$. Then the vector x has the form

$$x = \sum_{i=1}^{m} \alpha_i x_i, \quad \text{where } \alpha_i \in \mathbb{C}, x_i \in \bigcup_{\lambda\in\mathbb{T}} \ker(T-\lambda I) \text{ and } m \in \mathbb{N}.$$

Then, for all $i \in \{1, ..., m\}$, there exists $\lambda_i \in \mathbb{T}$ such that

$$Tx_i = \lambda_i x_i + T(0).$$

Since $T^2(0)$ is a linear subspace of X and $T(0) \subset T^2(0)$, then we get

$$\begin{aligned}
T^2 x_i &= T(\lambda_i x_i + T(0)) \\
&= \lambda_i^2 x_i + T(0) + T^2(0) \\
&= \lambda_i^2 x_i + T^2(0).
\end{aligned}$$

Therefore, by induction, we obtain

$$T^n x_i = \lambda_i^n x_i + T^n(0), \quad \text{for all } n \in \mathbb{N}.$$

Using the fact $\lambda_i \in \mathbb{T}$, then there exists a strictly increasing sequence of positive integers $\{k_n\}$ such that $\{\lambda_i^{k_n}\}$ converges to 1 as $n \longrightarrow \infty$. Let I be a subset of $\{1, 2, ..., m\}$ defined by $I := \{i : \alpha_i \neq 0 \text{ and } 1 \leqslant i \leqslant m\}$. Therefore

$$\begin{aligned}
T^{k_n} x &= T^{k_n}(\sum_{i=1}^{m} \alpha_i x_i) \\
&= T^{k_n}(\sum_{i\in I} \alpha_i x_i) \\
&= \sum_{i\in I} \alpha_i T^{k_n}(x_i) \\
&= \sum_{i\in I} \alpha_i \lambda^{k_n} x_i + \sum_{i\in I} \alpha_i T^{k_n}(0) \\
&= \lambda^{k_n} \sum_{i=1}^{m} \alpha_i x_i + T^{k_n}(0) \\
&= \lambda^{k_n} x + T^{k_n}(0) \\
&= y_{k_n} + T^{k_n}(0),
\end{aligned}$$

where $y_{k_n} := \lambda^{k_n} x$. Since $\{y_{k_n}\}$ converges to x as $n \longrightarrow \infty$ and $y_{k_n} \in T^{k_n} x$, then x is a recurrent vector for T. Thus $\mathcal{D}$ is a subset of $Rec(T)$. As $\mathcal{D}$ is dense in X, then also $Rec(T)$ is dense in X. From Theorem 4.11, we conclude that T is a recurrent linear relation. $\square$

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