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Left dual (b,c)-core inverses in rings

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Abstract. Let R be a ring with an involution $* : R \to R$ satisfying $(x^*)^* = x$ and $(xy)^* = y^*x^*$ for all $x, y \in R$, and let $a, b, c \in R$. We call a *left dual* (b, c)*-core invertible* if there exists $x \in Rc$ such that $bxab = b$ and $(xab)^* = xab$. Such an x is called a left dual (b, c)-core inverse of a. In this paper, characteriztions of left dual (b, c)-core invertible element are introduced. We characterize left dual (b, c)-core inverses in terms of properties of the left annihilators and ideals. Moreover, we prove that a is left dual (b, c)-core invertible if and only if a is left (b, c) invertible and b is {1,4} invertible. Also, properties of left dual (b, c)-core invertible elements are examined. We present the matrix representations of left dual (b, c)-core inverses by the Pierce decomposition. Furthermore, relations between left dual (b, c)-core inverses and the other generalized inverses are given.

1. Introduction

The Moore-Penrose inverse [13] and the Drazin inverse [6] are two important classes of generalized inverses. After those, the inverse along an element [11] and the (b, c)-inverse [7] which recover the Moore-Penrose and Drazin inverses were introduced. There are several researches about those inverses (see [2], [3], [4], [12]). Shortly afterwards, one-sided inverses along an element were given [16]. Later, one-sided (b, c)-inverses were introduced by Drazin [8] which extend one-sided inverses along an element and (b, c)-inverses.

A map $* : R \to R$ is an *involution of* R if it satisfies $(x^*)^* = x$, $(xy)^* = y^*x^*$ and $(x + y)^* = x^* + y^*$ for all $x, y \in R$. A $*$-ring is a ring R together with an involution on R. Throughout this paper, any ring R is assumed to be a unital $*$-ring unless otherwise specified.

The Moore-Penrose invertible elements are presented in [13]. An element $a \in R$ is said to be *Moore-Penrose invertible* if there exists some $x \in R$ such that $axa = a$, $xax = x$, $(ax)^* = ax$ and $(xa)^* = xa$. Such an x is called a Moore-Penrose inverse of a. It is unique if it exists and is denoted by $a^\dagger$.

Generally, any solution x satisfying the equations $axa = a$ and $(ax)^* = ax$ (respectively, $(xa)^* = xa$) is called a {1,3}-inverse (respectively, a {1,4}-inverse) of a. The symbols $a^{(1,3)}$ and $a^{(1,4)}$ denote a {1,3}-inverse and a

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{1,4}-inverse of a, respectively. We denote by $a\{1,3\}$ and $a\{1,4\}$, the set of all {1,3}-inverses and {1,4}-inverses of a, respectively. In general, the set of all {1,3}-invertible, {1,4}-invertible and Moore-Penrose invertible elements in R will be denoted by $R^{\{1,3\}}$, $R^{\{1,4\}}$ and $R^{\dagger}$, respectively. It is known that a is Moore-Penrose invertible if and only if it is both {1,3}-invertible and {1,4}-invertible.

In 2018, in a $*$-ring, the pseudo core invertible and the dual pseudo core invertible elements are introduced in [9].

In 2023, Zhu introduced the (b,c)-core and the dual (b,c)-core inverses in the context of $*$-semigroups which generalize the core inverse [1], the core-EP inverse [10], the Moore-Penrose inverse, the right and left w-core inverse [17, 18].

Also, in 2023, Zhu et al. introduced right w-core invertible and left dual v-core invertible elements in a $*$-ring.

Recently, Dong et al. introduced the right (b,c)-core inverses in a $*$-ring, [5].

Motivated by the works on the concepts mentioned above it is of interest to investigate the dual concept of the right (b,c)-core inverses. In this regard, we introduce the concept of the left dual (b,c)-core inverses in a $*$-ring. We give its relations with other classes of generalized inverses. This offers a contribution to the theory of generalized inverses. Let $a,b,c \in R$ where R is a $*$-ring. We call a *left dual (b,c)-core invertible* if there exists $x \in Rc$ such that $bxab = b$ and $(xab)^* = xab$. Such an x is called a left dual (b,c)-core inverse of a and denoted by $a_{l,\textcircled{\#}(b,c)}$.

In Section 3, several fundamental results for characterizing left dual (b,c)-core inverses are established and it is shown that a is left dual (b,c)-core invertible if and only if a is left (b,c) invertible and b is {1,4} invertible. Moreover, $a_{l,\textcircled{\#}(b,c)} = b^{(1,4)}a_l^{(b,c)}$. Then we characterize left dual (b,c)-core inverses in terms of the properties of the left annihilators and ideals. Further, we present the matrix representations of left dual (b,c)-core inverses by the Pierce decomposition.

After that, in Section 4, we show that several generalized inverses, such as left inverses, left dual core inverses, left dual pseudo core inverses, left dual v-core inverses, and Moore-Penrose inverses, are special cases of left dual (b,c)-core inverses. Precisely, for any nonnegative integers m and n satisfying $m+n=1$, we establish the following equivalences in a ring:

1. a is left invertible if and only if a is left dual $(1,1)$-core invertible.
2. a is left dual pseudo core invertible if and only if a^m is left dual (a^k,a^n)-core invertible for some positive integer k.
3. a is left dual core invertible if and only if a^m is left dual (a,a^n)-core invertible.
4. a is left dual v-core invertible if and only if v is left dual (a,a)-core invertible.
5. a is Moore-Penrose invertible if and only if a is dual left (a^*,a^*)-core invertible if and only if a^* is dual left (a,a)-core invertible.

Moreover, we explore multiple characterizations of left dual core inverses and left dual v-core inverses. Additionally, we investigate the relationship between left dual pseudo core inverses and left dual v-core inverses.

2. Preliminary

Now, let us recall a few concepts about generalized inverses.

Definition 2.1. Let R be an associative ring with unity 1. An element $a \in R$ is called *(von Neumann) regular* if there exists some $x \in R$ such that $axa = a$. Such an x is called an *inner inverse* or a {1}-inverse of a and is denoted by a^-. The symbol $a\{1\}$ means the set of all inner inverses of a. The set of all regular elements in R is denoted by R^-.

In [16], Zhu et al. extended inverses along an element to one-sided cases.

Definition 2.2. Let $a, d \in R$. An element a is called *left invertible along d* if there exists some $x \in R$ such that $xad = d$ and $x \in Rd$. Such an element x is called a *left inverse of a along d* and is denoted by a_l^d. Dually, an element a is called right invertible along d if there exists some $y \in R$ such that $day = d$ and $y \in dR$. Such an element y is called a right inverse of a along d and is indicated by a_r^d.

Theorem 2.3. [16, Theorem 2.3] *Let S be a semigroup and $a, d \in S$. Then a is left invertible along d if and only if $S^1 d \subseteq S^1 dad$ (S^1 denotes the monoid generated by S.)*

In 2016, Drazin defined left and right (b, c)-inverses, [8].

Definition 2.4. Let S be a semigroup and let $a, b, c \in S$. Then we shall say that a is *left (b, c)-invertible* if $b \in Scab$, or equivalently if there exists $x \in Sc$ such that $xab = b$, in which case any such x will be called a *left (b, c)-inverse of a*. We will denote it by $a_l^{(b,c)}$.

Dually, a is called *right (b, c)-invertible* if $c \in cabS$, or equivalently if there exists $z \in bS$ such that $caz = c$, and any such z will be called a right (b, c)-inverse of a. We will denote it by $a_r^{(bc)}$.

In particular, a is called (b, c)-invertible if it is both left and right (b, c)-invertible.

We denote by $R_l^{(b,c)}$, $R_r^{(b,c)}$ and $R^{(b,c)}$, the set of all left (b, c)-invertible, the set of all right (b, c)-invertible and the set of all (b, c)-invertible elements in R, respectively. It should be noted that a is left (d, d)-invertible if and only if it is left invertible along d. Moreover, the left (d, d)-inverse of a is exactly the left inverse of a along d.

Definition 2.5. [18, Definition 2.1] Let $a, w \in R$. Then a is called *right w-core invertible* if there exists some $x \in R$ such that $awxa = a$, $awx^2 = x$, $awx = (awx)^*$.

Any such x is called a *right w-core inverse* of a and denoted by $a^{\circledast}_{r,w}$. Also the set of all right w-core invertible elements in R denoted by $R^{\circledast}_{r,w}$.

Theorem 2.6. [18, Theorem 2.21] *Let $a \in R$. Then the following conditions are equivalent:*

(i) $a \in R^{\dagger}$.
(ii) $a \in R^{\circledast}_{r,a^*}$.
(iii) $a^* \in R^{\circledast}_{r,a}$.

In this case, $(a^{\dagger})^* a^{\dagger}$ is a right a^*-core inverse of a, and $a^{\dagger}(a^{\dagger})^*$ is a right a-core inverse of a^*.

Lemma 2.7. [18, Lemma 2.20] *Let $a \in R$. Then the following conditions are equivalent:*

(i) $a \in R^{\dagger}$.
(ii) $a \in aa^*aR$.
(iii) $a \in Raa^*a$.

In this case, $a^{\dagger} = (ax)^* = (ya)^*$, where $x, y \in R$ satisfy $a = aa^*ax = yaa^*a$.

In [18], the authors introduce left dual v-core invertible elements in a $*$-ring.

Definition 2.8. For any $a, v \in R$, a is called *left dual v-core invertible* if there exists some $x \in R$ such that $axva = a$, $xva = (xva)^*$ and $x^2va = x$. Such an x is called a left dual v-core inverse of a and denoted by $a_{l,v\circledast}$. The symbol $R_{l,v\circledast}$ denotes the set of all left dual v-core invertible elements in R.

The right pseudo core invertible element in a $*$-ring is introduced in [14].

Definition 2.9. An element $a \in R$ is *right pseudo core invertible* if there exist $x \in R$ and a positive integer k such that

$$a^k = axa^k, \quad (ax)^* = ax, \quad \text{and} \quad ax^2 = x.$$

Such an x is called a *right pseudo core inverse* of a and is denoted by $a_r^{\circledcirc}$. The smallest positive integer k, denoted by $I(a)$, is called the *right pseudo core index* of a. In particular, a is called *right core invertible* when a is right pseudo core invertible with $I(a) = 1$.

As a dual of this topic, we give the following:

Definition 2.10. An element $a \in R$ is called *left dual pseudo core invertible* if there exist $x \in R$ and a positive integer k such that $a^k xa = a^k$, $(xa)^* = xa$ and $x^2 a = x$. Such an x is called a left dual pseudo core inverse of a and denoted by $a_{l,\textcircled{\scriptsize D}}$. In this case, the smallest positive integer k is called left dual pseudo core index of a and denoted by $I(a)$. In particular, a is called left dual core invertible if $I(a) = 1$.

Generally, $R_{l,\textcircled{\scriptsize D}}$ and $R_{l,\textcircled{\scriptsize \#}}$ denote the set of all left dual pseudo core invertible and left dual core invertible elements in R, respectively.

The (b, c)-core inverse was defined in a $*$-monoid S in [15]. For convenience, we next state this notion in a ring R.

Definition 2.11. Let $a, b, c \in R$. The element a is called (b, c)*-core invertible* if there exists some $x \in R$ such that:

$$caxc = c, \quad xR = bR \quad \text{and} \quad Rx = Rc^*.$$

Dually, an element $a \in R$ is called *dual* (b, c)*-core invertibl*e if there exists some $y \in R$ such that:

$$byab = b, \quad yR = b^*R \quad \text{and} \quad Ry = Rc.$$

The (b, c)-core inverse and dual (b, c)-core inverse of a are unique if they exist and are denoted by $a^{\textcircled{\scriptsize \#}}_{(b,c)}$ and $a_{\textcircled{\scriptsize \#}(b,c)}$, respectively. As usual, we denote by $R^{\textcircled{\scriptsize \#}}_{(b,c)}$ and $R_{\textcircled{\scriptsize \#}(b,c)}$, the set of all (b, c)-core invertible and dual (b, c)-core invertible elements in R, respectively.

Lemma 2.12. [15, Lemma 3.1] *Let $a, b, c \in R$. We have the following results.*

(I) *The following statements are equivalent:*
  (i) $c \in cabR \cap Rc^*c$.
  (ii) $c \in R(cab)^*c$.
  (iii) $R = R(cab)^* \oplus l(c)$.
  (iv) $R = R(cab)^* + l(c)$.

(II) *The following statements are equivalent:*
  (i) $b \in Rcab \cap bb^*R$.
  (ii) $b \in b(cab)^*R$.
  (iii) $R = (cab)^*R \oplus r(b)$.
  (iv) $R = (cab)^*R + r(b)$.

Theorem 2.13. [15, Theorem 3.5] *Let $a, b, c \in R$. Then the following statements are equivalent:*

(i) $a \in R^{\textcircled{\scriptsize \#}}_{(b,c)} \cap R_{\textcircled{\scriptsize \#}(b,c)}$.
(ii) $b \in b(cab)^*R$ *and* $c \in R(cab)^*c$.
(iii) $a \in R^{(b,c)}$ *and* $cab \in R^{\dagger}$.
(iv) $R = R(cab)^* \oplus l(c) = (cab)^*R \oplus r(b)$.
(v) $R = R(cab)^* + l(c) = (cab)^*R + r(b)$.

In this case, $a^{\textcircled{\scriptsize \#}}_{(b,c)} = br^*$, $a_{\textcircled{\scriptsize \#}(b,c)} = s^*c$, $a^{(b,c)} = bs^*c = br^*c$, *where* $s, r \in R$ *satisfy* $b = b(cab)^*s$ *and* $c = r(cab)^*c$.

The right (b, c)-core invertible elements, in a $*$-ring, which is given below is introduced in [5].

Definition 2.14. For any a, b, c of a ring R, the element a is called *right* (b, c)*-core invertible* if there exists some $x \in bR$ such that $caxc = c$ and $(cax)^* = cax$.

Theorem 2.15. [5, Theorem 2.6] *Let $a, b, c \in R$. The following conditions are equivalent:*

(i) $a \in R^{\textcircled{\scriptsize \#}}_{r,(b,c)}$.
(ii) $a \in R^{(b,c)}_r$ *and* $c \in R^{\{1,3\}}$.
(iii) $a \in R^{(b,c)}_r$ *and* $ca \in R^{\{1,3\}}$.
(iv) $a \in R^{(b,c)}_r$ *and* $cab \in R^{\{1,3\}}$.

In this case, $a^{\textcircled{\scriptsize \#}}_{r,(b,c)} = a^{(b,c)}_r c^{(1,3)} = a^{(b,c)}_r a(ca)^{(1,3)} = b(cab)^{(1,3)}cab(cab)^{(1,3)}$.

3. Left dual (b,c)-core inverses

In this section, we define left dual (b, c)-core invertible elements and we explore multiple characterizations of them.

Definition 3.1. Let $a, b, c \in R$. We call a left dual (b, c)-core invertible if there exists some $x \in Rc$ such that $bxab = b$ and $(xab)^* = xab$. Such an x is called a left dual (b, c)-core inverse of a. The set of all left dual (b, c)-core invertible elements in R is denoted by $R_{l,\oplus(b,c)}$.

We will denote the left dual (b, c)-core inverse of a with $a_{l,\oplus(b,c)}$. An element a in R could have different left dual (b, c)-core inverses. For example, let R be a $*$-ring, $a \in R$, $b = 0$ and $c = 1$. Then every $x \in R$ is the left dual $(0, 1)$-core inverse of a.

It is obvious that every dual (b, c)-core invertible element is left dual (b, c)-core invertible. The converse statement is not always true. For example, let R be a $*$-ring, $a \in R$, $b = 0 \neq c$. Clearly, a is left dual (b, c)-core invertible. But a is not dual (b, c)-core invertible since $cabx = 0$ for every $x \in R$.

Recall that an element $p \in R$ is called a *projection* if $p^2 = p = p^*$.

Theorem 3.2. *Let R be a ring and $a, b, c \in R$. The following conditions are equivalent:*

1. $a \in R_{l,\oplus(b,c)}$.
2. *There exists some $x \in Rc$ such that $bxab = b$, $(xab)^* = xab$ and $xabx = x$.*
3. *There exists some $x \in R$ such that $bxab = b$, $xR = b^*R$ and $Rx \subseteq Rc$.*
4. *There exists some $x \in R$ such that $bxab = b$, $l(x) = l(b^*)$ and $Rx \subseteq Rc$.*
5. *There exists some $x \in R$ such that $bxab = b$, $Rx \subseteq Rc$ and $xR \subseteq b^*R$.*
6. *There exists some $x \in R$ such that $bxab = b$, $Rx \subseteq Rc$ and $l(b^*) \subseteq l(x)$.*
7. *There exists a projection $q \in R$ and an idempotent $p \in R$ such that $Rb \subseteq Rq \subseteq Rab$, $Rp \subseteq Rc$ and $abR \subseteq pR$.*

In this case, $a_{l,\oplus(b,c)} = q(ab)^- p$ for any $(ab)^- \in (ab)\{1\}$.

Proof. (1) ⇒ (2) Assume $a \in R_{l,\oplus(b,c)}$. Then there exists some $y \in Rc$ such that $byab = b$ and $(yab)^* = yab$. Let $x = yaby$. We get $xab = (yaby)ab = ya(byab) = yab$. And so $(xab)^* = xab$. Also,

$$bxab = b(yaby)ab = (byab)yab = byab = b$$

and

$$xabx = (yaby)ab(yaby) = ya(byab)yaby = ya(byab)y = yaby = x.$$

(2) ⇒ (3) Since $bxab = b$ and $(xab)^* = xab$, we have $b^* = xabb^* \in xR$. Also, $xabx = x$ implies $x = (xab)^*x = b^*a^*x^*x$. Therefore, $x \in b^*R$.

(3) ⇒ (4) Obvious.

(4) ⇒ (5) Since $b = bxab$, $b^* = b^*a^*x^*b^*$. Therefore $b^* - b^*a^*x^*b^* = (1 - b^*a^*x^*)b^* = 0$. Since $l(x) = l(b^*)$, $(1 - b^*a^*x^*)x = 0$. And so $x = b^*a^*x^*x \in b^*R$, as desired.

(5) ⇒ (6) Obvious.

(6) ⇒ (7) Since $b^* = (xab)^*b^*$ and $l(b^*) \subseteq l(x)$, we get $x = (xab)^*x$. Then we have $xab = (xab)^*(xab)$ which implies $(xab)^* = (xab)$. Set $q = xab$ and $p = abx$. Hence $q^2 = (xab)(xab) = xa(bxab) = xab$ and $q^* = (xab)^* = xab = q$. So q is a projection. Also $p^2 = (abx)(abx) = a(bxab)x = abx = p$, so p is an idempotent. Therefore, $Rb = Rbq \subseteq Rq = R(xab) \subseteq Rab$, $Rp = R(abx) \subseteq Rx \subseteq Rc$ and $abR = a(bxab)R = (abx)abR = pabR \subseteq pR$.

(7) ⇒ (1) Since $abR \subseteq pR$, $pab = ab$. From $Rb \subseteq Rq \subseteq Rab$, it follows that $b = bq$ and $q = zab$ for some $z \in R$. Therefore, $ab = a(bq) = ab(zab) = (ab)z(ab)$ and so $ab \in R^-$. Let $x = q(ab)^-p$ for any $(ab)^- \in (ab)\{1\}$. Then, $x = q(ab)^-p \in Rp \subseteq Rc$. Also, $xab = q(ab)^-pab = q(ab)^-ab = zab(ab)^-(ab) = zab = q = (xab)^*$. Moreover, we have $bxab = bq = b$, as asserted. □

Lemma 3.3. [19, Lemma 2.2] *Let R be a ring and $a \in R$. Then $a^{(1,4)}$ exists if and only if $a \in aa^*R$. If $aa^*y = a$ for some $y \in R$, then $y^* = a^{(1,4)}$.*

Assume that $a \in R_{l,\oplus(b,c)}$ and a left dual (b,c)-core inverse of a is x. Then since $bxab = b$, we get $(xab)^n = xab$ for any positive integer n. It is concluded that $a \in R_{l,\oplus(b,c)}$ implies $x \in Rc$, $b(xab)^n = b$ and $((xab)^n)^* = (xab)^n$ for any positive integer n. It is natural to ask if the converse implication is true. The following theorem provides a solution to that question.

Theorem 3.4. *Let R be a ring and $a, b, c \in R$. The following statements are equivalent:*

1. $a \in R_{l,\oplus(b,c)}$.
2. $b \in b(cab)^*R$.
3. $b \in Rcab \cap bb^*R$.
4. *There exists some $x \in Rc$ such that $b(xab)^n = b$ and $((xab)^n)^* = (xab)^n$ for any positive integer n.*
5. *There exists some $x \in Rc$ such that $b(xab)^n = b$ and $((xab)^n)^* = (xab)^n$ for some positive integer n.*

Proof. (1) ⇒ (2) Suppose $a \in R_{l,\oplus(b,c)}$. Then there exists some $x \in Rc$ such that $bxab = b$, $(xab)^* = xab$. Let $x = rc$ for some $r \in R$. Then $b = bxab = b(xab)^* = b(rcab)^* = b(cab)^*r^* \in b(cab)^*R$, as desired.

(2) ⇒ (3) It is clear by Lemma 2.12 II.

(3) ⇒ (4) Since $b \in Rcab \cap bb^*R$, we have $b = rcab = bb^*s$ for some $r, s \in R$. In this case, $s^* = b^{(1,4)}$ by Lemma 3.3. Let $x = s^*rc$. Then $x \in Rc$, $xab = (s^*rc)ab = s^*(rcab) = s^*b = (s^*b)^* = (xab)^*$ and $bxab = bs^*b = b$. Multiplying the latter equation by xa from left, we get $(xab)^2 = xab$, and so $(xab)^n = xab$ for any positive integer n. As a consequence $b(xab)^n = bxab$ and $((xab)^n)^* = (xab)^* = xab = (xab)^n$.

(4) ⇒ (5) Obvious.

(5) ⇒ (1) Suppose $x \in Rc$ such that $b(xab)^n = b$ and $((xab)^n)^* = (xab)^n$ for some positive integer n. Set $y = (xab)^{(n-1)}x$. We claim that $y = a_{l\oplus(b,c)}$. Indeed, $y = (xab)^{(n-1)}x \in Rc$. Also, $byab = b((xab)^{(n-1)}x)ab = b(xab)^n = b$. Moreover, $yab = ((xab)^{(n-1)}x)ab = (xab)^n = ((xab)^n)^* = (yab)^*$. □

Let $a, b, c \in R$. Zhu showed that a is dual (b,c)-core invertible if and only if a is (b,c)-invertible and b (ab or cab) is {1,4}-invertible. An analogous result on dual right (b,c)-core inverses can be obtained.

Theorem 3.5. *Let R be a ring and $a, b, c \in R$. The following statements are equivalent:*

1. $a \in R_{l,\oplus(b,c)}$.
2. $a \in R_l^{(b,c)}$ *and* $b \in R^{(1,4)}$.
3. $a \in R_l^{(b,c)}$ *and* $ab \in R^{(1,4)}$.
4. $a \in R_l^{(b,c)}$ *and* $cab \in R^{(1,4)}$.

In this case, $a_{l,\oplus(b,c)} = b^{(1,4)}a_l^{(b,c)} = (ab)^{(1,4)}aa_l^{(b,c)} = (cab)^{(1,4)}c$.

Proof. (1) ⇒ (2) It is a direct consequence of Lemma 3.3 and Theorem 3.4.

(2) ⇒ (3) Let $a \in R_l^{(b,c)}$ and $b \in R^{(1,4)}$. Then there exists some $s \in R$ such that $scab = b$. Let $x = b^{(1,4)}$. Now, $abxscab = abxb = ab$. So ab is regular and $xscab = xb = (xb)^* = (xscab)^*$. Hence, $ab \in R^{(1,4)}$.

(3) ⇒ (4) Let $a \in R_l^{(b,c)}$ and $ab \in R^{(1,4)}$. Then there exists some $s \in R$ such that $scab = b$. Suppose that $x = ab^{(1,4)}$. Now $cabxascab = cabxab = cab$. So cab is regular and $xascab = xab = (xab)^* = (xascab)^*$. Hence $cab \in R^{(1,4)}$.

(4) ⇒ (1) Let $a \in R_l^{(b,c)}$ and $cab \in R^{(1,4)}$. Then there exists $s \in R$ such that $b = scab$. Let $x = (cab)^{(1,4)}$. Set $y = xc$. Then $y \in Rc$, $yab = xcab = (xcab)^* = (yab)^*$ and $byab = bxcab = scabxcab = scab = b$. Therefore, $a \in R_{l,\oplus(b,c)}$. The other equalities can be seen in a similar way. □

Suppose that $a \in R_{l,\oplus(b,c)}$. Theorem 3.5 guarantees that $cab \in R^{(1,4)}$. Therefore, $cab \in R^-$. Also by Theorem 3.4, there exists some $s \in R$ such that $b = scab$. It follows that $b = scab(cab)^-cab = b(cab)^-cab$ for any $(cab)^- \in (cab)\{1\}$. This implies $a_l^{(b,c)} = b(cab)^-c$. Now another representation of $a_{l,\oplus(b,c)}$ can be presented.

Proposition 3.6. *Let R be a ring and $a, b, c \in R$ with $a \in R_{l,\oplus(b,c)}$. Then $a_{l,\oplus(b,c)} = b^{(1,4)}b(cab)^-c$ for any $(cab)^- \in (cab)\{1\}$ and $b^{(1,4)} \in b\{1,4\}$.*

Theorem 3.7. *Let R be a ring and $a, b, c \in R$. Then the following statements are equivalent:*

1. $a \in R_{l,\circledast(b,c)}$.
2. $R = (cab)^*R \oplus r(b)$.
3. $R = (cab)^*R + r(b)$.

Proof. (1) ⇒ (2) Let $a \in R_{l,\circledast(b,c)}$. By Theorem 3.4, we have $b \in b(cab)^*R$. Hence $b = bb^*a^*c^*r$ for some $r \in R$. So $1 - b^*a^*c^*r = 1 - (cab)^*r \in r(b)$. For any $s \in R$, we have $s = [(1 - (cab)^*r) + (cab)^*r]s = (1 - (cab)^*)s + (cab)^*rs \in (1 - (cab)^*)R + (cab)^*R$. Since $b = bb^*a^*c^*r$ and Lemma 3.3, $r^*ca \in b\{1, 4\}$. Then for any $z \in (cab)^*R \cap r(b)$, $bz = 0$ and there exists some $s \in R$ such that $z = (cab)^*s = (cab(r^*ca)b)^*s = (r^*cab)^*(cab)^*s = r^*cabz = 0$. Hence, $R = (cab)^*R \oplus r(b)$.

(2) ⇒ (3) Obvious.

(3) ⇒ (1) Let $R = (cab)^*R + r(b)$. Then $b \in b(cab)^*R$. By Theorem 3.4, $a \in R_{l,\circledast(b,c)}$ as desired. □

For any $p \in R$, any $a \in R$ can be written as

$$a = pap + pa(1 - p) + (1 - p)ap + (1 - p)a(1 - p)$$

or the matrix form

$$a = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}_p,$$

where $a_1 = pap$, $a_2 = pa(1 - p)$, $a_3 = (1 - p)ap$ and $a_4 = (1 - p)a(1 - p)$. This decomposition is well known as the Pierce decomposition.

If $p^2 = p = p^*$, then

$$a^* = \begin{bmatrix} a_1{}^* & a_2{}^* \\ a_3{}^* & a_4{}^* \end{bmatrix}_p.$$

We next give the matrix represantations of left dual (b, c)-core inverses.

Theorem 3.8. *Let R be a ring, $a, b, c \in R$. Then the following statements are equivalent:*

1. *$a \in R_{l,\circledast(b,c)}$ and x is a left dual (b, c)-core inverse of a.*
2. *There exists a projection $q \in R$ such that*

$$a = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}_q,\ b = \begin{bmatrix} b_1 & 0 \\ b_3 & 0 \end{bmatrix}_q,\ c = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}_p \text{ and } x = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}_q$$

where $(x_1a_1 + x_2a_3)b_1 + (x_1a_2 + x_2a_4)b_3 = q$, $(x_3a_1 + x_4a_3)b_1 + (x_3a_2 + x_4a_4)b_3 = 0$ and $\boldsymbol{R}(x) \subseteq \boldsymbol{R}(c)$ ($\boldsymbol{R}(x)$ and $\boldsymbol{R}(c)$ denotes the column space of x and c respectively).

3. *There exists a projection $p \in R$ such that*

$$a = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}_p,\ b = \begin{bmatrix} 0 & b_2 \\ 0 & b_4 \end{bmatrix}_p,\ c = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}_p \text{ and } x = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}_p$$

where $(x_1a_1 + x_2a_3)b_2 + (x_1a_2 + x_2a_4)b_4 = 1 - p$, $(x_3a_1 + x_4a_3)b_2 + (x_3a_2 + x_4a_4)b_4 = 0$ and $\boldsymbol{R}(x) \subseteq \boldsymbol{R}(c)$.

Proof. (1) ⇒ (2) Suppose $a \in R_{l,\circledast(b,c)}$ with a left dual (b, c)-core inverse x. Then $x \in Rc$, $(xab)^* = xab$ and $bxab = b$. Let $q = xab$. Then $q^2 = q = q^*$. So, a, b, c can be represented as in (2). Since $x \in Rc$, $\mathbf{R}(x) \subseteq \mathbf{R}(c)$. By the Pierce decomposition, we have

$$(x_1a_1 + x_2a_3)b_1 + (x_1a_2 + x_2a_4)b_3$$

$$= (qxqqaq + qx(1 - q)(1 - q)aq)qbq + (qxqqa(1 - q) + qx(1 - q)(1 - q)a(1 - q))(1 - q)bq$$

$= qxqaqbq + qxaqbq - qxqaqbq + qxqa(1-q)bq + qxa(1-q)bq - qxqa(1-q)bq$
$= qxaqbq + qxqabq - qxqaqbq + qxabq - qxaqbq - qxqabq + qxqaqbq$
$= qxabq$
$= q$.
The equality $(x_3a_1 + x_4a_3)b_1 + (x_3a_2 + x_4a_4)b_3 = 0$ can be shown in a similar way.
(2) ⇒ (1) Since

$$xab = \begin{bmatrix} (x_1a_1 + x_2a_3)b_1 + (x_1a_2 + x_2a_4)b_3 & 0 \\ (x_3a_1 + x_4a_3)b_1 + (x_3a_2 + x_4a_4)b_3 & 0 \end{bmatrix}_q = \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix}_q = q,$$

here it is obvious that $bxab = b$ and $(xab)^* = xab$. Besides, since $\mathbf{R}(x) \subseteq \mathbf{R}(c)$, $x \in Rc$. Hence, $a \in R_{l,\oplus(b,c)}$ and x is a left dual (b,c)-core inverse of a.

(1) ⇔ (3) It can be shown in a similar way of the proof of (1) ⇔ (2) by taking $p = 1 - q$. □

4. Applications

In this section, we explore multiple characterizations of left dual core inverses and left dual v-core inverses. Additionally, we investigate the relationship between left dual pseudo core inverses, left dual w-core inverses and left dual (b,c)-core inverses.

Theorem 4.1. *Let R be a ring and $a, w \in R$ and m, n be nonnegative integers such that $m + n \geq 1$. Then*

1. *a is left invertible if and only if a is left dual $(1,1)$-core invertible,*
2. *a is left dual pseudo core invertible if and only if a^m is left dual (a^k, a^n)-core invertible, for some positive integer k,*
3. *a is left dual core invertible if and only if a^m is left dual (a, a^n)-core invertible,*
4. *a is left dual v-core invertible if and only if v is left dual (a,a)-core invertible.*

Proof. (1) Let a be left dual $(1,1)$-core invertible. Then there exists some $x \in R$ such that $(xa)^* = xa$ and $xa = 1$. Converse is clear.

(2) Let a be left dual pseudo core invertible with $I(a) = k$. Then there exists some $x \in R$ such that $a^kxa = a^k$, $(xa)^* = xa$ and $x^2a = x$, whence $xa = x^2aa = x^2a^2 = \cdots = x^na^n$ for any positive integer n. Let $y = x^{k+m}$. Then

$$y = x^{k+m} = x^{k+m-1}x^2a = x^{k+m-1}x^{n+1}a^n = x^{k+m+n}a^n \in Ra^n,$$

$$ya^ma^k = ya^{m+k} = x^{m+k}a^{m+k} = xa = (xa)^* = (ya^ma^k)^*,$$

$$a^kya^ma^k = a^kx^{k+m}a^{m+k} = a^kxa = a^k.$$

Hence, $a^m \in R_{l,\oplus(a^k,a^n)}$ and $(a^m)_{l,\oplus(a^k,a^n)} = (a_{l,\textcircled{D}})^{k+m}$.

Conversely, let $a^m \in R_{l\oplus(a^k,a^n)}$. Then there exists some $x \in Ra^n$ such that $(xa^ma^k)^* = (xa^{k+m})^* = xa^{k+m}$ and $a^kxa^ma^k = a^k$. Let $z = xa^{k+m-1}$. Then $a^kza = a^kxa^{k+m-1}a = a^kxa^{k+m} = a^kxa^ma^k = a^k$. Also, $za = xa^{k+m-1}a = xa^{k+m} = (xa^{k+m})^* = (za)^*$. Moreover, we have $z^2a = xa^{k+m-1}xa^{k+m-1}a = xa^{k+m-1}xa^{k+m} = xa^{m-1}a^kxa^ma^k = xa^{m-1}a^k = xa^{k+m-1} = z$. Therefore $a \in R_{l\textcircled{D}}$ and $a_{l,\textcircled{D}} = (a^m)_{l,\oplus(a^k,a^n)}a^{k+m-1}$.

(3) It is obvious from (2).

(4) Let $a \in R_{l,v,\oplus}$. Then there exists some $y \in R$ such that $ayva = a$, $yva = (yva)^*$ and $y^2va = y$. Let $x = y$. Then, $x = x^2va \in Ra$ and $(xva)^* = xva$. Also, $axva = a$. So $v \in R_{l,\oplus(a,a)}$ and $v_{l,\oplus(a^k,a^n)} = a_{l,v,\oplus}$. Now let $v \in R_{l,\oplus(a,a)}$. Then there exists some $x \in Ra$ such that $(xva)^* = (xva)$ and $axva = a$. Let $y = x$. Then there exists some $r \in R$ such that $y = ra$. Also, $ayva = a$ and $(yva)^* = yva$. Moreover, we have $y^2va = yyva = rayva = ra = y$, as asserted. □

It is proved in Theorem 4.1 that a is left dual v-core invertible if and only if v is left dual (a,a)-core invertible. As a result of Theorems 3.8 and 4.1, we get the matrix representation of the left dual v-core inverses as follows.

Corollary 4.2. *Let R be a ring and $a, v \in R$. Then the following statements are equivalent:*

1. *$a \in R_{l,v,\oplus}$ and x is a left dual v-core inverse of a.*
2. *There exists a projection $q \in R$ such that*
 $v = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}_q$, $a = \begin{bmatrix} a_1 & 0 \\ a_3 & 0 \end{bmatrix}_q$ $x = \begin{bmatrix} x_1 & 0 \\ x_3 & 0 \end{bmatrix}_q$
 where $x_1v_1a_1 + x_1v_2a_3 = q$ and $x_3v_1a_1 + x_3v_2a_3 = 0$.
3. *There exists a projection $p \in R$ such that*
 $v = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}_p$, $a = \begin{bmatrix} 0 & a_2 \\ 0 & a_4 \end{bmatrix}_p$ $x = \begin{bmatrix} 0 & x_2 \\ 0 & x_4 \end{bmatrix}_p$
 where $x_2v_3a_2 + x_2v_4a_4 = 1 - p$ and $x_4v_3a_2 + x_4v_4a_4 = 0$.

Theorem 4.3. *Let R be a ring and $a, v \in R$ with $a \in R_{l,v,\oplus}$. Then $va = a_1 + a_2$ where*

1. *$a_1 \in R_{l,\oplus}$,*
2. *${a_2}^2 = 0$,*
3. *${a_2}^* a_1 = 0 = a_1a_2$. In addition, $a_{l,v,\oplus}(va)^2 \in R_{l,\oplus}$ with a left dual core inverse $a_{l,v,\oplus}$.*

Proof. Suppose $a \in R_{l,v,\oplus}$ with a left dual v-core inverse x. Then $axva = a$, $(xva)^* = xva$ and $x^2va = x$. Let $a_1 = x(va)^2$ and $a_2 = (1 - xva)va$. Then $va = a_1 + a_2$. Now,

(1) $xa_1 = xx(va)^2 = x^2(va)(va) = xva = (xva)^* = (xa_1)^*$. Also, we have $a_1xa_1 = a_1(xva) = x(va)^2xva = x(va)(va)xva = xvava = x(va)^2 = a_1$. Moreover, $x^2a_1 = xxa_1 = x(xva) = x^2va = x$. Hence, $a_1 \in R_{l,\oplus}$ with a left dual core inverse $a_{l,\oplus}$.

(2) ${a_2}^2 = (va - xvava)^2 = vava - vax(va)^2 - x(va)^3 + x(va)^2x(va)^2 = vava - vava - x(va)^3 + x(va)^3 = 0$.

(3) ${a_2}^*a_1 = (va - x(va)^2)^*x(va)^2 = ((va)^* - (va)^*xva)x(va)^2 = (va)^*(1 - xva)x(va)^2 = (va)^*(x(va)^2 - xv(axva)va) = (va)^*(x(va)^2 - x(va)^2) = 0$. $a_1a_2 = 0$ can be shown in a similar way. □

Let $a \in R$ and m, n be nonnegative integers such that $m + n \geq 1$. From Theorem 4.1, we get that a^k is left dual core invertible if and only if a^m is left dual (a^k, a^n)-core invertible, for some positive integer k.

According to Theorem 4.1, we can establish the relation between left dual pseudo core inverses and left dual v-core inverses.

Proposition 4.4. *Let R be a ring, $a \in R$ and m be a nonnegative integer. Then a is left dual pseudo core invertible if and only if a^k is left dual a^m-core invertible for some positive integer k. In this case, $a_{l\circledcirc} = (a^k)_{l,a^m,\oplus}a^{k+m-1}$ and $(a^k)_{l,a^m,\oplus} = (a_{l,\circledcirc})^{k+m}$.*

Proof. a is left dual pseudo core invertible if and only if a^m left dual (a^k, a^n)-core invertible for some positive integer k by Theorem 4.1(2). For the case $n = k$, a^m is left dual (a^k, a^k)-core invertible if and only if a^k is left dual a^m-core invertible by Theorem 4.1(4). □

As a result of Proposition 4.4, if we take $k = 1$, then we have following.

Corollary 4.5. *Let R be a ring, $a \in R$ and m be a nonnegative integer. Then a is left dual core invertible if and only if a is left dual a^m-core invertible.*

The following corollary follows as a special case of Theorem 4.1(3) and Corollary 4.5.

Corollary 4.6. *Let R be a ring and $a \in R$. Then following statements are equivalent:*

1. *$a \in R_{l,\oplus}$.*
2. *a is left dual (a, a)-core invertible.*
3. *a is left dual $(a, 1)$-core invertible.*
4. *1 is left dual (a, a)-core invertible.*
5. *a is left dual a-core invertible.*
6. *a is left dual 1-core invertible.*

7. *a is left* (a^*, a)*- invertible.*

It is noteworthy that any $A \in M_n(\mathbb{C})$ is left dual pseudo core invertible. By applying Theorem 4.1(2) together with Proposition 4.4, we obtain the following result for complex matrices.

Corollary 4.7. *Let* $A \in M_n(\mathbb{C})$ *with* $I(A) = k$*. Then*

1. A^m *is left dual* (A^k, A^n)*-core invertible for any nonnegative integers* m, n *satisfying* $m + n \geq 1$*. In this case,* $(A_{l,\textcircled{D}})^{m+k}$ *is a left dual* (A^k, A^n)*-core inverse of* A^m*.*
2. A^k *is left dual* A^m*-core invertible, for any nonnegative integer* m*. In this case,* $(A_{l,\textcircled{D}})^{k+m}$ *is a left dual* (A^k, A^n)*-core inverse of* A^m*.*

As established in Theorem 4.1(4), left dual v-core inverses of a coincide with left dual (a, a)- core inverses of v. This leads to the following existence criterion for left dual v-core inverses in rings.

Corollary 4.8. *Let* R *be a ring and* $a, v \in R$*. The following statements are equivalent:*

1. $a \in R_{l,v,\textcircled{\#}}$*.*
2. *There exists some* $x \in Ra$ *such that* $axva = a$*,* $(xav)^* = xav$ *and* $xvax = x$*.*
3. *There exists some* $x \in R$ *such that* $axva = a$*,* $Rx \subseteq Ra$ *and* $a^*R \subseteq xR$*.*
4. *There exists some* $x \in R$ *such that* $axva = a$*,* $Rx \subseteq Ra$ *and* $axva = a$*,* $l(x) = l(a^*)$ *and* $Ra \subseteq Ra$*.*
5. *There exists some* $x \in R$ *such that* $axva = a$*,* $Rx \subseteq Ra$ *and* $xR \subseteq a^*R$*.*
6. *There exists some* $x \in R$ *such that* $axva = a$*,* $Rx \subseteq Ra$ *and* $l(a^*) \subseteq l(x)$*.*
7. *There exists a projection* $q \in R$ *and an idempotent* $p \in R$ *such that* $Ra \subseteq Rq \subseteq Rva$*,* $Rp \subseteq Ra$ *and* $vaR \subseteq pR$*. In this case* $a_{l,v,\textcircled{\#}} = q(va)^- p$ *for any* $(va)^- \in (va)\{1\}$*.*

As a consequence of above corollary, we have the following result.

Corollary 4.9. *Let* R *be a ring and* $a \in R$*. Then the following statements are equivalent:*

1. $a \in R_{l,\textcircled{\#}}$*.*
2. *There exists some* $x \in Ra$ *such that* $axa = a$*,* $(xa)^* = xa$ *and* $xax = x$*.*
3. *There exists some* $x \in R$ *such that* $axa = a$*,* $Rx \subseteq Ra$ *and* $a^*R \subseteq xR$*.*
4. *There exists some* $x \in R$ *such that* $axa = a$*,* $Rx \subseteq Ra$ *and* $axa = a$*,* $l(x) = l(a^*)$ *and* $Ra \subseteq Ra$*.*
5. *There exists some* $x \in R$ *such that* $axa = a$*,* $Rx \subseteq Ra$ *and* $xR \subseteq a^*R$*.*
6. *There exists some* $x \in R$ *such that* $axa = a$*,* $Rx \subseteq Ra$ *and* $l(a^*) \subseteq l(x)$*.*
7. *There exists a projection* $q \in R$ *and an idempotent* $p \in R$ *such that* $Ra \subseteq Rq \subseteq Ra$*,* $Rp \subseteq Ra$ *and* $aR \subseteq pR$*. In this case* $a_{l,\textcircled{\#}} = q(a)^- p$ *for any* $(a)^- \in (a)\{1\}$*.*

Theorem 4.10. *Let* R *be a ring and* $a \in R$*. Then the following statements are equivalent:*

1. *a is Moore-Penrose invertible.*
2. *a is left dual* a^**-core invertible.*
3. a^* *is left dual a-core invertible.*
4. *a is left dual* (a^*, a^*)*-core invertible.*
5. *a is left* (a^*, a^*) *invertible.*
6. a^* *is left dual* (a, a)*-core invertible.*
7. a^* *is left* (a, a) *invertible.*

Proof. (1) ⇔ (2) ⇔ (3) by Theorem 2.6.

(1) ⇒ (4) Let $a \in R^\dagger$. Then we have $a^* \in R^{\{1,4\}}$. Again, since $a \in R^\dagger$, a is left invertible along a^* by [11, Theorem 11]. Also, $a^* \in Ra^*aa^*$ by Theorem 2.3. Hence, $a \in R_l^{(a^*,a^*)}$. Consequently, $a \in R_{l,\textcircled{\#}(a^*,a^*)}$ by Theorem 3.5.

(4) ⇒ (5) It is clear from Theorem 3.5.

(5) ⇒ (1) Let $a \in R_l^{(a^*,a^*)}$. Then $a^* \in Ra^*aa^*$. Hence, we get $a \in aa^*aR$ and by [18, Lemma 2.20], $a \in R^\dagger$.

(1) ⇔ (6) ⇔ (7) If we consider the proof of (1) ⇔ (4) ⇔ (5) and the fact $a \in R^\dagger$ if and only $a^* \in R^\dagger$, then the proof is clear. □

Theorem 4.11. *Let R be a ring and $a, b, c \in R$. Then a is left dual (b, c)-core invertible if and only if ab is left (b^*, c) invertible. In this case, the left dual (b, c)-core inverse of a coincides with the left (b^*, c) inverse of ab.*

Proof. Let $a \in R_{l,\textcircled{\#}(b,c)}$ and $a_{l,\textcircled{\#}(b,c)} = x$. Then $x \in Rc$, $bxab = b$ and $(xab)^* = xab$. Thus, $b^* = (bxab)^* = (xab)^* b^* = xabb^* \in Rcabb^*$ as required.

Conversely, let $x = (ab)_l^{(b^*,c)}$. Then $x \in Rc$ and $xabb^* = b^*$. So since $b = b(xab)^*$, $xab = xab(xab)^*$. Hence $xab = (xab)^*$. Finally $bxab = b(xab)^* = b$ as desired. □

Recall that an element $a \in R$ is *strongly left (b, c) invertible* if $b \in Rcab$ and cab is regular, or equivalently if there exists $x \in R$ such that $xax = x$, $xR = bR$, and $Rx \subseteq Rc$. In which case any such x will be called strongly left (b, c) inverse of a. Clearly, if x is strongly left (b, c) inverse of a, then so is xax. It is clear that every strongly left (b, c) invertible element is left (b, c) invertible. Moreover, every strongly left (b, c) inverse of a is a left (b, c) inverse of a.

Now we have the following theorem.

Theorem 4.12. *Let R be a ring and $a, b, c \in R$. Then a is left dual (b, c)-core invertible if and only if ab is strongly left (b^*, c) invertible.*

Proof. Assume that ab is strongly left (b^*, c) invertible. So ab is left (b^*, c) invertible. By Theorem 4.11, a is left dual (b, c)-core invertible. For the other direction, let $a \in R_{l,\textcircled{\#}(b,c)}$ and $a_{l,\textcircled{\#}(b,c)} = y$. Then $y \in Rc$, $byab = b$ and $(yab)^* = yab$, whence $y = tc$ for some $t \in R$. Consequently, $b^* = yabb^* = tcabb^* \in Rc(ab)b^*$. Now, $c(ab)b^* = c(ab)(tcabb^*) = ca(tcabb^*)^*(tcabb^*) = cabb^*(tca)^* tcabb^*$. So, $cabb^*$ is regular and this completes the proof. □

In light of Corollary 4.6 and Theorems 4.10 and 4.12, the following corollaries are established.

Corollary 4.13. *Let R be a ring and $a \in R$. The following statements are equivalent:*

1. $a \in R_{l,\oplus}$.
2. a^2 *is strongly left* (a^*, a) *invertible.*
3. a^2 *is strongly left* $(a^*, 1)$ *invertible.*
4. a *is strongly left* (a^*, a) *invertible.*

Corollary 4.14. *Let R be a ring and $a \in R$. The following statements are equivalent:*

1. *a is Moore-Penrose invertible.*
2. a^*a *is strongly left* (a^*, a) *invertible.*
3. aa^* *is strongly left* (a, a^*) *invertible.*

The next corollary is a direct consequence of Theorem 2.6 and Theorems 4.1 and 4.3.

Corollary 4.15. *Let R be a ring and $a \in R^\dagger$. Then $aa^* = a_1 + a_2$ where*

1. $a_1 \in R_{l,\oplus}$,
2. $a_2^2 = 0$,
3. $a_2^* a_1 = 0 = a_1 a_2$. *In addition,* $a^*a \in R_{l,\oplus}$ *with a left dual core inverse* $a^\dagger a^{\dagger *}$.

By Proposition 4.4 and Theorem 4.3, we have following corollary.

Corollary 4.16. *Let R be a ring and $a \in R_{l,\textcircled{D}}$ with $I(a) = k$. Then $a^k = a_1 + a_2$ where*

1. $a_1 \in R_{l,\oplus}$,
2. $a_2^2 = 0$,
3. $a_2^* a_1 = 0 = a_1 a_2$. *In addition,* $a_{l,\textcircled{D}} a^{k+1} \in R_{l,\oplus}$ *with a left dual core inverse* $(a_{l,\textcircled{D}})^k$.

Theorem 4.17. *Let R be a ring and* $a, d \in R_l^{(b,c)}$. *Then the following hold.*

1. $a_l^{(b,c)} = d_l^{(b,c)} d a_l^{(b,c)}$,
2. $d_l^{(b,c)} = a_l^{(b,c)} a d_l^{(b,c)}$.

Proof. Let $a, d \in R_l^{(b,c)}$, $x = a_l^{(b,c)}$ and $y = d_l^{(b,c)}$. Then we have $x, y \in Rc$ and $b = xab = ydb$.

(1) Let $z = ydx$. Now, $z \in Rc$ since $x \in Rc$. Also, $zab = ydxab = ydb = b$. This implies that ydx is one of the (b, c) inverses of a.

(2) Let $t = xay$. Since $y \in Rc$, we have $t \in Rc$. Also, $tdb = xaydb = xab = b$. This implies that xay is one of the (b, c) inverses of d, as desired. □

Theorem 4.18. *Let R be a ring and* $a, d \in R_{l,\circledast(b,c)}$. *Then we have the following.*

1. $a_{l,\circledast(b,c)} = d_{l,\circledast(b,c)} d a_l^{(b,c)}$.
2. $d_{l,\circledast(b,c)} = a_{l,\circledast(b,c)} a d_l^{(b,c)}$.

Proof. Let $a, d \in R_{l,\circledast(b,c)}$. Then by Theorem 3.5, $a_{l,\circledast(b,c)} = b^{(1,4)} a_l^{(b,c)}$ and $d_{l,\circledast(b,c)} = b^{(1,4)} d_l^{(b,c)}$. Also we have $a_l^{(b,c)} = d_l^{(b,c)} d a_l^{(b,c)}$ and $d_l^{(b,c)} = a_l^{(b,c)} a d_l^{(b,c)}$ by Theorem 4.17. By Theorem 3.5,

(1) $a_{l,\circledast(b,c)} = b^{(1,4)} a_l^{(b,c)} = b^{(1,4)} d_l^{(b,c)} d a_l^{(b,c)} = d_{l,\circledast(b,c)} d a_l^{(b,c)}$.

(2) $d_{l,\circledast(b,c)} = b^{(1,4)} d_l^{(b,c)} = b^{(1,4)} a_l^{(b,c)} a d_l^{(b,c)} = a_{l,\circledast(b,c)} a d_l^{(b,c)}$. □

Theorem 4.19. *Let R be a ring and* $a, b, c \in R$. *Then following are equivalent:*

1. $a \in R_{l,\circledast(b,c)} \cap R^{\circledast}_{r(b,c)}$.
2. $a \in R_{\circledast(b,c)} \cap R^{\circledast}_{(b,c)}$.
3. $cab \in R^{\dagger}$.
4. $b \in (cab)^* R$ *and* $c \in R(cab)^* c$.
5. $R = R(cab)^* \oplus l(c) = (cab)^* R \oplus r(b)$.
6. $R = R(cab)^* + l(c) = (cab)^* R + r(b)$.

Proof. It is sufficient to show (1) ⇒ (2) because (2) ⇒ (1) is obvious and (2) ⇔ (3) ⇔ (4) ⇔ (5) ⇔ (6) follows from Theorem 2.13. Now let $a \in R_{l,\circledast(b,c)} \cap R^{\circledast}_{r,(b,c)}$. Then by Theorems 3.5 and 2.15, we have $a \in R_l^{(b,c)} \cap R_r^{(b,c)}$, $b \in R^{(1,4)}$ and $c \in R^{(1,3)}$. Since $a \in R_l^{(b,c)} \cap R_r^{(b,c)}$, $a \in R^{(b,c)}$. Hence, $a \in R_{\circledast(b,c)} \cap R^{\circledast}_{(b,c)}$ as desired. □

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