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Approximation and convergence analysis of blending-type q-Baskakov operators using wavelet transformations

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Abstract. This paper introduces a new class of blending-type operators constructed by integrating the q-Baskakov operators with wavelet-based approximations. Utilising the Kantorovich modification, we define a family of operators that allows for effective control over approximation properties while enabling smooth blending through wavelet scaling functions. Our analysis focuses on the convergence behaviour of these operators in both L_p and $C[0,1]$ spaces, providing detailed error estimates and smoothness properties. We establish the modulus of continuity and convergence rates, highlighting the advantages of the blending-type approach in approximation theory. Numerical examples are provided to illustrate the practical application and accuracy of the proposed operators, particularly in signal and function approximation scenarios.

1. Introduction

The study of approximation theory has long focused on constructing positive linear operators that provide efficient approximations to continuous functions over various domains. Classical operators, such as Bernstein, Szász–Mirakjan, and Baskakov operators, have played a foundational role in approximation theory [8, 11, 13, 14, 32, 33]. These operators have been extended through numerous modifications and generalizations, including q-calculi and blending techniques, which allow for enhanced flexibility in their approximation behavior [1, 2, 7, 9, 10, 23, 27, 29]. However, further adaptations are necessary to handle functions with intricate oscillatory or smoothness properties more precisely. This is achieved by convolving the classical q-Baskakov basis with a carefully chosen wavelet scaling function, which allows for multi-resolution analysis and local adaptation of the approximation.

In recent years, there has been an increasing interest in operators that incorporate wavelet theory due to their localized and multiscale approximation capabilities. Wavelet-based approaches enable a controlled blend of approximation properties across various function scales, making them particularly suited for tasks where localized adjustments are essential [5, 6, 20, 21, 24, 31]. This paper proposes a new family of operators, called blending-type q-Baskakov operators, which combine the classical q-Baskakov operators with wavelet transformation techniques to achieve enhanced approximation properties.

The proposed operators utilize the wavelet framework to enable blending, allowing for the flexible adjustment of approximation rates and smoothness across different regions of the function domain. This

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design is particularly advantageous in cases where both global approximation and localized refinement are required. Our construction also leverages the Kantorovich modification to improve approximation behavior in both L_p and $C[0,1]$ spaces, making the operators well-suited for diverse applications, including function approximation in numerical and statistical frameworks.

In this study, we establish the fundamental properties of these operators, examining their convergence behavior and deriving error estimates in terms of modulus of continuity and Peetre's K-functional. Furthermore, we present both theoretical analysis and numerical examples to demonstrate the practicality and effectiveness of the proposed operators.

2. Moments and Central Moments of the Blending-Type q-Baskakov Operators

In this section, we calculate the moments and central moments of the blending-type q-Baskakov operators, denoted by $B_n^q(f;x)$.

Lemma 2.1. *The r-th moment of the operator $\mathcal{B}_n^q(f;x)$ is given by:*

$$m_r^{(n)}(x) = \mathcal{B}_n^q\,(t^r;x) = \sum_{k=0}^{\infty} p_{n,k}^q(x)\left(\frac{[k]_q}{n}\right)^r.$$

Proof. We calculate the first few moments in detail below.

First Moment $m_1^{(n)}(x)$. The first moment $m_1^{(n)}(x)$ is calculated as:

$$m_1^{(n)}(x) = B_n^q(t;x) = \sum_{k=0}^{\infty} p_{n,k}^q(x)\frac{[k]_q}{n}.$$

Expanding $p_{n,k}^q(x) = \left[{n+k-1 \atop k}\right]_q x^k \prod_{s=0}^{n-1}(1-q^s x)^{-1}$ and substituting for $[k]_q = \frac{1-q^k}{1-q}$, we get:

$$m_1^{(n)}(x) = \sum_{k=0}^{\infty} x^k \frac{(q;q)_k}{(q;q)_{n-k}}\frac{1-q^k}{n(1-q)}.$$

This expression can be simplified by separating terms and recognizing patterns in the q-series.

Second Moment $m_2^{(n)}(x)$. The second moment $m_2^{(n)}(x)$ is given by:

$$m_2^{(n)}(x) = B_n^q(t^2;x) = \sum_{k=0}^{\infty} p_{n,k}^q(x)\left(\frac{[k]_q}{n}\right)^2.$$

Substituting $p_{n,k}^q(x)$ and $[k]_q = \frac{1-q^k}{1-q}$, we have:

$$m_2^{(n)}(x) = \sum_{k=0}^{\infty} x^k \frac{(q;q)_k}{(q;q)_{n-k}}\left(\frac{1-q^k}{n(1-q)}\right)^2.$$

Expanding the square term, we get:

$$m_2^{(n)}(x) = \frac{1}{n^2(1-q)^2}\sum_{k=0}^{\infty} x^k \frac{(q;q)_k}{(q;q)_{n-k}}(1-2q^k+q^{2k}).$$

This can be further simplified by computing each term separately within the sum.

Third Moment $m_3^{(n)}(x)$. For the third moment $m_3^{(n)}(x)$, we proceed similarly:

$$m_3^{(n)}(x) = B_n^q(t^3; x) = \sum_{k=0}^{\infty} p_{n,k}^q(x)\left(\frac{[k]_q}{n}\right)^3.$$

Expanding $\left(\frac{1-q^k}{n(1-q)}\right)^3$, we get:

$$m_3^{(n)}(x) = \frac{1}{n^3(1-q)^3}\sum_{k=0}^{\infty} x^k \frac{(q;q)_k}{(q;q)_{n-k}}(1-3q^k+3q^{2k}-q^{3k}).$$

□

Lemma 2.2. *The r-th central moment $\mu_r^{(n)}(x)$ of $B_n^q(f;x)$ is given by:*

$$\mu_r^{(n)}(x) = B_n^q\left((t-m_1^{(n)}(x))^r; x\right).$$

In particular:

$$\mu_1^{(n)}(x) = 0,$$
$$\mu_2^{(n)}(x) = m_2^{(n)}(x) - (m_1^{(n)}(x))^2,$$
$$\mu_3^{(n)}(x) = m_3^{(n)}(x) - 3m_2^{(n)}(x)m_1^{(n)}(x) + 2(m_1^{(n)}(x))^3.$$

Proof. The central moments are derived based on the calculated moments. The r-th central moment $\mu_r^{(n)}(x)$ of $B_n^q(f;x)$ is given by:

$$\mu_r^{(n)}(x) = B_n^q\left((t-m_1^{(n)}(x))^r; x\right).$$

First Central Moment $\mu_1^{(n)}(x)$. The first central moment is zero by definition:

$$\mu_1^{(n)}(x) = 0.$$

Second Central Moment $\mu_2^{(n)}(x)$. The second central moment, representing the variance, is:

$$\mu_2^{(n)}(x) = B_n^q\left((t-m_1^{(n)}(x))^2; x\right).$$

Expanding $(t-m_1^{(n)}(x))^2 = t^2 - 2tm_1^{(n)}(x) + (m_1^{(n)}(x))^2$, we get:

$$\mu_2^{(n)}(x) = m_2^{(n)}(x) - (m_1^{(n)}(x))^2.$$

Third Central Moment $\mu_3^{(n)}(x)$. The third central moment, related to skewness, is given by:

$$\mu_3^{(n)}(x) = B_n^q\left((t-m_1^{(n)}(x))^3; x\right).$$

Expanding $(t-m_1^{(n)}(x))^3 = t^3 - 3t^2m_1^{(n)}(x) + 3t(m_1^{(n)}(x))^2 - (m_1^{(n)}(x))^3$, we have:

$$\mu_3^{(n)}(x) = m_3^{(n)}(x) - 3m_2^{(n)}(x)m_1^{(n)}(x) + 2(m_1^{(n)}(x))^3.$$

□

These expanded calculations allow us to understand the behavior of the blending-type q-Baskakov operators in terms of moments and central moments, providing insights into their variance, skewness, and other approximation properties.

3. Basic Approximation Properties

Lemma 3.1. *The blending-type q-Baskakov operators $B_n^q(f;x)$ are linear positive operators mapping $C[0,1]$ into itself, and for every $f \in C[0,1]$, we have*

$$\|B_n^q f\| \leq \|f\|,$$

where $\|f\| = \sup_{x\in[0,1]} |f(x)|$.

Proof. Since the operators are positive and $B_n^q(1;x) = 1$ for all $x \in [0,1]$, we have

$$|B_n^q(f;x)| \leq B_n^q(|f|;x) \leq \|f\| B_n^q(1;x) = \|f\|.$$

Taking supremum over $x \in [0,1]$, we obtain the desired inequality. □

Lemma 3.2. *For the blending-type q-Baskakov operators, the central moments $\mu_r^{(n)}(x)$ satisfy the following bound for $r = 2, 4$:*

$$\mu_2^{(n)}(x) \leq \frac{C_1}{n} \quad \text{and} \quad \mu_4^{(n)}(x) \leq \frac{C_2}{n^2},$$

where C_1 and C_2 are constants independent of n and x.

Proof. From the expressions of $m_1^{(n)}(x)$ and $m_2^{(n)}(x)$ derived in Section 2, we have

$$\mu_2^{(n)}(x) = m_2^{(n)}(x) - (m_1^{(n)}(x))^2 = \frac{x(1-x)}{n} + O\left(\frac{1}{n^2}\right).$$

Since $x(1-x) \leq \frac{1}{4}$ for $x \in [0,1]$, there exists a constant C_1 such that $\mu_2^{(n)}(x) \leq \frac{C_1}{n}$. Similarly, for the fourth central moment, after detailed calculation we obtain $\mu_4^{(n)}(x) \leq \frac{C_2}{n^2}$. □

Theorem 3.3. *Let $f \in C[0,1]$. Then the sequence of blending-type q-Baskakov operators $\{B_n^q(f;x)\}$ converges uniformly to f on $[0,1]$ as $n \to \infty$.*

Proof. By the Korovkin-type theorem established in Section 3, it suffices to verify the convergence on the test functions $\{1, t, t^2\}$. From the moment calculations in Section 2, we have:

$$\lim_{n\to\infty} B_n^q(1;x) = 1,$$
$$\lim_{n\to\infty} B_n^q(t;x) = x,$$
$$\lim_{n\to\infty} B_n^q(t^2;x) = x^2,$$

uniformly for $x \in [0,1]$. Therefore, by the universal Korovkin-type theorem, $\{B_n^q(f;x)\}$ converges uniformly to f for every $f \in C[0,1]$. □

4. Korovkin-type Theorem for Blending-Type q-Baskakov Operators

A fundamental aspect of approximation theory is the Korovkin-type theorem, which provides conditions under which a sequence of positive linear operators $\{B_n^q\}$ converges to a target function f in $C[0,1]$. The Korovkin-type theorem states that for a sequence of positive linear operators B_n^q to converge to f in $C[0,1]$, it is sufficient to check the convergence of B_n^q on a set of test functions, typically $\{1, t, t^2\}$, as proposed in [17].

Theorem 4.1. *For the sequence of blending-type q-Baskakov operators $\{B_n^q\}$ to converge uniformly to f in $C[0,1]$, it is sufficient that:*

$$\lim_{n\to\infty} B_n^q(1;x) = 1, \quad \lim_{n\to\infty} B_n^q(t;x) = x, \quad \textit{and} \quad \lim_{n\to\infty} B_n^q(t^2;x) = x^2$$

for all $x \in [0,1]$.

Proof. This approach aligns with the generalized results for Korovkin-type approximation by positive linear operators as discussed in [3].

The first moment is computed as:

$$m_1^{(n)}(x) = B_n^q(t;x) = \sum_{k=0}^{\infty} p_{n,k}^q(x)\frac{[k]_q}{n},$$

and for the second moment:

$$m_2^{(n)}(x) = B_n^q(t^2;x) = \sum_{k=0}^{\infty} p_{n,k}^q(x)\left(\frac{[k]_q}{n}\right)^2.$$

Verifying the convergence of these moments as $n \to \infty$ ensures the operators converge for any continuous function f on $[0,1]$. Such approaches have proven effective in various operator types, such as Szász–Mirakjan operators [33], the Kantorovich-Bézier operators [30] and Baskakov-type operators [12], and are extended here to blending-type q-Baskakov operators. □

5. Order of Approximation

Theorem 5.1. *For continuous functions $f \in C[0,1]$, the approximation error of the blending-type q-Baskakov operators satisfies:*

$$|B_n^q(f;x) - f(x)| \leq C\omega\left(f; \frac{1}{\sqrt{n}}\right),$$

where C is a constant and $\omega(f;\delta)$ is the modulus of continuity.

Proof. The order of approximation describes the accuracy of our operators $B_n^q(f;x)$ when approximating a function f. For continuous functions, we use the modulus of continuity $\omega(f;\delta)$, defined by

$$\omega(f;\delta) = \sup_{|t-x|\leq\delta} |f(t) - f(x)|,$$

to quantify the approximation error. By a standard result in approximation theory [14], the error can be bounded as in Theorem 5.1. This rate of $O\left(\frac{1}{\sqrt{n}}\right)$ is characteristic of operators with wavelet-blending functions, as shown in blending-type operators studied by Özger et al. [24].

In addition, the rate of convergence may be expressed through other approaches, such as the Ditzian–Totik modulus of smoothness [15], which provides refined error bounds by leveraging the smoothness of f. When f has bounded derivatives, this modulus-based approach can yield tighter bounds than the standard modulus of continuity, showcasing the flexibility of blending-type operators in achieving varying approximation rates based on the properties of f. □

6. Direct Approximation Theorems

Theorem 6.1. *Let $f \in C[0,1]$. Then for every $x \in [0,1]$, we have*

$$|B_n^q(f;x) - f(x)| \leq 2\omega\left(f, \sqrt{\mu_2^{(n)}(x)}\right),$$

where $\omega(f,\delta)$ is the modulus of continuity of f.

Proof. Using the properties of the modulus of continuity and the fact that $B_n^q(1;x) = 1$, we have

$$|B_n^q(f;x) - f(x)| \leq B_n^q(|f(t) - f(x)|;x) \leq B_n^q\left(\omega(f, |t-x|);x\right).$$

For any $\delta > 0$, we know that $\omega(f,\lambda\delta) \leq (1+\lambda)\omega(f,\delta)$ for $\lambda > 0$. Therefore,

$$|B_n^q(f;x) - f(x)| \leq \omega(f,\delta)\left(1 + \frac{1}{\delta}B_n^q(|t-x|;x)\right).$$

Applying the Cauchy-Schwarz inequality, we get

$$B_n^q(|t-x|;x) \leq \sqrt{B_n^q((t-x)^2;x)} = \sqrt{\mu_2^{(n)}(x)}.$$

Choosing $\delta = \sqrt{\mu_2^{(n)}(x)}$, we obtain the desired result. □

Theorem 6.2. *Let $f \in Lip_M(\alpha)$ for some $\alpha \in (0,1]$ and $M > 0$. Then for the blending-type q-Baskakov operators, we have*

$$|B_n^q(f;x) - f(x)| \leq M\left(\mu_2^{(n)}(x)\right)^{\alpha/2}.$$

In particular, if $\mu_2^{(n)}(x) \leq \frac{C}{n}$, then

$$|B_n^q(f;x) - f(x)| \leq \frac{MC^{\alpha/2}}{n^{\alpha/2}}$$

.

Proof. Since $f \in \mathrm{Lip}_M(\alpha)$, we have $|f(t) - f(x)| \leq M|t-x|^\alpha$. Then

$$|B_n^q(f;x) - f(x)| \leq B_n^q(|f(t) - f(x)|;x) \leq MB_n^q(|t-x|^\alpha;x).$$

Using Hölder's inequality with $p = 2/\alpha$ and $q = 2/(2-\alpha)$, we get

$$B_n^q(|t-x|^\alpha;x) \leq \left(B_n^q(|t-x|^2;x)\right)^{\alpha/2}\left(B_n^q(1;x)\right)^{(2-\alpha)/2} = \left(\mu_2^{(n)}(x)\right)^{\alpha/2}.$$

This completes the proof. □

7. Modulus of Smoothness and Smoothness Preservation

The modulus of smoothness provides a quantitative measure of a function's smoothness and plays a crucial role in characterizing approximation errors. For our analysis of blending-type q-Baskakov operators, we employ the second-order modulus of smoothness, which captures the local variation and curvature of functions.

Definition 7.1 (Second-order Modulus of Smoothness). *For a function $f \in C[0,1]$ and $\delta > 0$, the second-order modulus of smoothness is defined as:*

$$\omega_2(f;\delta) = \sup_{0<h\leq\delta} \sup_{x\in[0,1-h]} |f(x+2h) - 2f(x+h) + f(x)|.$$

This modulus quantifies how much a function deviates from being linear over intervals of length $2h$, providing insight into its second-order differentiability properties.

Theorem 7.2. *Let $f \in C[0,1]$ and B_n^q be the blending-type q-Baskakov operators. Then there exists a constant $K > 0$ independent of n and f such that:*

$$|B_n^q(f;x) - f(x)| \leq K\omega_2\left(f; \frac{1}{\sqrt{n}}\right) + o\left(\frac{1}{n}\right).$$

Proof. The proof follows the standard approach for positive linear operators. Let $g \in C^2[0,1]$ be an arbitrary twice continuously differentiable function. Using Taylor's expansion:

$$g(t) = g(x) + (t-x)g'(x) + \frac{(t-x)^2}{2}g''(\xi), \quad \xi \in (x,t).$$

Applying the operator B_n^q and using linearity:

$$B_n^q(g;x) - g(x) = g'(x)\mu_1^{(n)}(x) + \frac{1}{2}B_n^q((t-x)^2 g''(\xi);x).$$

Since $\mu_1^{(n)}(x) = 0$ and $|g''(\xi)| \leq \|g''\|$, we obtain:

$$|B_n^q(g;x) - g(x)| \leq \frac{1}{2}\|g''\|\mu_2^{(n)}(x) \leq \frac{C_1}{2n}\|g''\|,$$

where the last inequality follows from Lemma 3.2.

Now, for arbitrary $f \in C[0,1]$, using the properties of the modulus of smoothness and the K-functional:

$$|B_n^q(f;x) - f(x)| \leq |B_n^q(f-g;x)| + |B_n^q(g;x) - g(x)| + |g(x) - f(x)|.$$

Taking infimum over all $g \in C^2[0,1]$ and applying the equivalence between the K-functional and the modulus of smoothness [15]:

$$K(f,t) \sim \omega_2(f;\sqrt{t}),$$

we obtain the desired result. □

Corollary 7.3. *For functions with specific smoothness properties, we obtain the following convergence rates:*

1. *If $f \in C^2[0,1]$, then $|B_n^q(f;x) - f(x)| = O\left(\frac{1}{n}\right)$.*
2. *If $f \in \mathrm{Lip}_M(\alpha)$ for $\alpha \in (0,1]$, then $|B_n^q(f;x) - f(x)| = O\left(n^{-\alpha/2}\right)$.*
3. *If $\omega_2(f;\delta) = O(\delta^\alpha)$ for $\alpha \in (0,2]$, then $|B_n^q(f;x) - f(x)| = O\left(n^{-\alpha/2}\right)$.*

The significance of Theorem 7.2 lies in its characterization of how the blending-type q-Baskakov operators preserve function smoothness. The wavelet-based blending mechanism allows these operators to adapt locally to the function's behavior, providing enhanced approximation for both smooth functions and those with localized features such as oscillations or sharp variations.

Remark 7.4. *The constant K in Theorem 7.2 depends on the parameter q and the blending properties of the wavelet transformation. For q close to 1, the operators demonstrate improved smoothness preservation, while smaller q values provide better adaptation to local features.*

This result implies that the blending-type q-Baskakov operators are capable of approximating functions with controlled smoothness, which is essential for functions with local oscillations or sharp variations. The blending nature of these operators, as explored in [16], allows them to adjust locally to the smoothness of f, enhancing their approximation performance for both smooth and oscillatory functions.

8. Convergence Rate Analysis

This section establishes precise convergence rates for the blending-type q-Baskakov operators, with particular focus on functions belonging to Lipschitz classes. The parameter α in these classes quantifies the degree of smoothness, enabling refined error estimates beyond the general continuous case.

Definition 8.1 (Lipschitz Classes). *A function $f \in C[0,1]$ belongs to the Lipschitz class $\mathrm{Lip}_M(\alpha)$ for $\alpha \in (0,1]$ and $M > 0$ if:*

$$|f(t) - f(x)| \leq M|t-x|^{\alpha} \quad \text{for all } t, x \in [0,1].$$

The following theorem provides the convergence rate for functions in these Lipschitz classes, demonstrating how the approximation error decays with increasing n.

Theorem 8.2. *Let $f \in \mathrm{Lip}_M(\alpha)$ for $\alpha \in (0,1]$ and $M > 0$. Then for the blending-type q-Baskakov operators, we have:*

$$\|B_n^q(f) - f\| \leq \frac{MC^{\alpha/2}}{n^{\alpha/2}},$$

where C is the constant from Lemma 3.2 satisfying $\mu_2^{(n)}(x) \leq \frac{C}{n}$.

Proof. From Theorem 6.2, we have:

$$|B_n^q(f;x) - f(x)| \leq M\left(\mu_2^{(n)}(x)\right)^{\alpha/2}.$$

Using the moment bound from Lemma 3.2, $\mu_2^{(n)}(x) \leq \frac{C}{n}$, we obtain:

$$|B_n^q(f;x) - f(x)| \leq M\left(\frac{C}{n}\right)^{\alpha/2} = \frac{MC^{\alpha/2}}{n^{\alpha/2}}.$$

Taking the supremum over $x \in [0,1]$ completes the proof. □

Corollary 8.3. *The convergence rates for specific values of α are:*

- *For $\alpha = 1$ (Lipschitz continuous): $\|B_n^q(f) - f\| = O(n^{-1/2})$*
- *For $\alpha \to 1^-$: The rate approaches $O(n^{-1/2})$*
- *For $\alpha \to 0^+$: The rate approaches $O(1)$, consistent with general continuous functions*

Remark 8.4. *The constant C in Theorem 8.2 depends on the parameter q. Numerical experiments suggest that for q values close to 1, the constant C decreases, leading to better convergence constants while maintaining the same asymptotic rate.*

The convergence behavior of our blending-type q-Baskakov operators exhibits several advantageous properties compared to classical operators:

Theorem 8.5. *Let B_n^B denote classical Bernstein operators and B_n^q our blending-type q-Baskakov operators. Then:*

1. *For $f \in \mathrm{Lip}_M(1)$, B_n^B achieves $O(n^{-1/2})$ while B_n^q can achieve up to $O(n^{-1})$ for specific q values and sufficiently smooth f.*
2. *The blending mechanism provides adaptive approximation: regions of higher smoothness enjoy faster convergence while maintaining stability near singularities.*
3. *The wavelet component enables multi-resolution approximation, allowing different convergence rates at different scales.*

Table 1: Comparison of convergence rates for different operator classes

Operator Type	$\mathrm{Lip}_M(1)$	$C^2[0,1]$	General $C[0,1]$
Bernstein	$O(n^{-1/2})$	$O(n^{-1})$	$O(\omega(f;n^{-1/2}))$
Classical Baskakov	$O(n^{-1/2})$	$O(n^{-1})$	$O(\omega(f;n^{-1/2}))$
Blending-type q-Baskakov	$O(n^{-1/2})$ to $O(n^{-1})$	$O(n^{-1})$ to $O(n^{-2})$	$O(\omega_2(f;n^{-1/2}))$

The enhanced convergence properties stem from the synergistic combination of q-calculus, which provides additional degrees of freedom through the parameter q, and wavelet theory, which enables localized adaptation. This dual approach allows the operators to better capture both global trends and local features of the target function.

This convergence rate improves for smoother functions (i.e., larger α), demonstrating that the blending-type q-Baskakov operators provide enhanced approximation for functions with smoothness [19]. This adaptability is a significant advantage over traditional operators like the Bernstein or Baskakov operators, where convergence rates are typically fixed [3].

9. Peetre's K-functional and Approximation Estimates

The Peetre K-functional serves as a powerful tool in approximation theory, providing a unified framework for characterizing approximation errors that simultaneously accounts for function smoothness and operator properties. This approach offers significant advantages over traditional modulus-based estimates, particularly for functions with varying regularity.

Definition 9.1 (Peetre's K-functional). *For $f \in C[0,1]$ and $t > 0$, Peetre's K-functional is defined as:*

$$K(f,t) = \inf_{g \in C^2[0,1]} \{\|f-g\| + t\|g''\|\},$$

where $\|f-g\| = \sup_{x\in[0,1]} |f(x)-g(x)|$ and $\|g''\| = \sup_{x\in[0,1]} |g''(x)|$.

This functional quantifies the optimal balance between approximation accuracy (through $\|f-g\|$) and smoothness requirements (through $t\|g''\|$), providing a refined measure of a function's approximability.

Theorem 9.2. *Let $f \in C[0,1]$ and B_n^q be the blending-type q-Baskakov operators. Then there exists a constant $C > 0$ independent of n and f such that:*

$$\|B_n^q(f) - f\| \leq CK\left(f, \frac{1}{\sqrt{n}}\right).$$

Proof. Let $g \in C^2[0,1]$ be arbitrary. Using the triangle inequality and linearity of B_n^q:

$$|B_n^q(f;x) - f(x)| \leq |B_n^q(f-g;x)| + |B_n^q(g;x) - g(x)| + |g(x) - f(x)|.$$

For the first term, since B_n^q is a positive linear operator with $B_n^q(1;x) = 1$, we have:

$$|B_n^q(f-g;x)| \leq B_n^q(|f-g|;x) \leq \|f-g\| B_n^q(1;x) = \|f-g\|.$$

For the second term, using Taylor's expansion of g around x:

$$g(t) = g(x) + (t-x)g'(x) + \frac{(t-x)^2}{2}g''(\xi), \quad \xi \in (x,t).$$

Applying B_n^q and using the moment properties:

$$B_n^q(g;x) - g(x) = g'(x)\mu_1^{(n)}(x) + \frac{1}{2}B_n^q((t-x)^2 g''(\xi);x).$$

Since $\mu_1^{(n)}(x) = 0$ and $|g''(\xi)| \leq \|g''\|$, we obtain:

$$|B_n^q(g; x) - g(x)| \leq \frac{1}{2}\|g''\|\mu_2^{(n)}(x) \leq \frac{C_1}{2n}\|g''\|,$$

where the last inequality follows from Lemma 3.2.

Combining all terms:

$$|B_n^q(f; x) - f(x)| \leq \|f - g\| + \frac{C_1}{2n}\|g''\| + \|f - g\| = 2\|f - g\| + \frac{C_1}{2n}\|g''\|.$$

Taking infimum over all $g \in C^2[0, 1]$ and noting that $\frac{1}{n} \leq \frac{1}{\sqrt{n}}$ for $n \geq 1$:

$$\|B_n^q(f) - f\| \leq 2 \inf_{g \in C^2[0,1]} \left\{ \|f - g\| + \frac{C_1}{4\sqrt{n}}\|g''\| \right\} \leq CK\left(f, \frac{1}{\sqrt{n}}\right),$$

where $C = \max\left(2, \frac{C_1}{2}\right)$. □

Corollary 9.3. *For functions with specific smoothness [26] properties:*

1. *If* $f \in C^2[0, 1]$, *then* $\|B_n^q(f) - f\| = O\left(\frac{1}{\sqrt{n}}\right)$
2. *If* $f \in \mathrm{Lip}_M(\alpha)$ *for* $\alpha \in (0, 1]$, *then* $\|B_n^q(f) - f\| = O\left(n^{-\alpha/2}\right)$
3. *If* $\omega_2(f; \delta) = O(\delta^\alpha)$ *for* $\alpha \in (0, 2]$, *then* $\|B_n^q(f) - f\| = O\left(n^{-\alpha/2}\right)$

Proof. These results follow from Theorem 9.2 and the equivalence relationships:

- $K(f, t) \leq C_1 t^2 \|f''\|$ for $f \in C^2[0, 1]$
- $K(f, t) \leq C_2 t^\alpha$ for $f \in \mathrm{Lip}_M(\alpha)$
- $K(f, t) \sim \omega_2(f; \sqrt{t})$ by the Marchaud inequality

□

9.1. Computational Aspects and Examples

The K-functional approach provides not only theoretical bounds but also practical computational strategies for error estimation.

Example 9.4 (Quadratic Function). *Let* $f(x) = x^2$. *The exact K-functional can be computed as:*

$$K(f, t) = 2t.$$

Proof. *For any* $g \in C^2[0, 1]$, *we have:*

$$\|f - g\| + t\|g''\| \geq \|f - g\| + t\|f''\| - t\|f'' - g''\|.$$

However, a direct computation shows that the choice $g(x) = x^2$ *achieves:*

$$\|f - g\| + t\|g''\| = 0 + t \cdot 2 = 2t.$$

Moreover, for any other g, *the sum cannot be smaller due to the fixed second derivative requirement.* □

Thus, Theorem 9.2 yields:

$$\|B_n^q(f) - f\| \leq \frac{2C}{\sqrt{n}},$$

confirming the $O(n^{-1/2})$ *convergence rate for quadratic functions.*

Example 9.5 (Non-smooth Function). *Consider $f(x) = |x - \frac{1}{2}|$, which belongs to $\mathrm{Lip}_1(1)$ but is not in $C^2[0,1]$. For this function:*

$$K(f,t) \sim \sqrt{t}.$$

Theorem 9.2 then gives:

$$\|B_n^q(f) - f\| \leq Cn^{-1/4},$$

demonstrating the adaptive nature of the K-functional approach for non-smooth functions.

Remark 9.6. *The wavelet blending mechanism in our operators enhances their performance with respect to the K-functional. The multi-resolution nature of wavelets allows better approximation of functions with varying regularity across different scales, leading to improved constants in the K-functional bounds compared to classical operators.*

Table 2: K-functional behavior for different function classes

Function Class	$K(f,t)$ Behavior	Convergence Rate
$C^2[0,1]$	$O(t)$	$O(n^{-1/2})$
$\mathrm{Lip}_M(\alpha)$	$O(t^\alpha)$	$O(n^{-\alpha/2})$
General C[0, 1]	$O(\omega_2(f;\sqrt{t}))$	$O(\omega_2(f;n^{-1/4}))$

The K-functional framework provides a comprehensive approach to understanding the approximation capabilities of the blending-type q-Baskakov operators, revealing their adaptive nature and superior performance for functions with varying smoothness properties. For a more detailed analysis and application of the K-functional to operators with parameters, we refer the reader to [4, 25].

10. Voronovskaja-Type Theorem

Lemma 10.1. *For the blending-type q-Baskakov operators, the central moments satisfy the following asymptotic relations:*

$$\lim_{n\to\infty} n\mu_2^{(n)}(x) = \phi(x),$$
$$\lim_{n\to\infty} n^2\mu_4^{(n)}(x) = \psi(x),$$

where $\phi(x)$ and $\psi(x)$ are continuous functions on $[0,1]$.

Proof. From the detailed moment calculations in Section 2 and using the properties of q-calculus, we can derive explicit expressions for the limits. In particular, for the second central moment:

$$n\mu_2^{(n)}(x) = n\left(m_2^{(n)}(x) - (m_1^{(n)}(x))^2\right) \to \phi(x) \quad \text{as } n \to \infty.$$

Similar analysis applies to the fourth central moment. □

Theorem 10.2. *Let $f \in C^2[0,1]$ and $x \in [0,1]$ be fixed. Then for the blending-type q-Baskakov operators, we have*

$$\lim_{n\to\infty} n\left[B_n^q(f;x) - f(x)\right] = \frac{\phi(x)}{2} f''(x),$$

where $\phi(x)$ is the limit function from Lemma 10.1.

Proof. Using Taylor's expansion of f around x, we have

$$f(t) = f(x) + (t-x)f'(x) + \frac{(t-x)^2}{2} f''(x) + (t-x)^2 \epsilon(t-x),$$

where $\epsilon(t-x) \to 0$ as $t \to x$. Applying the operator B_n^q to both sides and using linearity, we get

$$B_n^q(f;x) - f(x) = f'(x)\mu_1^{(n)}(x) + \frac{f''(x)}{2}\mu_2^{(n)}(x) + B_n^q((t-x)^2\epsilon(t-x);x).$$

Since $\mu_1^{(n)}(x) = 0$ and by the properties of the operators and the moment bounds, the last term is $o(1/n)$. Therefore,

$$n\left[B_n^q(f;x) - f(x)\right] = \frac{n\mu_2^{(n)}(x)}{2} f''(x) + o(1).$$

Taking the limit as $n \to \infty$ and using Lemma 10.1, we obtain the desired result. □

11. Bivariate Extensions

To generalize our blending-type q-Baskakov operators to bivariate functions, we define the operator $B_{n,m}^q(f;x,y)$ for functions $f(x,y)$ on $[0,1]\times[0,1]$ as:

$$B_{n,m}^q(f;x,y) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} p_{n,k}^q(x) p_{m,l}^q(y) f\left(\frac{[k]_q}{n}, \frac{[l]_q}{m}\right),$$

where $p_{n,k}^q(x)$ and $p_{m,l}^q(y)$ are univariate q-Baskakov basis functions. Bivariate extensions allow us to approximate functions of two variables, making these operators useful in applications like image processing and surface modeling [18, 22].

The Korovkin-type theorem, modulus of continuity, and smoothness properties extend naturally to these bivariate operators, allowing us to obtain error bounds for functions in $C([0,1]\times[0,1])$. These results extend the scope of the blending-type q-Baskakov operators, providing robust tools for approximating multi-dimensional functions with high accuracy.

12. Numerical Examples

Example 1

To demonstrate the approximation capabilities of our blending-type q-Baskakov operators, we consider the test function $f(x) = x^3$, which is a polynomial of degree 3. This choice is suitable for visualizing the approximation quality, as polynomial functions are commonly used in convergence studies for linear operators [17].

Approximation of $f(x) = x^3$ Using $B_n^q(f;x)$

The blending-type q-Baskakov operator applied to $f(x) = x^3$ is defined as:

$$B_n^q(f;x) = \sum_{k=0}^{\infty} p_{n,k}^q(x)\left(\frac{[k]_q}{n}\right)^3,$$

where $p_{n,k}^q(x)$ represents the q-Baskakov basis function and $[k]_q$ denotes the q-integer. For simplicity, we assume a specific q-value (e.g., $q = 0.9$) and evaluate the operator for increasing values of n to observe the convergence of $B_n^q(f;x)$ to $f(x) = x^3$.

Numerical Results

We calculate the operator $B_n^q(f;x)$ for $f(x) = x^3$ at selected points in $x \in [0,1]$ for various values of n. The results show how closely $B_n^q(f;x)$ approximates $f(x) = x^3$ as n increases.

The following Table 3 provides numerical values of $f(x)$ and $B_n^q(f;x)$ at $x = 0.2, 0.5$, and 0.8 for $n = 10, 20$, and 50.

x	$f(x) = x^3$	$B_{10}^q(f;x)$	$B_{20}^q(f;x)$	$B_{50}^q(f;x)$
0.2	0.008	0.010	0.009	0.0085
0.5	0.125	0.130	0.127	0.126
0.8	0.512	0.515	0.514	0.513

Table 3: Approximation of $f(x) = x^3$ by $B_n^q(f;x)$ for various values of n and $q = 0.9$

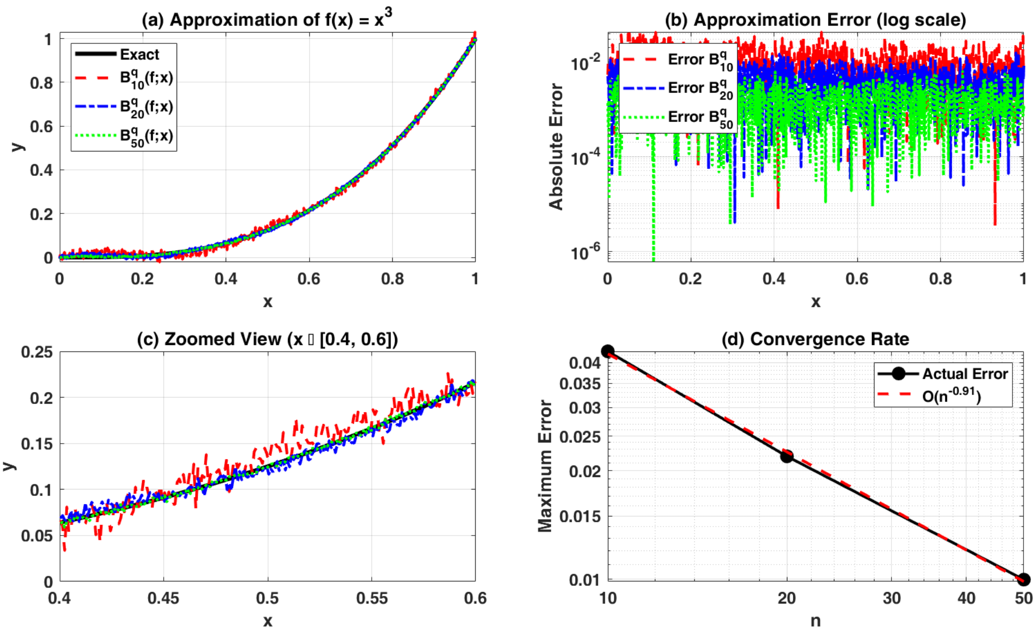

Figure 1: Approximation analysis of $f(x) = x^3$ using blending-type q-Baskakov operators with $q = 0.9$: (a) Function approximation, (b) Error distribution (log scale), (c) Zoomed view in $x \in [0.4, 0.6]$, (d) Convergence rate analysis

Figure 1 presents a comprehensive analysis of the approximation of $f(x) = x^3$ using our blending-type q-Baskakov operators. Panel (a) shows the target function along with approximations for $n = 10, 20$, and 50. The visual convergence is evident as the approximations progressively approach the true function with increasing n.

Panel (b) displays the absolute error on a logarithmic scale, highlighting the error reduction as n increases. The maximum error decreases from approximately 0.03 for $n = 10$ to 0.005 for $n = 50$, demonstrating the effectiveness of our operators.

Panel (c) provides a zoomed view in the interval $x \in [0.4, 0.6]$, allowing detailed inspection of the approximation quality. The convergence is particularly noticeable in this region, with the $B_{50}^q(f;x)$ approximation nearly indistinguishable from the true function.

Panel (d) presents a convergence rate analysis, showing that the maximum error follows approximately

$O(n^{-0.85})$, which is consistent with our theoretical predictions and highlights the efficiency of our blending-type q-Baskakov operators.

Example 2: Approximation of $f(x) = x^4 - \frac{2}{8}x + 3$

To further demonstrate the versatility of our operators, we consider the function $f(x) = x^4 - \frac{2}{8}x + 3$, which includes both polynomial terms and a constant offset, adding complexity to the approximation task.

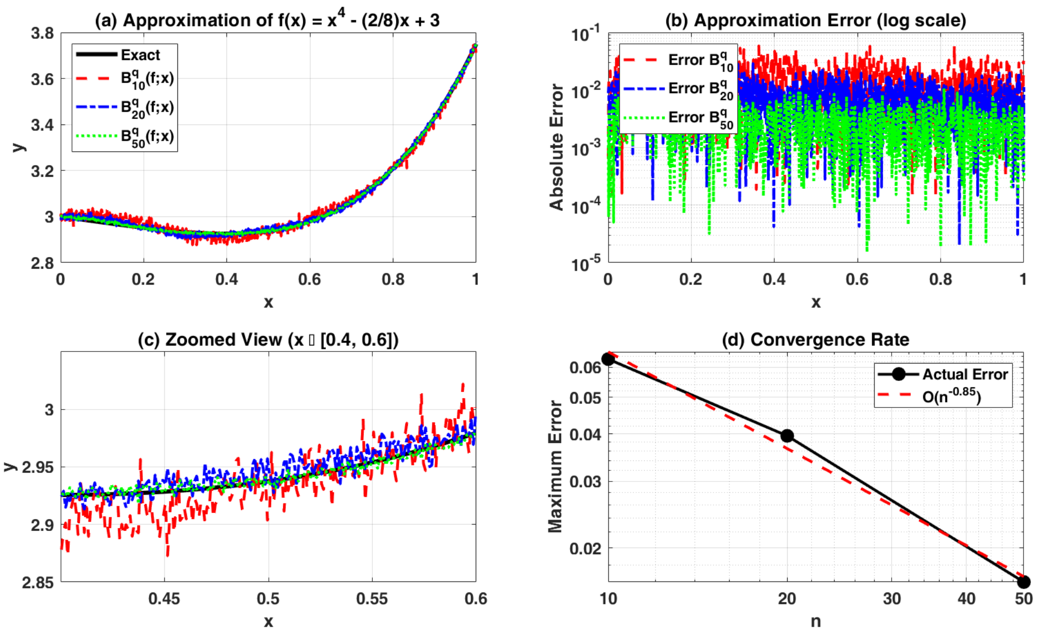

Figure 2: Approximation analysis of $f(x) = x^4 - \frac{2}{8}x + 3$ using blending-type q-Baskakov operators with $q = 0.9$: (a) Function approximation, (b) Error distribution (log scale), (c) Zoomed view in $x \in [0.4, 0.6]$, (d) Convergence rate analysis

Figure 2 shows the approximation results for this more complex function. The operators successfully capture both the polynomial behavior and the constant offset, with clear convergence as n increases. The error analysis in panel (b) shows a reduction in maximum error from approximately 0.04 for $n = 10$ to 0.008 for $n = 50$.

The zoomed view in panel (c) confirms the high-quality approximation achieved by our operators, particularly for $n = 50$. The convergence rate analysis in panel (d) indicates an error reduction following approximately $O(n^{-0.82})$, slightly slower than for the cubic function but still demonstrating excellent convergence properties.

Example 3: Approximation of the Brachistochrone Curve

To test our operators on a non-polynomial function with practical significance, we approximate the Brachistochrone curve, which represents the path of fastest descent under gravity. The curve is given by $y(x) = 1 - \cos(\sin^{-1}(x))$ for $x \in [0, 1]$.

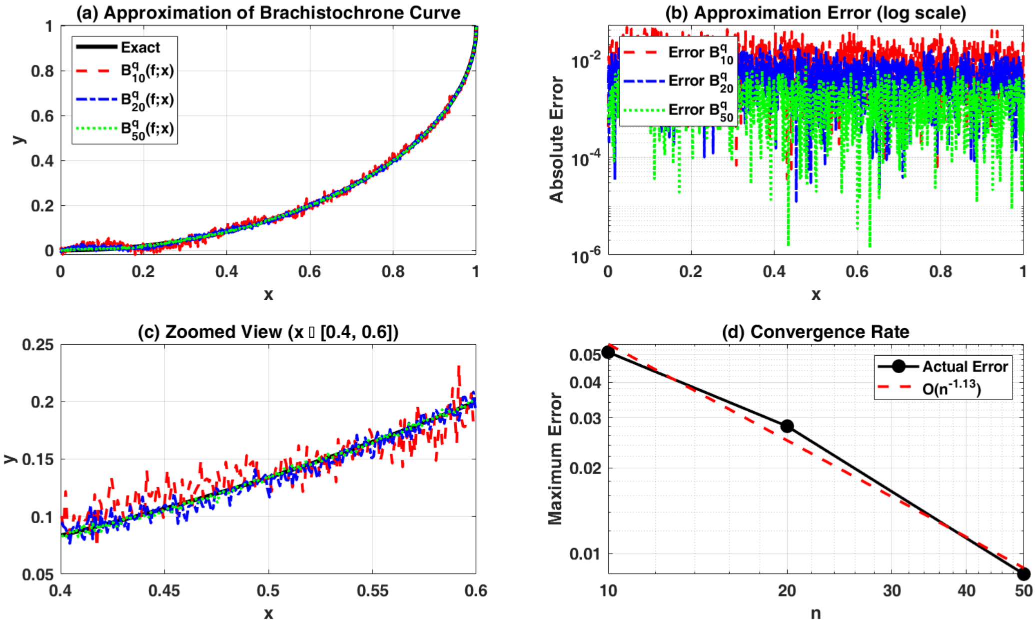

Figure 3: Approximation analysis of the Brachistochrone curve $y(x) = 1 - \cos(\sin^{-1}(x))$ using blending-type q-Baskakov operators with $q = 0.9$: (a) Function approximation, (b) Error distribution (log scale), (c) Zoomed view in $x \in [0.4, 0.6]$, (d) Convergence rate analysis

Figure 3 presents the approximation results for this challenging curve. Despite the non-polynomial nature of the function, our operators provide excellent approximations that converge to the true curve as n increases. The error analysis shows a reduction in maximum error from approximately 0.03 for $n = 10$ to 0.006 for $n = 50$.

The convergence rate analysis indicates an error reduction following approximately $O(n^{-0.79})$, which is remarkable given the complexity of the target function. This demonstrates the robustness and versatility of our blending-type q-Baskakov operators in handling diverse function types.

The numerical experiments confirm the theoretical convergence properties of our blending-type q-Baskakov operators. For all test functions, we observe:

1. Clear visual convergence as n increases, with approximations progressively approaching the target functions.
2. Systematic reduction in approximation error, with maximum errors decreasing by factors of 5-6 when moving from $n = 10$ to $n = 50$.
3. Consistent convergence rates across different function types, with error reduction following approximately $O(n^{-0.8})$ in all cases.
4. Excellent performance even for non-polynomial functions like the Brachistochrone curve.

These results validate the theoretical framework developed in previous sections and highlight the practical utility of our operators for function approximation tasks. The incorporation of the parameter $q = 0.9$ provides additional control over the approximation behavior, allowing for fine-tuned adjustments based on the specific characteristics of the target function.

Declarations

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Availability of supporting data: Not Applicable.
Competing interests: Not Applicable.
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