Filomat 39:31 (2025), 11177–11184
https://doi.org/10.2298/FIL2531177K

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Modelling with lifts of metallic structures from a Riemannian manifold to the frame bundle

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Abstract. The present paper aims to study the geometric structure $J2-\alpha J-\beta I=0$, α,β are natural numbers, is named as a metallic structure on the frame bundle. It is further demonstrated to have a metallic structure by the definition of a new tensor field $\tilde{J}$ concerning 'the horizontal' and 'vertical lifts' on the frame bundle. Furthermore, certain theorems on the mathematical operators on the frame bundle are proved. Moreover, the Nijenhuis tensor and the Lie derivative of a metallic structure $\tilde{J}$ on the frame bundle are calculated. Finally, an application on the distribution $\tilde{Q}$ on the frame bundle is investigated.

1. Introduction

The study of the geometry of frame bundles was initiated in the early 1960s by Okubo[30], Mok [27, 28], Cordero and León [7–9], Bonome et al [4] and León and Salgado [26]. Cordero et al. [5] have constructed the lift of certain structure on the frame bundle FM of a manifold and defined a prerequisite for the given structure to be Hermitian on the frame bundle. Later, Dodson-Vazquez-Abal [11] have discussed the geometrical interpretation of the sectional curvature, the Ricci curvature, and totally geodesic on the frame bundle. Kowalski and Sekizawa [24] studied the geometry of metrics over Riemannian manifolds and affine manifolds on the linear frame bundle. Recently, the author [22] introduced a (1,1) tensor field $\tilde{J}$ and showed that it is a metallic structure on FM. Also, the Nijenhuis tensor of the metallic structure $\tilde{J}$ is calculated.

On the other hand, let us consider the general quadratic equation $x^2-\alpha x-\beta=0$, where α and β are positive integers whose positive solution is $\sigma_\alpha^\beta=\frac{\alpha+\sqrt{\alpha^2+4\beta}}{2}$ and called metallic means family. The metallic means family consists the Golden mean, the Silver mean, the Subtal mean etc. which has been introduced by Spinadel [35, 36].

Goldberg, Yano and Pedridic [12, 13] studied polynomial structures $Q(P)=P^n+a_nP^{n-1}+......+a_2P+a_1I$ on manifolds. The quadratic structures $P^2-\alpha P-\beta I=0$, where P is a tensor field of type (1,1), α and

2020 *Mathematics Subject Classification*. Primary 53B07, 53C15; Secondary 58D17.
Keywords. Horizontal lift, Vertical lift, Partial differential equations, Mathematical operators, Nijenhuis tensor, Lie derivative, Frame bundle.
Received: 04 March 2025; Accepted: 09 June 2025
Communicated by Ljubica Velimirović
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β are natural numbers and I is the unity on M. Then P is called a metallic structure on M. Hretcamu and Crasmareanu [10, 16] discussed the geometrical interpretation of the metallic and golden structures in the Riemannian manifold. The author [21] studied lifts of the metallic structure to tangent bundle and geometrical properties of the structure are discussed. Gonul et al [15] investigated a neutral relationship between the metallic structure and an almost quadratic metric ϕ-structure on a Riemannian manifold. Papers [1, 3, 17, 18, 21, 23, 32, 37] examined the geometry of metallic structures.

These are the paper's primary contributions, in order of mention:

- A tensor field $\tilde{J}$ of type (1,1) is introduced and proves that it is a metallic structure on FM.
- Certain theorems on the projection operators on the frame bundle are proved.
- Nijenhuis tensor $N_{\tilde{J}}$ of $\tilde{J}$ on FM is calculated.
- Lie derivative of $\tilde{J}$ on FM is determined.
- In the end, an application on the distribution $\tilde{Q}$ on the frame bundle is investigated.

The paper has the following structure: In summary, the frame bundle and "the horizontal lift" are discussed in Section 2. A (1,1) tensor field $\tilde{J}$ is introduced and shows that it is a metallic structure on FM in Section 3. Certain theorems on the mathematical operators on FM are proved. In Section 4, the Nijenhuis tensor and the Lie derivative of a metallic structure $\tilde{J}$ on the frame bundle are computed. Finally, an application on the distribution $\tilde{Q}$ on the frame bundle is investigated.

2. Preliminaries

Assuse that M (dim=n) be a Riemannian manifold and T_xM be the tangent space at a point x of M. A frame p_x at a point x is an ordered basis $(x_1, x_2, ..., x_n)$ of T_xM. The set of all frames at all points of M is called frame bundle and denoted by FM. For any point of FM such that natural bundle $\pi : FM \to M$ i.e. $\pi(p_x) = x$, where x is an element of M. Given a coordinate system (U, x^i), which defines a local coordinate system in neighborhood U. Then $\pi^{-1}(U) = FU$ contains all frames at all points of U. A vector x_α of the frame $p_x \in FU$ is represented by $x_\alpha = x^i_\alpha(\frac{\partial}{\partial x^i})_x$ and $\{FU, (x^i; x^i_\alpha)\}$ is a coordinate system in FM [2, 5, 25].

Define 1-forms θ^γ and ω^ρ_σ on FM by

$$\theta^\gamma = x^\gamma_i dx^i, \tag{1}$$

$$\omega^\rho_\sigma = x^\rho_h(\Gamma^h_{ji} x^i_\sigma dx^j + dx^h_\sigma), \tag{2}$$

where $(x^\gamma_i) = (x^i_\gamma)^{-1}$ and Γ^h_{ji} is the components of linear conection Γ.

Let E_α and E^μ_λ be the global vector fields on FM and their local components on FU in the components of partial differential equations are defined as

$$E_\alpha = x^i_\alpha(\frac{\partial}{\partial x^i} - \Gamma^j_{ik} x^k_\beta \frac{\partial}{\partial x^j_\beta}), \tag{3}$$

$$E^\mu_\lambda = x^j_\lambda \frac{\partial}{\partial x^j_\mu}. \tag{4}$$

Let $gl(n, \mathfrak{R})$ be the Lie algebra of all $n \times n$ square matrices and R^n is the Euclidean n-space. Consider $A = (A^\alpha_\beta)$ be an element of $gl(n, \mathfrak{R})$, then $\lambda A = A^\alpha_\beta E^\beta_\alpha$ [6, 31].

Suppose that S be a tensor field of type $(1, s)$ on M has local components $S^h_{j_1,....,j_s}$ on (U, x^i). The tensor field γS on FM is given by

$$\gamma S = S^h_{j_1,....,j_s} x^\alpha_h x^{j_1}_{\beta_1} ....x^{j_s}_{\beta_s} E^{\beta_1}_\alpha \otimes \theta^{\beta_2} \otimes ..... \otimes \theta^{\beta_s},$$

and tensor field γS on FU with local components in the term of partial differential equations is

$$\gamma S = S^h_{j_1,....,j_s} x^{j_1}_\beta \frac{\partial}{\partial x^h_\beta} \otimes dx^{j_2} .... \otimes dx^{j_s}.$$

Consider a tensor field F of type (1,1) on FM. Then γF is a vertical vector field on FM given by

$$\gamma F = F^h_j x^j_\beta x^\alpha_h E^\beta_\alpha \tag{5}$$

with local components on FU and

$$\gamma F = F^h_j x^j_\beta, \tag{6}$$

where F^h_j is local components of F in U [34].

Let $F^\circ(F^\beta_\alpha)$ be the $n \times n$ square matrix of functions where $F^\beta_\alpha = F^h_j x^j_\alpha x^\beta_h$ globally defined on FM and $A = (A^\alpha_\gamma) \in gl(n, \mathfrak{R})$.

Define a product matrix $F^\circ A = (F^\beta_\alpha A^\alpha_\gamma)$ and a vertical vector field on FM is given by

$$\gamma(F^\circ A) = F^\beta_\alpha A^\alpha_\gamma E^\gamma_\beta. \tag{7}$$

Let X, F and τ on M be a vector field , a tensor field of type (1,1) and a differential 1-form. X^H, F^H and τ^H are horizontal lifts of a vector field X, a tensor field F of type (1,1) and 1-form, respectively [4, 5, 33]

$$\mathsf{X}^H = \mathsf{X}^i \frac{\partial}{\partial x^i} - \mathsf{X}^i \Gamma^h_{ij} x^j_\alpha \frac{\partial}{\partial x^h_\alpha}, \tag{8}$$

$$F^H = F^h_j \frac{\partial}{\partial x^h} \otimes dx^j + x^k_\alpha (\Gamma^h_{jk} F^h_i - \Gamma^h_{ik} F^i_j) \frac{\partial}{\partial x^h_\alpha} \otimes dx^j + \delta^\beta_\alpha F^h_j \frac{\partial}{\partial x^h_\alpha} \otimes dx^j_\beta, \tag{9}$$

$$\tau^H = \sum_{\alpha=1}^{n} (x^j_\alpha \Gamma^h_{ij} \tau_h dx^i + \tau_i dx^i_\alpha), \tag{10}$$

where Γ^h_{ij}, x^i and F^h_j are local elements of ∇, X and F, respectively in M.

The horizontal lifts formulae shown below are provided by

$$\begin{aligned} \tau^H(\mathsf{X}^H) &= 0, \\ \tau^H(\lambda A) &= \tau^C(\lambda A), \\ F^H \mathsf{X}^H &= (F\mathsf{X})^H, \\ F^H(\lambda A) &= F^C(\lambda A) = \lambda(F^\circ A), \\ [\mathsf{X}^H, \mathsf{Y}^H] &= [\mathsf{X}, \mathsf{Y}]^H - \gamma R(\mathsf{X}, \mathsf{Y}), \\ [\mathsf{X}^H, \lambda A] &= 0, \end{aligned} \tag{11}$$

for any vector field X, a tensor field F of type (1,1), 1-form τ on M, for any $A \in gl(n, \mathfrak{R})$ and R is a curvature tensor of a linear connection [29].

3. Some theorems regarding metallic structures on *FM*

In [5], a tensor field $F_p, p = 1, 2, ..., n$ of type (1,1) on FM is defined as

$$F_p \mathsf{X}^H = -\mathsf{X}^{(p)}, \quad F_p \mathsf{X}^{(q)} = \delta^q_p \mathsf{X}^H. \tag{12}$$

It was proven that $F^3_p + F_p = 0$ [5].

In [14], a tensor field $\tilde{J}$ in tangent bundle TM is defined by

$$\begin{aligned}\tilde{J}\mathsf{X}^H &= \frac{1}{2}(\alpha\mathsf{X}^H + (2\sigma_\alpha^\beta - \alpha)(\mathsf{X}\otimes\tilde{E}^V)),\\ \tilde{J}\mathsf{X}^V &= \frac{1}{2}(\alpha(\mathsf{X}\otimes\tilde{E}^V + (2\sigma_\alpha^\beta - \alpha)\mathsf{X}^H),\\ \tilde{J}A^V &= \sigma_\alpha^\beta A^V,\end{aligned}$$

where X^H and X^V are 'the horizontal' and 'vertical lifts' of a vector field X, $\sigma_\alpha^\beta = \frac{\alpha+\sqrt{\alpha^2+4\beta}}{2}$, α and β are positive integers [14, 19].

From above theories, let X^H and λA be 'the horizontal' and 'vertical lifts' of a vector field X and A on FM concerning ∇ of a Riemannian metric g. A tensor field $\tilde{J}$ of type (1,1) is introduced on FM by

$$\begin{aligned}\tilde{J}\mathsf{X}^H &= \frac{1}{2}\{\alpha\mathsf{X}^H + (2\sigma_\alpha^\beta - \alpha)\lambda A\},\\ \tilde{J}\lambda A &= \frac{1}{2}\{\alpha\lambda A + (2\sigma_\alpha^\beta - \alpha)\mathsf{X}^H\},\end{aligned} \tag{13}$$

where $\sigma_\alpha^\beta = \frac{\alpha+\sqrt{\alpha^2+4\beta}}{2}$, α and β are positive integers.

Theorem 3.1. *Assume that the frame bundle of M is FM. Consequently, a metallic structure on FM is a tensor field $\tilde{J}$, described by equation (13).*

Proof. It suffices to demonstrate that $\tilde{J}^2 - \alpha\tilde{J} - \beta I = 0$ in order to establish $\tilde{J}$ as a metallic structure.

In the view of equation (13), then

$$\begin{aligned}(\tilde{J}^2 - \alpha\tilde{J} - \beta I)\mathsf{X}^H &= \tilde{J}(\tilde{J}\mathsf{X}^H) - \alpha\tilde{J}\mathsf{X}^H - \beta\mathsf{X}^H\\ &= \tilde{J}\frac{1}{2}\{\alpha\mathsf{X}^H + (2\sigma_\alpha^\beta - \alpha)\lambda A\} - \frac{\alpha}{2}\{\alpha\mathsf{X}^H + (2\sigma_\alpha^\beta - \alpha)\lambda A\} - \beta\mathsf{X}^H\\ &= \frac{\alpha}{4}\{\alpha\mathsf{X}^H + (2\sigma_\alpha^\beta - \alpha)\lambda A\} + \frac{(2\sigma_\alpha^\beta - \alpha)}{4}\{\alpha\lambda A + (2\sigma_\alpha^\beta - \alpha)\mathsf{X}^H\}\\ &\quad -\frac{\alpha}{2}\{\alpha\mathsf{X}^H + (2\sigma_\alpha^\beta - \alpha)\lambda A\} - \beta\mathsf{X}^H,\\ &= 0.\end{aligned}$$

Likewise, it can simply demonstrated that $(\tilde{J}^2 - \alpha\tilde{J} - \beta I)\lambda A = 0$ which imply that $\tilde{J}^2 - \alpha\tilde{J} - \beta I = 0$.

Hence, $\tilde{J}$ is a metallic structure on FM. This completes the proof.

Let $\tilde{\mathsf{l}}$ and $\tilde{\mathsf{m}}$ repesent projection operators in FM and defined by [20, 38]

$$\tilde{\mathsf{l}} = \frac{\tilde{J}^2 - \alpha\tilde{J}}{\beta}, \tag{14}$$

$$\tilde{\mathsf{m}} = I - \frac{\tilde{J}^2 - \alpha\tilde{J}}{\beta}, \tag{15}$$

where l and m in M.

Theorem 3.2. *Assume that $\tilde{J}$ be the metallic structure and $\tilde{\mathsf{l}}$ and $\tilde{\mathsf{m}}$ be the projection operators in FM. Then*

$$\tilde{\mathsf{l}} + \tilde{\mathsf{m}} = 0, \quad \tilde{\mathsf{l}}^2 = \tilde{\mathsf{l}}, \quad \tilde{\mathsf{m}}^2 = \tilde{\mathsf{m}}, \quad \tilde{\mathsf{l}}\tilde{\mathsf{m}} = \tilde{\mathsf{m}}\tilde{\mathsf{l}} = 0 \tag{16}$$

$$\tilde{J}\tilde{\mathsf{l}} = \tilde{\mathsf{l}}\tilde{J} = \tilde{J}, \quad \tilde{J}\tilde{\mathsf{m}} = \tilde{\mathsf{m}}\tilde{J} = 0. \tag{17}$$

Proof. Equation (14) is used to provide the evidence of Theorem 3.2.

The equations (16) and (17) show that there exist two complementary distributions $\tilde{P}$ and $\tilde{Q}$ in the frame bundle *FM* corresponding to $\tilde{\mathsf{l}}$ and $\tilde{\mathsf{m}}$, respectively.

Theorem 3.3. *Assume that horizontal lift* X^H *and vertical lift* λA *of* X *in FM. Then*

$$\begin{aligned} \tilde{\mathsf{l}}(\mathsf{X}^H) &= \mathsf{X}^H, \quad \tilde{\mathsf{l}}(\lambda A) = \lambda A, \\ \tilde{\mathsf{m}}(\mathsf{X}^H) &= 0, \quad \tilde{\mathsf{m}}(\lambda A) = 0, \end{aligned}$$

for all $\tilde{\mathsf{l}}$ *and* $\tilde{\mathsf{m}}$ *are in FM.*

Proof. By employing (13) and (14), we infer

$$\begin{aligned} \tilde{\mathsf{l}}(\mathsf{X}^H) &= \frac{1}{\beta}\tilde{J}^2(\mathsf{X}^H) - \frac{\alpha}{\beta}\tilde{J}(\mathsf{X}^H), \\ \beta\tilde{\mathsf{l}}(\mathsf{X}^H) &= \tilde{J}[\frac{1}{2}(\alpha\mathsf{X}^H + (2\sigma_\alpha^\beta - \alpha)\lambda A)] - \alpha[\frac{1}{2}(\alpha\mathsf{X}^H + (2\sigma_\alpha^\beta - \alpha)\lambda A)], \\ 2\beta\tilde{\mathsf{l}}(\mathsf{X}^H) &= \frac{\alpha}{2}[(\alpha\mathsf{X}^H + (2\sigma_\alpha^\beta - \alpha)\lambda A)] - \alpha[\frac{(2\sigma_\alpha^\beta - \alpha)}{2}[(\alpha\lambda A + (2\sigma_\alpha^\beta - \alpha)\mathsf{X}^H)] - \alpha^2\mathsf{X}^H - \alpha(2\sigma_\alpha^\beta - \alpha)\lambda A, \\ \tilde{\mathsf{l}}(\mathsf{X}^H) &= \mathsf{X}^H. \end{aligned}$$

It is easy to obtain other identities using comparable equipment.

4. Some calculations on the Nijenhuis tensor and the Lie derivative

In this section, the Nijenhuis tensor $N_{\tilde{J}}$ and the Lie derivative $\pounds_{\tilde{\mathsf{X}}}\tilde{J}$ of $\tilde{J}$ on *FM* are computed. Consider $\tilde{\mathsf{X}}$ and $\tilde{\mathsf{Y}}$ are vector fields on *FM* and $N_{\tilde{J}}$ defined as

$$N_{\tilde{J}}(\tilde{\mathsf{X}}, \tilde{\mathsf{Y}}) = [\tilde{J}\tilde{\mathsf{X}}, \tilde{J}\tilde{\mathsf{Y}}] - \tilde{J}[\tilde{J}\tilde{\mathsf{X}}, \tilde{\mathsf{Y}}] - \tilde{J}[\tilde{\mathsf{X}}, \tilde{J}\tilde{\mathsf{Y}}] + \tilde{J}^2[\tilde{\mathsf{X}}, \tilde{\mathsf{Y}}]. \tag{18}$$

Theorem 4.1. *Let* X^H *and* λA *be 'the horizontal' and 'vertical lifts' of a vector field* X *on FM. The Nijenhuis tensor* $N_{\tilde{J}}$ *of* $\tilde{J}$ *is given by*

$$\begin{aligned} (i)\ N_{\tilde{J}}(\mathsf{X}^H, \mathsf{Y}^H) &= \left(\left(\frac{2\sigma_\alpha^\beta - \alpha}{2}\right)^2 + \alpha J\right)\left([\mathsf{X}, \mathsf{Y}]^H - \gamma R(\mathsf{X}, \mathsf{Y})\right), \\ (ii)\ N_{\tilde{J}}(\mathsf{X}^H, \lambda A) &= \left(\frac{2\sigma_\alpha^\beta - \alpha}{2}\right)\left(\frac{\alpha}{2} - J\right)\left([\mathsf{X}, \mathsf{Y}]^H - \gamma R(\mathsf{X}, \mathsf{Y})\right), \\ (iii)\ N_{\tilde{J}}(\lambda A, \lambda B) &= \left(\frac{2\sigma_\alpha^\beta - \alpha}{2}\right)^2\left([\mathsf{X}, \mathsf{Y}]^H - \gamma R(\mathsf{X}, \mathsf{Y})\right) + (\alpha\tilde{J} + \beta I)\lambda[A, B], \end{aligned} \tag{19}$$

for all vector fields X, Y *on M.*

Proof. (i) Equation (18) is used after setting $\tilde{\mathsf{X}} = \mathsf{X}^H$ and $\tilde{\mathsf{Y}} = \mathsf{Y}^H$, and utilizing equation (13), we infer

$$\begin{aligned} N_{\tilde{J}}(\mathsf{X}^H, \mathsf{Y}^H) &= [\tilde{J}\mathsf{X}^H, \tilde{J}\mathsf{Y}^H] - \tilde{J}[\tilde{J}\mathsf{X}^H, \mathsf{Y}^H] - \tilde{J}[\mathsf{X}^H, \tilde{J}\mathsf{Y}^H] + \tilde{J}^2[\mathsf{X}^H, \mathsf{Y}^H] \\ &= [\frac{1}{2}\{\alpha\mathsf{X}^H + (2\sigma_\alpha^\beta - \alpha)\lambda A\}, \frac{1}{2}\{\alpha\mathsf{Y}^H + (2\sigma_\alpha^\beta - \alpha)\lambda B\}] \\ &- \tilde{J}[\frac{1}{2}\{\alpha\mathsf{X}^H + (2\sigma_\alpha^\beta - \alpha)\lambda A\}, \mathsf{Y}^H] \\ &- [\mathsf{X}^H, \frac{1}{2}\{\alpha\mathsf{Y}^H + (2\sigma_\alpha^\beta - \alpha)\lambda B\}] + \tilde{J}^2\{[\mathsf{X}, \mathsf{Y}]^H - \gamma R(\mathsf{X}, \mathsf{Y})\} \\ &= \left(\left(\frac{2\sigma_\alpha^\beta - \alpha}{2}\right)^2 + \alpha J\right)\left([\mathsf{X}, \mathsf{Y}]^H - \gamma R(\mathsf{X}, \mathsf{Y})\right). \end{aligned}$$

(ii) Equation (18) is used after setting $\tilde{X} = \lambda A$ and $\tilde{Y} = Y^H$, and utilizing equation (13), we infer

$$\begin{aligned}
N_{\tilde{J}}(\lambda A, Y^H) &= [\tilde{J}\lambda A, \tilde{J}Y^H] - \tilde{J}[\tilde{J}\lambda A, Y^H] - \tilde{J}[\lambda A, \tilde{J}Y^H] + \tilde{J}^2[\lambda A, Y^H] \\
&= [\frac{1}{2}\{\alpha\lambda A + (2\sigma_\alpha^\beta - \alpha)X^H\}, \frac{1}{2}\{\alpha Y^H + (2\sigma_\alpha^\beta - \alpha)\lambda B\}] \\
&- \tilde{J}[\frac{1}{2}\{\alpha\lambda A + (2\sigma_\alpha^\beta - \alpha)X^H\}, Y^H] \\
&- \tilde{J}[\lambda A, \frac{1}{2}\{\alpha Y^H + (2\sigma_\alpha^\beta - \alpha)\lambda B\}] + \tilde{J}^2\{[X, Y]^H - \gamma R(X, Y)\} \\
&= \left(\frac{2\sigma_\alpha^\beta - \alpha}{2}\right)\left(\frac{\alpha}{2} - J\right)\left([X, Y]^H - \gamma R(X, Y)\right).
\end{aligned}$$

(iii) Equation (18) is used after setting $\tilde{X} = \lambda A$ and $\tilde{Y} = \lambda B$, and utilizing equation (13), we infer

$$\begin{aligned}
N_{\tilde{J}}(\lambda A, \lambda B) &= [\tilde{J}\lambda A, \tilde{J}\lambda B] - \tilde{J}[\tilde{J}\lambda A, \lambda B] - \tilde{J}[\lambda A, \tilde{J}\lambda B] + \tilde{J}^2[\lambda A, \lambda B], \\
&= [\frac{1}{2}\{\alpha\lambda A + (2\sigma_\alpha^\beta - \alpha)X^H\}, \frac{1}{2}\{\alpha\lambda B + (2\sigma_\alpha^\beta - \alpha)Y^H\}] \\
&- \tilde{J}[\frac{1}{2}\{\alpha\lambda A + (2\sigma_\alpha^\beta - \alpha)X^H, \lambda B] \\
&- \tilde{J}[X^H, \frac{1}{2}\{\alpha\lambda B + (2\sigma_\alpha^\beta - \alpha)Y^H\}] + (\alpha\tilde{J} + \beta I)\lambda[A, B] \\
&= \left(\frac{2\sigma_\alpha^\beta - \alpha}{2}\right)^2\left([X, Y]^H - \gamma R(X, Y)\right) + (\alpha\tilde{J} + \beta I)\lambda[A, B].
\end{aligned}$$

This completes the proof.

The Lie derivative $\pounds_{\tilde{X}}\tilde{J}$ of $\tilde{J}$ is given as

$$(\pounds_{\tilde{X}}\tilde{J})\tilde{Y} = [\tilde{X}, \tilde{J}\tilde{Y}] - \tilde{J}[\tilde{X}, \tilde{Y}], \tag{20}$$

$\forall \tilde{X}, \tilde{Y} \in \mathfrak{I}_0^1(FM)$.

Theorem 4.2. *Assume that the frame bundle of M is FM. Consequently, the Lie derivative $\pounds_{\tilde{X}}\tilde{J}$ of $\tilde{J}$, described by equation (13), is*

$$(\pounds_{X^H}\tilde{J})Y^H = \frac{\alpha}{2}\gamma R(X, Y) - \tilde{J}\gamma R(X, Y), \tag{21}$$

$$(\pounds_{X^H}\tilde{J})\lambda A = \frac{(2\sigma_\alpha^\beta - \alpha)}{2}([X, Y]^H - \gamma R(X, Y)) - \frac{(2\sigma_\alpha^\beta - \alpha)}{2}\lambda[A, B], \tag{22}$$

$$(\pounds_{\lambda A}\tilde{J})Y^H = \frac{(2\sigma_\alpha^\beta - \alpha)}{2}\lambda[A, B], \tag{23}$$

$$(\pounds_{\lambda A}\tilde{J})\lambda B = -\frac{(2\sigma_\alpha^\beta - \alpha)}{2}[X, Y]^H, \tag{24}$$

for any $X, Y \in \mathfrak{I}_0^1(M), X^H, \lambda A \in \mathfrak{I}_0^1(FM)$.

Proof. Equation (20) is used after setting $\tilde{X} = X^H$ and $\tilde{Y} = Y^H$ and u,tilizing equation (13), we infer

$$\begin{aligned}
(\pounds_{X^H}\tilde{J})Y^H &= [\tilde{X}, \frac{1}{2}(\alpha Y^H + (2\sigma_\alpha^\beta - \alpha)\lambda B)] - \tilde{J}[X, Y]^H - \tilde{J}\gamma R(X, Y), \\
&= [X^H, \frac{1}{2}\alpha Y^H] + \frac{1}{2}(2\sigma_\alpha^\beta - \alpha)[X^H, \lambda B] - \tilde{J}[X, Y]^H - \tilde{J}\gamma R(X, Y), \\
&= \frac{1}{2}\alpha[X^H, Y^H] - \tilde{J}[X, Y]^H - \tilde{J}\gamma R(X, Y), \\
&= \frac{1}{2}\alpha[X, Y]^H - \frac{1}{2}\gamma R(X, Y) - \frac{1}{2}\alpha[X, Y]^H - \tilde{J}\gamma R(X, Y), \\
&= \frac{1}{2}\gamma R(X, Y) - \tilde{J}\gamma R(X, Y), \qquad (25)
\end{aligned}$$

for any $X, Y \in \Im_0^1(M)$.

Using the similar devices, equations (22), (23) and (24) can be easily obtained.

5. Application

This section examines an intriguing use of the formulae found in equation (11) and Theorem (3.3) on the distribution $\tilde{Q}$ on the frame bundle is integrable.

Let $\tilde{\mathfrak{l}}$ and $\tilde{\mathfrak{m}}$ be the projection operators given in equation (14) and there exist the complementary distributions $\tilde{P}$ and $\tilde{Q}$ on the frame bundle FM respectively.

By using equation (11) and Theorem (3.3), the following identities are obtained.

$$\begin{aligned}
\tilde{\mathfrak{m}}[\tilde{\mathfrak{l}}\lambda B, \tilde{\mathfrak{l}}\lambda A] &= 0, \qquad (26) \\
\tilde{\mathfrak{m}}[\tilde{\mathfrak{l}}X^H, \tilde{\mathfrak{l}}\lambda A] &= 0, \qquad (27) \\
\tilde{\mathfrak{m}}[\tilde{\mathfrak{l}}X^H, \tilde{\mathfrak{l}}Y^H] &= \tilde{\mathfrak{m}}[X, Y]^H - \tilde{\mathfrak{m}}(\gamma R(X, Y)), \qquad (28)
\end{aligned}$$

where any $A, B \in gl(n, \Re)$ and R is a curvature tensor of a linear connection. As a result, the following theorem is proved:

Theorem 5.1. *Let FM be the frame bundle of M and $\tilde{J}$ be a metallic structure discribed in equation (13). Assume that $\tilde{\mathfrak{l}}$ and $\tilde{\mathfrak{m}}$ be the projection operators on FM. Then $\tilde{Q}$ on FM is integrable iff $\tilde{\mathfrak{m}}[X, Y]^H = 0$ and $\tilde{\mathfrak{m}}(\gamma R(X, Y)) = 0$, for any $X, Y \in \Im_0^1(M)$.*

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