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New additive results for the Drazin inverse of two matrices and its applications

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Abstract. In this paper, we investigate additive properties of the Drazin inverse of two matrices. A formula is given for the Drazin inverse of the sum of two matrices $P, Q \in \mathbf{C}^{n\times n}$ under the conditions that $P^2QP = 0$, $Q^2P = 0$ and $(QP)^d = 0$. As an application, we give some new representations for the Drazin inverse of a 2×2 block matrix.

1. Introduction

Let $A \in \mathbb{C}^{n\times n}$ and I be the identity matrix. The index of a matrix A is the smallest positive integer such that $\operatorname{rank}(A^{k+1}) = \operatorname{rank}(A^k)$. The index of A is denoted by $\operatorname{ind}(A)$. The Drazin inverse of A is a matrix $A^d \in \mathbb{C}^{n\times n}$ satisfying:

$$AA^d = A^dA, \quad A^dAA^d = A^d, \quad A^{k+1}A^d = A^k,$$

where $k = \operatorname{ind}(A)$. It is well known that A^d always exists and is unique. The $\operatorname{ind}(A)$ is the smallest nonnegative integer k such that the equation $A^{k+1}A^d = A^k$ holds. We write $A^\pi = I - AA^d$.

The Drazin inverse was first studied by Drazin [5] in associative rings and semigroups. The Drazin inverse of matrices is very important in various applied mathematical fields such as singular differential equations, singular difference equations, Markov chains, iterative methods and so on [2].

Let $P, Q \in \mathbf{C}^{n\times n}$. Finding the explicit formulas for the Drazin inverse of the sums of two matrices P and Q was first considered by Drazin in 1958 [5]. Herein, it was proved that $(P+Q)^d = P^d + Q^d$ under $PQ = QP = 0$. In 2001, Hartwig, Wang and Wei [9] studied $(P+Q)^d$ under the condition $PQ = 0$. In 2011, Yang and Liu [17] gave the formulas for $(P+Q)^d$ under the condition $PQP = 0$ and $PQ^2 = 0$. In 2019, Guo and Chen gave the formulas for $(P+Q)^d$ under the condition $PQ^iP = 0, i = 1, 2, \ldots\ldots, n$ [6]. Recently, Shakoor et al. [14] studied $(P+Q)^d$ under the condition $P^2QP = 0, PQ^2 = 0$. Until now, there has been no explicit formula for the Drazin inverse of the sum of two general matrices P and Q in terms of P, Q, P^d and Q^d. More representations for the Drazin inverse of the sum of two block matrices were considered in

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[3, 4, 7, 11, 12] under some restrictive assumptions. In this paper, we derive the formula for $(P+Q)^d$ under the condition $P^2QP=0$, $Q^2P=0$ and $(QP)^d=0$, which extends some existing results in [9, 17].

Let M denote a 2×2 block complex matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A and D are square matrices not necessarily with the same orders. Campbell and Meyer [2] posed the following open problem: how to establish an explicit representation for the Drazin inverse of M in terms of the blocks of the partition. Miao and Wei gave an expression for the Drazin inverse of M with the conditions $A^\pi B=0, CA^\pi=0$ and the generalized Schur complement $S=D-CA^DB=0$ (see [13, 16]). In 2006, Hartwig et al. [8] obtained an expression for the Drazin inverse of M with the conditions $AA^\pi B=0, CA^\pi B=0$ and $S=0$. In 2016, the expressions for the Drazin inverse of M with the conditions $A^2A^\pi BC=0, BCA^\pi BC=0, CAA^\pi BC=0, S=0$ *and* $BCA^2A^\pi=0, BCAA^\pi B=0, BCA^\pi BC=0, S=0$ were deduced [15]. Shakoor et al. [14] gave an expression for the Drazin inverse of M with the conditions $A^2A^\pi BC=0, CAA^\pi BC=0, ABCA^\pi B=0$ and $S=0$. There are also some results on the representations for the Drazin inverse of M (see [10, 12, 18]). In this paper, we derive some new representations for M^d under the conditions $A^2A^\pi BC=0$, $A^dBCA^\pi BC=0$, $CBCA^\pi BC=0$ and $CAA^\pi BC=0$, which generalize the results in [8, 12–15].

Next, we first present some useful lemmas.

Lemma 1.1. *[1] Let $M=\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}\in\mathbb{C}^{n\times n}$, where A and C are square, $s=\mathrm{ind}(A)$, and $l=\mathrm{ind}(C)$. Then*

$$M^d=\begin{pmatrix} A^d & X \\ 0 & C^d \end{pmatrix},$$

where $X=\sum_{i=0}^{l-1}(A^d)^{i+2}BC^iC^\pi+\sum_{i=0}^{s-1}A^\pi A^iB(C^d)^{i+2}-A^dBC^d$.

Lemma 1.2. *[9] Let $P,Q\in\mathbb{C}^{n\times n}$. If $PQ=0$, then*

$$(P+Q)^d=(I-QQ^d)\sum_{i=0}^{l-1}Q^i(P^d)^{i+1}+\sum_{i=0}^{s-1}(Q^d)^{i+1}P^i(I-PP^d),$$

where $l=\mathrm{ind}(Q)$ and $s=\mathrm{ind}(P)$.

Lemma 1.3. *[6] Let $B\in\mathbb{C}^{m\times n}$ and $C\in\mathbb{C}^{n\times m}$. Then $(BC)^dB=B(CB)^d$, $C(BC)^d=(CB)^dC$ and $((BC)^d)^i=B((CB)^d)^{i+1}C$, for any positive integer i.*

Lemma 1.4. *[11] Let $M=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A and D are square. If $ABC=0$, $DCA=0$, $DCB=0$, $t_1=\mathrm{ind}(BC)$, $\nu_1=\mathrm{ind}(A^2)$, $\nu_2=\mathrm{ind}(D^2)$ and $\mu_2=\mathrm{ind}(A^2+BC)$, then*

$$M^d=\begin{pmatrix} \Phi A+(U_1+U_2)C & \Phi B+(U_1+U_2)D \\ \begin{array}{c} C\Phi+C(U_1(Q^d)^2+\Phi U_2)DC \\ +(Q^d)^2C-C\Phi^2(AB+BD)Q^dC \end{array} & D^d+C(U_1+U_2) \end{pmatrix},$$

where

$$\Phi=(A^2+BC)^d=\sum_{i=0}^{t_1-1}(BC)^\pi(BC)^i(A^d)^{2i+2}+\sum_{i=0}^{\nu_1-1}((BC)^d)^{i+1}A^{2i}A^\pi,$$

$$U_1=\sum_{i=0}^{\mu_2-1}(BC)^\pi(A^2+BC)^i(AB+BD)(Q^d)^{2i+4}-\sum_{i=0}^{\mu_2}\Phi A^{2i}(AB+BD)(Q^d)^{2i+2},$$

$$U_2=\sum_{i=0}^{\nu_2-1}\Phi^{i+2}(AB+BD)D^{2i}D^\pi.$$

2. The Drazin inverse of the sum of two matrices

In this section, we mainly investigate the representation of $(P+Q)^d$ under the new conditions as follows.

Theorem 2.1. *Let $P, Q \in \mathbf{C}^{n\times n}$, $r_1 = \operatorname{ind}(QP)$, $r_2 = \operatorname{ind}(P^2)$, $s_1 = \operatorname{ind}(Q^2)$ and $s_2 = \operatorname{ind}(P^2+QP)$. If*

$$P^2QP = 0, Q^2P = 0 \text{ and } (QP)^d = 0,$$

then

$$\begin{aligned}(P+Q)^d &= (\Psi_1)^2(P+Q) + Q\Psi_1 P^d + \Psi_1(\Phi_1+\Phi_2)Q^2 + \Psi_1 Q^d Q + \Phi_1 Q \\ &\quad + Q\Psi_1(P^d)^2 Q + Q\Psi_1 P^d(\Phi_1+\Phi_2)Q^2 + Q^d + Q\Phi_1.\end{aligned}$$

where

$$\begin{aligned}\Psi_1 &= \sum_{i=0}^{r_1-1} P(QP)^i(P^d)^{2i+2},\\ \Phi_1 &= \sum_{i=0}^{s_2-1} P(P^2+QP)^i(P+Q)(Q^d)^{2i+4} - \sum_{i=0}^{s_2}\Psi_1 P^{2i}(P+Q)(Q^d)^{2i+2},\\ \Phi_2 &= \sum_{i=0}^{s_1-1}\Psi_1(P^d)^{2i+2}(P+Q)Q^{2i}Q^\pi.\end{aligned}$$

Proof. Let $P = CB$ be a full rank decomposition of the matrix P. Applying Lemma 1.3, we can deduce that

$$(P+Q)^d = (CB+Q)^d = \left((C\ I)\begin{pmatrix} B \\ Q\end{pmatrix}\right)^d = (C\ I)\left(\begin{pmatrix} BC & B \\ QC & Q\end{pmatrix}^d\right)^2\begin{pmatrix} B \\ Q\end{pmatrix}. \tag{2.1}$$

From $P^2QP = 0$ and $Q^2P = 0$, we have $BCBQC = 0$ and $Q^2C = 0$. In view of Lemma 1.4, we get that

$$\begin{pmatrix} BC & B \\ QC & Q\end{pmatrix}^d = \begin{pmatrix} \widetilde{\Phi}BC + (\widetilde{U}_1+\widetilde{U}_2)QC & \widetilde{\Phi}B + (\widetilde{U}_1+\widetilde{U}_2)Q \\ \triangle & Q^d + QC(\widetilde{U}_1+\widetilde{U}_2)\end{pmatrix},$$

where

$$r_1 = \operatorname{ind}(BQC),\ r_2 = \operatorname{ind}((BC)^2),\ s_1 = \operatorname{ind}(Q^2),\ s_2 = \operatorname{ind}((BC)^2 + BQC)$$

and

$$\begin{aligned}\triangle &= QC\widetilde{\Phi} + QC(\widetilde{U}_1(Q^d)^2 + \widetilde{\Phi}\widetilde{U}_2)Q^2C + (Q^d)^2QC - QC\widetilde{\Phi}^2(BCB+BQ)Q^dQC,\\ \widetilde{\Phi} &= ((BC)^2+BQC)^d = \sum_{i=0}^{r_1-1}(BQC)^\pi(BQC)^i((BC)^d)^{2i+2} + \sum_{i=0}^{r_2-1}((BQC)^d)^{i+1}(BC)^{2i}(BC)^\pi,\\ \widetilde{U}_1 &= \sum_{i=0}^{s_2-1}(BQC)^\pi((BC)^2+BQC)^i(BCB+BQ)(Q^d)^{2i+4} - \sum_{i=0}^{s_2}\widetilde{\Phi}(BC)^{2i}(BCB+BQ)(Q^d)^{2i+2},\\ \widetilde{U}_2 &= \sum_{i=0}^{s_1-1}\widetilde{\Phi}^{i+2}(BCB+BQ)Q^{2i}Q^\pi.\end{aligned} \tag{2.2}$$

According to the conditions $BCBQC = 0$ and $Q^2C = 0$, we get $\triangle = QC\widetilde{\Phi}$ and $(\widetilde{U}_1+\widetilde{U}_2)QC = 0$. Since $(QP)^d = 0$, we have $(BQC)^d = 0$, then $\widetilde{\Phi} = \sum_{i=0}^{r_1-1}(BQC)^i((BC)^d)^{2i+2}$. Thus

$$\begin{pmatrix} BC & B \\ QC & Q\end{pmatrix}^d = \begin{pmatrix} \widetilde{\Phi}BC & \widetilde{\Phi}B + (\widetilde{U}_1+\widetilde{U}_2)Q \\ QC\widetilde{\Phi} & Q^d + QC(\widetilde{U}_1+\widetilde{U}_2)\end{pmatrix}.$$

Let

$$\left(\begin{pmatrix} BC & B \\ QC & Q \end{pmatrix}^d\right)^2 := \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix},$$

where

$$\begin{aligned}
F_1 &= (\widetilde{\Phi}BC)^2 + (\widetilde{\Phi}B + (\widetilde{U}_1 + \widetilde{U}_2)Q)QC\widetilde{\Phi},\\
F_2 &= \widetilde{\Phi}BC(\widetilde{\Phi}B + (\widetilde{U}_1 + \widetilde{U}_2)Q) + (\widetilde{\Phi}B + (\widetilde{U}_1 + \widetilde{U}_2)Q)(Q^d + QC(\widetilde{U}_1 + \widetilde{U}_2)),\\
F_3 &= QC\widetilde{\Phi}^2BC + (Q^d + QC(\widetilde{U}_1 + \widetilde{U}_2))QC\widetilde{\Phi},\\
F_4 &= QC\widetilde{\Phi}(\widetilde{\Phi}B + (\widetilde{U}_1 + \widetilde{U}_2)Q) + (Q^d + QC(\widetilde{U}_1 + \widetilde{U}_2))^2.
\end{aligned}$$

From $Q^2C = 0$, $\widetilde{\Phi}BQC = 0$, $\widetilde{U}_1QQ^d = \widetilde{U}_1$ *and* $\widetilde{U}_2Q^d = 0$, it is easy to verify that

$$\left(\begin{pmatrix} BC & B \\ QC & Q \end{pmatrix}^d\right)^2 = \begin{pmatrix} (\widetilde{\Phi}BC)^2 & \widetilde{\Phi}BC\widetilde{\Phi}B + \widetilde{\Phi}BC(\widetilde{U}_1 + \widetilde{U}_2)Q + \widetilde{\Phi}BQ^d + \widetilde{U}_1 \\ QC\widetilde{\Phi}^2BC & QC\widetilde{\Phi}^2B + QC\widetilde{\Phi}(\widetilde{U}_1 + \widetilde{U}_2)Q + (Q^d)^2 + QC\widetilde{U}_1Q^d \end{pmatrix}. \tag{2.3}$$

Substituting (2.3) into (2.1), we have

$$\begin{aligned}
(P+Q)^d = {}& C(\widetilde{\Phi}BC)^2B + QC\widetilde{\Phi}^2BCB + (C\widetilde{\Phi}B)^2Q + C\widetilde{\Phi}BC(\widetilde{U}_1 + \widetilde{U}_2)Q^2 + C\widetilde{\Phi}BQ^dQ\\
&+ C\widetilde{U}_1Q + QC\widetilde{\Phi}^2BQ + QC\widetilde{\Phi}(\widetilde{U}_1 + \widetilde{U}_2)Q^2 + Q^d + QC\widetilde{U}_1.
\end{aligned} \tag{2.4}$$

Applying Lemma 1.3, $(BC)^dB = B(CB)^d$, $C(BC)^d = (CB)^dC$ and $B((CB)^d)^{i+1}C = ((BC)^d)^i$, for any positive integer i. Let $\Psi_1 = C\widetilde{\Phi}B$. Using (2.2) and induction hypothesis, we see that $C\widetilde{\Phi}^iB = \Psi_1(P^d)^{2i-2}$ for any positive integer i. Furthermore, we can deduce that

$$\begin{aligned}
C\widetilde{U}_1 &= \sum_{i=0}^{s_2-1} C((BC)^2 + BQC)^iB(CB+Q)(Q^d)^{2i+4} - \sum_{i=0}^{s_2} C\widetilde{\Phi}(BC)^{2i}B(CB+Q)(Q^d)^{2i+2}\\
&= \sum_{i=0}^{s_2-1} CB((CB)^2 + QCB)^i(CB+Q)(Q^d)^{2i+4} - \sum_{i=0}^{s_2} C\widetilde{\Phi}B(CB)^{2i}(CB+Q)(Q^d)^{2i+2}\\
&= \sum_{i=0}^{s_2-1} P(P^2 + QP)^i(P+Q)(Q^d)^{2i+4} - \sum_{i=0}^{s_2} \Psi_1P^{2i}(P+Q)(Q^d)^{2i+2},\\
\Psi_1 &= C\sum_{i=0}^{r_1-1}(BQC)^i((BC)^d)^{2i+2}B = \sum_{i=0}^{r_1-1} C(BQC)^iB((CB)^d)^{2i+3}CB\\
&= \sum_{i=0}^{r_1-1} CB(QCB)^i((CB)^d)^{2i+2} = \sum_{i=0}^{r_1-1} P(QP)^i(P^d)^{2i+2},\\
C\widetilde{U}_2 &= C\sum_{i=0}^{s_1-1}\widetilde{\Phi}^{i+2}(BCB+BQ)Q^{2i}Q^{\pi} = \sum_{i=0}^{s_1-1} C\widetilde{\Phi}^{i+2}B(CB+Q)Q^{2i}Q^{\pi}\\
&= \sum_{i=0}^{s_1-1}\Psi_1(P^d)^{2i+2}(P+Q)Q^{2i}Q^{\pi},
\end{aligned}$$

and

$$\begin{aligned}
C\widetilde{\Phi}(\widetilde{U}_1 + \widetilde{U}_2) &= C\widetilde{\Phi}BC(BC)^d(\widetilde{U}_1 + \widetilde{U}_2) = C\widetilde{\Phi}BCB((CB)^d)^2C(\widetilde{U}_1 + \widetilde{U}_2)\\
&= \Psi_1P^dC(\widetilde{U}_1 + \widetilde{U}_2).
\end{aligned}$$

Combining the upper equations and (2.4), we can deduce that

$$\begin{aligned}(P+Q)^d &= (\Psi_1)^2(P+Q) + Q\Psi_1 P^d + \Psi_1(\Phi_1+\Phi_2)Q^2 + \Psi_1 Q^d Q + \Phi_1 Q \\ &\quad + Q\Psi_1(P^d)^2 Q + Q\Psi_1 P^d(\Phi_1+\Phi_2)Q^2 + Q^d + Q\Phi_1,\end{aligned}$$

where $r_1 = \operatorname{ind}(PQ)$, $r_2 = \operatorname{ind}(P^2)$, $s_1 = \operatorname{ind}(Q^2)$, $s_2 = \operatorname{ind}(P^2+PQ)$ and

$$\begin{aligned}\Psi_1 &= \sum_{i=0}^{r_1-1} P(QP)^i(P^d)^{2i+2}, \\ \Phi_1 &= C\widetilde{U}_1 = \sum_{i=0}^{s_2-1} P(P^2+QP)^i(P+Q)(Q^d)^{2i+4} - \sum_{i=0}^{s_2} \Psi_1 P^{2i}(P+Q)(Q^d)^{2i+2}, \\ \Phi_2 &= C\widetilde{U}_2 = \sum_{i=0}^{s_1-1} \Psi_1(P^d)^{2i+2}(P+Q)Q^{2i}Q^{\pi}.\end{aligned}$$

Similarly, we have the following theorem.

Theorem 2.2. *Let $P, Q \in \mathbf{C}^{n\times n}$, $r_1 = \operatorname{ind}(PQ)$, $r_2 = \operatorname{ind}(P^2)$, $s_1 = \operatorname{ind}(Q^2)$ and $s_2 = \operatorname{ind}(P^2+PQ)$. If*

$$PQP^2 = 0, PQ^2 = 0 \text{ and } (PQ)^d = 0,$$

then

$$\begin{aligned}(P+Q)^d &= (P+Q)(\Psi_1)^2 + P^d\Psi_1 Q + Q^2(\Phi_1+\Phi_2)\Psi_1 + QQ^d\Psi_1 + Q\Phi_1 \\ &\quad + Q(P^d)^2\Psi_1 Q + Q^2(\Phi_1+\Phi_2)P^d\Psi_1 Q + Q^d + \Phi_1 Q,\end{aligned}$$

where

$$\begin{aligned}\Psi_1 &= \sum_{i=0}^{r_1-1} (P^d)^{2i+2}(PQ)^i P, \\ \Phi_1 &= \sum_{i=0}^{s_2-1} (Q^d)^{2i+4}(P+Q)(P^2+PQ)^i P - \sum_{i=0}^{s_2} (Q^d)^{2i+2}(P+Q)P^{2i}\Psi_1, \\ \Phi_2 &= \sum_{i=0}^{s_1-1} Q^{\pi}Q^{2i}(P+Q)(P^d)^{2i+2}\Psi_1.\end{aligned}$$

Next, we consider some specializations of our main result.

Corollary 2.3. *[17] Let $P, Q \in \mathbf{C}^{n\times n}$, $\operatorname{ind}(P) = r$ and $\operatorname{ind}(Q) = s$. If*

$$PQP = 0 \text{ and } Q^2P = 0,$$

then

$$\begin{aligned}(P+Q)^d &= P^{\pi}\sum_{i=0}^{r-1} P^i(Q^d)^{i+1} + \sum_{i=0}^{s-1}(P^d)^{i+1}Q^iQ^{\pi} + Q\sum_{i=0}^{s-1}(P^d)^{i+2}Q^iQ^{\pi} \\ &\quad + Q\sum_{i=0}^{s-2} P^{\pi}P^{i+1}(Q^d)^{i+3} - QP^dPQ^{2d} - QP^dQ^d.\end{aligned} \tag{2.5}$$

Proof. According to Theorem 2.1, under the assumption $PQP = 0$ and $Q^2P = 0$, we have

$$\begin{aligned}
\Psi_1 &= P^d,\\
\Phi_1 &= \sum_{i=0}^{s_2-1} P^{2i+1}(P+Q)(Q^d)^{2i+4} - \sum_{i=0}^{s_2} P^d P^{2i}(P+Q)(Q^d)^{2i+2}\\
&= \sum_{i=0}^{s_2-1} P^\pi P^{2i+2}(Q^d)^{2i+4} + \sum_{i=0}^{s_2-1} P^\pi P^{2i+1}(Q^d)^{2i+3} - P^d P(Q^d)^2 - P^d Q^d\\
&= \sum_{i=0}^{s_2-1} P^\pi P^{i+1}(Q^d)^{i+3} - P^d P(Q^d)^2 - P^d Q^d,\\
\Phi_2 &= \sum_{i=0}^{s_1-1} (P^d)^{2i+2} Q^{2i} Q^\pi + \sum_{i=0}^{s_1-1} (P^d)^{2i+3} Q^{2i+1} Q^\pi,
\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}
(P+Q)^d = {} & P^d + (P^d)^2 Q + P^d(\Phi_1+\Phi_2)Q^2 + P^d Q^d Q + \Phi_1 Q + Q^d\\
& + Q(P^d)^2 + Q(P^d)^3 Q + Q(P^d)^2(\Phi_1+\Phi_2)Q^2 + Q\Phi_1.
\end{aligned} \tag{2.7}$$

Substituting (2.6) to (2.7), we get that

$$\begin{aligned}
(P+Q)^d = {} & P^\pi \sum_{i=0}^{r-1} P^i (Q^d)^{i+1} + \sum_{i=0}^{s-1} (P^d)^{i+1} Q^i Q^\pi + Q(P^d)^2 + Q(P^d)^3 Q - Q(P^d)^3 P(Q^d)^2 Q^2\\
& - Q(P^d)^3 Q^d Q^2 + \sum_{i=0}^{s_1-1} Q(P^d)^{2i+4} Q^{2i+2} Q^\pi + \sum_{i=0}^{s_1-1} Q(P^d)^{2i+5} Q^{2i+3} Q^\pi\\
& + \sum_{i=0}^{s_2-1} QP^\pi P^{i+1}(Q^d)^{i+3} - QP^d P(Q^d)^2 - QP^d Q^d\\
= {} & P^\pi \sum_{i=0}^{r-1} P^i (Q^d)^{i+1} + \sum_{i=0}^{s-1} (P^d)^{i+1} Q^i Q^\pi + Q(P^d)^2 + Q(P^d)^3 Q - Q(P^d)^2 Q^d Q\\
& - Q(P^d)^3 Q^d Q^2 + \sum_{i=0}^{s_1-1} Q(P^d)^{2i+4} Q^{2i+2} Q^\pi + \sum_{i=0}^{s_1-1} Q(P^d)^{2i+5} Q^{2i+3} Q^\pi\\
& + \sum_{i=0}^{s_2-1} QP^\pi P^{i+1}(Q^d)^{i+3} - QP^d P Q^{2d} - QP^d Q^d\\
= {} & P^\pi \sum_{i=0}^{r-1} P^i (Q^d)^{i+1} + \sum_{i=0}^{s-1} (P^d)^{i+1} Q^i Q^\pi + Q\sum_{i=0}^{s-1} (P^d)^{i+2} Q^i Q^\pi\\
& + Q\sum_{i=0}^{s-2} P^\pi P^{i+1}(Q^d)^{i+3} - QP^d P Q^{2d} - QP^d Q^d
\end{aligned}$$

Corollary 2.4. *[9] Let $P, Q \in \mathbf{C}^{n\times n}$, $\operatorname{ind}(P) = r$ and $\operatorname{ind}(Q) = s$. If $QP = 0$, then*

$$(P+Q)^d = P^\pi \sum_{i=0}^{r-1} P^i (Q^d)^{i+1} + \sum_{i=0}^{s-1} (P^d)^{i+1} Q^i Q^\pi. \tag{2.8}$$

Proof. From Theorem 2.1 and $QP = 0$, we have

$$\begin{aligned}
\Psi_1 &= P^d, \\
\Phi_1 &= \sum_{i=0}^{s_2-1} P^{2i+1}(P+Q)(Q^d)^{2i+4} - \sum_{i=0}^{s_2} P^d P^{2i}(P+Q)(Q^d)^{2i+2} \\
&= \sum_{i=0}^{s_2-1} P^\pi P^{2i+1}(P+Q)(Q^d)^{2i+4} - P^d(P+Q)(Q^d)^2 \\
&= \sum_{i=0}^{s_2-1} P^\pi P^{2i+2}(Q^d)^{2i+4} + \sum_{i=0}^{s_2-1} P^\pi P^{2i+1}(Q^d)^{2i+3} - P^d P(Q^d)^2 - P^d Q^d, \\
\Phi_2 &= \sum_{i=0}^{s_1-1} (P^d)^{2i+2} Q^{2i} Q^\pi + \sum_{i=0}^{s_1-1} (P^d)^{2i+3} Q^{2i+1} Q^\pi,
\end{aligned} \tag{2.9}$$

and

$$(P+Q)^d = P^d + (P^d)^2 Q + P^d(\Phi_1 + \Phi_2)Q^2 + P^d Q^d Q + \Phi_1 Q + Q^d. \tag{2.10}$$

Substituting (2.9) to (2.10), we get that

$$\begin{aligned}
(P+Q)^d &= P^d + (P^d)^2 Q - (P^d)^2 P(Q^d)^2 Q^2 - (P^d)^2 Q^d Q^2 + \sum_{i=0}^{s_1-1} (P^d)^{2i+3} Q^{2i+2} Q^\pi \\
&\quad + \sum_{i=0}^{s_1-1} (P^d)^{2i+4} Q^{2i+3} Q^\pi + P^d Q^d Q + \sum_{i=0}^{s_2-1} P^\pi P^{2i+2} (Q^d)^{2i+4} Q \\
&\quad + \sum_{i=0}^{s_2-1} P^\pi P^{2i+1} (Q^d)^{2i+3} Q - P^d P (Q^d)^2 Q - P^d Q^d Q + Q^d \\
&= P^d Q^\pi + \sum_{i=0}^{s_1-1} (P^d)^{2i+3} Q^{2i+2} Q^\pi + \sum_{i=0}^{s_1-1} (P^d)^{2i+4} Q^{2i+3} Q^\pi \\
&\quad + \sum_{i=0}^{s_2-1} P^\pi P^{2i+2} (Q^d)^{2i+3} + \sum_{i=0}^{s_2-1} P^\pi P^{2i+1} (Q^d)^{2i+2} - P^\pi Q^d \\
&= P^\pi \sum_{i=0}^{r-1} P^i (Q^d)^{i+1} + \sum_{i=0}^{s-1} (P^d)^{i+1} Q^i Q^\pi.
\end{aligned}$$

Now, we give an example to illustrate our result.

Example 2.5. Let

$$P = \begin{pmatrix} 2 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } Q = \begin{pmatrix} 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

Clearly, $\mathrm{ind}(P^2) = 2$ and $\mathrm{ind}(Q^2) = 1$.

$$P^d = \begin{pmatrix} -1 & 0 & 0 & 0 & -3/2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ -3/2 & 0 & 0 & -3/2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, P^\pi = \begin{pmatrix} 0 & 0 & -1/2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$Q^d = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 \end{pmatrix} \text{ and } Q^\pi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The matrices P and Q do not satisfy the conditions given in Corollary 2.3, but satisfy that in Theorem 2.1. Thus, from Theorem 2.1 it follows that

$$(P+Q)^d = \begin{pmatrix} -1/4 & 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & -3/4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -3/2 & 0 & 0 & -3/2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 \end{pmatrix}.$$

3. Applications

In this section, we first derive some representations for the Drazin inverse of M as applications of Theorem 2.1.

Theorem 3.1. *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{C}^{n\times n}$, $A \in \mathbb{C}^{r\times r}$, *and* $D = CA^dB$. *If* M *satisfies the conditions*

$$A^2A^\pi BC = 0,\ A^dBCA^\pi BC = 0\ , CBCA^\pi BC = 0 \text{ and } CAA^\pi BC = 0,$$

then

$$M^d = M^4(P_1^d)^6[I + \sum_{i=0}^{s-1}(P_1^d)^{i+1}\begin{pmatrix} A^{i+1}A^\pi & 0 \\ CA^iA^\pi & 0 \end{pmatrix}]M,$$

where $s = \operatorname{ind}(A)$ *and for any positive integer* t,

$$(P_1^d)^t = \begin{pmatrix} I \\ CA^d \end{pmatrix}((AW)^d)^{t+1}A\begin{pmatrix} I & A^dB \end{pmatrix}, W = AA^d + A^dBCA^d.$$

Proof. Note that $M = \begin{pmatrix} A & AA^dB \\ C & CA^dB \end{pmatrix} + \begin{pmatrix} 0 & A^\pi B \\ 0 & 0 \end{pmatrix} := P + Q$. Obviously, $Q^2 = 0$. The conditions $A^2A^\pi BC = 0$, $A^dBCA^\pi BC = 0, CAA^\pi BC = 0$ *and* $CBCA^\pi BC = 0$ imply that $P^2QP = 0$ and $(QP)^3 = 0$. According to Theorem 2.1, it yields that

$$M^d = (\Psi_1)^2(P+Q) + Q\Psi_1P^d(I + P^dQ), \tag{3.11}$$

where

$$\Psi_1 = P^d + PQ(P^d)^3 + (PQ)^2(P^d)^5.$$

We consider $P = \begin{pmatrix} A^2A^d & AA^dB \\ CAA^d & CA^dB \end{pmatrix} + \begin{pmatrix} AA^\pi & 0 \\ CA^\pi & 0 \end{pmatrix} := P_1 + P_2$. Observe that $P_2P_1 = 0$ *and* $P_2^{s+1} = 0$, where $s = \mathrm{ind}(A)$. It follows from Lemma 1.2 that

$$P^d = \sum_{i=0}^{s} (P_1^d)^{i+1} P_2^i. \tag{3.12}$$

Decompose P_1^d into the following form

$$P_1^d = \begin{pmatrix} A^2A^d & AA^dB \\ CAA^d & CA^dB \end{pmatrix}^d = \begin{pmatrix} I & 0 \\ CA^d & I \end{pmatrix} \begin{pmatrix} AW & AA^dB \\ 0 & 0 \end{pmatrix}^d \begin{pmatrix} I & 0 \\ -CA^d & I \end{pmatrix},$$

where $W = AA^d + A^dBCA^d$. It follows from Lemma 1.1 that

$$\begin{aligned} \begin{pmatrix} AW & AA^dB \\ 0 & 0 \end{pmatrix}^d &= \begin{pmatrix} (AW)^d & ((AW)^d)^2AA^dB \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} I \\ 0 \end{pmatrix} \begin{pmatrix} (AW)^d & ((AW)^d)^2AA^dB \end{pmatrix}. \end{aligned}$$

Since $(AW)^dA^2A^d = (AW)^dA$, we have

$$P_1^d = \begin{pmatrix} I \\ CA^d \end{pmatrix} ((AW)^d)^2 A \begin{pmatrix} I & A^dB \end{pmatrix}.$$

Computation shows that, for any positive integer t,

$$(P_1^d)^t = \begin{pmatrix} I \\ CA^d \end{pmatrix} ((AW)^d)^{t+1} A \begin{pmatrix} I & A^dB \end{pmatrix}. \tag{3.13}$$

By (3.12), for any positive integer j, we obtain the expression of $(P^d)^j$

$$(P^d)^j = (P_1^d)^j [I + \sum_{i=0}^{s-1} (P_1^d)^{i+1} \begin{pmatrix} A^{i+1}A^\pi & 0 \\ CA^iA^\pi & 0 \end{pmatrix}]. \tag{3.14}$$

We can verify that

$$\begin{aligned} (\Psi_1)^2 =& (P^d)^2 + PQ(P^d)^4 + (PQ)^2(P^d)^6, \\ Q\Psi_1P^d =& Q(P^d)^2 + QPQ(P^d)^4. \end{aligned}$$

Substituting the above two equations and (3.14) into (3.11), we have

$$\begin{aligned} M^d =& ((P^d)^2 + PQ(P^d)^4 + (PQ)^2(P^d)^6)(P+Q) + (Q(P^d)^2 + QPQ(P^d)^4)(I + P^dQ) \\ =& ((P^d)^2 + PQ(P^d)^4 + (PQ)^2(P^d)^6 + Q(P^d)^3 + QPQ(P^d)^5)(P+Q) \\ =& (P+Q)^4(P^d)^6(P+Q) \\ =& M^4(P_1^d)^6[I + \sum_{i=0}^{s-1} (P_1^d)^{i+1} \begin{pmatrix} A^{i+1}A^\pi & 0 \\ CA^iA^\pi & 0 \end{pmatrix}]M. \end{aligned}$$

So we obtain the expression of M^d, as required.

Applying Theorem 3.1, we can get

Corollary 3.2. *[15] Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{C}^{n\times n}$, $A \in \mathbb{C}^{r\times r}$, *and* $D = CA^dB$. *If* M *satisfies the conditions*

$$A^2A^\pi BC = 0,\ BCA^\pi BC = 0 \text{ , and } CAA^\pi BC = 0,$$

then

$$M^d = M^2(P_1^d)^4[I + \sum_{i=0}^{s-1}(P_1^d)^{i+1}\begin{pmatrix} A^{i+1}A^\pi & 0 \\ CA^iA^\pi & 0 \end{pmatrix}]M,$$

where $s = \mathrm{ind}(A)$ *and for any positive integer* t,

$$(P_1^d)^t = \begin{pmatrix} I \\ CA^d \end{pmatrix}((AW)^d)^{t+1}A\begin{pmatrix} I & A^dB \end{pmatrix}, W = AA^d + A^dBCA^d.$$

Proof. The proof follows a decomposition similar to Theorem 3.1. The conditions $A^2A^\pi BC = 0$, $BCA^\pi BC = 0$ and $CAA^\pi BC = 0$ lead to $P^2QP = 0$ and $(QP)^2 = 0$. Applying Theorem 2.1, it yields that

$$M^d = (\Psi_1)^2(P + Q) + Q\Psi_1P^d(I + P^dQ),$$

where

$$\Psi_1 = P^d + PQ(P^d)^3.$$

Similarly as in the proof of Theorem 3.1, we can get

$$\begin{aligned}
(\Psi_1)^2 &= (P^d)^2 + PQ(P^d)^4,\\
Q\Psi_1P^d &= Q(P^d)^2 + QPQ(P^d)^4\\
\text{and } M^d &= ((P^d)^2 + PQ(P^d)^4)(P + Q) + (Q(P^d)^2 + QPQ(P^d)^4)(I + P^dQ)\\
&= ((P^d)^2 + PQ(P^d)^4 + Q(P^d)^3)(P + Q)\\
&= M^2(P_1^d)^4[I + \sum_{i=0}^{s-1}(P_1^d)^{i+1}\begin{pmatrix} A^{i+1}A^\pi & 0 \\ CA^iA^\pi & 0 \end{pmatrix}]M.
\end{aligned}$$

So we obtain the expression of M^d.

At the end of this section, we present two examples to compute the Drazin inverses of the matrices by applying Theorem 3.1.

Example 3.3. Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{C}^{n\times n}$,

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We have $\mathrm{ind}(A) = 3$,

$$A^d = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A^\pi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$P_1^d = \begin{pmatrix} 1/4 & 0 & 0 & 0 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & 0 & 0 & 0 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

From $A^2A^\pi BC = 0, A^dBCA^\pi BC = 0, CBCA^\pi BC = 0, CAA^\pi BC = 0$ and $D = CA^dB$, we get

$$M^d = \begin{pmatrix} 1/4 & 0 & 0 & 1/8 & 1/4 & 1/16 & 0 \\ 3/16 & 0 & 0 & 3/32 & 3/16 & 3/64 & 0 \\ 1/8 & 0 & 0 & 1/16 & 1/8 & 1/32 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & 0 & 0 & 1/8 & 1/4 & 1/16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/16 & 0 & 0 & 1/32 & 1/16 & 1/64 & 0 \end{pmatrix}.$$

Example 3.4. Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{C}^{n\times n}$,

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix} \text{ and } D = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \end{pmatrix}.$$

We have $\mathrm{ind}(A^2) = 1, \mathrm{ind}(CB) = 2$ and

$$A^d = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A^\pi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, (CB)^d = 0, (CB)^\pi = I.$$

From $A^2A^\pi BC = 0$, $A^dBCA^\pi BC = 0$, $CBCA^\pi BC = 0$, $CAA^\pi BC = 0$ and $D = CA^dB$, we get

$$M^d = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \end{pmatrix}.$$

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