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Stochastic heat equation with a special generalized fractional noise

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Abstract. We investigate a novel stochastic heat equation driven by a special generalized fractional Gaussian noise. We establish the existence, mixed-self-similarity, and regularity properties of the mild solution. Additionally, we explore the fractal dimensions of the solution sample paths' graphs and ranges. Our findings have potential applications in modeling complex physical phenomena with long-range correlations.

1. Introduction

Stochastic heat equations driven by various noise processes have garnered significant attention in recent years, leading to a wealth of valuable insights (see, for example, [1], [17], [18], [20]). Notably, [17] investigated solutions to stochastic heat equations driven by fractional-white noise, a Gaussian noise exhibiting Brownian behavior in space and fractional Brownian behavior in time. These equations offer a powerful tool for modeling physical phenomena subject to random fluctuations. The introduction of noise into partial differential equations aims to capture the inherent stochasticity often observed in real-world processes. However, selecting the appropriate noise term is not a straightforward task. The choice of a suitable stochastic process must be carefully considered based on the specific equation of motion and the underlying physical interpretation.

The purpose of the present paper is to investigate a new stochastic partial differential equation with Laplacian operator, driven by a White-colored noise, which behaves as a Wiener process in space variable and as a generalized fractional Brownian motion in time. The generalized fractional Brownian motion considered here was introduced by [27] as an extension of both fractional and sub-fractional brownian motions.

While numerous extensions of fractional Brownian motion (fBm) and sub-fractional Brownian motion (sfBm) have emerged in recent decades, including multifractional Brownian motion [13], mixed fractional Brownian motion [24], bifractional Brownian motion [5], mixed sub-fractional Brownian motion [3, 11, 26], generalized sub-fractional Brownian motion [15], and the newly introduced generalized fractional Brownian motion [12], these extensions are typically limited to extending either fBm or sfBm individually. In contrast, Zili Generalized fractional Brownian motion (ZgfBm) [27] offers a unique approach by simultaneously extending both fractional Brownian motion (fBm) and sub-fractional Brownian motion (sfBm). This versatility allows for modeling a wider range of random physical phenomena, including both stationary

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and non-stationary behaviors. By adjusting its three parameters, ZgfBm can capture complex systems with heterogeneous time-dependent characteristics, offering greater flexibility compared to traditional fBm and sfBm models, which are limited by a single Hurst parameter. For further details on ZgfBm, please refer to [27–29].

Our study generalizes existing works on more particular Gaussian noises such as fBm or sfBm. By considering ZgfBm, we provide a more general framework for the analysis of stochastic partial differential equations and thus allow the study of more varied and complex physical phenomena.

This paper is organized as follows. We begin with a review of the definition and key properties of our generalized fractional Brownian motion (ZgfBm), introducing new characteristics relevant to our study. Next, we introduce our special generalized fractional heat equation and examine the existence and mixed-self-similarity properties of its mild solution. We then delve into the regularity of the mild solution, both in time and space, and investigate the Hausdorff and Packing dimensions of the solution sample paths' graphs and ranges. Finally, we provide a conclusion discussing potential future research directions. The last section is devoted to an appendix providing a technical lemma and its proof.

2. Generalized fractional Brownian motion: Zili version

Before introducing our generalized fractional Brownian motion, let us recall that the two-sided fractional Brownian motion (tsfBm) of Hurst parameter $H \in (0,1)$ is a centered Gaussian process $B^H = \{B_t^H, t \in \mathbb{R}\}$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with the covariance function:

$$Cov(B_t^H, B_s^H) = \frac{1}{2}\Big(\mid s \mid^{2H} + \mid t \mid^{2H} - \mid t-s \mid^{2H} \Big);\ t, s \in \mathbb{R}. \tag{1}$$

The restriction of the tsfBm to the set $[0, +\infty)$ is the well known fractional Brownian motion (fBm), which in turn is an extension of the Brownian motion (Bm) because $Cov(B_s^{1/2}, B_t^{1/2}) = t \wedge s$ for every $s, t \geq 0$. Both tsfBm and fBm have been considered as an important tool in modeling due to their properties of long-range dependence, self-similarity and stationarity of their increments. For more information on tsfBm and fBm see, e.g. [9, 11, and references therein].

In [2], the authors introduced another kind extension of the Brownian motion, referred as sub-fractional Brownian motion (sfBm), preserving most of the properties of the fBm, but not the stationarity of the increments. It was introduced as a centered Gaussian process $\xi^H = \{\xi_t^H, t \geq 0\}$, with the covariance function

$$Cov(\xi_t^H, \xi_s^H) = s^{2H} + t^{2H} - \frac{1}{2}\left((t+s)^{2H} + \mid t-s \mid^{2H}\right),\ s, t \in [0, +\infty). \tag{2}$$

We refer to [2, 16, and references therein] for further information on the sfBm.

Our research centers around the following more general process:

Definition 2.1. *The Zili Generalized fractional Brownian motion, ZgfBm for short, with parameters* $(a, b, H) \in (\mathbb{R}^2 \setminus \{(0,0)\}) \times (0,1)$*, denoted by* $(Z_t^H(a,b))_{t \in \mathbb{R}_+}$*, is a centered Gaussian process, defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, with the covariance function:*

$$R^{H,a,b}(t,s) := Cov\Big(Z_t^H(a,b), Z_s^H(a,b)\Big) = \frac{1}{2}(a+b)^2\left(s^{2H} + t^{2H}\right) - ab(t+s)^{2H} - \frac{a^2+b^2}{2} \mid t-s \mid^{2H}. \tag{3}$$

From Equation (3), we see that the ZgfBm reduces to a fractional Brownian motion (fBm) when $a = 1$ and $b = 0$, and to a sub-fractional Brownian motion (sfBm) when $a = b = \dfrac{1}{\sqrt{2}}$.

This demonstrates the ZgfBm's versatility as a generalization of both fBm and sfBm, making it a valuable tool for modeling a wider range of natural phenomena.

The following proposition establishes the existence of the ZgfBm.

Proposition 2.2. *For every $(a,b,H) \in (\mathbb{R}^2 \setminus \{(0,0)\}) \times (0,1)$, the process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by:*

$$\forall t \in \mathbb{R}_+, \qquad Z_t^H(a,b) = aB_t^H + bB_{-t}^H \tag{4}$$

where $(B_t^H)_{t\in\mathbb{R}}$ is a two-sided fractional Brownian motion of parameter H, is a ZgfBm of parameters (a,b,H).

Proof. Let us denote $Y_t := aB_t^H + bB_{-t}^H$. Due to the Gaussianity of B^H and using Equation (1), it is straightforward to verify that Y is a centered Gaussian process with a covariance function identical to (3). This completes the proof of Proposition 2.2. □

Using the same technique as in Lemma 2.4 of [11], we obtain:

Lemma 2.3. *Let the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ be rich enough to support any of stochastic processes introduced below, and let $Z^H(a,b) = \{Z_t^H(a,b); t \geq 0\}$ be a ZgfBm. Then there exists a tsfBm $B^H = \{B_t^H; t \in \mathbb{R}\}$ on the whole real line such that $Z^H(a,b)$ admits the representaion (4).*

Before proceeding, we will revisit some interesting characteristics of the ZgfBm, which were thoroughly investigated in [25, 27].

Lemma 2.4.
1. *The ZgfBm is a self-similar process; that is the processes $\{Z_{ht}^H(a,b); t \geq 0\}$, and $\left\{h^H Z_t^H(a,b); t \geq 0\right\}$ have the same law.*
2. *There exist two positive constants, $\gamma(a,b,H)$ and $\nu(a,b,H)$ such that, for all $(s,t) \in \mathbb{R}_+^2$; $s \leq t$,*

$$\gamma(a,b,H)(t-s)^{2H} \leq \mathbb{E}\Big(Z_t^H(a,b) - Z_s^H(a,b)\Big)^2 \leq \nu(a,b,H)(t-s)^{2H}. \tag{5}$$

3. *The ZgfBm $Z^H(a,b)$ admits a version whose sample paths are almost surely Hölder continuous of order strictly less than H.*
4. $E\left(Z_t^H(a,b)^2\right) := C_H(a,b)t^{2H}$;

$$C_H(a,b) = a^2 + b^2 - (2^{2H} - 2)ab. \tag{6}$$

We will now introduce a moving average expression of $Z^H(a,b)$ for $H \in (1/2, 1)$.

Lemma 2.5. *For every $t \in [0, +\infty)$ and $H \in (1/2, 1)$,*

$$\begin{aligned} Z_t^H(a,b) \quad =: \quad & \frac{(H-1/2)}{2C(H)} \bigg[\int_{\mathbb{R}} \int_0^t (a+b) \Big(((u-s)_+)^{H-3/2} - ((u+s)_-)^{H-3/2} \Big) \\ + \quad & (a-b) \Big(((u-s)_+)^{H-3/2} + ((u+s)_-)^{H-3/2} \Big) du dM(s) \bigg] \end{aligned} \tag{7}$$

where M is the Brownian measure on $\mathbb{R}$. The constant $C(H)$ is defined as

$$C(H) = \left(\int_0^\infty \Big((1+x)^{H-1/2} - x^{H-1/2} \Big)^2 dx + \frac{1}{2H} \right)^{1/2},$$

and we use the notation $(x)_+ = \max(x,0)$ and $(x)_- = \max(-x,0)$.

Proof. Let us first recall that the tsfBm $\{B^H(t); t \in \mathbb{R}\}$ has the integral representation,

$$B^H(t) = \frac{1}{C(H)} \int_{\mathbb{R}} \Big(((t-s)_+)^{H-1/2} - ((-s)_+)^{H-1/2} \Big) dM(s), \quad t \in \mathbb{R}. \tag{8}$$

The proof of (8) can be found e.g. in [14].

Applying (4) and (8), we obtain, for every $t \in [0, +\infty)$,

$$\begin{aligned} Z_t^H(a,b) \quad =: \quad & \frac{1}{2C(H)} \left[\int_{\mathbb{R}} (a+b) \left(((t-s)_+)^{H-1/2} + ((t+s)_-)^{H-1/2} - 2\,((-s)_+)^{H-1/2} \right) \right. \\ + \quad & \left. (a-b) \left(((t-s)_+)^{H-1/2} - ((t+s)_-)^{H-1/2} \right) dM(s) \right]. \end{aligned} \tag{9}$$

This, together with the identities:

$$((t-s)_+)^{H-1/2} + ((t+s)_-)^{H-1/2} - 2\,((-s)_+)^{H-1/2} = \int_0^t ((u-s)_+)^{H-3/2} - ((u+s)_-)^{H-3/2}\, du$$

and

$$((t-s)_+)^{H-1/2} - ((t+s)_-)^{H-1/2} = \int_0^t ((u-s)_+)^{H-3/2} + ((u+s)_-)^{H-3/2}\, du,$$

for every $H \in (1/2, 1)$, complete the proof of Lemma 2.5. □

We will now delve deeper into the covariance function $R^{H,a,b}$.

Lemma 2.6. *Let $H \in (1/2, 1)$ and $\alpha_H = H(2H-1)$. For every $(a,b) \in \mathbb{R}^2$ we have:*

$$\frac{\partial^2 R^{a,b,H}(s,t)}{\partial s \partial t} = \alpha_H \left[(a^2+b^2)|t-s|^{2H-2} - 2ab(t+s)^{2H-2} \right]. \tag{10}$$

Moreover, for every $s, t \in [0,T]$ with $s \neq t$, we have

$$C_1 |t-s|^{2H-2} \leq \frac{\partial^2 R^{a,b,H}(s,t)}{\partial s \partial t} \leq C_2 |t-s|^{2H-2}, \tag{11}$$

with the constants given by

$$C_1 = \min(\alpha_H(a^2+b^2), \alpha_H(a-b)^2) \quad \text{and} \quad C_2 = \alpha_H(|a|+|b|)^2.$$

Proof. The explicit expression of $\frac{\partial^2 R^{H,a,b}}{\partial s \partial t}$ can be easily obtained. We will just prove the two stated estimates (11).

Since $|t-s| \leq (t+s)$ and $x \longmapsto x^{2H-2}$ is decreasing, we have

$$(t+s)^{2H-2} \leq |t-s|^{2H-2},$$

and consequently

$$\frac{\partial^2 R^{H,a,b}(s,t)}{\partial s \partial t} \leq \alpha_H(|a|+|b|)^2 |t-s|^{2H-2}.$$

For the lower bound, if $ab < 0$ then,

$$\alpha_H \left[(a^2+b^2)|t-s|^{2H-2} - 2ab(t+s)^{2H-2} \right] \geq \alpha_H(a^2+b^2)|t-s|^{2H-2}.$$

And if $ab \geq 0$ then, we can write

$$\frac{\partial^2 R^{H,a,b}(s,t)}{\partial s \partial t} = \alpha_H \left[(a-b)^2 |t-s|^{2H-2} + 2ab \left[|t-s|^{2H-2} - (t+s)^{2H-2} \right] \right]$$

which clearly implies that

$$\frac{\partial^2 R^{H,a,b}(s,t)}{\partial s \partial t} \geq \alpha_H(a-b)^2 |t-s|^{2H-2}.$$

□

Corollary 2.7. *For $1/2 < H < 1$, the ZgfBm process admits a covariance measure $\mu_{a,b,H}$ on $[0,T]^2$ with the density given by*

$$d\mu_{a,b,H}(u,v) = \frac{\partial^2 R^{a,b,H}(u,v)}{\partial u \partial v} du dv.$$

Proof. From the estimates in Lemma 2.6, we have that for $s,t \in [0,T]$ with $s \neq t$,

$$\left| \frac{\partial^2 R^{a,b,H}(s,t)}{\partial s \partial t} \right| \leq C_2 |t-s|^{2H-2}.$$

Since $H > 1/2$, we have $2H-2 > -1$. Therefore, the function $|t-s|^{2H-2}$ is integrable on $[0,T]^2$, and consequently, $\frac{\partial^2 R^{a,b,H}(s,t)}{\partial s \partial t} \in L^1([0,T]^2)$, which is a sufficient condition for the existence of the covariance measure $\mu_{a,b,H}$ (see [7, 20]). □

3. Stochastic partial differential Equation

For the remainder of this paper, we assume that the Hurst parameter H satisfies $H > 1/2$. As a consequence of Corollary 2.7, this assumption ensures that the noise ZgfBm has a covariance measure structure. This property is crucial for the existence and well-definedness of the Wiener integral, which serves as the primary tool for analyzing the mild solution to the stochastic partial differential equation below.

Let $d \geq 1$. The aim of this paper is to study the stochastic partial differential equation

$$\begin{cases} \frac{\partial u_{a,b,H}}{\partial t} &= \frac{1}{2}\Delta u_{a,b,H} + \dot{W}_{a,b,H}, \; t \in (0,T], \; x \in \mathbb{R}^d \\ u_{a,b,H}(x,0) &= 0, \end{cases} \tag{12}$$

where $\dot{W}_{a,b,H}$ is the formal derivative of the centered random noise $W_{a,b,H} = \{W_{a,b,H}(t,A); t \in [0,T], A \in \mathcal{B}_b(\mathbb{R}^d)\}$, with covariance given by:

$$\mathbb{E}(W_{a,b,H}(t,A)W_{a,b,H}(s,B)) = R^{H,a,b}(t,s)\lambda(A \cap B), \tag{13}$$

where λ is the Lebesgue measure, and $R^{a,b,H}$ is the covariance function of $Z^H(a,b)$, defined by (3).

The canonical Hilbert space associated to the noise $W_{a,b,H}$ is defined as follows. First, consider $\mathcal{E}$ the set of linear combinations of elementary functions $\mathbf{1}_{[0,t]} \times A$, $t \in [0,T]$, $A \in \mathcal{B}_b(\mathbb{R}^d)$, and $\mathcal{H}_{a,b,H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the inner product

$$< \mathbf{1}_{[0,t]} \times A, \mathbf{1}_{[0,s]} \times B >_{\mathcal{H}_{a,b,H}} := \mathbb{E}(W_{a,b,H}(t,A)W_{a,b,H}(s,B)).$$

We have for $g, h \in \mathcal{H}_{a,b,H}$, smooth enough,

$$< g,h >_{\mathcal{H}_{a,b,H}} = \int_0^T \int_0^T du dv \int_{\mathbb{R}^d} dy \frac{\partial^2 R^{a,b,H}}{\partial u \partial v}(u,v) g(y,u) h(y,v). \tag{14}$$

By a routine extension of the construction described, for example in [17] and [20], it is possible to define Wiener integrals with respect to the process $W_{a,b,H}$. This Wiener integral will act as an isometry between the Hilbert space $\mathcal{H}_{a,b,H}$ and $L^2(\Omega)$ by

$$\mathbb{E} \int_0^T \int_{\mathbb{R}^d} \varphi(u,y) W_{a,b,H}(du,dy) \int_0^T \int_{\mathbb{R}^d} \psi(u,y) W_{a,b,H}(du,dy) = \int_0^T \int_0^T \mu_{a,b,H}(du,dv) \int_{\mathbb{R}^d} dy \varphi(u,y)\psi(v,y) \tag{15}$$

for any function φ, ψ such that

$$\int_0^T \int_0^T d \mid \mu_{a,b,H} \mid (u,v) \int_{\mathbb{R}^d} dy \mid \varphi(u,y) \mid \mid \varphi(v,y) \mid < \infty$$

and

$$\int_0^T \int_0^T d \mid \mu_{a,b,H} \mid (u,v) \int_{\mathbb{R}^d} dy \mid \psi(u,y) \mid \mid \psi(v,y) \mid < \infty,$$

where $\mu_{a,b,H}$ is the measure defined by

$$d\mu_{a,b,H}(u,v) = \frac{\partial^2 R^{a,b,H}}{\partial u \partial v}(u,v) du dv, \tag{16}$$

and $|\mu_{a,b,H}|$ denotes the total variation measure associated to $\mu_{a,b,H}$.

The following transfer formula will be useful in the sequel.

Proposition 3.1. *For every $g \in \mathcal{H}_{a,b,H}$ we have*

$$\int_0^T \int_{\mathbb{R}^d} g(s,y) dW_{a,b,H}(s,y) = d_H \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{\mathbb{R}} \mathbf{1}_{(0,T)}(u) g(u,y) K_{a,b,H}(u,s) du dW(s,y), \tag{17}$$

where $d_H = \dfrac{(H-1/2)}{2C(H)}$, W is a space-time white noise with covariance

$$\mathbb{E}(W(s,A)W(t,B)) = (t \wedge s)\lambda(A \cap B),$$

and

$$K_{a,b,H}(u,s) = (a+b)\left(((u-s)_+)^{H-3/2} - ((u+s)_-)^{H-3/2}\right) + (a-b)\left(((u-s)_+)^{H-3/2} + ((u+s)_-)^{H-3/2}\right).$$

Proof. Proposition 3.1 is straightforward consequence of the moving average expression of the ZgfBm (7) □

Now we will define the mild solution of the SPDE (12).

Definition 3.2. *If we denote by $\{u_{a,b,H}(t,x); t \in [0,T], x \in \mathbb{R}^d\}$, the process defined by*

$$u_{a,b,H}(t,x) := \int_0^t \int_{\mathbb{R}^d} G(t-v, x-y) W_{a,b,H}(dv, dy), \tag{18}$$

where the above integral is a Wiener integral with respect to the noise $W_{a,b,H}$ and G is the Green kernel of the heat equation given by:

$$G(t,x) = \begin{cases} \dfrac{1}{(2\pi t)^{\frac{d}{2}}} \exp\left(-\dfrac{\|x\|^2}{2t}\right) & \text{if} \quad t > 0, x \in \mathbb{R}^d, \\ 0 & \text{if} \quad t \leq 0, x \in \mathbb{R}^d \end{cases} \tag{19}$$

where $\|x\|$ denotes the Euclidean norm of the vector x in $\mathbb{R}^d$, then the process u is called the mild solution of the SPDE (12).

In the following proposition, we will give a necessary and sufficient condition for existence of the mild solution to equation (12).

Proposition 3.3. *The solution to the generalized fractional heat equation exists if and only if $d < 4H$.*

Proof. By (15) and (18),

$$\mathbb{E}\Big(u_{a,b,H}(t,x)^2\Big) = \int_0^t \int_0^t \mu_{a,b,H}(du,dv) \int_{\mathbb{R}^d} dy G(t-u,x-y)G(t-v,x-y). \tag{20}$$

It follows from (18) and (11) that

$$C_1 I(t,x) \leq \mathbb{E}\Big(u_{a,b,H}(t,x)^2\Big) \leq C_2 I(t,x) \tag{21}$$

where

$$I(t,x) = \int_0^t \int_0^t dudv \mid u-v \mid^{2H-2} \int_{\mathbb{R}^d} dy G(t-u,x-y)G(t-v,x-y). \tag{22}$$

Using (19) and making a suitable change of variables, we obtain

$$\int_{\mathbb{R}^d} dy G(t-u,x-y)G(t-v,x-y) = \left(\frac{\pi}{2(2t-u-v)}\right)^{d/2}. \tag{23}$$

Hence, from (21), (22) and (23), we deduce that the solution to Equation (12) exists if and only if

$$\int_0^t \int_0^t dudv \mid u-v \mid^{2H-2} (2t-u-v)^{-d/2} < \infty. \tag{24}$$

And we easliy check that (24) is true if, and only if $d < 4H$. □

Throughout the rest of this paper, we will assume that

$$d < 4H. \tag{25}$$

In the following proposition we will give an explicit expression of the covariance of the mild solution.

Proposition 3.4. *For fixed $x \in \mathbb{R}^d$, and for $s \leq t$,*

$$\mathbb{E}\left[u_{a,b,H}(t,x)u_{a,b,H}(s,x)\right] = \left(\frac{\pi}{2}\right)^{d/2} \alpha_H \int_0^t \int_0^s dudv \frac{(a^2+b^2)|u-v|^{2H-2} - 2ab(u+v)^{2H-2}}{(t+s-u-v)^{d/2}}. \tag{26}$$

Proof. By applying the Isometry identity (15), then using (23) we obtain

$$\begin{aligned}
\mathbb{E}\left[u_{a,b,H}(t,x)u_{a,b,H}(s,x)\right] &= \int_0^t \int_0^s \mu_{a,b,H}(du,dv) \int_{\mathbb{R}^d} dy G(t-u,x-y)G(s-v,x-y) \\
&= \int_0^t \int_0^s dudv \frac{\partial^2 R^{a,b,H}(u,v)}{\partial u \partial v} \left(\frac{\pi}{2(t+s-u-v)}\right)^{d/2}.
\end{aligned}$$

This with (10) allow us to conclude the proof of the proposition. □

The following proposition deals with the mixed-self-similarity of the solution sample paths.

Proposition 3.5. *The process $u_{a,b,H} : t \longmapsto u_{a,b,H}(t,x)$ is mixed-self-similar of order $H - \frac{d}{4}$, in the sense that, for every $h > 0$, the processes $(u_{a,b,H}(ht,x))_{t \in \mathbb{R}_+}$ and $(u_{ah^{H-\frac{d}{4}},bh^{H-\frac{d}{4}},H}(t,x))_{t \in \mathbb{R}_+}$ have the same law.*

Proof. For fixed $h > 0$, the processes $(u_{a,b,H}(ht,x))_{t\in\mathbb{R}_+}$ and $(u_{ah^{H-\frac{d}{4}},bh^{H-\frac{d}{4}},H}(t,x))_{t\in\mathbb{R}_+}$ are Gaussian and centered. Consequently, to prove the desired result, it is sufficient to demonstrate that they possess the same covariance function. Indeed, as a direct consequence of Proposition 3.4, we have

$$\mathbb{E}\left[u_{a,b,H}(ht,x)u_{a,b,H}(hs,x)\right] = \left(\frac{\pi}{2}\right)^{d/2}\alpha_H \int_0^{ht}\int_0^{hs} dudv \frac{(a^2+b^2)|u-v|^{2H-2} - 2ab(u+v)^{2H-2}}{(ht+hs-u-v)^{d/2}}. \tag{27}$$

So, by the change of variables $u' = \frac{u}{h}$, $v' = \frac{v}{h}$ in the integral $dudv$ we immediatly get

$$\begin{aligned} \mathbb{E}\left[u_{a,b,H}(ht,x)u_{a,b,H}(hs,x)\right] &= \left(\frac{\pi}{2}\right)^{d/2}\alpha_H\, h^{2H-d/2}\int_0^{t}\int_0^{s} dudv \frac{(a^2+b^2)|u-v|^{2H-2} - 2ab(u+v)^{2H-2}}{(t+s-u-v)^{d/2}} \\ &= \mathbb{E}\left[u_{ah^{H-\frac{d}{4}},bh^{H-\frac{d}{4}},H}(t,x)\, u_{ah^{H-\frac{d}{4}},bh^{H-\frac{d}{4}},H}(s,x)\right], \end{aligned} \tag{28}$$

which completes the proof of Proposition 3.5. □

4. Regularity and fractal properties

4.1. Study of the regularity of the solution in time

In this section, we will examine the behavior of the increments of the solution $u_{a,b,H}(t,x)$ to (12) with respect to the time variable t. We will establish sharp upper and lower bounds for the L^2-norm of these increments.

The key result of this section is the following theorem.

Theorem 4.1. *There exist two positive constants C_3 and C_4 such that, for any $s,t\in[0,T]$ and for any $x\in\mathbb{R}^d$,*

$$C_3|t-s|^{2H-\frac{d}{2}} \le \mathbb{E}\left|u_{a,b,H}(t,x)-u_{a,b,H}(s,x)\right|^2 \le C_4|t-s|^{2H-\frac{d}{2}}.$$

Proof. We have

$$\mathbb{E}\left|u_{a,b,H}(t,x)-u_{a,b,H}(s,x)\right|^2 = R_u(t,t) - 2R_u(t,s) + R_u(s,s),$$

where R_u denotes the covariance of the process $u_{a,b,H}$ with respect to the time variable for fixed $x\in\mathbb{R}^d$

$$R_u(t,s) = \mathbb{E}\left[u_{a,b,H}(t,x)u_{a,b,H}(s,x)\right] = \left(\frac{\pi}{2}\right)^{d/2}\int_0^t\int_0^s dudv\frac{\partial^2 R^{a,b,H}(u,v)}{\partial u\partial v}(t+s-u-v)^{-d/2}$$

for every $s,t\in[0,T]$. So,

$$\mathbb{E}\left|u_{a,b,H}(t,x)-u_{a,b,H}(s,x)\right|^2 = \left(\frac{\pi}{2}\right)^{d/2}\Bigg(\int_0^t\int_0^t dudv\frac{\partial^2 R^{a,b,H}(u,v)}{\partial u\partial v}(2t-u-v)^{-d/2}$$
$$-2\int_0^t\int_0^s dudv\frac{\partial^2 R^{a,b,H}(u,v)}{\partial u\partial v}(t+s-u-v)^{-d/2} + \int_0^s\int_0^s dudv\frac{\partial^2 R^{a,b,H}(u,v)}{\partial u\partial v}(2s-u-v)^{-d/2}\Bigg),$$

which can also be written as

$$\mathbb{E}\left|u_{a,b,H}(t,x)-u_{a,b,H}(s,x)\right|^2 = \left(\frac{\pi}{2}\right)^{d/2}\left(A_{a,b,H}(t,s)+B_{a,b,H}(t,s)+C_{a,b,H}(t,s)\right),$$

$$A_{a,b,H}(t,s) = \int_s^t\int_s^t dudv\frac{\partial^2 R^{a,b,H}(u,v)}{\partial u\partial v}(2t-u-v)^{-d/2},$$

$$B_{a,b,H}(t,s) = \int_0^s \int_0^s dudv \frac{\partial^2 R^{a,b,H}(u,v)}{\partial u \partial v} \left[(2t-u-v)^{-d/2} - 2(t+s-u-v)^{-d/2} + (2s-u-v)^{-d/2} \right]$$

and

$$C_{a,b,H}(t,s) = 2 \int_s^t \int_0^s dudv \frac{\partial^2 R^{a,b,H}(u,v)}{\partial u \partial v} \left[(2t-u-v)^{-d/2} - 2(t+s-u-v)^{-d/2} \right].$$

Since, $C_{a,b,H}(t,s) \le 0$ and since

$$\frac{\partial^2 R^{a,b,H}(u,v)}{\partial u \partial v} \le C_2 \mid u-v \mid^{2H-2},$$

we have

$$\begin{aligned} \mathbb{E} \left| u_{a,b,H}(t,x) - u_{a,b,H}(s,x) \right|^2 \le c \Bigg[& \int_s^t \int_s^t dudv \mid u-v \mid^{2H-2} (2t-u-v)^{-d/2} \\ & + \int_0^s \int_0^s dudv \mid u-v \mid^{2H-2} \left((2t-u-v)^{-d/2} - 2(t+s-u-v)^{-d/2} + (2s-u-v)^{-d/2} \right) \Bigg], \end{aligned} \tag{29}$$

where c denote a positive constant. Consequently, by (25), (29) and Lemma 6.1 we get the upper bound.

Let us now prove the lower bound. The mild solution has the form given in Equation (18). So, for every $x \in \mathbb{R}^d$ and $(s,t) \in [0,T]^2$,

$$u_{a,b,H}(t,x) - u_{a,b,H}(s,x) = \int_0^T \int_{\mathbb{R}^d} \left(G(t-\sigma, x-y) \mathbf{1}_{(0,t)}(\sigma) - G(s-\sigma, x-y) \mathbf{1}_{(0,s)}(\sigma) \right) dW_{a,b,H}(\sigma, y). \tag{30}$$

By the transfer formula (17) we get:

$$u_{a,b,H}(t,x) - u_{a,b,H}(s,x) = d_H \int_{\mathbb{R}} \int_{\mathbb{R}^d} dW(\sigma,y) F_{t,s}(\sigma,y), \tag{31}$$

where

$$F_{t,s}(\sigma,y) = \int_{\mathbb{R}} du K_{a,b,H}(u,\sigma) \left[G(t-u, x-y) \mathbf{1}_{(0,t)}(u) - G(s-u, x-y) \mathbf{1}_{(0,s)}(u) \right]$$

with

$$K_{a,b,H}(u,\sigma) = (a+b) \left(((u-\sigma)_+)^{H-3/2} - ((u+\sigma)_-)^{H-3/2} \right) + (a-b) \left(((u-\sigma)_+)^{H-3/2} + ((u+\sigma)_-)^{H-3/2} \right).$$

Equation (31) implies that

$$\mathbb{E} \left[u_{a,b,H}(t,x) - u_{a,b,H}(s,x) \right]^2 = d_H^2 \mathbb{E} \left[\int_{\mathbb{R}} \int_{\mathbb{R}^d} dW(\sigma,y) F_{t,s}(\sigma,y) \right]^2. \tag{32}$$

By the isometry of the Wiener process W we get

$$\mathbb{E} \left[\int_{\mathbb{R}} \int_{\mathbb{R}^d} dW(\sigma,y) F_{t,s}(\sigma,y) \right]^2 = \int_{\mathbb{R}} \int_{\mathbb{R}^d} d\sigma dy F_{t,s}^2(\sigma,y) \ge \int_s^t \int_{\mathbb{R}^d} d\sigma dy F_{t,s}^2(\sigma,y). \tag{33}$$

Therefore, for every $\sigma \in [s,t]$ and $u \ge 0$, we have $(u+\sigma)_- = 0$,

$$K_{a,b,H}(u,\sigma) = (a+b)(u-\sigma)_+^{H-3/2} + (a-b)(u-\sigma)_+^{H-3/2} = 2a(u-\sigma)_+^{H-3/2},$$

and as consequence:

$$\begin{aligned} F_{t,s}(\sigma,y) &= \int_0^t du G(t-u, x-y) K_{a,b,H}(u,\sigma) - \int_0^s du G(s-u, x-y) K_{a,b,H}(u,\sigma) \\ &= 2a \left[\int_0^t du G(t-u, x-y)(u-\sigma)_+^{H-3/2} - \int_0^s du G(s-u, x-y)(u-\sigma)_+^{H-3/2} \right]. \end{aligned}$$

As $\sigma \in [s,t]$, the term $(u-\sigma)_+^{H-3/2}$ is only non-zero when $u > \sigma$. This means the second integral, $\int_0^s duG(s-u,x-y)(u-\sigma)_+^{H-3/2}$, is zero. So, the second integral vanishes, and the first integral simplifies to an integral over the interval $[\sigma,t]$. Therefore,

$$F_{t,s}(\sigma,y) = 2a\int_\sigma^t duG(t-u,x-y)(u-\sigma)^{H-3/2}.$$

Hence,

$$\mathbb{E}\left[\int_{\mathbb{R}}\int_{\mathbb{R}^d} dW(\sigma,y)F_{t,s}(\sigma,y)\right]^2 \geq 4a^2\int_s^t\int_{\mathbb{R}^d} d\sigma dy\left(\int_\sigma^t duG(t-u,x-y)(u-\sigma)^{H-3/2}\right)^2. \tag{34}$$

So, for every $s,t \in [0,T]$; $s \leq t$,

$$\begin{aligned}
&\mathbb{E}\left[\int_{\mathbb{R}}\int_{\mathbb{R}^d} dW(\sigma,y)F_{t,s}(\sigma,y)\right]^2 \\
\geq\ & 4a^2\int_s^t d\sigma\int_{\mathbb{R}^d} dy\int_\sigma^t\int_\sigma^t dvduG(t-u,x-y)(u-\sigma)^{H-3/2}G(t-v,x-y)(v-\sigma)^{H-3/2} \\
=\ & 4a^2\int_s^t du\int_s^t dv\int_{\mathbb{R}^d} dyG(t-u,x-y)G(t-v,x-y)\int_s^{u\wedge v}(u-\sigma)_+^{H-3/2}(v-\sigma)_+^{H-3/2}d\sigma
\end{aligned} \tag{35}$$

where in the last equality we have used the fact that

$$(s \leq \sigma \leq t,\ \sigma \leq u \leq t,\ \sigma \leq v \leq t) \Longleftrightarrow (s \leq u \leq t,\ s \leq v \leq t,\ s \leq \sigma \leq u \wedge v).$$

Equations (35) and (23) imply

$$\mathbb{E}\left[\int_{\mathbb{R}}\int_{\mathbb{R}^d} dW(\sigma,y)F_{t,s}(\sigma,y)\right]^2 \geq 4a^2\Big(\frac{\pi}{2}\Big)^{d/2}\int_s^t du\int_s^t dv(2t-u-v)^{-d/2}\int_s^{u\wedge v}(u-\sigma)_+^{H-3/2}(v-\sigma)_+^{H-3/2}d\sigma. \tag{36}$$

By the change of variable $z = \dfrac{u\wedge v-\sigma}{u\vee v-\sigma}$ and by an easy calculus we get

$$\int_s^{u\wedge v}(u-\sigma)^{H-3/2}(v-\sigma)^{H-3/2}d\sigma = \mid u-v\mid^{2H-2}\int_0^{\frac{u\wedge v-s}{u\vee v-s}}(1-z)^{1-2H}z^{H-3/2}dz. \tag{37}$$

Then, by (36) and (37),

$$\begin{aligned}
&\mathbb{E}\left[\int_{\mathbb{R}}\int_{\mathbb{R}^d} dW(\sigma,y)F_{t,s}(\sigma,y)\right]^2 \\
\geq\ & 4a^2\Big(\frac{\pi}{2}\Big)^{d/2}\int_s^t du\int_s^t dv(2t-u-v)^{-d/2}\mid u-v\mid^{2H-2}\int_0^{\frac{u\wedge v-s}{u\vee v-s}}(1-z)^{1-2H}z^{H-3/2}dz.
\end{aligned} \tag{38}$$

Now, by the change of variable $u-s=u'$ and $v-s=v'$ we obtain:

$$\begin{aligned}
&\mathbb{E}\left[\int_{\mathbb{R}}\int_{\mathbb{R}^d} dW(\sigma,y)F_{t,s}(\sigma,y)\right]^2 \\
\geq\ & 4a^2\Big(\frac{\pi}{2}\Big)^{d/2}\int_0^{t-s} du\int_0^{t-s} dv(2(t-s)-u-v)^{-d/2}\mid u-v\mid^{2H-2}\int_0^{\frac{u\wedge v}{u\vee v}}(1-z)^{1-2H}z^{H-3/2}dz.
\end{aligned} \tag{39}$$

Finally, by the change of variable $\tilde{u} = \frac{u}{t-s}$ and $\tilde{v} = \frac{v}{t-s}$ we get:

$$\mathbb{E}\left[\int_{\mathbb{R}}\int_{\mathbb{R}^d} dW(\sigma,y)F_{t,s}(\sigma,y)\right]^2 \geq D(d,H)(t-s)^{2H-\frac{d}{2}}, \tag{40}$$

where $D(d,H)$ is the constant defined by

$$D(d,H) = Cte\int_0^1 du\int_0^1 dv(2-u-v)^{-d/2} \mid u-v\mid^{2H-2}\int_0^{\frac{u\wedge v}{u\vee v}}(1-z)^{1-2H}z^{H-3/2}dz.$$

The constant $D(d,H)$ is clearly finite since $H>\frac{1}{2}$. □

Remark 4.2. *Proposition 4.1 establishes that the process $(u_{a,b,H}(.,x))$ is an infinite-dimensional quasi-helix (in the sense of [6]) with index $\kappa = H-\dfrac{d}{4}$. Quasi-helices possess a range of well-studied properties, as detailed in [6] .*

In particular, as an immediate consequence of Proposition 4.1, we get:

Corollary 4.3. *For any $x\in\mathbb{R}^d$, the process $t\to u_{a,b,H}(t,x)$ is Hölder continuous of order $\delta\in(0,H-\frac{d}{4})$.*

Proof. Proposition 4.1 yields the following inequality:

$$\mathbb{E}\left|u_{a,b,H}(t,x)-u_{a,b,H}(s,x)\right|^2 \leq C_4|t-s|^{2H-\frac{d}{2}},$$

for every $(s,t)\in\mathbb{R}_+^2; s\leq t$ and $(a,b)\in\mathbb{R}^2\setminus\{(0,0)\}$. In conjunction with the Kolmogorov continuity theorem (see e.g. Theorem 1.2, page 14 in [11]), this inequality yields Corollary 4.3. □

As a second consequence of Proposition 4.1, by proceeding as in the proof of Proposition 3.2 in [5], we get

Corollary 4.4. *For every $x\in\mathbb{R}^d$,*

$$\lim_{\epsilon\to 0}\sup_{t\in[t_0-\epsilon,t_0+\epsilon]}\left|\frac{u_{a,b,H}(t,x)-u_{a,b,H}(t_0,x)}{t-t_0}\right| = +\infty$$

with probability one for every t_0. And consequently, the trajectories of the process $u_{a,b,H}(.,x)$ are not differentiable.

4.2. Sharp regularity of the solution in space

In the spirit of [19], in this section we fix $t>0$ and analyze the space regularity of the solution $\{u_{a,b,H}(t,x), x\in\mathbb{R}^d\}$. We will first prove the following lemma.

Lemma 4.5. *The Gaussian random field $\{u_{a,b,H}(t,x), x\in\mathbb{R}^d\}$ is stationary with spectral measure*

$$\Delta(d\xi) = \alpha_H(2\pi)^{-d}\int_0^t\int_0^t\left[(a^2+b^2)|u-v|^{2H-2}-2ab(u+v)^{2H-2}\right]\exp(-\frac{(2t-u-v)}{2}\mid\xi\mid^2)d\xi.$$

Proof. By the Fourier transform of the Green kernel and Parseval's identity we get

$$\mathbb{E}\left[u_{a,b,H}(t,x)u_{a,b,H}(t,y)\right] = \int_0^t\int_0^t dudv\frac{\partial^2 R^{a,b,H}(u,v)}{\partial u\partial v}\int_{\mathbb{R}^d} dzG(t-u,x-z)G(t-v,y-z)$$
$$= (2\pi)^{-d}\int_0^t\int_0^t dudv\frac{\partial^2 R^{a,b,H}(u,v)}{\partial u\partial v}\int_{\mathbb{R}^d} d\xi\exp\left(-\frac{(t-u)}{2}|\xi|^2\right)\exp(i\langle x,\xi\rangle)\exp\left(-\frac{(t-v)}{2}|\xi|^2\right)\exp(-i\langle y,\xi\rangle),$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product on $\mathbb{R}^d$ defined by $\langle x, y \rangle = \sum_{i=1}^{d} x_i y_i$. Thus,

$$\mathbb{E}\left[u_{a,b,H}(t,x)u_{a,b,H}(t,y)\right] = \int_{\mathbb{R}^d} \exp(i\langle x-y, \xi\rangle)\left((2\pi)^{-d}\int_0^t\int_0^t dudv \frac{\partial^2 R^{a,b,H}(u,v)}{\partial u \partial v} \exp\left(-\frac{(2t-u-v)}{2}|\xi|^2\right)\right)d\xi$$
$$= \alpha_H (2\pi)^{-d}\int_{\mathbb{R}^d} \exp(i\langle x-y, \xi\rangle)\left(\int_0^t\int_0^t dudv \left[(a^2+b^2)|u-v|^{2H-2} - 2ab(u+v)^{2H-2}\right]\exp\left(-\frac{(2t-u-v)}{2}|\xi|^2\right)\right)d\xi.$$

In the last equality, we are used the expression for $\frac{\partial^2 R_{a,b,H}}{\partial s \partial t}$ given in (10). □

Corollary 4.6. *There exist two positive constants $c_1(t,H,a,b)$ and $c_2(t,H,a,b)$, depending only on t, H, a and b, such that:*

$$c_1(t,H,a,b) \mid \xi \mid^{-4H} d\xi \leq \Delta(d\xi) \leq c_2(t,H,a,b) \mid \xi \mid^{-4H} d\xi$$

for all $\xi \in \mathbb{R}^d$ with $\mid \xi \mid \geq 1$.

Proof. By Equation (11), we have

$$\begin{aligned} & C_1(2\pi)^{-d}\int_0^t\int_0^t \mid u-v \mid^{2H-2} dudv \exp(-\frac{(2t-u-v)}{2} \mid \xi \mid^2) d\xi \\ \leq \; & \Delta(d\xi) \\ \leq \; & C_2(2\pi)^{-d}\int_0^t\int_0^t \mid u-v \mid^{2H-2} dudv \exp(-\frac{(2t-u-v)}{2} \mid \xi \mid^2) d\xi, \end{aligned}$$

and by [1] (Propostion 4.3), there exist two strictly positive constants $c_{1,H}, c_{2,H}$ such that

$$c_{1,H}(t^{2H}\wedge 1)\Big(\frac{1}{1+\mid \xi \mid^2}\Big)^{2H} \leq \int_0^t\int_0^t dudv \mid u-v \mid^{2H-2} \exp(-\frac{(u+v)}{2} \mid \xi \mid^2) \leq c_{2,H}(t^{2H}\wedge 1)\Big(\frac{1}{1+\mid \xi \mid^2}\Big)^{2H}. \quad (41)$$

This allows to see the stated result. □

Corollary 4.6 means, among other things, that the spectral measure $\Delta(d\xi)$ is comparable with an absolutely continuous measure with density function that is comparable to $\mid \xi \mid^{-4H}$ for all $\xi \in \mathbb{R}^d$ with $\mid \xi \mid \geq 1$. This is very interesting for the study of the regularity of $\{u_{a,b,H}(t,x), x \in \mathbb{R}^d\}$. Indeed, as first consequence of Corollary 4.6, we get the folowing theorem.

Theorem 4.7. *If we denote $\beta = \min\{1, 2H - \frac{d}{2}\}$, $\rho = \begin{cases} 1 & if \quad \beta = 1 \\ 0 & otherwise \end{cases}$ and*

$$I(x,y) = \mathbb{E}\left[\mid u_{a,b,H}(t,x) - u_{a,b,H}(t,y) \mid^2\right],$$

then, for any $M > 0$, there exist positive and finite constants c_3, c_4 such that for any $x, y \in [-M, M]^d$,

$$c_3 \mid x-y \mid^{2\beta} \left(Log\frac{1}{\mid x-y \mid}\right)^{\rho} \leq I(x,y) \leq c_4 \mid x-y \mid^{2\beta} \left(Log\frac{1}{\mid x-y \mid}\right)^{\rho}. \quad (42)$$

Proof. Take $x, y \in [-M, M]^d$ and let $z := y - x \in \mathbb{R}^d$. By Parseval's identity, we can write

$$
\begin{aligned}
&\mathbb{E}\Big[\mid u_{a,b,H}(t,y)-u_{a,b,H}(t,x)\mid^2\Big]\\
&=\int_0^t\int_0^t \mu_{a,b,H}(du,dv)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} dy'\Big[G(t-u,x+z-y')-G(t-u,x-y')\Big]\Big[G(t-v,x+z-y')-G(t-v,x-y')\Big]\\
&=(2\pi)^{-d}\int_0^t\int_0^t \mu_{a,b,H}(du,dv)\int_{\mathbb{R}^d} d\xi\mathcal{F}\Big(G(t-u,x+z-.)-G(t-u,x-.)\Big)(\xi)\\
&\qquad\qquad\times\overline{\mathcal{F}}\Big(G(t-v,x+z-.)-G(t-v,x-.)\Big)(\xi)\\
&=(2\pi)^{-d}\int_0^t\int_0^t \mu_{a,b,H}(du,dv)\int_{\mathbb{R}^d} d\xi\exp\Big(-(2t-u-v)\frac{\mid\xi\mid^2}{2}\Big)(2-2\cos<\xi,z>),
\end{aligned}
$$

where in the last equality we have used that

$$
\mathcal{F}G(t,x-.)(\xi)=\exp\Big(i<x,\xi>-\frac{t\mid\xi\mid^2}{2}\Big)\mathbf{1}_{t>0}(\xi),\ \xi\in\mathbb{R}^d.
$$

Therefore,

$$
\mathbb{E}\Big[\mid u_{a,b,H}(t,x)-u_{a,b,H}(t,y)\mid^2\Big]=2(2\pi)^{-d}\int_{\mathbb{R}^d} d\xi(1-\cos<\xi,z>)\theta(t,\xi) \tag{43}
$$

where

$$
\theta(t,\xi)=\int_0^t\int_0^t dudv\frac{\partial^2 R_{a,b,H}}{\partial u\partial v}(u,v)\exp\Big(-(2t-u-v)\frac{\mid\xi\mid^2}{2}\Big).
$$

By Equation (11),

$$
\begin{aligned}
&C_1\int_0^t\int_0^t dudv\mid u-v\mid^{2H-2}\exp\Big(-(2t-u-v)\frac{\mid\xi\mid^2}{2}\Big)\\
\leq\ &\theta(t,\xi) \qquad\qquad (44)\\
\leq\ &C_2\int_0^t\int_0^t dudv\mid u-v\mid^{2H-2}\exp\Big(-(2t-u-v)\frac{\mid\xi\mid^2}{2}\Big)
\end{aligned}
$$

where C_1 and C_2 are two positive constants. Following the same lines as those of the proof of Theorem 4 in [19], we show that there exist two strictly positive constants C_5 and C_6 such that

$$
\begin{aligned}
&C_5\mid x-y\mid^{2\beta}\Big(Log\frac{1}{\mid x-y\mid}\Big)^{\rho}\\
\leq\ &\int_{\mathbb{R}^d}(1-\cos<\xi,z>)d\xi\int_0^t\int_0^t dudv\mid u-v\mid^{2H-2}\exp\Big(-(2t-u-v)\frac{\mid\xi\mid^2}{2}\Big) \qquad (45)\\
\leq\ &C_6\mid x-y\mid^{2\beta}\Big(Log\frac{1}{\mid x-y\mid}\Big)^{\rho}.
\end{aligned}
$$

Hence, the result is a straightforward consequence of Equations (43), (44) and (45). □

As direct consequence of Theorem 1 we get:

Corollary 4.8. *If $2H-\frac{d}{2}>1$, then $\{u_{a,b,H}(t,x), x\in\mathbb{R}^d\}$ has a modification (still denoted by the same notation) such that almost surely the sample function $x\longmapsto u_{a,b,H}(t,x)$ is continuously differentiable on $\mathbb{R}^d$. Moreover, for* *any $M > 0$, there exists a positive random variable K with all moments such that for every $j = 1, ..., d$, the partial derivative $\frac{\partial}{\partial x_j} u_{a,b,H}(t,x)$ has the following modulus of continuity on $[-M,M]^d$:*

$$\sup_{x,y\in[-M,M]^d, |x-y|\le\epsilon} \left|\frac{\partial}{\partial x_j} u_{a,b,H}(t,x) - \frac{\partial}{\partial y_j} u_{a,b,H}(t,y)\right| \le K\epsilon^{2H-\frac{d}{2}-1}\sqrt{Log\frac{1}{\epsilon}}. \tag{46}$$

Proof. With Equation (11), we come back to apply exactly the same steps of the proof of Theorem 5 in [19]. □

By Lemma 3 and Equation (11), and with the results of [23] we obtain the following result corresponding to the case where $2H - \frac{d}{2} < 1$.

Lemma 4.9. *Suppose $2H - \frac{d}{2} < 1$. Then, for every fixed $t > 0$, the Gaussian field $\{u_{a,b,H}(t,x), x \in \mathbb{R}^d\}$ is strongly locally nondeterministic. Namely, for every $M > 0$, there exits a constant $C_7 > 0$ (depending on t and M) such that for every $n \ge 1$ and for every $x, y_1, ..., y_n \in [-M,M]^d$,*

$$Var(u_{a,b,H}(t,x)|u_{a,b,H}(t,y_1), ..., u_{a,H}(t,y_n)) \ge C_7 \min_{0\le j\le n}\{| x - y_j |^{4H-d}\},$$

where $y_0 = 0$.

As a consequence of this lemma, and by [[10], Theorems 4.1 and 5.1], we get the following uniform and local moduli of continuity characteristic.

Corollary 4.10. *Suppose $2H - \frac{d}{2} < 1$. Let $t > 0$ and $M > 0$ be fixed. Then, if we denote $\beta = 2H - \frac{d}{2}$, we have almost surely*

$$\lim_{\epsilon\to 0} \frac{\max_{x\in[-M,M]^d, |h|\le\epsilon} | u_{a,b,H}(t,x+h) - u_{a,b,H}(t,x) |}{\epsilon^{\beta}\sqrt{LogLog(1/\epsilon)}} = C_8.$$

For $x_0 \in \mathbb{R}^d$,

$$\limsup_{\epsilon\to 0} \frac{\max_{|h|\le\epsilon} | u_{a,b,H}(t,x_0+h) - u_{a,b,H}(t,x_0) |}{\epsilon^{\beta}\sqrt{LogLog(1/\epsilon)}} = C_9,$$

where C_8 and C_9 are positive constants.

Now by Lemma 4.9 and [8] we get this Chung's LIL characteristic:

Corollary 4.11. *Suppose $2H - \frac{d}{2} < 1$. For every $t > 0$ and $x_0 \in \mathbb{R}^d$,*

$$\liminf_{\epsilon\to 0} \frac{\max_{|h|\le\epsilon} | u_{a,b,H}(t,x_0+h) - u_{a,b,H}(t,x_0) |}{\epsilon^{\beta}(LogLog(1/\epsilon))^{-\beta}} = C_{10},$$

where C_{10} is a positive constant.

4.3. Fractal characteristics of the sample paths

For fixed $x \in \mathbb{R}^d$, we denote the range of the restriction of the process $u_{a,b,H}(.,x)$ on $[0,T]$ by

$$u_{a,b,H}([0,T],x) = \{u_{a,b,H}(t,x); t \in [0,T]\}, \tag{47}$$

and its graph by

$$Grf_T u_{a,b,H}(.,x) = \{(t, u_{a,b,H}(t,x)); t \in [0,T]\}, \tag{48}$$

where $T > \epsilon > 0$. The aim of this paragraph is to study Hausdorff and Packing dimensions of the sets defined just above. These dimensions have been extensively used in describing thin sets and fractals. We only recall briefly their definitions. The Hausdorff dimension of a set $E \subset \mathbb{R}^d$ is defined by

$$dim_H E = \inf\{\alpha > 0; M^\alpha(E) = 0\} = \sup\{\alpha > 0; M^\alpha(E) = +\infty\}, \tag{49}$$

where, for α >0, $M^\alpha(E)$ denotes the $\alpha-$ dimensional Hausdorff measure of E, defined by

$$M^\alpha(E) = \lim_{\delta \to 0} \inf\Big\{ \mid E \mid^\alpha; E \subset \bigcup_{k=1}^\infty E_k; \mid E_k \mid < \delta\Big\}, \tag{50}$$

where $\mid E_k \mid$ is the diameter of the set E_k and the inifinimum is taken over all coverings $(E_k)_{k\in\mathbb{N}}$ of E.

The packing dimension of a bounded set $F \subset \mathbb{R}^d$ is defined by:

$$dim_P F = \inf\Big\{ \sup_n \overline{dim_B} F_n : F \subset \bigcup_{n=1}^\infty F_n \Big\}. \tag{51}$$

where, $\overline{dim_B} F_n$ is the *upper box-counting dimension* of F_n defined by

$$\overline{dim_B} F_n = \lim_{\epsilon \to 0} sup \frac{log N(F_n, \epsilon)}{-log\epsilon}, \tag{52}$$

and for any $\epsilon > 0$, $N(F_n, \epsilon)$ is the smallest number of balls of radius ϵ [in Euclidean metric] needed to cover F_n.

Among the properties of such dimensions we recall that, for any bounded set $F \subset \mathbb{R}^d$,

$$dim_H F \leq dim_P F \leq \overline{dim_B} F \leq d. \tag{53}$$

For more information on Hausdorff and Packing dimensions see [4].

Let us start our study by the set $Grf_T u_{a,b,H}(.,x)$. Throughout all the sequel of the paper, c denotes a generic positive constant that may be different from line to line.

Lemma 4.12. *For any $T > 0$, with probability* 1,

$$dim_H Grf_T u_{a,b,H}(.,x) = dim_P Grf_T u_{a,b,H}(.,x) = 2 - H + \frac{d}{4}.$$

Proof. By Corollary 4.3, for any $T > 0$ and $x \in \mathbb{R}^d$, $u_{a,b,H}(.,x)$ has a modification which sample-paths have a Hölder continuity, with order $\gamma < H - \frac{d}{4}$ on the interval $[0,T]$. So by Lemmas 2.1 and 2.2 in [22], for any $T > 0$, with probability 1,

$$dim_H Grf_T u_{a,bH}(.,x) \leq 2 - H + \frac{d}{4} \quad \text{and} \quad dim_P Grf_T u_{a,bH}(.,x) \leq 2 - H + \frac{d}{4}.$$

Now, in order to get the lower bound, by (53) and by the Frostman's Theorem (see e.g. [4]), we only need to show that for any $T > 0$, the occupation measure ν of $t \longmapsto (t, u_{a,b,H}(t,x))$, when t is restricted to the interval $[0, T]$, has, with probability 1, a finite $\gamma-$dimensional energy, for any $\gamma \in \left(1, 2 - H + \frac{d}{4}\right)$. More precisely, for any Borel set $A \subset \mathbb{R}^2$, $\nu(A)$ is defined as the integral

$$\nu(A) = \int_0^T \mathbf{1}_{\{(t,u_{a,b,H}(t,x))\in A\}} dt, \tag{54}$$

where, for every set $V \subset \mathbb{R}^2$, $\mathbf{1}_V$ denotes the characteristic function of the set V, and we need to prove that with probability 1 the integral

$$\int_{Grf_T u_{a,b,H}(.,x)} \int_{Grf_T u_{a,b,H}(.,x)} \mid x - y \mid^{-\gamma} \nu(dx)\nu(dy) \tag{55}$$

is finite. By a monotone class argument this is easily seen to be equivalent to

$$\int_0^T \int_0^T (\mid s - t \mid + \mid u_{a,b,H}(s,x) - u_{a,b,H}(t,x) \mid)^{-\gamma} \, dsdt < +\infty. \tag{56}$$

In order to get (56), it suffices to show

$$\int_0^T \int_0^T \mathbb{E}\left(((\mid s - t \mid + \mid u_{a,b,H}(s,x) - u_{a,b,H}(t,x) \mid)^{-\gamma}\right) dsdt < +\infty. \tag{57}$$

Since the process $u_{a,b,H}(., x)$ is centered Gaussian, we easly check that for all $(s,t) \in \mathbb{R}^2, s \neq t$ and for every real $\gamma > 1$, we have

$$\mathbb{E}\left((\mid s - t \mid + \mid u_{a,b,H}(t,x) - u_{a,bH}(s,x) \mid)^{-\gamma}\right) \leq C_{11} \mid t - s \mid^{1-\gamma} \sigma^{-1}_{a,b,H,x}(s,t), \tag{58}$$

where

$$\sigma^2_{a,b,H,x}(s,t) = \mathbb{E}\left[u_{a,b,H}(t,x) - u_{a,b,H}(s,x)\right]^2$$

and C_{11} is a positive constant.

Now, by (58) and by Proposition 4.1, we get

$$\begin{aligned} & \int_0^T \int_0^T \mathbb{E}\Big(\mid s - t \mid + \mid u_{a,b,H}(s,x) - u_{a,b,H}(t,x) \mid^{-\gamma} \Big) dsdt \\ \leq \quad & C_{11} \int_0^T \int_0^T \mid t - s \mid^{1-\gamma} \sigma^{-1}_{a,b,H,x}(s,t) dsdt \\ \leq \quad & C_{12} \int_0^T \int_0^T \mid t - s \mid^{1+\frac{d}{4}-H-\gamma} \, dsdt \end{aligned}$$

where C_{12} is a positive constant. And since $\gamma \in (1, 2 - H + \frac{d}{4})$ the last double integral is finite, which achieves the proof. □

In the following lemma, we will give the Hausdorff and Packing dimensions of the set $u_{a,b,H}([0,T], x)$.

Lemma 4.13. *For any $T > 0$, with probability* 1,

$$dim_H u_{a,b,H}([0,T], x) = 1 \;\text{ and }\; dim_P u_{a,b,H}([0,T], x) = 1.$$

Proof. By Lemmas 2.1 and 2.2 in [22], we clearly have

$$dim_H u_{a,b,H}([0,T],x) \leq 1 \text{ and } dim_P u_{a,b,H}([0,T],x) \leq 1 \ a.s..$$

So, by (53), we only need to prove that

$$1 \leq dim_H \ u_{a,b,H}([0,T],x) \ a.s.$$

Note that for $\epsilon \in (0,T)$,

$$dim u_{a,b,H}([0,T],x) \geq dim u_{a,b,H}([\epsilon,T],x),$$

and that for any standard normal variable X and $0 < \gamma < 1$, we have

$$\mathbb{E}\,[|\ X\ |^{-\gamma}] < \infty. \tag{59}$$

Hence by Frostman's theorem (see e.g. [4]), it is sufficient to show that for all $0 < \gamma < 1$,

$$E_\gamma = \int_\epsilon^T \int_\epsilon^T \mathbb{E}\,[|\ u_{a,b,H}(s,x) - u_{a,b,H}(t,x)\ |^{-\gamma}]\, dsdt < +\infty. \tag{60}$$

By Proposition 8 and (60), there exists a positive and finite constant C_{13} such that

$$E_\gamma \leq C_{13} \int_\epsilon^T \int_\epsilon^T |\ s-t\ |^{-\gamma(H-\frac{d}{4})}\, dsdt. \tag{61}$$

Since $0 < \gamma(H - \dfrac{d}{4}) < 1$, the second member of the inequality (61) is finite and we get the result. □

5. Conclusion

This paper has presented a comprehensive analysis of a new stochastic heat equation driven by a generalized fractional Brownian motion (ZgfBm). By leveraging the versatile nature of ZgfBm, which simultaneously extends both fractional Brownian motion (fBm) and sub-fractional Brownian motion (sfBm), we have established a more general framework for studying stochastic partial differential equations.

Our findings have revealed the existence and mixed-self-similarity properties of the mild solution to the equation. Additionally, we have investigated the regularity of the solution in both time and space, as well as the Hausdorff and Packing dimensions of its sample paths' graphs and ranges.

These results contribute to a deeper understanding of stochastic heat equations driven by non-standard noise processes and provide valuable insights for modeling complex physical phenomena with heterogeneous time-dependent characteristics. Future research directions may include:

- Parameter estimation and inference: Developing statistical methods to estimate the parameters of the ZgfBm (a, b, and H) from observed data. This would enable real-world applications of the model.

- Investigating different boundary conditions: Exploring the impact of various boundary conditions on the solution's behavior and properties.

- Examining the long-time behavior of the solutions: Analyzing the asymptotic properties of the solution as time tends to infinity, which can provide insights into the system's stability and long-term dynamics.

By addressing these future research directions, we can further expand the applicability and understanding of stochastic heat equations driven by the Generalized fractional Gaussian noise introduced in this paper.

6. Appendix

The following technical lemma plays a crucial role in Section 4.

Lemma 6.1. *For every $\gamma \in (d/4, 1)$, there exist two positive constants $c_1(d,\gamma)$ and $c_2(d,\gamma)$, depending only on d and γ, such that, for every $s, t \in [0, T]$,*

1. $\int_s^t \int_s^t dudv \mid u - v \mid^{2\gamma-2} (2t - u - v)^{-d/2} = c_1(d,\gamma) \mid t - s \mid^{2\gamma-d/2}$.
2. $\int_0^s \int_0^s dvdu \mid u - v \mid^{2\gamma-2} \left[(2t - u - v)^{-d/2} - 2(t + s - u - v)^{-d/2} + (2s - u - v)^{-d/2}\right] \leq c_2(d,\gamma) \mid t - s \mid^{2\gamma-d/2}$.

Proof. The proofs of both assertions use in their first stage the change of variables $u' = t - u, v' = s - v$ and then $u' = \frac{u}{t-s}, v' = \frac{v}{t-s}$. For the second assertion, we use also the fact that the integral

$$\int_0^\infty \int_0^\infty dudv \mid u - v \mid^{2\gamma-2} \left[(2 + u + v)^{-d/2} - (1 + u + v)^{-d/2} + (u + v)^{-d/2}\right]$$

is finite. □

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