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Φ-moment B-valued martingale inequalities on Lorentz-Karamata spaces

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Abstract. In this article, several Φ-moment Banach spaces valued (briefly by **B**-valued) martingale inequalities on Lorentz-Karamata spaces are established by the tool of atomic decompositions, which are new versions of the basic inequalities in **B**-valued martingale setting associated with concave functions Φ. It is to be mentioned that the results obtained herein are closely connected with the geometric properties of the underlying Banach spaces. In particular, we present several novel characterizations of the geometric properties of Banach spaces by using the Φ-moment **B**-valued martingale inequalities in the context of Lorentz-Karamata spaces. Our conclusions obtained here generalize the previous conclusions for **B**-valued martingale inequalities. Moreover, we remove the condition that the slowly varying function b is nondecreasing in [Bull. Malays. Math. Sci. Soc., 2019, 42(5): 2395-2422].

1. Introduction

Martingale inequalities play a key role in martingale theory, probability theory and other fields in mathematics. The classical martingale inequalities lay the foundation for studying the geometrical properties of the martingale space, such as the Φ-moment martingale inequalities. In 1970's, Burkholder and Gundy [4] firstly researched the Φ-moment martingale inequalities. Later, the Φ-moment version of the Burkholder-Davis-Gundy inequality was established by Burkholder et al. in [3] as follows:

$$\mathbb{E}\left(\Phi\left(S(f)\right)\right) \lesssim \mathbb{E}\left(\Phi\left(M(f)\right)\right) \lesssim \mathbb{E}\left(\Phi\left(S(f)\right)\right), \tag{1}$$

where Φ is a strictly convex Orlicz function satisfying the Δ_2-condition, $M(f) = \sup_{m\geq 0} |f_m|$ and $S(f) = \left(\sum_{m=0}^{\infty} |df_m|^2\right)^{1/2}$ denote the maximal function and the square function of a martingale f, respectively. Kikuchi [14] generalized (1) to the rearrangement invariant Banach function spaces by the tool of Doob's decomposition and the averaging operators. Jiao and Yu [13] investigated some Φ-moment martingale inequalities associated with concave functions. Subsequently, the results in [13] were promoted to the Lorentz spaces by

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Peng and Li [25]. Quite recently, Jiao et al. [12] proved the Φ-moment version of Burkholder's inequalities in rearrangement invariant spaces. For more information on Φ-moment inequalities, we refer to [9, 16, 28].

In this paper, we shall discuss the Φ-moment martingale inequalities for Banach spaces valued (**B**-valued) martingale on Lorentz-Karamata spaces. Pisier [26] firstly proved that the inequalities of **B**-valued martingale essentially depend on the geometric properties of Banach spaces. From then on, **B**-valued martingale theory has attracted more attentions and flourished in the last decades. For instance, based on [26], the p-variations S_p and the conditional p-variations s_p of **B**-valued martingales were introduced by Liu in [19]. Subsequently, Liu and Hou [22] delved into the atomic decompositions for **B**-valued martingales in Hardy spaces and derived certain inequalities using these decompositions. In the following period, **B**-valued martingale theory has seen extensive applications in Banach space theory and Hardy spaces. We refer the reader to [1, 2, 17, 18, 20, 29, 30] and the monographs [21, 27] for more information on martingales and Fourier analysis in Banach spaces. What needs to be mentioned here is that the results on **B**-valued martingale inequalities are closely connected with the geometric properties of the underlying Banach space and we shall discuss the main topic in the context of Lorentz-Karamata spaces.

The family of Lorentz-Karamata space is a generalization of the Lorentz space, the Lorentz-Zygmund space and even the generalized Lorentz-Zygmund space [5, 24], which is defined by the so called slowly varying functions. The Lorentz-Karamata space plays an irreplaceable and crucial role in several key fields such as harmonic analysis, operator theory, and so on. We refer the reader to [5, 6, 11] for more details on the Lorentz-Karamata spaces. In 2014, Ho [10] introduced the martingale theory to the Lorentz-Karamata spaces, in which the atomic decompositions, dualities and interpolations on Hardy-Lorentz-Karamata martingale spaces were established. Liu et al. [17] studied **B**-valued martingale Hardy-Lorentz-Karamata spaces. One of their important conclusions is as follows. Let **B** be a Banach space, $1 < p \leq 2$, $0 < r_1 < p$, $0 < r_2 \leq \infty$ and b be a nondecreasing slowly varying function. Then the following assertions are equivalent:
(i) **B** is isomorphic to a p-uniformly smooth space.
(ii) For each **B**-valued martingale $f = (f_n)_{n\geq 0} \in H^{s^p}_{r_1,r_2,b}(\mathbf{B})$, there exists a constant C such that

$$\|M(f)\|_{r_1,r_2,b} \leq C\|f\|_{H^{s^p}_{r_1,r_2,b}}. \tag{2}$$

It is noteworthy that the condition of the nondecreasing of the slowly varying function b is necessary in the above conclusion.

Inspired by the above results, we have a question that whether it is possible to extend (2) to the **B**-valued Φ-moment version and remove the condition that the slowly varying function b is nondecreasing. The answer is affirmative. Let us briefly state the corresponding results as follows (see Theorem 4.1 in Section 4). Let $\Phi \in \mathcal{G}$ be a concave function, **B** be a Banach space, $1 < r \leq 2$, $0 < p < r$, $0 < q \leq \infty$ and b be a slowly varying function. Then the following statements are equivalent:
(i) **B** is isomorphic to a r-uniformly smooth space.
(ii) If the **B**-valued martingale $f = (f_n)_{n\geq 0}$ satisfies $\|\Phi(s^r(f))\|_{p,q,b} < \infty$, then

$$\|\Phi(M(f))\|_{p,q,b} \lesssim \|\Phi(s^r(f))\|_{p,q,b}. \tag{3}$$

The above results indicate that the geometric properties of Banach spaces can be characterized by studying the Φ-moment **B**-valued martingale inequalities, and vice versa. Notice that if $\Phi(t) = t$ and b is nondecreasing, then item (ii) arrives at [17, Theorem 5.4 (ii)], obviously, the condition that b is nondecreasing is not necessary here; if we take $b \equiv 1$, then item (ii) returns to the very recently work [15, Theorem 4 (ii)], namely, the Φ-moment **B**-valued martingale inequalities on Lorentz spaces; if $r = 2$, $b \equiv 1$ and $\mathbf{B} = \mathbb{R}$, then item (ii) recovers the main result in [25]. Moreover, if $b \equiv 1$ and $\Phi(t) = t$ in item (ii), then we recover [18, Theorem 5.4 (ii)] while item (ii) gives [22, Theorem 5 (ii)] when $\Phi(t) = t$, $b \equiv 1$ and $p = q$.

Compared with the above results, the topic we discuss here, namely the Φ-moment **B**-valued martingale inequalities for Lorentz-Karamata spaces, needs to consider both the geometric properties of the Banach spaces in which the martingales take values and the involving of the slowly varying function b. Moreover, we characterized the uniform convexity and uniform smoothness of Banach spaces by using the Φ-moment **B**-valued martingale inequalities on Lorentz-Karamata spaces. It is observed that we need new technique to establish these martingale inequalities, by the way, the technique we mainly rely on is the atomic decompositions in the framework of **B**-valued Lorentz-Karamata martingale spaces.

The paper is organized as follows. Some preliminary lemmas and notations will be introduced in Section 2. In Section 3, we present the atomic decompositions for **B**-valued martingale Hardy-Lorentz-Karamata spaces. As usual, the theorems depend on the geometric properties of the underlying Banach space. We note that here the slowly varying function b does not require to be nondecreasing. In the last section, employing the atomic decompositions, we establish some Φ-moment **B**-valued martingale inequalities on Lorentz-Karamata spaces. Compared with the relevant conclusions in [17], the slowly varying function b is not necessarily nondecreasing. These are new versions of the basic inequalities in **B**-valued martingale setting associated with concave functions.

Throughout this paper, the set of nonnegative integers, the set of integers, the real number field and the complex number field are denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$, respectively. The letter C stands for a positive real number, which may vary from line to line. $f \lesssim g$ means there exists a positive constant C such that $f \leq Cg$. We write $f \approx g$ if $f \lesssim g \lesssim f$.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. For a Banach space **B** with norm $\|\cdot\|$ and any **B**-valued function $f : \Omega \to \mathbf{B}$, let

$$\|f\|_{L_p(\mathbf{B})} = \left(\int_\Omega \|f\|^p d\mathbb{P} \right)^{\frac{1}{p}} \ (0 < p < \infty) \text{ and } \|f\|_{L_\infty(\mathbf{B})} = ess\ \sup \|f\|.$$

When the Banach space **B** in the definition above is the scalar field, for a scalar-valued function $f : \Omega \to \mathbb{R}$ (or $\mathbb{C}$), let

$$\|f\|_p = \left(\int_\Omega |f|^p d\mathbb{P} \right)^{\frac{1}{p}} \ (0 < p < \infty) \text{ and } \|f\|_\infty = ess\ \sup |f|.$$

2.1. *Lorentz-Karamata spaces.*

In this subsection, we recall the definition of slowly varying function in order to define the Lorentz-Karamata spaces.

Definition 2.1 ([5]). *A Lebesgue measurable function $b : [1, \infty) \to (0, \infty)$ is said to be a slowly varying function if for any given $\varepsilon > 0$, the function $t^\varepsilon b(t)$ is equivalent to a nondecreasing function and the function $t^{-\varepsilon} b(t)$ is equivalent to a nonincreasing function on $[1, \infty)$.*

If b is a slowly varying function, we define γ_b on $(0, \infty)$ by

$$\gamma_b(t) = b(\max\{t, t^{-1}\}), \ t > 0.$$

This definition can be found in [24]. We introduce the following properties for slowly varying functions, which will be used in this paper.

Proposition 2.2 ([5]). *Let b be a slowly varying function on $[1, \infty)$. Then the following properties hold:*
(i) *For each $\varepsilon > 0$, the function $t^\varepsilon \gamma_b(t)$ is equivalent to a nondecreasing function and the function $t^{-\varepsilon} \gamma_b(t)$ is equivalent to a nonincreasing function on $(0, \infty)$.*
(ii) *For $r \in \mathbb{R}$, b^r is a slowly varying function on $[1, \infty)$ and $\gamma_{b^r}(t) = \gamma_b^r(t)$.*
(iii) *For all $r > 0$ and $t > 0$, we have $\gamma_b(rt) \approx \gamma_b(t)$.*
(iv) *For all $r > 0$, denote $b_1(t) = b(t^r)$ on $[1, \infty)$. Then b_1 is also a slowly varying function.*

Lemma 2.3 ([10]). *Let $0 \leq p < \infty$ and b be a slowly varying function. For any constants $\alpha, \beta > 0$, we have*

$$(\alpha + \beta)^p \gamma_b(\alpha + \beta) \lesssim \alpha^p \gamma_b(\alpha) + \beta^p \gamma_b(\beta).$$

Lemma 2.4 ([7]). *Let b be a slowly varying function and α_i ($i \in \mathbb{N}$) be positive constants. If $0 < p < 1$, then*

$$\left(\sum_{i\in\mathbb{N}} \alpha_i\right)^p \gamma_b\left(\sum_{i\in\mathbb{N}} \alpha_i\right) \lesssim \sum_{i\in\mathbb{N}} \alpha_i^p \gamma_b(\alpha_i).$$

If $1 < p < \infty$, then

$$\sum_{i\in\mathbb{N}} \alpha_i^p \gamma_b(\alpha_i) \lesssim \left(\sum_{i\in\mathbb{N}} \alpha_i\right)^p \gamma_b\left(\sum_{i\in\mathbb{N}} \alpha_i\right).$$

Given a measurable function f on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the decreasing rearrangement of f is the function f^* defined on $[0, \infty)$ by

$$f^*(t) = \inf\{s \geq 0 : \mathbb{P}(|f| > s) \leq t\} \ (\inf \emptyset = \infty).$$

Definition 2.5. *Let $0 < p < \infty$, $0 < q \leq \infty$ and b be a slowly varying function. The Lorentz-Karamata space $L_{p,q,b}$ is defined to be the set of all measurable functions f such that*

$$\|f\|_{p,q,b} = \begin{cases} \left(\int_0^1 \left(t^{\frac{1}{p}}\gamma_b(t) f^*(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}}, & \text{if } 0 < q < \infty; \\ \sup\limits_{0\leq t\leq 1} t^{\frac{1}{p}}\gamma_b(t) f^*(t), & \text{if } q = \infty \end{cases}$$

is finite.

Remark 2.6 ([8],[10]). (i) *If $b \equiv 1$, then the Lorentz-Karamata space $L_{p,q,b}$ is the usual Lorentz space $L_{p,q}$. If $b(t) = 1 + \log t$, then the Lorentz-Karamata space is the Lorentz-Zygmund space. If $b \equiv 1$ and $0 < p = q < \infty$, then the Lorentz-Karamata space $L_{p,q,b}$ is the classical Lebesgue space L_p.*
(ii) *The following is an equivalent characterization of the quasi-norm $\|\cdot\|_{p,q,b}$:*

$$\|f\|_{p,q,b} \approx \begin{cases} \left(\int_0^\infty \left(t\mathbb{P}(|f| > t)^{\frac{1}{p}}\gamma_b\left(\mathbb{P}(|f| > t)\right)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}}, & \text{if } 0 < q < \infty; \\ \sup\limits_{t>0} t\mathbb{P}(|f| > t)^{\frac{1}{p}}\gamma_b\left(\mathbb{P}(|f| > t)\right), & \text{if } q = \infty. \end{cases}$$

(iii) *Another equivalent discrete characterization of the quasi-norm $\|\cdot\|_{p,q,b}$ is given below:*

$$\|f\|_{p,q,b} \approx \begin{cases} \left(\sum\limits_{k\in\mathbb{Z}} 2^{kq} \left\|\chi_{\{|f|>2^k\}}\right\|_p^q \gamma_b^q\left(\left\|\chi_{\{|f|>2^k\}}\right\|_p^p\right)\right)^{\frac{1}{q}}, & \text{if } 0 < q < \infty; \\ \sup\limits_{k\in\mathbb{Z}} 2^k \left\|\chi_{\{|f|>2^k\}}\right\|_p \gamma_b\left(\left\|\chi_{\{|f|>2^k\}}\right\|_p^p\right), & \text{if } q = \infty. \end{cases}$$

Lemma 2.7 ([5]). *Let $0 < p_1, p_2 < \infty$, $0 < q_1, q_2 \leq \infty$ and b_1, b_2 be slowly varying functions. If $p_1 > p_2$, then $L_{p_1,q_1,b_1} \subset L_{p_2,q_2,b_2}$.*

2.2. *Orlicz functions*

An Orlicz function is a nondecreasing function $\Phi : [0, \infty) \to [0, \infty)$ such that $\Phi(t) > 0$ for all $t > 0$, $\Phi(0) = 0$ and $\Phi(t) \to \infty$ as $t \to \infty$. Let $\mathcal{G}$ be the set of all Orlicz functions. We now give the following simple but useful lemmas.

Lemma 2.8 ([23, 25]). *Let $\Phi \in \mathcal{G}$ be a concave function. Then the following conclusions hold:*
(i) $s\Phi(t) \leq \Phi(st)$ *for* $0 < s \leq 1$ *and* $t \geq 0$.
(ii) $\Phi(st) \leq s\Phi(t)$ *for* $s \geq 1$ *and* $t \geq 0$.
In addition, it can be proved that Φ is subadditive, continuous and bijective from $[0,\infty)$ *to* $[0,\infty)$.

Lemma 2.9 ([25]). *Let $\Phi \in \mathcal{G}$ be a concave function, $0 < p < \infty$, $0 < q \leq \infty$ and $p, q < r \leq \infty$. Fix $f \in L_r$ and $A \in \mathcal{F}$ with $\mathbb{P}(A) \neq 0$. If $\{f \neq 0\} \subset A$, then*

$$\|\Phi(|f|)\|_{p,q} \lesssim \mathbb{P}(A)^{\frac{1}{p}} \Phi\left(\frac{\|f\|_r}{\mathbb{P}(A)^{\frac{1}{r}}}\right).$$

2.3. B-valued martingales

We now introduce some standard notations for **B**-valued martingale theory. Let $\{\mathcal{F}_n\}_{n\geq 0}$ be a nondecreasing sequence of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F} = \sigma\Big(\bigcup_{n\geq 0} \mathcal{F}_n\Big)$. The expectation operator and the conditional expectation operator related to $\mathcal{F}_n$ are denoted by $\mathbb{E}$ and $\mathbb{E}_n$, respectively. A sequence $f = (f_n)_{n\geq 0}$ in $L_1(\mathbf{B})$ is called a **B**-valued martingale if f_n is $\mathcal{F}_n$-measurable and satisfies $\mathbb{E}_n(f_{n+1}) = f_n$ for each $n \geq 0$. Denote by $\mathcal{M}$ the set of all **B**-valued martingales $f = (f_n)_{n\geq 0}$ relative to $\{\mathcal{F}_n\}_{n\geq 0}$ such that $f_0 = 0$. For $f = (f_n)_{n\geq 0} \in \mathcal{M}$, we define its martingale difference by $df_n = f_n - f_{n-1}$ $(n \geq 0)$, with convention $f_{-1} = 0$ and $\mathcal{F}_{-1} = \{\emptyset, \Omega\}$. Let $\mathcal{T}$ be the set of all stopping times relative to $\{\mathcal{F}_n\}_{n\geq 0}$. For $f \in \mathcal{M}$ and $\tau \in \mathcal{T}$, the stopped martingale $f^\tau = (f_n^\tau)_{n\geq 0}$ is defined by

$$f_n^\tau = \sum_{m=1}^{n} \chi_{\{m\leq\tau\}} df_m.$$

The **B**-valued martingale $f = (f_n)_{n\geq 0} \in \mathcal{M}$ is said to be $L_p(\mathbf{B})$-bounded if $f_n \in L_p(\mathbf{B})$ for all $n \geq 0$ and

$$\|f\|_{L_p(\mathbf{B})} = \sup_{n\geq 0} \|f_n\|_{L_p(\mathbf{B})} < \infty.$$

Define the maximal function, the r-variation function and the conditional r-variation function $(1 \leq r < \infty)$ of a **B**-valued martingale $f = (f_n)_{n\geq 0}$, respectively, as follows:

$$M_n(f)(\omega) = \sup_{0\leq m\leq n} \|f_m(\omega)\|, \quad M(f)(\omega) = \sup_{m\geq 0} \|f_m(\omega)\|;$$

$$S_n^r(f)(\omega) = \left(\sum_{m=0}^{n} \|df_m(\omega)\|^r\right)^{\frac{1}{r}}, \quad S^r(f)(\omega) = \left(\sum_{m=0}^{\infty} \|df_m(\omega)\|^r\right)^{\frac{1}{r}};$$

$$s_n^r(f)(\omega) = \left(\sum_{m=0}^{n} \mathbb{E}_{m-1}\Big(\|df_m\|^r\Big)(\omega)\right)^{\frac{1}{r}}, \quad s^r(f)(\omega) = \left(\sum_{m=0}^{\infty} \mathbb{E}_{m-1}\Big(\|df_m\|^r\Big)(\omega)\right)^{\frac{1}{r}}.$$

Denote by Λ the set of all sequences $(\lambda_n)_{n\geq 0}$ of nondecreasing, nonnegative and adapted functions with respect to $\{\mathcal{F}_n\}_{n\geq 0}$. Set $\lambda_\infty = \lim_{n\to\infty} \lambda_n$. Let $f \in \mathcal{M}$, $\Phi \in \mathcal{G}$, $0 < p < \infty$, $0 < q \leq \infty$, $1 \leq r < \infty$ and b be a slowly varying function. We define

$$\Lambda[Q^{S^r}_{p,q,b,\Phi}(f)](\mathbf{B}) = \Big\{(\lambda_n)_{n\geq 0} \in \Lambda : S_n^r(f) \leq \lambda_{n-1}\ (n \geq 1), \Phi(\lambda_\infty) \in L_{p,q,b}\Big\}$$

and

$$\Lambda[\mathcal{D}_{p,q,b,\Phi}(f)](\mathbf{B}) = \Big\{(\lambda_n)_{n\geq 0} \in \Lambda : \|f_n\| \leq \lambda_{n-1}\ (n \geq 1), \Phi(\lambda_\infty) \in L_{p,q,b}\Big\}.$$

Set

$$\|\Phi(f)\|_{Q^{S^r}_{p,q,b}(\mathbf{B})} = \inf\Big\{\|\Phi(\lambda_\infty)\|_{p,q,b} : (\lambda_n)_{n\geq 0} \in \Lambda[Q^{S^r}_{p,q,b,\Phi}(f)](\mathbf{B})\Big\}$$

and

$$\|\Phi(f)\|_{\mathcal{D}_{p,q,b}(\mathbf{B})} = \inf\Big\{\|\Phi(\lambda_\infty)\|_{p,q,b} : (\lambda_n)_{n\geq 0} \in \Lambda[\mathcal{D}_{p,q,b,\Phi}(f)](\mathbf{B})\Big\}.$$

Remark 2.10. *If $\Phi(t) = t$ with the notations above, then we obtain the definitions of $Q^{S^r}_{p,q,b}(\mathbf{B})$ and $\mathcal{D}_{p,q,b}(\mathbf{B})$, respectively; see Liu et al. [17].*

It is well known that the inequalities of **B**-valued martingales are closely related with the geometrical properties of Banach spaces. Now, we are going to introduce definitions of p-uniformly smooth, q-uniformly convex and Radon-Nikodým property (in short **RNP**) of Banach spaces.

Definition 2.11 ([27]). *Let **B** be a Banach space and $t > 0$. The modulus of uniform smoothness $\rho_{\mathbf{B}}(t)$ is defined as*

$$\rho_{\mathbf{B}}(t) = \sup\left\{\frac{\|x+ty\| + \|x-ty\|}{2} - 1 : x, y \in \mathbf{B}, \|x\| = \|y\| = 1\right\}.$$

*We claim that **B** is p-uniformly smooth if there exists a constant $C > 0$ such that $\rho_{\mathbf{B}}(t) \le Ct^p$ for all $t > 0$.*

Definition 2.12 ([27]). *Let **B** be a Banach space and $0 < \theta \le 2$. The modulus of uniform convexity $\delta_{\mathbf{B}}(\theta)$ is defined as*

$$\delta_{\mathbf{B}}(\theta) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in \mathbf{B}, \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \theta\right\}.$$

*We say that **B** is q-uniformly convex if there exists a constant $C > 0$ such that $\delta_{\mathbf{B}}(\theta) \ge C\theta^q$ for all $0 < \theta \le 2$.*

Definition 2.13 ([27]). *A Banach space **B** is said to have the **RNP** if for every measurable space $(\Omega, \mathcal{F})$, for every positive, finite, countably additive measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ and for every $\mathbb{P}$-continuous, countably additive vector measure μ of bounded variation, there exists a function $f \in L_1(\mathbf{B})$ such that $\mu(A) = \int_A f d\mathbb{P}$ for all $A \in \mathcal{F}$. A Banach space **B** has the **RNP** if **B** has the **RNP** with respect to every finite measure space.*

Remark 2.14 ([21]). *If **B** is isomorphic to a p-uniformly smooth* (q-uniformly convex) space, then **B** has the **RNP**.

The useful lemmas below were frequently used in this paper.

Lemma 2.15. *Let **B** be a Banach space and $1 < r \le 2$. Then the following properties are equivalent:*
(i) ***B** is isomorphic to a r-uniformly smooth space.*
(ii) *For every **B**-valued martingale $f = (f_n)_{n\ge0}$ with $\mathbb{E}\Big(\sum_{m=0}^{\infty}\|df_m\|^r\Big) < \infty$* (or $\sum_{m=0}^{\infty}\|df_m\|^r \in L_\infty$), $f = (f_n)_{n\ge0}$ converges in probability.
(iii) *There exists a constant $C > 0$ such that all **B**-valued martingales $f = (f_n)_{n\ge0}$ in $L_p(\mathbf{B})$ $(1 \le p < \infty)$ satisfy*

$$\|M(f)\|_p \le C\|S^r(f)\|_p.$$

(iv) *There exists a constant $C > 0$ such that all **B**-valued martingales $f = (f_n)_{n\ge0}$ in $L_r(\mathbf{B})$ satisfy*

$$\sup_{n\ge0}\mathbb{E}\Big(\|f_n\|^r\Big) \le C^r\sum_{n=0}^{\infty}\mathbb{E}\Big(\|df_n\|^r\Big).$$

(v) *Same as* (iv) *for all **B**-valued dyadic martingales.*

For Lemma 2.15, the proof of (i) ⇔ (ii) was proved in [21] and the proof of (i) ⇔ (iii) ⇔ (iv) ⇔ (v) comes from [27].

Lemma 2.16. *Let **B** be a Banach space and $2 \le r < \infty$. Then the following conclusions are equivalent:*
(i) ***B** is isomorphic to a r-uniformly convex space.*
(ii) *There exists a constant $C > 0$ such that all **B**-valued martingales $f = (f_n)_{n\ge0}$ in $L_p(\mathbf{B})$ $(1 \le p < \infty)$* satisfy

$$\|S^r(f)\|_p \le C\|M(f)\|_p.$$

(iii) *For any* **B***-valued martingale* $f = (f_n)_{n\geq 0}$ *with* $\sup_{n\geq 0} \|f_n\|_{L_\infty(\mathbf{B})} < \infty$, $S^r(f) < \infty$ *a.e..*
(iv) *There exists a constant* $C > 0$ *such that all* **B***-valued martingales* $f = (f_n)_{n\geq 0}$ *in* $L_r(\mathbf{B})$ *satisfy*

$$\sum_{n=0}^{\infty} \mathbb{E}\Big(\|df_n\|^r\Big) \leq C^r \sup_{n\geq 0} \mathbb{E}\Big(\|f_n\|^r\Big).$$

(v) *Same as* (iv) *for all* **B***-valued dyadic martingales.*

For Lemma 2.16, we refer to [21] for the proof of (i) ⇔ (iii), the proof of (i) ⇔ (ii) ⇔ (iv) ⇔ (v) can be found in [27].

Lemma 2.17. *Let* **B** *be a Banach space. Then the following properties of* **B** *are equivalent:*
(i) **B** *has the* **RNP**.
(ii) *If* $1 < p < \infty$, *every* **B***-valued bounded martingale in* $L_p(\mathbf{B})$ *converges almost surely* (*a.s.*) and in $L_p(\mathbf{B})$.
(iii) *If there is a constant* $C > 0$ *such that* $\sup_{n\geq 0} \|f_n\|_{L_\infty(\mathbf{B})} \leq C$ *for any* **B***-valued martingale* $f = (f_n)_{n\geq 0}$, *then* f_n *converges a.e..*

For Lemma 2.17, the proof of (i) ⇔ (ii) was showed in [27]; the proof of (i) ⇔ (iii) can be seen in [21].

3. Atomic decompositions

In this section, we investigate atomic decompositions for **B**-valued martingale Hardy Lorentz-Karamata spaces. First, let us introduce the definition of atoms.

Definition 3.1. *Let* $\Phi \in \mathcal{G}$ *be a concave function,* $1 \leq r < \infty$, $0 < p < \infty$ *and* $0 < \ell \leq \infty$. *A* **B***-valued measurable function a is called a* $(\Phi, p, \ell)^{s^r}$*-atom (resp.* $(\Phi, p, \ell)^{S^r}$*-atom,* $(\Phi, p, \ell)^{M}$*-atom), if there exists a stopping time* $\tau \in \mathcal{T}$ *such that*
(i) $a_n = \mathbb{E}_n(a) = 0$, *if* $\tau \geq n$;
(ii) $\|s^r(a)\|_\ell$ ($\|S^r(a)\|_\ell$ *or* $\|M(a)\|_\ell$, *respectively*) $\leq \mathbb{P}(\tau < \infty)^{\frac{1}{\ell}} \Phi^{-1}(\mathbb{P}(\tau < \infty)^{-\frac{1}{p}})$.

Let $0 < q \leq \infty$ and b be a slowly varying function. Denote by $\mathcal{A}^{s^r}(\Phi, p, q, \ell)$ (resp. $\mathcal{A}^{S^r}(\Phi, p, q, \ell)$, $\mathcal{A}^{M}(\Phi, p, q, \ell)$) the set of all sequences of triples (μ^k, a^k, τ^k), where

$$\mu^k = \frac{\Phi^{-1}(2^{k+1})}{\Phi^{-1}\Big(\mathbb{P}(\tau^k < \infty)^{-\frac{1}{p}}\Big)},$$

a^k are $(\Phi, p, \ell)^{s^r}$-atoms (resp. $(\Phi, p, \ell)^{S^r}$-atoms, $(\Phi, p, \ell)^{M}$-atoms) satisfying (i) and (ii), τ^k are the stopping times corresponding to a^k, satisfying

$$\left\|\left\{\mathbb{P}(\tau^k < \infty)^{\frac{1}{p}} \Phi\left(\mu^k \Phi^{-1}\left(\mathbb{P}(\tau^k < \infty)^{-\frac{1}{p}}\right)\right) \gamma_b\left(\mathbb{P}(\tau^k < \infty)\right)\right\}_{k\in\mathbb{Z}}\right\|_{l_q} < \infty.$$

Let

$$\mathcal{W}\left(\{\mu^k\}, \{a^k\}, \{\tau^k\}\right) = \left\|\left\{\mathbb{P}(\tau^k < \infty)^{\frac{1}{p}} \Phi\Big(\mu^k \Phi^{-1}\Big(\mathbb{P}(\tau^k < \infty)^{-\frac{1}{p}}\Big)\Big) \gamma_b\Big(\mathbb{P}(\tau^k < \infty)\Big)\right\}_{k\in\mathbb{Z}}\right\|_{l_q}.$$

Theorem 3.2. *Let* $\Phi \in \mathcal{G}$ *be a concave function,* **B** *be a Banach space,* $1 < r \leq 2$, $0 < p \leq r$, $0 < q \leq \infty$, $\max\{1, p\} < \ell \leq \infty$ *and b be a slowly varying function. Then the following statements are equivalent:*
(i) **B** *is isomorphic to a r-uniformly smooth space.*
(ii) *Assume that* **B***-valued martingale* $f = (f_n)_{n\geq 0}$ *satisfies* $\|\Phi(s^r(f))\|_{p,q,b} < \infty$. *Then there exists a sequence of triples* $(\mu^k, a^k, \tau^k) \in \mathcal{A}^{s^r}(\Phi, p, q, \ell)$ *such that for all* $n \in \mathbb{N}$,

$$f_n = \sum_{k\in\mathbb{Z}} \mu_k \mathbb{E}_n(a^k) \quad a.e. \tag{4}$$

and

$$\sup_{k\in\mathbb{Z}} \|M(a^k)\|_p < \infty. \tag{5}$$

Moreover,

$$\|\Phi(s^r(f))\|_{p,q,b} \approx \inf \mathcal{W}(\{\mu^k\},\{a^k\},\{\tau^k\}), \tag{6}$$

where the infimum is taken over all the preceding decompositions of f of the form (4).

Proof. (i) ⇒ (ii). Suppose that $f = (f_n)_{n\geq 0}$ is a $\mathbf{B}$-valued martingale which satisfies $\|\Phi(s^r(f))\|_{p,q,b} < \infty$. For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, we define the following stopping times:

$$\tau^k = \inf\{n \in \mathbb{N} : s^r_{n+1}(f) > \Phi^{-1}(2^k)\} \quad (\inf\emptyset = \infty).$$

Obviously, the sequence of these stopping times is nondecreasing. For each $n \in \mathbb{N}$, it is easy to see that

$$\begin{aligned} f_n &= \sum_{i=0}^{n} d_i f = \sum_{i=0}^{n}\sum_{k\in\mathbb{Z}} \Big(\chi_{\{i\leq\tau^{k+1}\}} - \chi_{\{i\leq\tau^k\}}\Big) d_i f \\ &= \sum_{k\in\mathbb{Z}} \left(\sum_{i=0}^{n} \chi_{\{i\leq\tau^{k+1}\}} d_i f - \sum_{i=0}^{n} \chi_{\{i\leq\tau^k\}} d_i f \right) \\ &= \sum_{k\in\mathbb{Z}} (f_n^{\tau^{k+1}} - f_n^{\tau^k}) \quad a.e.. \end{aligned}$$

For all $k \in \mathbb{Z}$ and $n \in \mathbb{N}$, set

$$\mu^k = \frac{\Phi^{-1}(2^{k+1})}{\Phi^{-1}\Big(\mathbb{P}(\tau^k < \infty)^{-\frac{1}{p}}\Big)} \quad \text{and} \quad a_n^k = \frac{f_n^{\tau^{k+1}} - f_n^{\tau^k}}{\mu^k}.$$

Let $a_n^k = 0$ if $\mu^k = 0$. It is easy to check that

$$\sum_{k\in\mathbb{Z}} \mu_k a_n^k = \sum_{k\in\mathbb{Z}} (f_n^{\tau^{k+1}} - f_n^{\tau^k}) = f_n \quad a.e.$$

for all $n \geq 0$. Since

$$da_n^k = \frac{df_n^{\tau^{k+1}} - df_n^{\tau^k}}{\mu^k} = \frac{df_n \chi_{\{\tau^k<n\leq\tau^{k+1}\}}}{\mu^k}, \quad \forall\, n \in \mathbb{N},$$

for any fixed $k \in \mathbb{Z}$, we have

$$\mathbb{E}_{n-1}\Big(da_n^k\Big) = \frac{\mathbb{E}_{n-1}\Big(df_n\Big)\chi_{\{\tau^k<n\leq\tau^{k+1}\}}}{\mu^k} = 0.$$

Thus, $(a_n^k)_{n\geq 0}$ is a $\mathbf{B}$-valued martingale. From the definition of a_n^k, it is clear that $s^r\Big((a_n^k)_{n\geq 0}\Big) = 0$ on $\{\tau^k = \infty\}$. Furthermore, $s^r(f^{\tau^k}) = s^r_{\tau^k}(f) \leq \Phi^{-1}(2^k)$. Hence, we have

$$s^r\left((a_n^k)_{n\geq 0}\right) = \left(\sum_{m=0}^{\infty} \mathbb{E}_{m-1}\Big(\|da_m^k\|^r\Big)\right)^{\frac{1}{r}} \chi_{\{\tau^k<\infty\}} \leq \frac{s^r_{\tau^{k+1}}(f)}{\mu^k}\chi_{\{\tau^k<\infty\}} \leq \Phi^{-1}\Big(\mathbb{P}(\tau^k < \infty)^{-\frac{1}{p}}\Big)\chi_{\{\tau^k<\infty\}}. \tag{7}$$

It follows from Lemma 2.15 (iii) that

$$\Big\|M\Big((a_n^k)_{n\geq 0}\Big)\Big\|_r \leq C\,\Big\|S^r\Big((a_n^k)_{n\geq 0}\Big)\Big\|_r = C\,\Big\|s^r\Big((a_n^k)_{n\geq 0}\Big)\Big\|_r \leq C\mathbb{P}(\tau^k < \infty)^{\frac{1}{r}}\Phi^{-1}\Big(\mathbb{P}(\tau^k < \infty)^{-\frac{1}{p}}\Big) < \infty, \tag{8}$$

which means $(a_n^k)_{n\geq 0}$ is a $L_r(\mathbf{B})$-bounded martingale. Since condition (i) implies $\mathbf{B}$ has the **RNP** (see Remark 2.14), then by Lemma 2.17, we know that a_n^k converges *a.s.* to a limit a^k in $L_r(\mathbf{B})$. Therefore, $a_n^k = \mathbb{E}_n(a^k)$ (see [21, p.27]). Combining this fact with (7), we have

$$\left\|s^r(a^k)\right\|_\ell \leq \mathbb{P}(\tau^k < \infty)^{\frac{1}{\ell}}\Phi^{-1}\Big(\mathbb{P}(\tau^k < \infty)^{-\frac{1}{p}}\Big). \tag{9}$$

For $n \leq \tau^k$,

$$\mathbb{E}_n(a^k) = a_n^k = \frac{f_n^{\tau^{k+1}} - f_n^{\tau^k}}{\mu^k} = \frac{f_n - f_n}{\mu^k} = 0. \tag{10}$$

Consequently, a^k is a $(\Phi, p, \ell)^{s^r}$-atom and (4) holds. Since $0 < p \leq r$, by using (8) and Hölder's inequality, for any fixed $k \in \mathbb{Z}$, we conclude that

$$\begin{aligned}
\|M(a^k)\|_p^p &= \mathbb{E}\left(M(a^k)^p\right) = \mathbb{E}\left(M(a^k)^p\chi_{\{\tau^k<\infty\}}\right) \\
&\leq \|M(a^k)\|_r^p\mathbb{P}(\tau^k < \infty)^{1-\frac{p}{r}} \\
&\lesssim \mathbb{P}(\tau^k < \infty)\left(\Phi^{-1}\Big(\mathbb{P}(\tau^k < \infty)^{-\frac{1}{p}}\Big)\right)^p.
\end{aligned}$$

Thus, inequality (5) holds.

Next, we will prove that (6) holds. Apparently,

$$\Phi\Big(\mu^k\Phi^{-1}\Big(\mathbb{P}(\tau^k < \infty)^{-\frac{1}{p}}\Big)\Big) = 2^{k+1} \tag{11}$$

and

$$\{\tau^k < \infty\} = \Big\{\Phi\Big(s^r(f)\Big) > 2^k\Big\}.$$

For the case of $0 < q < \infty$. Proposition 2.2 (i) and Remark 2.6 (ii) guarantee that

$$\begin{aligned}
\mathcal{W}(\{\mu^k\}, \{a^k\}, \{\tau^k\}) &= \left\|\Big\{\mathbb{P}(\tau^k < \infty)^{\frac{1}{p}}\Phi\Big(\mu^k\Phi^{-1}\Big(\mathbb{P}(\tau^k < \infty)^{-\frac{1}{p}}\Big)\Big)\gamma_b\Big(\mathbb{P}(\tau^k < \infty)\Big)\Big\}_{k\in\mathbb{Z}}\right\|_{l_q} \\
&= \left\|\Big\{\mathbb{P}\Big(\Phi\Big(s^r(f)\Big) > 2^k\Big)^{\frac{1}{p}}2^{k+1}\gamma_b\Big(\mathbb{P}\Big(\Phi\Big(s^r(f)\Big) > 2^k\Big)\Big)\Big\}_{k\in\mathbb{Z}}\right\|_{l_q} \\
&= \left(\sum_{k\in\mathbb{Z}}\mathbb{P}\Big(\Phi\Big(s^r(f)\Big) > 2^k\Big)^{\frac{q}{p}}2^{(k+1)q}\gamma_b^q\Big(\mathbb{P}\Big(\Phi\Big(s^r(f)\Big) > 2^k\Big)\Big)\right)^{\frac{1}{q}} \\
&\approx \left(\sum_{k\in\mathbb{Z}}\mathbb{P}\Big(\Phi\Big(s^r(f)\Big) > 2^k\Big)^{\frac{q}{p}}\gamma_b^q\Big(\mathbb{P}\Big(\Phi\Big(s^r(f)\Big) > 2^k\Big)\Big)\int_{2^{k-1}}^{2^k} y^q\frac{dy}{y}\right)^{\frac{1}{q}} \\
&\lesssim \left(\sum_{k\in\mathbb{Z}}\int_{2^{k-1}}^{2^k}\mathbb{P}\Big(\Phi\Big(s^r(f)\Big) > y\Big)^{\frac{q}{p}}\gamma_b^q\Big(\mathbb{P}\Big(\Phi\Big(s^r(f)\Big) > y\Big)\Big)y^{q-1}dy\right)^{\frac{1}{q}} \\
&\approx \left\|\Phi\Big(s^r(f)\Big)\right\|_{p,q,b}.
\end{aligned}$$

Similarly, for the case of $q = \infty$. Applying Remark 2.6 (ii), we find that

$$\begin{aligned}
\mathcal{W}(\{\mu^k\}, \{a^k\}, \{\tau^k\}) &= \left\|\Big\{\mathbb{P}(\tau^k < \infty)^{\frac{1}{p}}\Phi\Big(\mu^k\Phi^{-1}\Big(\mathbb{P}(\tau^k < \infty)^{-\frac{1}{p}}\Big)\Big)\gamma_b\Big(\mathbb{P}(\tau^k < \infty)\Big)\Big\}_{k\in\mathbb{Z}}\right\|_\infty \\
&= \sup_{k\in\mathbb{Z}}\mathbb{P}\Big(\Phi\Big(s^r(f)\Big) > 2^k\Big)^{\frac{1}{p}}2^{k+1}\gamma_b\Big(\mathbb{P}\Big(\Phi\Big(s^r(f)\Big) > 2^k\Big)\Big)
\end{aligned}$$

$$= \sup_{t>0} \mathbb{P}\Big(\Phi\big(s^r(f)\big) > t\Big)^{\frac{1}{p}} 2t\gamma_b\Big(\mathbb{P}\Big(\Phi\big(s^r(f)\big) > t\Big)\Big)$$
$$\approx \Big\|\Phi\big(s^r(f)\big)\Big\|_{p,\infty,b}.$$

Thus $(\mu^k, a^k, \tau^k) \in \mathcal{A}^{s^r}(\Phi, p, q, \ell)$. Then we have the following inequality for $0 < q \le \infty$,

$$\mathcal{W}\left(\{\mu^k\}, \{a^k\}, \{\tau^k\}\right) \lesssim \Big\|\Phi\big(s^r(f)\big)\Big\|_{p,q,b}. \tag{12}$$

On the other hand, for an arbitrary integer k_0, set

$$T_1 = \sum_{k=-\infty}^{k_0-1} \Phi\Big(\mu^k s^r(a^k)\Big) \quad \text{and} \quad T_2 = \sum_{k=k_0}^{\infty} \Phi\Big(\mu^k s^r(a^k)\Big).$$

It follows from the sublinearity of the conditional r-variation s^r and the subadditivity of Φ that

$$\Phi\Big(s^r(f)\Big) \le \Phi\left(\sum_{k\in\mathbb{Z}} \mu^k s^r(a^k)\right) \le \sum_{k\in\mathbb{Z}} \Phi\Big(\mu^k s^r(a^k)\Big) \triangleq T_1 + T_2. \tag{13}$$

Firstly, we consider the case of $0 < q < \infty$. In view of (13) and Remark 2.6 (iii), we obtain

$$\begin{aligned} &\|\Phi(s^r(f))\|_{p,q,b} \le \|T_1 + T_2\|_{p,q,b} \lesssim \|T_1\|_{p,q,b} + \|T_2\|_{p,q,b} \qquad (14)\\ &\approx \left(\sum_{k_0\in\mathbb{Z}} 2^{k_0 q} \|\chi_{\{T_1 > 2^{k_0}\}}\|_p^q \gamma_b^q\left(\|\chi_{\{T_1>2^{k_0}\}}\|_p^p\right)\right)^{\frac{1}{q}} + \left(\sum_{k_0\in\mathbb{Z}} 2^{k_0 q} \|\chi_{\{T_2 > 2^{k_0}\}}\|_p^q \gamma_b^q\left(\|\chi_{\{T_2>2^{k_0}\}}\|_p^p\right)\right)^{\frac{1}{q}}. \end{aligned}$$

Estimation for $\sum_{k_0\in\mathbb{Z}} 2^{k_0 q} \|\chi_{\{T_1 > 2^{k_0}\}}\|_p^q \gamma_b^q\left(\|\chi_{\{T_1>2^{k_0}\}}\|_p^p\right)$**.**

Let $0 < \theta < \min\{p, q, 1\}$ and $1 < \lambda_1 < \min\{\frac{1}{\theta}, \frac{\ell}{p}\}$. Using Chebyshev's inequality and Lemma 2.9, we have

$$\begin{aligned} \|\chi_{\{T_1>2^{k_0}\}}\|_p &\le \frac{1}{2^{k_0\lambda_1}} \left\|\left[\sum_{k=-\infty}^{k_0-1} \Phi\Big(\mu^k s^r(a^k)\Big)\right]^{\lambda_1}\right\|_p = \frac{1}{2^{k_0\lambda_1}} \left\|\left[\sum_{k=-\infty}^{k_0-1} \Phi\Big(\mu^k s^r(a^k)\Big)\right]^{\lambda_1\theta}\right\|_{\frac{p}{\theta}}^{\frac{1}{\theta}} \\ &\le \frac{1}{2^{k_0\lambda_1}} \left\|\sum_{k=-\infty}^{k_0-1} \Phi\Big(\mu^k s^r(a^k)\Big)^{\lambda_1\theta}\right\|_{\frac{p}{\theta}}^{\frac{1}{\theta}} \le \frac{1}{2^{k_0\lambda_1}} \left\{\sum_{k=-\infty}^{k_0-1} \left\|\Phi\Big(\mu^k s^r(a^k)\Big)^{\lambda_1\theta}\right\|_{\frac{p}{\theta}}\right\}^{\frac{1}{\theta}} \\ &= \frac{1}{2^{k_0\lambda_1}} \left\{\sum_{k=-\infty}^{k_0-1} \Big\|\Phi\Big(\mu^k s^r(a^k)\Big)\Big\|_{p\lambda_1}^{\lambda_1\theta}\right\}^{\frac{1}{\theta}} \lesssim \frac{1}{2^{k_0\lambda_1}} \left\{\sum_{k=-\infty}^{k_0-1} \mathbb{P}(\tau^k < \infty)^{\frac{\theta}{p}} \Phi\left(\frac{\mu^k \|s^r(a^k)\|_\ell}{\mathbb{P}(\tau^k<\infty)^{\frac{1}{\ell}}}\right)^{\lambda_1\theta}\right\}^{\frac{1}{\theta}}. \end{aligned}$$

Since a^k is a $(\Phi, p, \ell)^{s^r}$-atom, it is easy to check that

$$\Phi\left(\frac{\mu^k \|s^r(a^k)\|_\ell}{\mathbb{P}(\tau^k<\infty)^{\frac{1}{\ell}}}\right) \le 2^{k+1}. \tag{15}$$

Choose η_1 such that $1 < \eta_1 < \lambda_1$. By (15) and Hölder's inequality with $\frac{\theta}{q} + \frac{q-\theta}{q} = 1$, we obtain

$$\|\chi_{\{T_1>2^{k_0}\}}\|_p \lesssim \frac{1}{2^{k_0\lambda_1}} \left\{\sum_{k=-\infty}^{k_0-1} \mathbb{P}(\tau^k < \infty)^{\frac{\theta}{p}} 2^{k\lambda_1\theta}\right\}^{\frac{1}{\theta}} \tag{16}$$

$$= \frac{1}{2^{k_0\lambda_1}} \left\{ \sum_{k=-\infty}^{k_0-1} \mathbb{P}(\tau^k < \infty)^{\frac{\theta}{p}} 2^{k\eta_1\theta} 2^{k(\lambda_1-\eta_1)\theta} \right\}^{\frac{1}{\theta}}$$

$$\leq \frac{1}{2^{k_0\lambda_1}} \left\{ \sum_{k=-\infty}^{k_0-1} \mathbb{P}(\tau^k < \infty)^{\frac{q}{p}} 2^{k\eta_1 q} \right\}^{\frac{1}{q}} \left\{ \sum_{k=-\infty}^{k_0-1} 2^{k(\lambda_1-\eta_1)\theta\frac{q}{q-\theta}} \right\}^{\frac{q-\theta}{q\theta}}$$

$$= \frac{2^{\eta_1-\lambda_1}}{2^{k_0\eta_1}\left(1-2^{\frac{q\theta(\eta_1-\lambda_1)}{q-\theta}}\right)^{\frac{q-\theta}{q\theta}}} \left\{ \sum_{k=-\infty}^{k_0-1} \mathbb{P}(\tau^k < \infty)^{\frac{q}{p}} 2^{k\eta_1 q} \right\}^{\frac{1}{q}}.$$

Define $b_1(t) = b(t^{\frac{p}{q}})$ for $t \in [1,\infty)$ and $0 < \varepsilon_1 < 1$. Applying (16), Proposition 2.2 (ii), (iv) and Lemma 2.4, we can further deduce that

$$\sum_{k_0\in\mathbb{Z}} 2^{k_0 q} \|\chi_{\{T_1>2^{k_0}\}}\|_p^q \gamma_b^q \left(\|\chi_{\{T_1>2^{k_0}\}}\|_p^p\right) \tag{17}$$

$$\lesssim \sum_{k_0\in\mathbb{Z}} 2^{k_0 q} \left(\sum_{k=-\infty}^{k_0-1} 2^{-k_0\eta_1 q} \mathbb{P}(\tau^k < \infty)^{\frac{q}{p}} 2^{k\eta_1 q} \right) \gamma_b^q \left[\left(\sum_{k=-\infty}^{k_0-1} 2^{-k_0\eta_1 q} \mathbb{P}(\tau^k < \infty)^{\frac{q}{p}} 2^{k\eta_1 q} \right)^{\frac{p}{q}} \right]$$

$$= \sum_{k_0\in\mathbb{Z}} 2^{k_0 q} \left(\sum_{k=-\infty}^{k_0-1} 2^{(k-k_0)\eta_1 q} \mathbb{P}(\tau^k < \infty)^{\frac{q}{p}} \right) \gamma_{b_1^q} \left(\sum_{k=-\infty}^{k_0-1} 2^{(k-k_0)\eta_1 q} \mathbb{P}(\tau^k < \infty)^{\frac{q}{p}} \right)$$

$$= \sum_{k_0\in\mathbb{Z}} 2^{k_0 q} \left[\left(\sum_{k=-\infty}^{k_0-1} 2^{(k-k_0)\eta_1 q} \mathbb{P}(\tau^k < \infty)^{\frac{q}{p}} \right)^{\varepsilon_1} \gamma_{b_1^{\varepsilon_1 q}} \left(\sum_{k=-\infty}^{k_0-1} 2^{(k-k_0)\eta_1 q} \mathbb{P}(\tau^k < \infty)^{\frac{q}{p}} \right) \right]^{\frac{1}{\varepsilon_1}}$$

$$\lesssim \sum_{k_0\in\mathbb{Z}} 2^{k_0 q} \left[\sum_{k=-\infty}^{k_0-1} 2^{(k-k_0)\eta_1\varepsilon_1 q} \mathbb{P}(\tau^k < \infty)^{\frac{\varepsilon_1 q}{p}} \gamma_{b_1^{\varepsilon_1 q}} \left(2^{(k-k_0)\eta_1 q} \mathbb{P}(\tau^k < \infty)^{\frac{q}{p}} \right) \right]^{\frac{1}{\varepsilon_1}}$$

$$= \sum_{k_0\in\mathbb{Z}} 2^{k_0 q} \left[\sum_{k=-\infty}^{k_0-1} 2^{(k-k_0)\eta_1\varepsilon_1 q} \mathbb{P}(\tau^k < \infty)^{\frac{\varepsilon_1 q}{p}} \gamma_b^{\varepsilon_1 q} \left(2^{(k-k_0)\eta_1 p} \mathbb{P}(\tau^k < \infty) \right) \right]^{\frac{1}{\varepsilon_1}}.$$

Set $0 < \delta_1 < \frac{\eta_1-1}{\eta_1}$. (17) and Hölder's inequality with $1-\varepsilon_1+\varepsilon_1 = 1$ guarantee that

$$\sum_{k_0\in\mathbb{Z}} 2^{k_0 q} \|\chi_{\{T_1>2^{k_0}\}}\|_p^q \gamma_b^q \left(\|\chi_{\{T_1>2^{k_0}\}}\|_p^p\right) \tag{18}$$

$$\lesssim \sum_{k_0\in\mathbb{Z}} 2^{k_0 q} \left[\sum_{k=-\infty}^{k_0-1} 2^{(k-k_0)\eta_1\varepsilon_1 q\delta_1} 2^{(k-k_0)\eta_1\varepsilon_1 q(1-\delta_1)} \mathbb{P}(\tau^k < \infty)^{\frac{\varepsilon_1 q}{p}} \gamma_b^{\varepsilon_1 q} \left(2^{(k-k_0)\eta_1 p} \mathbb{P}(\tau^k < \infty) \right) \right]^{\frac{1}{\varepsilon_1}}$$

$$\leq \sum_{k_0\in\mathbb{Z}} 2^{k_0 q} \left(\sum_{k=-\infty}^{k_0-1} 2^{(k-k_0)\eta_1\varepsilon_1 q\delta_1/(1-\varepsilon_1)} \right)^{\frac{1-\varepsilon_1}{\varepsilon_1}} \sum_{k=-\infty}^{k_0-1} 2^{(k-k_0)\eta_1 q(1-\delta_1)} \mathbb{P}(\tau^k < \infty)^{\frac{q}{p}} \gamma_b^q \left(2^{(k-k_0)\eta_1 p} \mathbb{P}(\tau^k < \infty) \right)$$

$$\lesssim \sum_{k_0\in\mathbb{Z}} 2^{k_0 q} \sum_{k=-\infty}^{k_0-1} 2^{(k-k_0)\eta_1 q(1-\delta_1)} \mathbb{P}(\tau^k < \infty)^{\frac{q}{p}} \gamma_b^q \left(2^{(k-k_0)\eta_1 p} \mathbb{P}(\tau^k < \infty) \right).$$

Given $0 < z_1 < \frac{\eta_1-\eta_1\delta_1-1}{\eta_1 p}$. By Proposition 2.2 (i), we find that $t^{z_1}\gamma_b(t)$ is equivalent to a nondecreasing function on $(0,\infty)$. When $k \leq k_0-1$, we conclude that

$$\gamma_b\left(2^{(k-k_0)\eta_1 p}\mathbb{P}(\tau^k < \infty)\right) = \left(2^{(k-k_0)\eta_1 p}\mathbb{P}(\tau^k < \infty)\right)^{-z_1} \left(2^{(k-k_0)\eta_1 p}\mathbb{P}(\tau^k < \infty)\right)^{z_1} \gamma_b\left(2^{(k-k_0)\eta_1 p}\mathbb{P}(\tau^k < \infty)\right) \tag{19}$$

$$\lesssim \left(2^{(k-k_0)\eta_1 p}\mathbb{P}(\tau^k<\infty)\right)^{-z_1}\left(\mathbb{P}(\tau^k<\infty)\right)^{z_1}\gamma_b\left(\mathbb{P}(\tau^k<\infty)\right)$$
$$= 2^{-(k-k_0)\eta_1 p z_1}\gamma_b\left(\mathbb{P}(\tau^k<\infty)\right).$$

It follows from (18), (19), Abel's transformation and (15) that

$$\begin{aligned}
&\sum_{k_0\in\mathbb{Z}} 2^{k_0 q}\|\chi_{\{T_1>2^{k_0}\}}\|_p^q\gamma_b^q\left(\|\chi_{\{T_1>2^{k_0}\}}\|_p^p\right) \qquad (20)\\
&\lesssim \sum_{k_0\in\mathbb{Z}} 2^{k_0 q}\sum_{k=-\infty}^{k_0-1} 2^{(k-k_0)\eta_1 q(1-\delta_1)}\mathbb{P}(\tau^k<\infty)^{\frac{q}{p}}2^{-(k-k_0)\eta_1 p z_1 q}\gamma_b^q\left(\mathbb{P}(\tau^k<\infty)\right)\\
&= \sum_{k\in\mathbb{Z}} 2^{k(1-\delta_1-z_1 p)\eta_1 q}\mathbb{P}(\tau^k<\infty)^{\frac{q}{p}}\gamma_b^q\left(\mathbb{P}(\tau^k<\infty)\right)\sum_{k_0=k+1}^{\infty} 2^{k_0 q[1+\eta_1(\delta_1-1)+\eta_1 z_1 p]}\\
&= \frac{2^{q\eta_1\delta_1-q\eta_1+q\eta_1 z_1 p}}{1-2^{q[1+\eta_1(\delta_1-1)+\eta_1 z_1 p]}}\sum_{k\in\mathbb{Z}}\mathbb{P}(\tau^k<\infty)^{\frac{q}{p}}2^{(k+1)q}\gamma_b^q\left(\mathbb{P}(\tau^k<\infty)\right)\\
&= \frac{2^{q\eta_1\delta_1-q\eta_1+q\eta_1 z_1 p}}{1-2^{q[1+\eta_1(\delta_1-1)+\eta_1 z_1 p]}}\sum_{k\in\mathbb{Z}}\mathbb{P}(\tau^k<\infty)^{\frac{q}{p}}\Phi\Big(\mu^k\Phi^{-1}\Big(\mathbb{P}(\tau^k<\infty)^{-\frac{1}{p}}\Big)\Big)^q\gamma_b^q\left(\mathbb{P}(\tau^k<\infty)\right).
\end{aligned}$$

Estimation for $\sum_{k_0\in\mathbb{Z}} 2^{k_0 q}\|\chi_{\{T_2>2^{k_0}\}}\|_p^q\gamma_b^q\left(\|\chi_{\{T_2>2^{k_0}\}}\|_p^p\right)$.

Let θ be as above and $0<\lambda_2<1$. Applying Chebyshev's inequality and Lemma 2.9, we get

$$\begin{aligned}
\|\chi_{\{T_2>2^{k_0}\}}\|_p &\le \frac{1}{2^{k_0\lambda_2}}\left\|\left[\sum_{k=k_0}^{\infty}\Phi\Big(\mu^k s^r(a^k)\Big)\right]^{\lambda_2}\right\|_p = \frac{1}{2^{k_0\lambda_2}}\left\|\left[\sum_{k=k_0}^{\infty}\Phi\Big(\mu^k s^r(a^k)\Big)\right]^{\lambda_2\theta}\right\|_{\frac{p}{\theta}}^{\frac{1}{\theta}}\\
&\le \frac{1}{2^{k_0\lambda_2}}\left\|\sum_{k=k_0}^{\infty}\Phi\Big(\mu^k s^r(a^k)\Big)^{\lambda_2\theta}\right\|_{\frac{p}{\theta}}^{\frac{1}{\theta}} \le \frac{1}{2^{k_0\lambda_2}}\left\{\sum_{k=k_0}^{\infty}\left\|\Phi\Big(\mu^k s^r(a^k)\Big)^{\lambda_2\theta}\right\|_{\frac{p}{\theta}}\right\}^{\frac{1}{\theta}}\\
&= \frac{1}{2^{k_0\lambda_2}}\left\{\sum_{k=k_0}^{\infty}\left\|\Phi\Big(\mu^k s^r(a^k)\Big)\right\|_{\lambda_2 p}^{\lambda_2\theta}\right\}^{\frac{1}{\theta}} \lesssim \frac{1}{2^{k_0\lambda_2}}\left\{\sum_{k=k_0}^{\infty}\mathbb{P}(\tau^k<\infty)^{\frac{\theta}{p}}\Phi\bigg(\frac{\mu^k\|s^r(a^k)\|_\ell}{\mathbb{P}(\tau^k<\infty)^{\frac{1}{\ell}}}\bigg)^{\lambda_2\theta}\right\}^{\frac{1}{\theta}}.
\end{aligned}$$

Set $0<\lambda_2<\eta_2<1$. It follows from (15) and Hölder's inequality with $\frac{\theta}{q}+\frac{q-\theta}{q}=1$ that

$$\begin{aligned}
\|\chi_{\{T_2>2^{k_0}\}}\|_p &\lesssim \frac{1}{2^{k_0\lambda_2}}\left\{\sum_{k=k_0}^{\infty}\mathbb{P}(\tau^k<\infty)^{\frac{\theta}{p}}2^{k\lambda_2\theta}\right\}^{\frac{1}{\theta}} \qquad (21)\\
&= \frac{1}{2^{k_0\lambda_2}}\left\{\sum_{k=k_0}^{\infty}\mathbb{P}(\tau^k<\infty)^{\frac{\theta}{p}}2^{k\eta_2\theta}2^{k(\lambda_2-\eta_2)\theta}\right\}^{\frac{1}{\theta}}\\
&\le \frac{1}{2^{k_0\lambda_2}}\left\{\sum_{k=k_0}^{\infty}\mathbb{P}(\tau^k<\infty)^{\frac{q}{p}}2^{k\eta_2 q}\right\}^{\frac{1}{q}}\left\{\sum_{k=k_0}^{\infty}2^{k(\lambda_2-\eta_2)\theta\frac{q}{q-\theta}}\right\}^{\frac{q-\theta}{q\theta}}\\
&= \frac{1}{2^{k_0\eta_2}\left(1-2^{\frac{q\theta(\lambda_2-\eta_2)}{q-\theta}}\right)^{\frac{q-\theta}{q\theta}}}\left\{\sum_{k=k_0}^{\infty}\mathbb{P}(\tau^k<\infty)^{\frac{q}{p}}2^{k\eta_2 q}\right\}^{\frac{1}{q}}.
\end{aligned}$$

Set $0 < \varepsilon_2 < 1$. (21), Proposition 2.2 (ii), (iv) and Lemma 2.4 ensure that

$$
\begin{aligned}
&\sum_{k_0\in\mathbb{Z}} 2^{k_0 q}\|\chi_{\{T_2>2^{k_0}\}}\|_p^q \gamma_b^q\left(\|\chi_{\{T_2>2^{k_0}\}}\|_p^p\right) \qquad (22)\\
&\lesssim \sum_{k_0\in\mathbb{Z}} 2^{k_0 q}\left(\sum_{k=k_0}^{\infty} 2^{-k_0\eta_2 q}\mathbb{P}(\tau^k<\infty)^{\frac{q}{p}} 2^{k\eta_2 q}\right)\gamma_b^q\left[\left(\sum_{k=k_0}^{\infty} 2^{-k_0\eta_2 q}\mathbb{P}(\tau^k<\infty)^{\frac{q}{p}} 2^{k\eta_2 q}\right)^{\frac{p}{q}}\right]\\
&= \sum_{k_0\in\mathbb{Z}} 2^{k_0 q}\left(\sum_{k=k_0}^{\infty} 2^{(k-k_0)\eta_2 q}\mathbb{P}(\tau^k<\infty)^{\frac{q}{p}}\right)\gamma_{b_1^q}\left(\sum_{k=k_0}^{\infty} 2^{(k-k_0)\eta_2 q}\mathbb{P}(\tau^k<\infty)^{\frac{q}{p}}\right)\\
&= \sum_{k_0\in\mathbb{Z}} 2^{k_0 q}\left[\left(\sum_{k=k_0}^{\infty} 2^{(k-k_0)\eta_2 q}\mathbb{P}(\tau^k<\infty)^{\frac{q}{p}}\right)^{\varepsilon_2}\gamma_{b_1^{\varepsilon_2 q}}\left(\sum_{k=k_0}^{\infty} 2^{(k-k_0)\eta_2 q}\mathbb{P}(\tau^k<\infty)^{\frac{q}{p}}\right)\right]^{\frac{1}{\varepsilon_2}}\\
&\lesssim \sum_{k_0\in\mathbb{Z}} 2^{k_0 q}\left[\sum_{k=k_0}^{\infty} 2^{(k-k_0)\eta_2\varepsilon_2 q}\mathbb{P}(\tau^k<\infty)^{\frac{\varepsilon_2 q}{p}}\gamma_{b_1^{\varepsilon_2 q}}\left(2^{(k-k_0)\eta_2 q}\mathbb{P}(\tau^k<\infty)^{\frac{q}{p}}\right)\right]^{\frac{1}{\varepsilon_2}}\\
&= \sum_{k_0\in\mathbb{Z}} 2^{k_0 q}\left[\sum_{k=k_0}^{\infty} 2^{(k-k_0)\eta_2\varepsilon_2 q}\mathbb{P}(\tau^k<\infty)^{\frac{\varepsilon_2 q}{p}}\gamma_b^{\varepsilon_2 q}\left(2^{(k-k_0)\eta_2 p}\mathbb{P}(\tau^k<\infty)\right)\right]^{\frac{1}{\varepsilon_2}}.
\end{aligned}
$$

Let $0 < \delta_2 < \frac{1-\eta_2}{\eta_2}$. By (22) and Hölder's inequality with $1-\varepsilon_2+\varepsilon_2 = 1$, we have

$$
\begin{aligned}
&\sum_{k_0\in\mathbb{Z}} 2^{k_0 q}\|\chi_{\{T_2>2^{k_0}\}}\|_p^q \gamma_b^q\left(\|\chi_{\{T_2>2^{k_0}\}}\|_p^p\right) \qquad (23)\\
&\lesssim \sum_{k_0\in\mathbb{Z}} 2^{k_0 q}\left[\sum_{k=k_0}^{\infty} 2^{-(k-k_0)\eta_2\varepsilon_2 q\delta_2}2^{(k-k_0)\eta_2\varepsilon_2 q(1+\delta_2)}\mathbb{P}(\tau^k<\infty)^{\frac{\varepsilon_2 q}{p}}\gamma_b^{\varepsilon_2 q}\left(2^{(k-k_0)\eta_2 p}\mathbb{P}(\tau^k<\infty)\right)\right]^{\frac{1}{\varepsilon_2}}\\
&\leq \sum_{k_0\in\mathbb{Z}} 2^{k_0 q}\left(\sum_{k=k_0}^{\infty} 2^{-(k-k_0)\eta_2\varepsilon_2 q\delta_2/(1-\varepsilon_2)}\right)^{\frac{1-\varepsilon_2}{\varepsilon_2}}\sum_{k=k_0}^{\infty} 2^{(k-k_0)\eta_2 q(1+\delta_2)}\mathbb{P}(\tau^k<\infty)^{\frac{q}{p}}\gamma_b^q\left(2^{(k-k_0)\eta_2 p}\mathbb{P}(\tau^k<\infty)\right)\\
&\approx \sum_{k_0\in\mathbb{Z}} 2^{k_0 q}\sum_{k=k_0}^{\infty} 2^{(k-k_0)\eta_2 q(1+\delta_2)}\mathbb{P}(\tau^k<\infty)^{\frac{q}{p}}\gamma_b^q\left(2^{(k-k_0)\eta_2 p}\mathbb{P}(\tau^k<\infty)\right).
\end{aligned}
$$

Choose z_2 such that $0 < z_2 < \frac{1-\eta_2\delta_2-\eta_2}{\eta_2 p}$. According to Proposition 2.2 (i), we know that $t^{-h}\gamma_b(t)$ is equivalent to a nonincreasing function on $(0,\infty)$. For $k \geq k_0$, we have

$$
\begin{aligned}
\gamma_b\left(2^{(k-k_0)\eta_2 p}\mathbb{P}(\tau^k<\infty)\right) &= \left(2^{(k-k_0)\eta_2 p}\mathbb{P}(\tau^k<\infty)\right)^{z_2}\left(2^{(k-k_0)\eta_2 p}\mathbb{P}(\tau^k<\infty)\right)^{-z_2}\gamma_b\left(2^{(k-k_0)\eta_2 p}\mathbb{P}(\tau^k<\infty)\right) \qquad (24)\\
&\lesssim \left(2^{(k-k_0)\eta_2 p}\mathbb{P}(\tau^k<\infty)\right)^{z_2}\left(\mathbb{P}(\tau^k<\infty)\right)^{-z_2}\gamma_b\left(\mathbb{P}(\tau^k<\infty)\right)\\
&= 2^{(k-k_0)\eta_2 p z_2}\gamma_b\left(\mathbb{P}(\tau^k<\infty)\right).
\end{aligned}
$$

Thus, we get the following inequality by (23), (24) and Abel's transformation

$$
\begin{aligned}
&\sum_{k_0\in\mathbb{Z}} 2^{k_0 q}\|\chi_{\{T_2>2^{k_0}\}}\|_p^q \gamma_b^q\left(\|\chi_{\{T_2>2^{k_0}\}}\|_p^p\right) \qquad (25)\\
&\lesssim \sum_{k_0\in\mathbb{Z}} 2^{k_0 q}\sum_{k=k_0}^{\infty} 2^{(k-k_0)\eta_2 q(1+\delta_2)}\mathbb{P}(\tau^k<\infty)^{\frac{q}{p}}2^{(k-k_0)\eta_2 p z_2 q}\gamma_b^q\left(\mathbb{P}(\tau^k<\infty)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k\in\mathbb{Z}} 2^{k(1+\delta_2+z_2p)\eta_2 q}\mathbb{P}(\tau^k<\infty)^{\frac{q}{p}}\gamma_b^q\left(\mathbb{P}(\tau^k<\infty)\right)\sum_{k_0=-\infty}^{k} 2^{k_0 q[1-\eta_2(\delta_2+1)-\eta_2 z_2 p]}\\
&= \frac{2^{q[-(\delta_2+1)\eta_2-\eta_2 z_2 p]}}{2^{q[1-\eta_2(\delta_2+1)-\eta_2 z_2 p]}-1}\sum_{k\in\mathbb{Z}}\mathbb{P}(\tau^k<\infty)^{\frac{q}{p}}2^{(k+1)q}\gamma_b^q\left(\mathbb{P}(\tau^k<\infty)\right)\\
&= \frac{2^{q[-(\delta_2+1)\eta_2-\eta_2 z_2 p]}}{2^{q[1-\eta_2(\delta_2+1)-\eta_2 z_2 p]}-1}\sum_{k\in\mathbb{Z}}\mathbb{P}(\tau^k<\infty)^{\frac{q}{p}}\Phi\Big(\mu^k\Phi^{-1}\Big(\mathbb{P}(\tau^k<\infty)^{-\frac{1}{p}}\Big)\Big)^q\gamma_b^q\left(\mathbb{P}(\tau^k<\infty)\right).
\end{aligned}
$$

As a consequence of (14), (20) and (25), we get

$$
\begin{aligned}
\|\Phi(s^r(f))\|_{p,q,b} &\approx \left(\sum_{k_0\in\mathbb{Z}} 2^{k_0 q}\|\chi_{\{T_1>2^{k_0}\}}\|_p^q\gamma_b^q\left(\|\chi_{\{T_1>2^{k_0}\}}\|_p^p\right)\right)^{\frac{1}{q}} + \left(\sum_{k_0\in\mathbb{Z}} 2^{k_0 q}\|\chi_{\{T_2>2^{k_0}\}}\|_p^q\gamma_b^q\left(\|\chi_{\{T_2>2^{k_0}\}}\|_p^p\right)\right)^{\frac{1}{q}} \\
&\lesssim \left\|\left\{\mathbb{P}(\tau^k<\infty)^{\frac{1}{p}}\Phi\Big(\mu^k\Phi^{-1}\Big(\mathbb{P}(\tau^k<\infty)^{-\frac{1}{p}}\Big)\Big)\gamma_b\Big(\mathbb{P}(\tau^k<\infty)\Big)\right\}_{k\in\mathbb{Z}}\right\|_{l_q}.
\end{aligned}
\tag{26}
$$

Next we prove the case of $q=\infty$. (13) and Remark 2.6 (iii) ensure that

$$
\begin{aligned}
\|\Phi(s^r(f))\|_{p,\infty,b} &\le \|T_1+T_2\|_{p,\infty,b} \lesssim \|T_1\|_{p,\infty,b}+\|T_2\|_{p,\infty,b}\\
&\approx \left(\sup_{k_0\in\mathbb{Z}} 2^{k_0}\|\chi_{\{T_1>2^{k_0}\}}\|_p\gamma_b\left(\|\chi_{\{T_1>2^{k_0}\}}\|_p^p\right)\right) + \left(\sup_{k_0\in\mathbb{Z}} 2^{k_0}\|\chi_{\{T_2>2^{k_0}\}}\|_p\gamma_b\left(\|\chi_{\{T_2>2^{k_0}\}}\|_p^p\right)\right).
\end{aligned}
\tag{27}
$$

Estimation for $\sup_{k_0\in\mathbb{Z}} 2^{k_0}\|\chi_{\{T_1>2^{k_0}\}}\|_p\gamma_b\left(\|\chi_{\{T_1>2^{k_0}\}}\|_p^p\right)$**.**

From (16) and Hölder's inequality with $\theta+1-\theta=1$, we obtain

$$
\begin{aligned}
\|\chi_{\{T_1>2^{k_0}\}}\|_p &\lesssim \frac{1}{2^{k_0\lambda_1}}\left\{\sum_{k=-\infty}^{k_0-1}\mathbb{P}(\tau^k<\infty)^{\frac{\theta}{p}}2^{k\eta_1\theta}2^{k(\lambda_1-\eta_1)\theta}\right\}^{\frac{1}{\theta}}\\
&\le \frac{1}{2^{k_0\lambda_1}}\sum_{k=-\infty}^{k_0-1}\mathbb{P}(\tau^k<\infty)^{\frac{1}{p}}2^{k\eta_1}\left\{\sum_{k=-\infty}^{k_0-1}2^{k(\lambda_1-\eta_1)\theta/(1-\theta)}\right\}^{\frac{1-\theta}{\theta}}\\
&\lesssim 2^{-k_0\eta_1}\sum_{k=-\infty}^{k_0-1}\mathbb{P}(\tau^k<\infty)^{\frac{1}{p}}2^{k\eta_1} = \sum_{k=-\infty}^{k_0-1}2^{(k-k_0)\eta_1}\mathbb{P}(\tau^k<\infty)^{\frac{1}{p}}.
\end{aligned}
$$

Let $b_2(t)=b(t^p)$ for $t\in[1,\infty)$. It follows from Proposition 2.2 (ii), (iv), Lemma 2.4 and Hölder's inequality with $1-\varepsilon_1+\varepsilon_1=1$ that

$$
\begin{aligned}
&\sup_{k_0\in\mathbb{Z}} 2^{k_0}\|\chi_{\{T_1>2^{k_0}\}}\|_p\gamma_b\left(\|\chi_{\{T_1>2^{k_0}\}}\|_p^p\right)\\
&\lesssim \sup_{k_0\in\mathbb{Z}} 2^{k_0}\left(\sum_{k=-\infty}^{k_0-1}2^{(k-k_0)\eta_1}\mathbb{P}(\tau^k<\infty)^{\frac{1}{p}}\right)\gamma_b\left[\left(\sum_{k=-\infty}^{k_0-1}2^{(k-k_0)\eta_1}\mathbb{P}(\tau^k<\infty)^{\frac{1}{p}}\right)^p\right]\\
&= \sup_{k_0\in\mathbb{Z}} 2^{k_0}\left(\sum_{k=-\infty}^{k_0-1}2^{(k-k_0)\eta_1}\mathbb{P}(\tau^k<\infty)^{\frac{1}{p}}\right)\gamma_{b_2}\left(\sum_{k=-\infty}^{k_0-1}2^{(k-k_0)\eta_1}\mathbb{P}(\tau^k<\infty)^{\frac{1}{p}}\right)\\
&= \sup_{k_0\in\mathbb{Z}} 2^{k_0}\left[\left(\sum_{k=-\infty}^{k_0-1}2^{(k-k_0)\eta_1}\mathbb{P}(\tau^k<\infty)^{\frac{1}{p}}\right)^{\varepsilon_1}\gamma_{b_2^{\varepsilon_1}}\left(\sum_{k=-\infty}^{k_0-1}2^{(k-k_0)\eta_1}\mathbb{P}(\tau^k<\infty)^{\frac{1}{p}}\right)\right]^{\frac{1}{\varepsilon_1}}
\end{aligned}
\tag{28}
$$

$$\lesssim \sup_{k_0\in\mathbb{Z}} 2^{k_0}\left[\sum_{k=-\infty}^{k_0-1} 2^{(k-k_0)\eta_1\varepsilon_1}\mathbb{P}(\tau^k<\infty)^{\frac{\varepsilon_1}{p}}\gamma_{b_2^{\varepsilon_1}}\left(2^{(k-k_0)\eta_1}\mathbb{P}(\tau^k<\infty)^{\frac{1}{p}}\right)\right]^{\frac{1}{\varepsilon_1}}$$

$$= \sup_{k_0\in\mathbb{Z}} 2^{k_0}\left[\sum_{k=-\infty}^{k_0-1} 2^{(k-k_0)\eta_1\varepsilon_1}\mathbb{P}(\tau^k<\infty)^{\frac{\varepsilon_1}{p}}\gamma_b^{\varepsilon_1}\left(2^{(k-k_0)\eta_1 p}\mathbb{P}(\tau^k<\infty)\right)\right]^{\frac{1}{\varepsilon_1}}$$

$$= \sup_{k_0\in\mathbb{Z}} 2^{k_0}\left[\sum_{k=-\infty}^{k_0-1} 2^{(k-k_0)\eta_1\varepsilon_1\delta_1}2^{(k-k_0)\eta_1\varepsilon_1(1-\delta_1)}\mathbb{P}(\tau^k<\infty)^{\frac{\varepsilon_1}{p}}\gamma_b^{\varepsilon_1}\left(2^{(k-k_0)\eta_1 p}\mathbb{P}(\tau^k<\infty)\right)\right]^{\frac{1}{\varepsilon_1}}$$

$$\le \sup_{k_0\in\mathbb{Z}} 2^{k_0}\left(\sum_{k=-\infty}^{k_0-1} 2^{(k-k_0)\eta_1\varepsilon_1\delta_1/(1-\varepsilon_1)}\right)^{\frac{1-\varepsilon_1}{\varepsilon_1}}\sum_{k=-\infty}^{k_0-1} 2^{(k-k_0)\eta_1(1-\delta_1)}\mathbb{P}(\tau^k<\infty)^{\frac{1}{p}}\gamma_b\left(2^{(k-k_0)\eta_1 p}\mathbb{P}(\tau^k<\infty)\right)$$

$$\lesssim \sup_{k_0\in\mathbb{Z}} 2^{k_0}\sum_{k=-\infty}^{k_0-1} 2^{(k-k_0)\eta_1(1-\delta_1)}\mathbb{P}(\tau^k<\infty)^{\frac{1}{p}}\gamma_b\left(2^{(k-k_0)\eta_1 p}\mathbb{P}(\tau^k<\infty)\right).$$

Combining (28) with (19) and Abel's transformation, we deduce that

$$\begin{aligned}
&\sup_{k_0\in\mathbb{Z}} 2^{k_0}\|\chi_{\{T_1>2^{k_0}\}}\|_p\gamma_b\left(\|\chi_{\{T_1>2^{k_0}\}}\|_p^p\right) \qquad (29)\\
&\lesssim \sup_{k_0\in\mathbb{Z}} 2^{k_0}\sum_{k=-\infty}^{k_0-1} 2^{(k-k_0)\eta_1(1-\delta_1)}\mathbb{P}(\tau^k<\infty)^{\frac{1}{p}}2^{-(k-k_0)\eta_1 p z_1}\gamma_b\left(\mathbb{P}(\tau^k<\infty)\right)\\
&= \sup_{k\in\mathbb{Z}} 2^{k(1-\delta_1-z_1p)\eta_1}\mathbb{P}(\tau^k<\infty)^{\frac{1}{p}}\gamma_b\left(\mathbb{P}(\tau^k<\infty)\right)\sum_{k_0=k+1}^{\infty} 2^{k_0[1+\eta_1(\delta_1-1)+\eta_1 z_1 p]}\\
&= \frac{2^{\eta_1\delta_1-\eta_1+\eta_1 z_1 p}}{1-2^{[1+\eta_1(\delta_1-1)+\eta_1 z_1 p]}}\sup_{k\in\mathbb{Z}}\mathbb{P}(\tau^k<\infty)^{\frac{1}{p}}2^{(k+1)}\gamma_b\left(\mathbb{P}(\tau^k<\infty)\right)\\
&= \frac{2^{\eta_1\delta_1-\eta_1+\eta_1 z_1 p}}{1-2^{[1+\eta_1(\delta_1-1)+\eta_1 z_1 p]}}\sup_{k\in\mathbb{Z}}\mathbb{P}(\tau^k<\infty)^{\frac{1}{p}}\Phi\Big(\mu^k\Phi^{-1}\Big(\mathbb{P}(\tau^k<\infty)^{-\frac{1}{p}}\Big)\Big)\gamma_b\left(\mathbb{P}(\tau^k<\infty)\right).
\end{aligned}$$

Estimation for $\sup_{k_0\in\mathbb{Z}} 2^{k_0}\|\chi_{\{T_2>2^{k_0}\}}\|_p\gamma_b\left(\|\chi_{\{T_2>2^{k_0}\}}\|_p^p\right)$.

Furthermore, by (21) and Hölder's inequality with $\theta+1-\theta=1$, we get

$$\begin{aligned}
\|\chi_{\{T_2>2^{k_0}\}}\|_p &\lesssim \frac{1}{2^{k_0\lambda_2}}\left\{\sum_{k=k_0}^{\infty}\mathbb{P}(\tau^k<\infty)^{\frac{\theta}{p}}2^{k\eta_2\theta}2^{k(\lambda_2-\eta_2)\theta}\right\}^{\frac{1}{\theta}}\\
&\le \frac{1}{2^{k_0\lambda_2}}\sum_{k=k_0}^{\infty}\mathbb{P}(\tau^k<\infty)^{\frac{1}{p}}2^{k\eta_2}\left\{\sum_{k=k_0}^{\infty}2^{k(\lambda_2-\eta_2)\theta/(1-\theta)}\right\}^{\frac{1-\theta}{\theta}}\\
&\lesssim 2^{-k_0\eta_2}\sum_{k=k_0}^{\infty}\mathbb{P}(\tau^k<\infty)^{\frac{1}{p}}2^{k\eta_2} = \sum_{k=k_0}^{\infty}2^{(k-k_0)\eta_2}\mathbb{P}(\tau^k<\infty)^{\frac{1}{p}}.
\end{aligned}$$

Applying Proposition 2.2 (ii), (iv), Lemma 2.4 and Hölder's inequality with $1-\varepsilon_2+\varepsilon_2=1$, we obtain

$$\begin{aligned}
&\sup_{k_0\in\mathbb{Z}} 2^{k_0}\|\chi_{\{T_2>2^{k_0}\}}\|_p\gamma_b\left(\|\chi_{\{T_2>2^{k_0}\}}\|_p^p\right) \qquad (30)\\
&\lesssim \sup_{k_0\in\mathbb{Z}} 2^{k_0}\left(\sum_{k=k_0}^{\infty}2^{(k-k_0)\eta_2}\mathbb{P}(\tau^k<\infty)^{\frac{1}{p}}\right)\gamma_b\left[\left(\sum_{k=k_0}^{\infty}2^{(k-k_0)\eta_2}\mathbb{P}(\tau^k<\infty)^{\frac{1}{p}}\right)^p\right]
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{k_0\in\mathbb{Z}} 2^{k_0}\left(\sum_{k=k_0}^{\infty} 2^{(k-k_0)\eta_2}\mathbb{P}(\tau^k<\infty)^{\frac{1}{p}}\right)\gamma_{b_2}\left(\sum_{k=k_0}^{\infty} 2^{(k-k_0)\eta_2}\mathbb{P}(\tau^k<\infty)^{\frac{1}{p}}\right)\\
&= \sup_{k_0\in\mathbb{Z}} 2^{k_0}\left[\left(\sum_{k=k_0}^{\infty} 2^{(k-k_0)\eta_2}\mathbb{P}(\tau^k<\infty)^{\frac{1}{p}}\right)^{\varepsilon_2}\gamma_{b_2^{\varepsilon_2}}\left(\sum_{k=k_0}^{\infty} 2^{(k-k_0)\eta_2}\mathbb{P}(\tau^k<\infty)^{\frac{1}{p}}\right)\right]^{\frac{1}{\varepsilon_2}}\\
&\lesssim \sup_{k_0\in\mathbb{Z}} 2^{k_0}\left[\sum_{k=k_0}^{\infty} 2^{(k-k_0)\eta_2\varepsilon_2}\mathbb{P}(\tau^k<\infty)^{\frac{\varepsilon_2}{p}}\gamma_{b_2^{\varepsilon_2}}\left(2^{(k-k_0)\eta_2}\mathbb{P}(\tau^k<\infty)^{\frac{1}{p}}\right)\right]^{\frac{1}{\varepsilon_2}}\\
&= \sup_{k_0\in\mathbb{Z}} 2^{k_0}\left[\sum_{k=k_0}^{\infty} 2^{(k-k_0)\eta_2\varepsilon_2}\mathbb{P}(\tau^k<\infty)^{\frac{\varepsilon_2}{p}}\gamma_b^{\varepsilon_2}\left(2^{(k-k_0)\eta_2 p}\mathbb{P}(\tau^k<\infty)\right)\right]^{\frac{1}{\varepsilon_2}}\\
&= \sup_{k_0\in\mathbb{Z}} 2^{k_0}\left[\sum_{k=k_0}^{\infty} 2^{-(k-k_0)\eta_2\varepsilon_2\delta_2}2^{(k-k_0)\eta_2\varepsilon_2(1+\delta_2)}\mathbb{P}(\tau^k<\infty)^{\frac{\varepsilon_2}{p}}\gamma_b^{\varepsilon_2}\left(2^{(k-k_0)\eta_2 p}\mathbb{P}(\tau^k<\infty)\right)\right]^{\frac{1}{\varepsilon_2}}\\
&\le \sup_{k_0\in\mathbb{Z}} 2^{k_0}\left(\sum_{k=k_0}^{\infty} 2^{-(k-k_0)\eta_2\varepsilon_2\delta_2/(1-\varepsilon_2)}\right)^{\frac{1-\varepsilon_2}{\varepsilon_2}}\sum_{k=k_0}^{\infty} 2^{(k-k_0)\eta_2(1+\delta_2)}\mathbb{P}(\tau^k<\infty)^{\frac{1}{p}}\gamma_b\left(2^{(k-k_0)\eta_2 p}\mathbb{P}(\tau^k<\infty)\right)\\
&\approx \sup_{k_0\in\mathbb{Z}} 2^{k_0}\sum_{k=k_0}^{\infty} 2^{(k-k_0)\eta_2(1+\delta_2)}\mathbb{P}(\tau^k<\infty)^{\frac{1}{p}}\gamma_b\left(2^{(k-k_0)\eta_2 p}\mathbb{P}(\tau^k<\infty)\right).
\end{aligned}
$$

Moreover, using (30), (24) and Abel's transformation, we have

$$
\begin{aligned}
&\sup_{k_0\in\mathbb{Z}} 2^{k_0}\|\chi_{\{T_2>2^{k_0}\}}\|_p\gamma_b\left(\|\chi_{\{T_2>2^{k_0}\}}\|_p^p\right) \\
&\lesssim \sup_{k_0\in\mathbb{Z}} 2^{k_0}\sum_{k=k_0}^{\infty} 2^{(k-k_0)\eta_2(1+\delta_2)}\mathbb{P}(\tau^k<\infty)^{\frac{1}{p}}2^{(k-k_0)\eta_2 p z_2}\gamma_b\left(\mathbb{P}(\tau^k<\infty)\right)\\
&= \sup_{k\in\mathbb{Z}} 2^{k(1+\delta_2+z_2p)\eta_2}\mathbb{P}(\tau^k<\infty)^{\frac{1}{p}}\gamma_b\left(\mathbb{P}(\tau^k<\infty)\right)\sum_{k_0=-\infty}^{k} 2^{k_0[1-\eta_2(\delta_2+1)-\eta_2z_2p]}\\
&= \frac{2^{\eta_2(-\delta_2-1-z_2p)}}{2^{[1-\eta_2(\delta_2+1)-\eta_2z_2p]}-1}\sup_{k\in\mathbb{Z}}\mathbb{P}(\tau^k<\infty)^{\frac{1}{p}}2^{(k+1)}\gamma_b\left(\mathbb{P}(\tau^k<\infty)\right)\\
&= \frac{2^{\eta_2(\delta_2-1-z_2p)}}{2^{[1+\eta_2(\delta_2-1)-\eta_2z_2p]}-1}\sup_{k\in\mathbb{Z}}\mathbb{P}(\tau^k<\infty)^{\frac{1}{p}}\Phi\Big(\mu^k\Phi^{-1}\Big(\mathbb{P}(\tau^k<\infty)^{-\frac{1}{p}}\Big)\Big)\gamma_b\left(\mathbb{P}(\tau^k<\infty)\right).
\end{aligned}
\tag{31}
$$

According to (27), (29) and (31), we further conclude that

$$
\begin{aligned}
\|\Phi(s^r(f))\|_{p,\infty,b} &\approx \left(\sup_{k_0\in\mathbb{Z}} 2^{k_0}\|\chi_{\{T_1>2^{k_0}\}}\|_p\gamma_b\left(\|\chi_{\{T_1>2^{k_0}\}}\|_p^p\right)\right)+\left(\sup_{k_0\in\mathbb{Z}} 2^{k_0}\|\chi_{\{T_2>2^{k_0}\}}\|_p\gamma_b\left(\|\chi_{\{T_2>2^{k_0}\}}\|_p^p\right)\right)\\
&\lesssim \left\|\left\{\mathbb{P}(\tau^k<\infty)^{\frac{1}{p}}\Phi\Big(\mu^k\Phi^{-1}\Big(\mathbb{P}(\tau^k<\infty)^{-\frac{1}{p}}\Big)\Big)\gamma_b\Big(\mathbb{P}(\tau^k<\infty)\Big)\right\}_{k\in\mathbb{Z}}\right\|_{l_\infty}.
\end{aligned}
\tag{32}
$$

Consequently, for $0<q\le\infty$, combining (26) with (32), we obtain that

$$
\begin{aligned}
\|\Phi(s^r(f))\|_{p,q,b} &\lesssim \left\|\left\{\mathbb{P}(\tau^k<\infty)^{\frac{1}{p}}\Phi\Big(\mu^k\Phi^{-1}\Big(\mathbb{P}(\tau^k<\infty)^{-\frac{1}{p}}\Big)\Big)\gamma_b\Big(\mathbb{P}(\tau^k<\infty)\Big)\right\}_{k\in\mathbb{Z}}\right\|_{l_q}\\
&= \mathcal{W}\left(\{\mu^k\},\{a^k\},\{\tau^k\}\right).
\end{aligned}
\tag{33}
$$

Taking the infimum over all decompositions of the form (4), we get (6).

(ii) ⇒ (i). Suppose that $f = (f_n)_{n\geq 0}$ is a **B**-valued martingale and satisfies

$$\mathbb{E}\left(\sum_{m=0}^{\infty} \|df_m\|^r\right) < \infty.$$

Given $1 < r \leq 2$, we have

$$\|s^r(f)\|_1 \leq \|s^r(f)\|_r = \left(\mathbb{E}\left(\sum_{m=0}^{\infty} \|df_m\|^r\right)\right)^{\frac{1}{r}} < \infty.$$

Let $\Phi(t) = t$. It is easy to find that $\|\Phi(s^r(f))\|_1 = \|s^r(f)\|_1 < \infty$ and $\Phi^{-1}(t) = t$. Hence, the **B**-valued martingale $f = (f_n)_{n\geq 0}$ has a decomposition as (4), where

$$(\mu^k)_{k\in\mathbb{Z}} \in l_1$$

and

$$\sup_{k\in\mathbb{Z}} \|M(a^k)\|_1 < \infty.$$

Consequently, there exists $k_0 \in \mathbb{Z}$ such that for any $\varepsilon > 0$,

$$\sum\nolimits_{|k|>k_0} \mu^k < \varepsilon.$$

Notice that $a_n^k = \mathbb{E}_n(a^k)$ converges to the function a^k as $n \to \infty$ in $L_1(\mathbf{B})$ for each $k \in \mathbb{Z}$ (see [21, p.27]). Then, there exists $M_k \in \mathbb{N}$ such that

$$\mathbb{E}\left(\left\|a_m^k - a_n^k\right\|\right) < \varepsilon$$

as $m, n > M_k$. Set $\mathbf{N} = \max_{|k|\leq k_0}\{M_k\}$. Thus, for $m, n > \mathbf{N}$, we can deduce that

$$\begin{aligned}\|f_m - f_n\|_{L_1(\mathbf{B})} &= \mathbb{E}\left(\left\|\sum_{k\in\mathbb{Z}} \mu^k a_m^k - \sum_{k\in\mathbb{Z}} \mu^k a_n^k\right\|\right) \leq \sum_{k\in\mathbb{Z}} \mu^k \mathbb{E}\Big(\|a_m^k - a_n^k\|\Big)\\ &= \sum_{|k|>k_0} \mu^k \mathbb{E}\Big(\|a_m^k - a_n^k\|\Big) + \sum_{|k|\leq k_0} \mu^k \mathbb{E}\Big(\|a_m^k - a_n^k\|\Big)\\ &\leq 2\sup_{k\in\mathbb{Z}} \|M(a^k)\|_1 \sum_{|k|>k_0} \mu^k + \varepsilon \sum_{|k|\leq k_0} \mu^k \lesssim \varepsilon.\end{aligned}$$

This yields that $(f_n)_{n\geq 0}$ is a Cauchy sequence in $L_1(\mathbf{B})$. Then $(f_n)_{n\geq 0}$ converges in probability (see [21, p.14]). It follows from Lemma 2.15 that **B** is isomorphic to a r-uniformly smooth space. The proof is finished. □

We see that the proof of (33) in Theorem 3.2 mainly relies on the subadditivity of Φ and the sublinearity of s^r. That is, for $\|\Phi(S^r(f))\|_{p,q,b}$ $\Big($resp. $\|\Phi(M(f))\|_{p,q,b}$, $\|\Phi(s^r(f))\|_{p,q,b}\Big)$ we obtain:

Corollary 3.3. *Let* **B** *be a Banach space,* $\Phi \in \mathcal{G}$ *be a concave function and* b *be a slowly varying function. If* $1 \leq r < \infty$, $0 < p < \infty$, $0 < q \leq \infty$, $\max\{1, p\} < \ell \leq \infty$ *and the* **B**-*valued martingale* $f = (f_n)_{n\geq 0}$ *has a decomposition of type* (4) *with* $(\mu^k, a^k, \tau^k) \in \mathcal{A}^{S^r}(\Phi, p, q, \ell)$ $\Big($*resp.* $(\mu^k, a^k, \tau^k) \in \mathcal{A}^M(\Phi, p, q, \ell), (\mu^k, a^k, \tau^k) \in \mathcal{A}^{s^r}(\Phi, p, q, \ell)\Big)$*, then*

$$\left\|\Phi(S^r(f))\right\|_{p,q,b} \lesssim \inf \mathcal{W}(\{\mu^k\}, \{a^k\}, \{\tau^k\})$$

$$\Bigg(\textit{resp. } \left\|\Phi(M(f))\right\|_{p,q,b} \lesssim \inf \mathcal{W}(\{\mu^k\}, \{a^k\}, \{\tau^k\}),$$

$$\left\|\Phi(s^r(f))\right\|_{p,q,b} \lesssim \inf \mathcal{W}(\{\mu^k\}, \{a^k\}, \{\tau^k\})\Bigg),$$

where the infimum is taken over all the decompositions of f *of the form* (4).

We prove the atomic decompositions for $\Phi(f)$ in $Q^{S^r}_{p,q,b}(\mathbf{B})$ and $\mathcal{D}_{p,q,b}(\mathbf{B})$ as follows.

Theorem 3.4. *Let $\Phi \in \mathcal{G}$ be a concave function, $\mathbf{B}$ be a Banach space, $1 < r \le 2$, $0 < p \le r$, $0 < q \le \infty$ and b be a slowly varying function. Then the following statements are equivalent:*
(i) *$\mathbf{B}$ is isomorphic to a r-uniformly smooth space.*
(ii) *Assume that the $\mathbf{B}$-valued martingale $f = (f_n)_{n\ge0}$ satisfies $\|\Phi(f)\|_{Q^{S^r}_{p,q,b}(\mathbf{B})} < \infty$. Then there exists a sequence of triples $(\mu^k, a^k, \tau^k) \in \mathcal{A}^{S^r}(\Phi, p, q, \infty)$ such that for all $n \in \mathbb{N}$, (4), (5) hold and*

$$\|\Phi(f)\|_{Q^{S^r}_{p,q,b}(\mathbf{B})} \approx \inf \mathcal{W}(\{\mu^k\}, \{a^k\}, \{\tau^k\}), \tag{34}$$

where the infimum is taken over all the decompositions of f of the form (4).

Proof. (i)⇒(ii). As the proof is similar to the proof of Theorem 3.2, we skip the details. Suppose that $f = (f_n)_{n\ge0}$ is a $\mathbf{B}$-valued martingale with $\|\Phi(f)\|_{Q^{S^r}_{p,q,b}(\mathbf{B})} < \infty$. For every $k \in \mathbb{Z}$, the stopping time is defined by

$$\tau^k = \inf\{n \in \mathbb{N} : \lambda_n > \Phi^{-1}(2^k)\} \quad (\inf \emptyset = \infty),$$

where $(\lambda_n)_{n\ge0} \in \Lambda[Q^{S^r}_{p,q,b,\Phi}(f)](\mathbf{B})$. Let μ^k and a^k_n be the same as in the proof of Theorem 3.2. Hence, $(a^k_n)_{n\ge0}$ is a $\mathbf{B}$-valued martingale. Moreover, $S^r_{\tau^k}(f) \le \lambda_{\tau^k-1} \le \Phi^{-1}(2^k)$. Similarly to the proof of (7), we have

$$S^r\left((a^k_n)_{n\ge0}\right) \le \Phi^{-1}\Big(\mathbb{P}(\tau^k < \infty)^{-\frac{1}{p}}\Big)\chi_{\{\tau^k<\infty\}}.$$

Lemma 2.15 (iii) guarantees that

$$\left\|M\Big((a^k_n)_{n\ge0}\Big)\right\|_r \le C\left\|S^r\Big((a^k_n)_{n\ge0}\Big)\right\|_r \le C\mathbb{P}(\tau^k < \infty)^{\frac{1}{r}}\Phi^{-1}\Big(\mathbb{P}(\tau^k < \infty)^{-\frac{1}{p}}\Big) < \infty.$$

Analogous to the method of the proof of Theorem 3.2, there exists a function a^k in $L_r(\mathbf{B})$ such that $a^k_n = \mathbb{E}_n(a^k)(n \in \mathbb{N})$. Moreover, a^k is a $(\Phi, p, \infty)^{S^r}$-atom, (4) and (5) hold.

Now we show (34). According to (11) and

$$\{\tau^k < \infty\} = \Big\{\Phi(\lambda_\infty) > 2^k\Big\},$$

for the case of $0 < q < \infty$, we get

$$\begin{aligned}
\mathcal{W}(\{\mu^k\}, \{a^k\}, \{\tau^k\}) &= \left\|\left\{\mathbb{P}(\tau^k < \infty)^{\frac{1}{p}}\Phi\Big(\mu^k\Phi^{-1}\Big(\mathbb{P}(\tau^k < \infty)^{-\frac{1}{p}}\Big)\Big)\gamma_b\Big(\mathbb{P}(\tau^k < \infty)\Big)\right\}_{k\in\mathbb{Z}}\right\|_{l_q} \\
&= \left\|\left\{\mathbb{P}\left(\Phi(\lambda_\infty) > 2^k\right)^{\frac{1}{p}} 2^{k+1}\gamma_b\Big(\mathbb{P}\left(\Phi(\lambda_\infty) > 2^k\right)\Big)\right\}_{k\in\mathbb{Z}}\right\|_{l_q} \\
&= \left(\sum_{k\in\mathbb{Z}} \mathbb{P}(\Phi(\lambda_\infty) > 2^k\Big)^{\frac{q}{p}} 2^{(k+1)q}\gamma^q_b\Big(\mathbb{P}\Big(\Phi(\lambda_\infty) > 2^k\Big)\Big)\right)^{\frac{1}{q}} \\
&\lesssim \left(\sum_{k\in\mathbb{Z}} \mathbb{P}\Big(\Phi(\lambda_\infty) > 2^k\Big)^{\frac{q}{p}}\gamma^q_b\Big(\mathbb{P}\Big(\Phi(\lambda_\infty) > 2^k\Big)\Big)\int_{2^{k-1}}^{2^k} y^q\frac{dy}{y}\right)^{\frac{1}{q}} \\
&\le \left(\sum_{k\in\mathbb{Z}} \int_{2^{k-1}}^{2^k} \mathbb{P}\Big(\Phi(\lambda_\infty) > y\Big)^{\frac{q}{p}}\gamma^q_b\Big(\mathbb{P}\Big(\Phi(\lambda_\infty) > y\Big)\Big)y^{q-1}dy\right)^{\frac{1}{q}} \\
&= \left\|\Phi\Big(\lambda_\infty\Big)\right\|_{p,q,b}.
\end{aligned}$$

Standard modifications can be made for the case of $q = \infty$. Taking the infimum over all $(\lambda_n)_{n\geq 0} \in \Lambda[Q^{S^r}_{p,q,,b,\Phi}(f)](\mathbf{B})$, we get

$$\mathcal{W}(\{\mu^k\}, \{a^k\}, \{\tau^k\}) \lesssim \|\Phi(f)\|_{Q^{S^r}_{p,q,b}(\mathbf{B})}.$$

On the other hand, let

$$\rho_n = \sum_{k\in\mathbb{Z}} \mu^k \|S^r(a^k)\|_\infty \chi_{\{\tau^k \leq n\}}.$$

Obviously, $(\rho_n)_{n\geq 0} \in \Lambda$ and $S^r_{n+1}(f) \leq \rho_n$ for every $n \geq 0$. In view of the definition of $(\Phi, p, \infty)^{S^r}$-atom, we find that $\{S^r(a^k) > 0\} \subset \{\tau^k < \infty\}$. For an arbitrary integer k_0, set

$$\rho^{(1)}_\infty = \sum_{k=-\infty}^{k_0-1} \Phi\Big(\mu^k \|S^r(a^k)\|_\infty \chi_{\{\tau^k<\infty\}}\Big),\ \rho^{(2)}_\infty = \sum_{k=k_0}^{\infty} \Phi\Big(\mu^k \|S^r(a^k)\|_\infty \chi_{\{\tau^k<\infty\}}\Big).$$

The subadditivity of Φ assures that

$$\Phi(\rho_\infty) \leq \rho^{(1)}_\infty + \rho^{(2)}_\infty.$$

Replacing T_1 and T_2 by $\rho^{(1)}_\infty$ and $\rho^{(2)}_\infty$ in Theorem 3.2, respectively. Then we deduce that

$$\|\Phi(f)\|_{Q^{S^r}_{p,q,b}(\mathbf{B})} \approx \inf \mathcal{W}(\{\mu^k\}, \{a^k\}, \{\tau^k\}),$$

where the infimum is taken over all decompositions of f of the form (4).

(ii) $\Rightarrow$ (i). If $f = (f_n)_{n\geq 0}$ is a $\mathbf{B}$-valued martingale with $S^r(f) \in L_\infty$. For every $n \geq 0$, set $\Phi(t) = t$ and $\lambda_n = \|S^r_{n+1}(f)\|_\infty$. It is easy to see that $(\lambda_n)_{n\geq 0} \in \Lambda$ and $S^r_{n+1}(f) \leq \lambda_n$. Consequently,

$$\|\Phi(f)\|_{Q^{S^r}_1(\mathbf{B})} \leq \|\Phi(\lambda_\infty)\|_1 = \|S^r(f)\|_\infty < \infty.$$

Therefore, $(f_n)_{n\geq 0}$ has a decomposition as (4). The remaining proof is similar to the proof of Theorem 3.2. This finishes the proof. □

Theorem 3.5. *Let* $\mathbf{B}$ *be a Banach space,* $\Phi \in \mathcal{G}$ *be a concave function,* $0 < p < \infty$, $0 < q \leq \infty$ *and* b *be a slowly varying function. Then the following assertions are equivalent:*
(i) $\mathbf{B}$ *has the* **RNP**.
(ii) *For every* $\mathbf{B}$*-valued martingale* $f = (f_n)_{n\geq 0}$ *which satisfies* $\|\Phi(f)\|_{\mathcal{D}_{p,q,b}(\mathbf{B})} < \infty$, *there exists a sequence of triples* $(\mu^k, a^k, \tau^k) \in \mathcal{A}^M(\Phi, p, q, \infty)$ *such that for* $n \geq 0$, (4), (5) *hold and*

$$\|\Phi(f)\|_{\mathcal{D}_{p,q,b}(\mathbf{B})} \approx \inf \mathcal{W}(\{\mu^k\}, \{a^k\}, \{\tau^k\}), \tag{35}$$

where the infimum is taken over all the decompositions of the form (4).

Proof. (i)$\Rightarrow$(ii). We omit some details of the proof, since it is similar to the proof of Theorem 3.4. If $f = (f_n)_{n\geq 0}$ is a $\mathbf{B}$-valued martingale with $\|\Phi(f)\|_{\mathcal{D}_{p,q,b}(\mathbf{B})} < \infty$, then the stopping times τ^k are defined by

$$\tau^k = \inf\{n \in \mathbb{N} : \lambda_n > \Phi^{-1}(2^k)\} \quad (\inf \emptyset = \infty),$$

where $(\lambda_n)_{n\geq 0} \in \Lambda[\mathcal{D}_{p,q,,b,\Phi}(f)](\mathbf{B})$. Let μ^k and a^k_n be the same as in the proof of Theorem 3.2. Thus it is obvious that

$$\begin{aligned} \|a^k_n\| = \frac{\|f^{\tau^{k+1}}_n - f^{\tau^k}_n\|}{\mu^k} &\leq \frac{\|f^{\tau^{k+1}}_n\| + \|f^{\tau^k}_n\|}{\mu^k} \chi_{\{\tau^k<\infty\}} \leq \frac{\lambda_{\tau^{k+1}-1} + \lambda_{\tau^k-1}}{\mu^k} \chi_{\{\tau^k<\infty\}} \\ &\leq \frac{\Phi^{-1}(2^{k+1}) + \Phi^{-1}(2^k)}{\mu^k} \chi_{\{\tau^k<\infty\}} \leq 2\Phi^{-1}\Big(\mathbb{P}(\tau^k<\infty)^{-\frac{1}{p}}\Big) \chi_{\{\tau^k<\infty\}}. \end{aligned} \tag{36}$$

It follows from (36) that

$$\left\| M\Big((a_n^k)_{n\geq 0}\Big)\right\|_\infty \lesssim \Phi^{-1}\Big(\mathbb{P}(\tau^k < \infty)^{-\frac{1}{p}}\Big).$$

From condition (i), there exists a **B**-valued integrable function a^k such that $a_n^k = \mathbb{E}_n(a^k)$ ($n \in \mathbb{N}$). It is clear that a^k is a $(\Phi, p, \infty)^M$-atom and (4) holds. Moreover, we have (5) and

$$\mathcal{W}(\{\mu^k\}, \{a^k\}, \{\tau^k\}) \lesssim \|\Phi(f)\|_{\mathcal{D}_{p,q,b}(\mathbf{B})}.$$

On the other hand, define

$$\rho_n = \sum_{k\in\mathbb{Z}} \mu^k \|M(a^k)\|_\infty \chi_{\{\tau^k \leq n\}}.$$

Thus, $(\rho_n)_{n\geq 0} \in \Lambda$ and $\|f_{n+1}\| \leq \rho_n$ for every $n \geq 0$. For an arbitrary integer k_0, let

$$\rho_\infty^{(1)} = \sum_{k=-\infty}^{k_0-1} \Phi\Big(\mu^k \|M(a^k)\|_\infty \chi_{\{\tau^k<\infty\}}\Big),\ \rho_\infty^{(2)} = \sum_{k=k_0}^{\infty} \Phi\Big(\mu^k \|M(a^k)\|_\infty \chi_{\{\tau^k<\infty\}}\Big).$$

It follows from the subadditivity of Φ that

$$\Phi(\rho_\infty) \leq \rho_\infty^{(1)} + \rho_\infty^{(2)}.$$

Replacing T_1 and T_2 by $\rho_\infty^{(1)}$ and $\rho_\infty^{(2)}$ in Theorem 3.2, respectively. Then we have

$$\|\Phi(f)\|_{\mathcal{D}_{p,q,b}(\mathbf{B})} \approx \inf \mathcal{W}(\{\mu^k\}, \{a^k\}, \{\tau^k\}),$$

where the infimum is taken over all decompositions of f of the form (4).

(ii) ⇒ (i). Assume that $f = (f_n)_{n\geq 0}$ is a **B**-valued martingale with

$$\sup_{n\geq 0} \|f_n\|_{L_\infty(\mathbf{B})} < \infty.$$

For every $n \geq 0$, set $\Phi(t) = t$ and $\lambda_n = \|M_{n+1}(f)\|_\infty$. It is obvious that $(\lambda_n)_{n\geq 0} \in \Lambda$ and $\|f_{n+1}\| \leq \lambda_n$. Consequently,

$$\|\Phi(f)\|_{\mathcal{D}_1(\mathbf{B})} \leq \|\Phi(\lambda_\infty)\|_1 \leq \sup_{n\geq 0} \|f_n\|_\infty < \infty.$$

Therefore, $(f_n)_{n\geq 0}$ has a decomposition as (4). Similar to the proof of Theorem 3.2, we find that $(f_n)_{n\geq 0}$ converges in $L_1(\mathbf{B})$. Then $(f_n)_{n\geq 0}$ converges a.e.. Lemma 2.17 guarantees that **B** has the **RNP**. Therefore, we complete the proof. □

Remark 3.6. *If b is a nondecreasing slowly varying function, $\Phi(t) = t$ and $\ell = \infty$ in Theorem 3.2, then we refer to Liu et al. [17] for the corresponding result; if b is a nondecreasing slowly varying function and $\Phi(t) = t$, then Theorems 3.4 and 3.5 recover the corresponding results in [17]. Moreover, the slowly varying function b is not necessarily nondecreasing in Theorems 3.2, 3.4 and 3.5 of this article. Hence, Theorems 3.2, 3.4 and 3.5 improve Theorems 3.2, 3.4 and 3.5 in [17], respectively.*

Remark 3.7. *Let $r = 2$, $\Phi(t) = t$ and $\mathbf{B} = \mathbb{R}$ in Theorems 3.2, 3.4 and 3.5, we obtain the atomic decomposition of Hardy-Lorentz-Karamata martingale spaces $H^s_{p,q,b}$, $Q_{p,q,b}$ and $\mathcal{D}_{p,q,b}$, respectively.*

Remark 3.8. *If $b \equiv 1$, $\Phi(t) = t$ and $\ell = \infty$, then Theorem 3.2 goes back to the corresponding result in [18]; if $b \equiv 1$ and $\Phi(t) = t$, then Theorems 3.4 and 3.5 reduce to the corresponding results in [18].*

Remark 3.9. *If we consider the special case $b \equiv 1$ in Theorems 3.2, 3.4 and 3.5, then we can return to Theorems 1, 2 and 3 in [15], respectively.*

4. Φ-moment B-valued martingale inequalities

In this section, with the help of the smoothness or convexity of Banach spaces and atomic decomposition theorems, we prove some basic Φ-moment **B**-valued martingale inequalities on Lorentz-Karamata spaces.

Theorem 4.1. *Let $\Phi \in \mathcal{G}$ be a concave function,* **B** *be a Banach space, $1 < r \le 2$, $0 < p < r$, $0 < q \le \infty$ and b be a slowly varying function. Then the following statements are equivalent:*
(i) **B** *is isomorphic to a r-uniformly smooth space.*
(ii) *Assume that the* **B***-valued martingale $f = (f_n)_{n\ge0}$ satisfies $\|\Phi(s^r(f))\|_{p,q,b} < \infty$, then*

$$\|\Phi(M(f))\|_{p,q,b} \lesssim \|\Phi(s^r(f))\|_{p,q,b}. \tag{37}$$

(iii) *Assume that the* **B***-valued martingale $f = (f_n)_{n\ge0}$ satisfies $\|\Phi(f)\|_{Q^{s^r}_{p,q,b}(\mathbf{B})} < \infty$, then*

$$\|\Phi(M(f))\|_{p,q,b} \lesssim \|\Phi(f)\|_{Q^{s^r}_{p,q,b}(\mathbf{B})}. \tag{38}$$

Proof. (i) ⇒ (ii). Suppose that $f = (f_n)_{n\ge0}$ is a **B**-valued martingale with

$$\|\Phi(s^r(f))\|_{p,q,b} < \infty.$$

It follows from Theorem 3.2 that

$$\frac{1}{C} f_n = \sum_{k\in\mathbb{Z}} \mu^k \mathbb{E}_n\Big(\frac{1}{C} a^k\Big) \quad a.e.$$

and

$$\Big\|\Phi(s^r(f))\Big\|_{p,q,b} \approx \mathcal{W}(\{\mu^k\},\{a^k\},\{\tau^k\}),$$

where $(\mu^k, a^k, \tau^k) \in \mathcal{A}^{s^r}(\Phi, p, q, r)$. By Lemma 2.15 (iii), there exists a constant $C > 1$ such that for any **B**-valued martingale g,

$$\|M(g)\|_r \le C\|S^r(g)\|_r = C\|s^r(g)\|_r.$$

Since $a^k = (a^k_n)_{n\ge0}$ is a **B**-valued martingale. It is easy to see that

$$\Big\|M\Big(\frac{1}{C}a^k\Big)\Big\|_r \le \|s^r(a^k)\|_r \le \mathbb{P}(\tau^k < \infty)^{\frac{1}{r}} \Phi^{-1}\Big(\mathbb{P}(\tau^k < \infty)^{-\frac{1}{p}}\Big).$$

This shows that $\Big(\mu^k, \frac{1}{C}a^k, \tau^k\Big) \in \mathcal{A}^M(\Phi, p, q, r)$. According to Corollary 3.3, we have

$$\Big\|\Phi\Big(M\Big(\frac{1}{C}f\Big)\Big)\Big\|_{p,q,b} \lesssim \mathcal{W}\Big(\{\mu^k\},\{\frac{1}{C}a^k\},\{\tau^k\}\Big).$$

Applying Lemma 2.8 (i), we deduce

$$\frac{1}{C}\Big\|\Phi(M(f))\Big\|_{p,q,b} \le \Big\|\Phi\Big(M\Big(\frac{1}{C}f\Big)\Big)\Big\|_{p,q,b} \lesssim \mathcal{W}\Big(\{\mu^k\},\{\frac{1}{C}a^k\},\{\tau^k\}\Big) \approx \|\Phi(s^r(f))\|_{p,q,b}.$$

Consequently, we obtain $\Big\|\Phi(M(f))\Big\|_{p,q,b} \lesssim \|\Phi(s^r(f))\|_{p,q,b}$.

(ii) ⇒ (i). Let $f = (f_n)_{n\ge0}$ be a **B**-valued martingale with

$$\mathbb{E}\Big(\sum_{m=0}^{\infty} \|df_m\|^r\Big) = \|s^r(f)\|_r^r < \infty.$$

Let $\Phi(t) = t$. By Lemma 2.7, we get $\|s^r(f)\|_{p,q,b} \le \|s^r(f)\|_{r,r} < \infty$ holds for $0 < p < r$. The **B**-valued martingale $g^n = (g^n_m)_{m\ge0}$ is defined by $g^n_m = f_{m+n} - f_n$ for each $n \in \mathbb{N}$. Actually, $[S^r(g^n)]^r = [S^r(f)]^r - [S^r_{n-1}(f)]^r \to 0$ as $n \to \infty$

and $S^r(g^n) \leq S^r(f)$. By the dominated convergence theorem, we have $\|s^r(g^n)\|_{p,q,b} \leq \|s^r(g^n)\|_r = \|S^r(g^n)\|_r \to 0$ as $n \to \infty$. Employing (37) for g^n, we obtain

$$\|f_{m+n} - f_n\|_{L_{p,q,b}(\mathbf{B})} \leq \|\Phi(M(g^n))\|_{p,q,b} \lesssim \|\Phi(s^r(g^n))\|_{p,q,b} \to 0, \ (n \to \infty).$$

Thus, $(f_n)_{n\geq 0}$ is a Cauchy sequence in $L_{p,q,b}(\mathbf{B})$. Then $(f_n)_{n\geq 0}$ converges in probability (see [21, p.14]). By Lemma 2.15, $\mathbf{B}$ is isomorphic to a r-uniformly smooth space.

(i) $\Rightarrow$ (iii). Let $f = (f_n)_{n\geq 0}$ be a $\mathbf{B}$-valued martingale with $\|\Phi(f)\|_{Q^{S^r}_{p,q,b}(\mathbf{B})} < \infty$. Applying Theorem 3.4, there exists a sequence of triples $(\mu^k, a^k, \tau^k) \in \mathcal{A}^{S^r}(\Phi, p, q, \infty)$ such that

$$\frac{1}{C} f_n = \sum\nolimits_{k\in\mathbb{Z}} \mu^k \mathbb{E}_n\Big(\frac{1}{C} a^k\Big) \quad a.e.$$

and

$$\big\|\Phi(f)\big\|_{Q^{S^r}_{p,q,b}(\mathbf{B})} \approx \mathcal{W}(\{\mu^k\}, \{a^k\}, \{\tau^k\}).$$

Since a^k is a $(\Phi, p, \infty)^{S^r}$-atom, we conclude that $\{S^r(a^k) > 0\} \subset \{\tau^k < \infty\}$ and $\|S^r(a^k)\|_r \leq \|S^r(a^k)\|_\infty \mathbb{P}(\tau^k < \infty)^{\frac{1}{r}}$. Lemma 2.15 (iii) assures that

$$\|M(a^k)\|_r \leq C\|S^r(a^k)\|_r \ (C > 1).$$

Thus, we find that

$$\Big\|M\Big(\frac{1}{C} a^k\Big)\Big\|_r \leq \mathbb{P}(\tau^k < \infty)^{\frac{1}{r}} \Phi^{-1}\Big(\mathbb{P}(\tau^k < \infty)^{-\frac{1}{p}}\Big).$$

So $\Big(\mu^k, \frac{1}{C} a^k, \tau^k\Big) \in \mathcal{A}^M(\Phi, p, q, r)$. Lemma 2.8 and Corollary 3.3 guarantee that

$$\frac{1}{C}\big\|\Phi(M(f))\big\|_{p,q,b} \leq \Big\|\Phi\Big(M\Big(\frac{1}{C} f\Big)\Big)\Big\|_{p,q,b} \lesssim \mathcal{W}\Big(\{\mu^k\}, \{\frac{1}{C} a^k\}, \{\tau^k\}\Big) \approx \big\|\Phi(f)\big\|_{Q^{S^r}_{p,q,b}(\mathbf{B})},$$

then we get the result.

(iii) $\Rightarrow$ (i). Assume that $f = (f_n)_{n\geq 0}$ is a $\mathbf{B}$-valued dyadic martingale with

$$\mathbb{E}\Big(\sum\nolimits_{m=0}^{\infty} \|df_m\|^r\Big) = \|s^r(f)\|_r^r < \infty.$$

Let $\Phi(t) = t$ and $\lambda_n = C s^r_{n+1}(f)$ for each $n \in \mathbb{N}$. Then $\|s^r(f)\|_{p,q,b} < \infty$ and $(\lambda_n)_{n\geq 0} \in \Lambda$. Notice that f is a $\mathbf{B}$-valued dyadic martingale, thus we have $S^r_n(f) \leq C s^r_n(f)$ for each $n \in \mathbb{N}$. Obviously, $S^r_{n+1}(f) \leq \lambda_n$. It is easy to check that $(\lambda_n)_{n\geq 0} \in \Lambda[\mathcal{Q}^{S^r}_{p,q,b,\Phi}(f)](\mathbf{B})$ and

$$\|\Phi(f)\|_{Q^{S^r}_{p,q,b}(\mathbf{B})} \leq \|\Phi(\lambda_\infty)\|_{p,q,b} = C\|s^r(f)\|_{p,q,b} < \infty. \tag{39}$$

Let us consider $g^n = (g^n_m)_{m\geq 0}$ as above. By (38) and (39), we obtain

$$\|f_{m+n} - f_n\|_{L_{p,q,b}(\mathbf{B})} \leq \|\Phi(M(g^n))\|_{p,q,b} \lesssim \|\Phi(g^n)\|_{Q^{S^r}_{p,q,b}(\mathbf{B})} \leq \|s^r(g^n)\|_{p,q,b} \to 0$$

when $n \to \infty$. The rest of the proof is similar to (ii) $\Rightarrow$ (i), we get $\mathbf{B}$ is isomorphic to a r-uniformly smooth space. This completes the proof. □

Theorem 4.2. *Let $\Phi \in \mathcal{G}$ be a concave function, $\mathbf{B}$ be a Banach space, $2 \leq r < \infty$, $0 < p < r$, $0 < q \leq \infty$ and b be a slowly varying function. Then the following statements are equivalent:*
(i) $\mathbf{B}$ *is isomorphic to a r-uniformly convex space.*
(ii) *Assume that the $\mathbf{B}$-valued martingale $f = (f_n)_{n\geq 0}$ satisfies $\|\Phi(f)\|_{\mathcal{D}_{p,q,b}(\mathbf{B})} < \infty$, then*

$$\|\Phi(S^r(f))\|_{p,q,b} \lesssim \|\Phi(f)\|_{\mathcal{D}_{p,q,b}(\mathbf{B})}. \tag{40}$$

(iii) *Assume that the $\mathbf{B}$-valued martingale $f = (f_n)_{n\geq 0}$ satisfies $\|\Phi(f)\|_{\mathcal{D}_{p,q,b}(\mathbf{B})} < \infty$, then*

$$\|\Phi(s^r(f))\|_{p,q,b} \lesssim \|\Phi(f)\|_{\mathcal{D}_{p,q,b}(\mathbf{B})}. \tag{41}$$

Proof. (i) ⇒ (ii). Consider that $f = (f_n)_{n\geq 0}$ is a **B**-valued martingale satisfies $\|\Phi(f)\|_{\mathcal{D}_{p,q,b}(\mathbf{B})} < \infty$. Condition (i) implies that **B** has the **RNP** (see Remark 2.14). It follows from Theorem 3.5 that

$$\frac{1}{C}f_n = \sum\nolimits_{k\in\mathbb{Z}} \mu^k \mathbb{E}_n\Big(\frac{1}{C}a^k\Big)\, a.e.$$

and

$$\|\Phi(f)\|_{\mathcal{D}_{p,q,b}(\mathbf{B})} \approx \mathcal{W}(\{\mu^k\}, \{a^k\}, \{\tau^k\}),$$

where $(\mu^k, a^k, \tau^k) \in \mathcal{A}^M(\Phi, p, q, \infty)$. By Lemma 2.16 (ii), we find that for any **B**-valued martingale g,

$$\|s^r(g)\|_r = \|S^r(g)\|_r \leq C\|M(g)\|_r,$$

where $C > 1$. Since $a^k = (a^k_n)_{n\geq 0}$ is a **B**-valued martingale. Similarly to the proof of (i) ⇒ (iii) in Theorem 4.1, we obtain

$$\Big\|S^r\Big(\frac{1}{C}a^k\Big)\Big\|_r \leq \mathbb{P}(\tau^k < \infty)^{\frac{1}{r}}\Phi^{-1}\Big(\mathbb{P}(\tau^k < \infty)^{-\frac{1}{p}}\Big).$$

So $\left(\mu^k, \frac{1}{C}a^k, \tau^k\right) \in \mathcal{A}^{S^r}(\Phi, p, q, r)$. Applying Lemma 2.8 (i) and Corollary 3.3, we get

$$\frac{1}{C}\left\|\Phi(S^r(f))\right\|_{p,q,b} \leq \left\|\Phi\Big(S^r\Big(\frac{1}{C}f\Big)\Big)\right\|_{p,q,b} \lesssim \mathcal{W}\left(\{\mu^k\}, \{\frac{1}{C}a^k\}, \{\tau^k\}\right) \approx \|\Phi(f)\|_{\mathcal{D}_{p,q,b}(\mathbf{B})}.$$

(i) ⇒ (iii). The proof is similar to the one of (i) ⇒ (ii) above.

(ii) ⇒ (i), (iii) ⇒ (i). Let $f = (f_n)_{n\geq 0}$ be an arbitrary **B**-valued dyadic martingale satisfying $\sup_{n\geq 0}\|f_n\|_{L_\infty(\mathbf{B})} < \infty$. For each $n \in \mathbb{N}$, set $\Phi(t) = t$ and $\lambda_n = \|M_{n+1}(f)\|_\infty$. This ensures that $(\lambda_n)_{n\geq 0} \in \Lambda$ and $\|f_{n+1}\| \leq \lambda_n$. Notice that f is a **B**-valued dyadic martingale, thus we have $S^r_n(f) \leq Cs^r_n(f)$ for each $n \in \mathbb{N}$. Therefore, we can conclude that

$$\|\Phi(f)\|_{\mathcal{D}_{p,q,b}(\mathbf{B})} \leq \|\Phi(\lambda_\infty)\|_{p,q,b} \leq \sup_{n\geq 0}\|f_n\|_{L_\infty(\mathbf{B})} < \infty.$$

Hence, by (40) we obtain $S^r(f) < \infty$ *a.e.* holds and by (41) we know that $s^r(f) < \infty$ *a.e.* holds. Applying Lemma 2.16, we get the desired results. □

Remark 4.3. *Let b be a nondecreasing slowly varying function and $\Phi(t) = t$ in Theorems 4.1 and 4.2. Then we refer to Liu et al. [17] for the corresponding results. Moreover, the slowly varying function b is not necessarily nondecreasing in Theorems 4.1 and 4.2 of this paper. Hence, Theorems 4.1 and 4.2 improve Theorems 5.4 and 5.6 in [17], respectively.*

Remark 4.4. *If $b \equiv 1$, $\Phi(t) = t$ and $0 < p = q \leq 1$, then Theorems 4.1 and 4.2 go back to Theorems 5 and 6 in [22], respectively.*

Remark 4.5. *If $b \equiv 1$ and $\Phi(t) = t$, then Theorems 4.1 and 4.2 recover the corresponding results in [18].*

Remark 4.6. *If we consider the special case $b \equiv 1$ in Theorems 4.1 and 4.2, then the conclusions return to Theorems 4 and 5 in [15], respectively.*

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References

[1] D. L. Burkholder, *Martingale transforms and the geometry of Banach spaces*, Probability in Banach spaces, III (Medford, Mass., 1980), Lecture Notes in Math. Springer, Berlin-New York, **860** (1981), 35–50.

[2] D. L. Burkholder, *Differential subordination of harmonic functions and martingales*, Harmonic analysis and partial differential equations (El Escorial, 1987), Lecture Notes in Math. Springer, Berlin, **1384** (1989), 1–23.

[3] D. L. Burkholder, B. J. Davis, R. F. Gundy, *Integral inequalities for convex functions of operators on martingales*, Proc. Sixth Berkeley Symp. Math. Statist. Prob. **2** (1972), 223–240.

[4] D. L. Burkholder, R. F. Gundy, *Extrapolation and interpolation of quasi-linear operators on martingales*, Acta Math. **124** (1970), 249–304.

[5] D. E. Edmunds, W. D. Evans, *Hardy operators, function spaces and embeddings*, Springer, Berlin, 2004.

[6] D. E. Edmunds, B. Opic, *Alternative characterisations of Lorentz-Karamata spaces*, Czech. Math. J. **58** (2008), 517–540.

[7] Z. Hao, X. Ding, L. Li, F. Weisz, *Variable martingale Hardy-Lorentz-Karamata spaces and their applications in Fourier analysis*, arXiv:2402.16133.

[8] Z. Hao, L. Li, L. Long, F. Weisz, *Orlicz-Lorentz-Karamata Hardy martingale spaces: inequalities and fractional integral operators*, Fract. Calc. Appl. Anal. **27** (2) (2024), 554–615.

[9] Z. Hao, L. Li, A. Yang, *Φ-moment martingale inequalities on Lorentz spaces with variable exponents*, Banach J. Math. Anal. **17** (1) (2023), 12.

[10] K.-P. Ho, *Atomic decompositions, dual spaces and interpolations of martingale Hardy-Lorentz Karamata spaces*, Q. J. Math. **65** (3) (2014), 985–1009.

[11] S. Janković, T. Ostrogorski, *Differences of slowly varying functions*, J. Math. Anal. Appl. **247** (2000), 478–488.

[12] Y. Jiao, X. Quan, L. Wu, *Burkholder's inequalities associated with Orlicz functions in rearrangement invariant spaces*, Math. Z. **289**, (2018), 943–960.

[13] Y. Jiao, W. Yu, *Some new Φ-moment martingale inequalities associated with concave functions*, Nonlinear Anal. **75** (11) (2012), 4271–4277.

[14] M. Kikuchi, *On some inequalities for Doob decompositions in Banach function spaces*, Math. Z. **265** (4) (2010), 865–887.

[15] L. Li, K. Liu, L. Wang, L. Yu, *Φ-moment B-valued martingale inequalities on Lorentz spaces*, Math. Inequal. Appl. **27** (2) (2024), 435–457.

[16] L. Li, K. Liu, Y. Wang, *Martingale inequalities in Orlicz-Karamata modular spaces*, Banach J. Math. Anal. **18** (54) (2024).

[17] K. Liu, W. Li, T. Yue, *B-valued martingale Hardy-Lorentz-Karamata spaces*, Bull. Malays. Math. Sci. Soc. **42** (5) (2019), 2395–2422.

[18] K. Liu, D. Zhou, Y. Jiao, *Hardy-Lorentz spaces for B-valued martingales*, J. Math. Anal. Appl. **450** (2) (2017), 1401–1420.

[19] P. Liu, *Martingale inequalities and convexity and smoothness of Banach spaces*, Acta Math. Sin. **32** (6) (1989), 765–775.

[20] P. Liu, *Martingale spaces and geometrical properties of Banach spaces*, Sci. China Ser. A **34** (5) (1991), 513–527.

[21] P. Liu, *Martingale and geometry of Banach spaces*, Science Press, Beijing, 2007.

[22] P. Liu, Y. Hou, *Atomic decompositions of Banach-space-valued martingales*, Sci. China Ser. A **42** (1) (1999), 38–47.

[23] T. Miyamoto, E. Nakai, G. Sadasue, *Martingale Orlicz-Hardy spaces*, Math. Nachr. **285** (2012), 670–686.

[24] J. S. Neves, *Lorentz-Karamata spaces, Bessel and Riesz potentials and embeddings*, Dissertationes Math. **405**, (2002), 1–46.

[25] L. Peng, J. Li, *A generalization of Φ-moment martingale inequalities*, Stat. Probab. Lett. **102** (2015), 61–68.

[26] G. Pisier, *Martingales with values in uniformly convex spaces*, Israel J. Math. **20** (3-4) (1975), 326–350.

[27] G. Pisier, *Martingales in Banach spaces*, Cambridge Studies in Advanced Mathematics, vol. **155**, Cambridge University Press, Cambridge, 2016.

[28] L. Wu, D. Zhou, Y. Jiao, *Modular inequalities in martingale Orlicz-Karamata spaces*, Math. Nachr. **291** (2018), 1450–1462.

[29] L. Yu, *Duality theorem for B-valued martingale Orlicz-Hardy spaces associated with concave functions*, Math. Nachr. **289** (5-6) (2016), 756–774.

[30] L. Yu, *Dual spaces of weak Orlicz-Hardy spaces for vector-valued martingales*, J. Math. Anal. Appl. **437** (2016), 71–89.