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Optimal solutions of a system of integro differential equations via measure of noncompactness

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Abstract. This article uses a new contraction operator and a generalization of the best proximity point (pair) theorem to investigate whether there is an optimal solution for the proposed system of integro differential equations. The application of our findings has also been illustrated with a suitable example.

1. Introduction

Different types of integral equations can be solved by using the best proximity point theorem and the measure of noncompactness (MNC). Ky Fan proved the first best proximity point theorem. We defined a map $\hat{\mathcal{G}} : Z \to \bar{U}$ for a nonempty subset Z of a normed linear space (NLS) $\bar{U}$. The best approximation point of $\hat{\mathcal{G}}$ in $\bar{U}$ is represented by $\bar{g} \in Z$, if the distance between $\bar{g} \in Z$ and $\hat{\mathcal{G}}(\bar{g})$ is as small as possible.

The format of the paper will be as follows: we start out by going over a few fundamental concepts and terms related to best proximity point theory. Next, we construct the best proximity point theorem for both cyclic and noncyclic contractive operators. We then provide examples of their specific situations. Finally, we apply our findings to investigate the optimal solutions of a system of Integro differential equations.

In order to accomplish the major extensions of the theory of compact operators, numerous researchers looked into the use of the concept of MNC, which was first introduced by Kuratowski and then further expanded by Hausdorff. Applying MNC to see if a mapping satisfies some important inequalities is a crucial skill. So, in order to aid the reader in comprehending our issue and goal, we have included a brief history. Using certain regularity assumptions from Schauder [2], we reexamine the basic fixed point problem in a Banach space $\hat{\mathtt{M}}$. The authors of [18] originally addressed the *BPP* results using MNC. They then used these findings to look into the possibility of optimal solutions for a system of second-order differential equations. Formulating theoretical results for the qualitative analysis of fractional-order integro differential equations with integral type conditions is the author's goal in [23]. Fractional calculus and fixed point theory are used to derive the concept. An example is being investigated as well for application

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and numerical verification purposes. To investigate the existence of best proximity points (best proximity pairs) with the help of a suitable measure of noncompactness, in [11], the authors introduced new classes of mappings, called cyclic (noncyclic) condensing operators, and they obtained some real generalizations of Schauder and Darbo's fixed point theorems. They applied their results to study the existence of optimum solutions to a system of differential equations. In [13], the authors used best proximity point methods and measure of noncompactness theory to investigate the existence of a solution for a system of differential equations. The authors of [12] introduced a new notion of cyclic (noncyclic) mappings involving measure of noncompactness and obtained some new existence results of best proximity points (pairs). In application part, they study the existence of an optimal solution for a system of integro differential equations under new sufficient conditions in a reflexive and Busemann convex space. In [21], the authors proved the existence of best proximity points (pairs) using the notion of measure of noncompactness. They introduced generalized classes of cyclic (noncyclic) F-contractive operators, and then derive best proximity point (pair) results in Banach (strictly convex Banach) spaces. At last, they applied their findings to investigate the existence of optimum solutions for a system of Hilfer fractional differential equations. The authors of [19] established the existence of *BPP* for Sadovskii type mappings, and they extended a Sadovskii type fixed point theorem. After that, they applied their results to investigate the existence of an optimal solution for a system of fractional differential equations.

Inspired by these research works, we created a newly defined contraction operator, then used MNC to build a best proximity point theorem and find out if there are any optimal solutions in Banach space for a system of integro differential equations.

Theorem 1.1. *[10] For a nonempty, bounded, closed and convex subset* A *of a Banach space* $\hat{\mathtt{M}}$*, consider* $\mathcal{T}:\mathsf{A}\to\mathsf{A}$ *be continuous and compact, then* $\mathcal{T}$ *admits atleast a fixed point.*

Clearly, it is the generalization of Brouwer fixed point theorem.

Consider a Banach space $\hat{\mathtt{M}}$ and a closed ball $\tilde{\mathcal{D}}(\bar{\mu},\bar{\nu})=\{\bar{h}\in\hat{\mathtt{M}}:\|\bar{h}-\bar{\mu}\|\ \leq\bar{\nu}\}$ in $\hat{\mathtt{M}}$. Suppose $\bar{I}$ (for all nonempty set I) denotes the closure of I, and $\overline{conv}\,(I)$ denotes the closed and convex hull of the nonempty set I which is the smallest convex and closed set containing I.

Also $\acute{\mathsf{V}}_{\hat{\mathtt{M}}}$ and $J_{\hat{\mathtt{M}}}$ represents the family of nonempty bounded subsets of $\hat{\mathtt{M}}$ and the subfamily of $\hat{\mathtt{M}}$ consisting all relatively compact sets respectively, $\tilde{\mathtt{R}}_{+}=[0,\infty)$, $\tilde{\mathtt{R}}=(-\infty,\infty)$, and $\mathbb{N}$ represents the set of natural numbers.

A measure of noncompactness (MNC) is defined axiomatically as follows:

Definition 1.2. *[1] A map* $\aleph:\acute{\mathsf{V}}_{\hat{\mathtt{M}}}\to\tilde{\mathtt{R}}_{+}$ *is a* MNC *(measure of noncompactness) in the Banach space* $\hat{\mathtt{M}}$*, if the following conditions are holds for* $\aleph$*:*

1. $ker\aleph=\{\hat{\mathsf{S}}\in\acute{\mathsf{V}}_{\hat{\mathtt{M}}}:\aleph\left(\hat{\mathsf{S}}\right)=0\}\neq\phi$,
2. $\hat{\mathsf{S}}\in ker\aleph$ *if and only if* $\hat{\mathsf{S}}$ *is relatively compact,*
3. $\hat{\mathsf{S}}_1\subseteq\hat{\mathsf{S}}_2\Rightarrow\aleph\left(\hat{\mathsf{S}}_1\right)\leq\aleph\left(\hat{\mathsf{S}}_2\right)$,
4. $\aleph\left(\bar{\hat{\mathsf{S}}}\right)=\aleph\left(\hat{\mathsf{S}}\right)$,
5. $\aleph\left(\overline{conv}(\hat{\mathsf{S}})\right)=\aleph\left(\hat{\mathsf{S}}\right)$,
6. $\aleph\left(\breve{\beta}\hat{\mathsf{S}}_1+\left(1-\breve{\beta}\right)\hat{\mathsf{S}}_2\right)\leq\breve{\beta}\aleph\left(\hat{\mathsf{S}}_1\right)+\left(1-\breve{\beta}\right)\aleph\left(\hat{\mathsf{S}}_2\right)$, *for* $\breve{\beta}\in[0,1]$,
7. $\max\{\aleph\left(\hat{\mathsf{S}}_1\right),\aleph\left(\hat{\mathsf{S}}_2\right)\}=\aleph\left(\hat{\mathsf{S}}_1\cup\hat{\mathsf{S}}_2\right)$,
8. *The set* $\hat{\mathsf{S}}_\infty=\cap_{n=1}^{\infty}\hat{\mathsf{S}}_n$ *is compact and non empty, if* $\left(\hat{\mathsf{S}}_n\right)$ *is a decreasing sequence of closed sets which are non empty in* $\acute{\mathsf{V}}_{\hat{\mathtt{M}}}$ *and* $\lim_{n\to\infty}\aleph\left(\hat{\mathsf{S}}_n\right)=0$.

In particular, the space $\hat{\mathtt{M}}=C(\mathtt{I})$, where $\mathtt{I}$ is the closed and bounded interval, is the set of real valued continuous functions on $\mathtt{I}$. Then $\hat{\mathtt{M}}$ is a Banach space with the norm

$\|V\| = \sup\{|V(\check{h})| : \check{h} \in \mathtt{I}\}, V \in \hat{\mathtt{M}}$.

Assume $\acute{N}(\neq \phi) \subseteq \hat{\mathtt{M}}$ be bounded. For $V \in \acute{N}$ and $s > 0$, the modulus of continuity of V, represented by $\hat{\mathcal{H}}(V, s)$ i.e., $\hat{\mathcal{H}}(V, s) = \sup\{|V(\acute{d}_1) - V(\acute{d}_2)| : \acute{d}_1, \acute{d}_2 \in \mathtt{I}, |\acute{d}_1 - \acute{d}_2| \leq s\}$.
Furthermore, we define
$\hat{\mathcal{H}}(\acute{N}, s) = \sup\{\hat{\mathcal{H}}(V, s) : V \in \acute{N}\}; \aleph_0(\acute{N}) = \lim_{s\to 0} \hat{\mathcal{H}}(\acute{N}, s)$.
A Housdorff MNC $\tilde{\mathtt{L}}$ is given by
$\tilde{\mathtt{L}}(\acute{N}) = \frac{1}{2}\aleph_0(\acute{N})$(see [2]).
It is widely known that the map $\aleph_0$ is a MNC in $\hat{\mathtt{M}}$.

2. preliminaries

We collect some fundamental definitions and notations needed for the paper.

Definition 2.1. *[10] Consider $\hat{\mathtt{M}}$ be a Banach space. Then $\hat{\mathtt{M}}$ is a strictly convex Banach space, if, for all $\tilde{r}, \tilde{t}, \tilde{s} \in \hat{\mathtt{M}}$ and $\tilde{\mathcal{L}} > 0$, the following conditions hold:*

$$\begin{cases} \|\tilde{r} - \tilde{s}\| \leq \tilde{\mathcal{L}}, \\ \|\tilde{t} - \tilde{s}\| \leq \tilde{\mathcal{L}}, \Longrightarrow \|\frac{\tilde{r}+\tilde{t}}{2} - \tilde{s}\| < \tilde{\mathcal{L}}. \\ \tilde{r} \neq \tilde{t}. \end{cases}$$

Example 2.2. *Hilbert spaces and l^p spaces $(1 < p < \infty)$ are strictly convex Banach spaces.*

Consider a normed linear space (NLS) $\bar{U}$. For any two nonempty subset $\mathtt{H}_1, \mathtt{H}_2$ of $\bar{U}$, the pair $(\mathtt{H}_1, \mathtt{H}_2)$ is closed $\Longleftrightarrow$ both $\mathtt{H}_1$, $\mathtt{H}_2$ are closed; $(\mathtt{H}_1, \mathtt{H}_2) \subseteq \left(\bar{\mathcal{E}}_1, \bar{\mathcal{E}}_2\right) \Longleftrightarrow \mathtt{H}_1 \subseteq \bar{\mathcal{E}}_1, \mathtt{H}_2 \subseteq \bar{\mathcal{E}}_2$.

In addition, we denote by $dist\left(\bar{\mathcal{E}}_1, \bar{\mathcal{E}}_2\right) = \inf\left\{\|\tilde{x} - \tilde{y}\| : (\tilde{x}, \tilde{y}) \in \bar{\mathcal{E}}_1 \times \bar{\mathcal{E}}_2\right\}$,
$\bar{\bar{\mathcal{E}}}_1 = \{\tilde{x} \in \bar{\mathcal{E}}_1 :$ *there exists* $\tilde{y}_1 \in \bar{\mathcal{E}}_2$ *so that* $\|\tilde{x} - \tilde{y}_1\| = dist\left(\bar{\mathcal{E}}_1, \bar{\mathcal{E}}_2\right)\}$,
$\bar{\bar{\mathcal{E}}}_2 = \{\tilde{y} \in \bar{\mathcal{E}}_2 :$ *there exists* $\tilde{x}_1 \in \bar{\mathcal{E}}_1$ *so that* $\|\tilde{x}_1 - \tilde{y}\| = dist\left(\bar{\mathcal{E}}_1, \bar{\mathcal{E}}_2\right)\}$.

Definition 2.3. *[10] Consider $\bar{U}$ as a* NLS. *A nonempty pair $\left(\bar{\mathcal{E}}_1, \bar{\mathcal{E}}_2\right)$ of $\bar{U}$ is proximinal, if $\bar{\mathcal{E}}_1 = \bar{\bar{\mathcal{E}}}_1$ and $\bar{\mathcal{E}}_2 = \bar{\bar{\mathcal{E}}}_2$. For a reflexive Banach space Q, if the pair $\left(\bar{\mathcal{E}}_1, \bar{\mathcal{E}}_2\right)$ be a closed, nonempty, convex and bounded in Q, then $\left(\bar{\bar{\mathcal{E}}}_1, \bar{\bar{\mathcal{E}}}_2\right)$ is also a closed, nonempty, convex and bounded pair.*

Consider a function $\hat{\mathcal{G}} : \bar{\mathcal{E}}_1 \cup \bar{\mathcal{E}}_2 \to \bar{\mathcal{E}}_1 \cup \bar{\mathcal{E}}_2$. We say that $\hat{\mathcal{G}}$ is,

1. *relatively nonexpansive, if $\|\hat{\mathcal{G}}(\tilde{a}) - \hat{\mathcal{G}}\left(\tilde{b}\right)\| \leq \|\tilde{a} - \tilde{b}\|$, for any $\left(\tilde{a}, \tilde{b}\right) \in \bar{\mathcal{E}}_1 \times \bar{\mathcal{E}}_2$,*
2. *cyclic, if $\hat{\mathcal{G}}\left(\bar{\mathcal{E}}_1\right) \subseteq \bar{\mathcal{E}}_2$ and $\hat{\mathcal{G}}\left(\bar{\mathcal{E}}_2\right) \subseteq \bar{\mathcal{E}}_1$,*
3. *noncyclic, if $\hat{\mathcal{G}}\left(\bar{\mathcal{E}}_1\right) \subseteq \bar{\mathcal{E}}_1$ and $\hat{\mathcal{G}}\left(\bar{\mathcal{E}}_2\right) \subseteq \bar{\mathcal{E}}_2$,*
4. *compact, if $\left(\overline{\hat{\mathcal{G}}\left(\bar{\mathcal{E}}_1\right)}, \overline{\hat{\mathcal{G}}\left(\bar{\mathcal{E}}_2\right)}\right)$ is compact.*

Definition 2.4. *[10] Consider $\left(\bar{\mathcal{E}}_1, \bar{\mathcal{E}}_2\right)$ as a nonempty pair in a Banach space $\hat{\mathtt{M}}$, and $\tilde{\mathtt{F}} : \bar{\mathcal{E}}_1 \cup \bar{\mathcal{E}}_2 \to \bar{\mathcal{E}}_1 \cup \bar{\mathcal{E}}_2$ be a cyclic function, then $\check{k} \in \bar{\mathcal{E}}_1 \cup \bar{\mathcal{E}}_2$ is called a BPP of $\tilde{\mathtt{F}}$, if $\|\check{k} - \tilde{\mathtt{F}}\left(\check{k}\right)\| = dist\left(\bar{\mathcal{E}}_1, \bar{\mathcal{E}}_2\right)$.*
If $\tilde{\mathtt{F}}$ is noncyclic, then the pair $\left(\check{k}, \check{d}\right) \in \bar{\mathcal{E}}_1 \times \bar{\mathcal{E}}_2$ is a best proximity pair, if $\|\check{k} - \check{d}\| = dist\left(\bar{\mathcal{E}}_1, \bar{\mathcal{E}}_2\right)$, where $\check{k} = \tilde{\mathtt{F}}\left(\check{k}\right), \check{d} = \tilde{\mathtt{F}}\left(\check{d}\right)$.

Corollary 2.5. *[10] Suppose a Banach space $\hat{\mathtt{M}}$, and a nonempty, convex and compact pair $\left(\bar{\mathcal{E}}_1, \bar{\mathcal{E}}_2\right)$ in $\hat{\mathtt{M}}$. Let we have a cyclic and relatively nonexpansive mapping $\hat{\mathcal{G}} : \bar{\mathcal{E}}_1 \cup \bar{\mathcal{E}}_2 \to \bar{\mathcal{E}}_1 \cup \bar{\mathcal{E}}_2$. Then $\hat{\mathcal{G}}$ has a BPP.*

Corollary 2.6. *[10] Suppose a strictly convex Banach space $\hat{\mathtt{M}}$, and a compact, nonempty and convex pair $\left(\bar{\mathcal{E}}_1, \bar{\mathcal{E}}_1\right)$ in $\hat{\mathtt{M}}$. Let we have a relatively nonexpansive and noncyclic mapping $\hat{\mathcal{G}} : \bar{\mathcal{E}}_1 \cup \bar{\mathcal{E}}_2 \to \bar{\mathcal{E}}_1 \cup \bar{\mathcal{E}}_2$. Then $\hat{\mathcal{G}}$ has a best proximity pair.*

The following theorems 2.7 and 2.8 are the extended form of corollaries 2.5 and 2.6.

Theorem 2.7. *[10] Suppose a reflexive Banach space $\hat{\mathtt{M}}$, and a convex, nonempty, closed and bounded pair $\left(\bar{\mathcal{E}}_1, \bar{\mathcal{E}}_2\right)$ in $\hat{\mathtt{M}}$. Let we have a relatively nonexpansive and cyclic mapping $\hat{\mathcal{G}} : \bar{\mathcal{E}}_1 \cup \bar{\mathcal{E}}_2 \to \bar{\mathcal{E}}_1 \cup \bar{\mathcal{E}}_2$. Then $\hat{\mathcal{G}}$ has a BPP, if $\hat{\mathcal{G}}$ is compact.*

Theorem 2.8. *[10] Suppose a reflexive, strictly convex Banach space $\hat{\mathtt{M}}$, and a convex, nonempty, closed and bounded pair $\left(\bar{\mathcal{E}}_1, \bar{\mathcal{E}}_2\right)$ in $\hat{\mathtt{M}}$. Let we have a relatively nonexpansive and noncyclic mapping $\hat{\mathcal{G}} : \bar{\mathcal{E}}_1 \cup \bar{\mathcal{E}}_2 \to \bar{\mathcal{E}}_1 \cup \bar{\mathcal{E}}_2$. Then $\hat{\mathcal{G}}$ has a best proximity pair, if $\hat{\mathcal{G}}$ is compact.*

3. Main result

Definition 3.1. *[4] Consider a nondecreasing, continuous function $\tilde{\mathtt{y}} : \tilde{\mathtt{R}}_+ \to \tilde{\mathtt{R}}_+$ such that $\tilde{\mathtt{y}}^{-1}(0) = 0$. Then $\tilde{\mathtt{y}}$ is said to be an alternating distance function.*

Definition 3.2. *[4] Let $\bar{\mathfrak{R}}$ be the set of all maps $\hat{\Xi} : \tilde{\mathtt{R}}_+ \times \tilde{\mathtt{R}}_+ \to \tilde{\mathtt{R}}_+$ with*

(i) $\hat{\Xi}(0,0) = 0 \geq 0$,

(ii) $\hat{\Xi}(\tilde{p}, \tilde{q}) < \tilde{q} - \tilde{p}, \quad \forall \tilde{p}, \tilde{q} > 0$,

(iii) If $\{\tilde{p}_n\}, \{\tilde{q}_n\}$ *are sequence in* $(0, \infty)$ *such that* $\lim_{n\to\infty} \tilde{p}_n = \tilde{p}$, $\lim_{n\to\infty} \tilde{q}_n = \tilde{q} > 0$, *then* $\lim_{n\to\infty} \sup \hat{\Xi}(\tilde{p}_n, \tilde{q}_n) < \tilde{q} - \tilde{p}$.

For example: Consider two alternating distance function $\tilde{\mathfrak{G}}_1, \tilde{\mathfrak{G}}_2$ such that $\tilde{\mathfrak{G}}_1(\tilde{p}) < \tilde{p} \leq \tilde{\mathfrak{G}}_2(\tilde{p})$ for all $\tilde{p} > 0$. Then $\hat{\Xi}_1(\tilde{p}, \tilde{q}) = \tilde{\mathfrak{G}}_1(\tilde{q}) - \tilde{\mathfrak{G}}_2(\tilde{p})$, for all $\tilde{p}, \tilde{q} > 0$ in $\bar{\mathfrak{R}}$.

If we consider $\tilde{\mathfrak{G}}_1(\tilde{p}) = \acute{\mu}\tilde{p}$, for all $\tilde{p} > 0$, $\acute{\mu} \in (0,1)$ and $\tilde{\mathfrak{G}}_2(\tilde{p}) = \tilde{p}$. Then we get the map $\hat{\Xi}_2(\tilde{p}, \tilde{q}) = \tilde{q} - \acute{\mu}\tilde{p}$ in $\bar{\mathfrak{R}}$, $\forall \tilde{p}, \tilde{q} > 0$.

Definition 3.3. *[4] Let $\bar{\mathtt{H}}$ be the set of all maps $\hat{\daleth} : \tilde{\mathtt{R}}_+ \times \tilde{\mathtt{R}}_+ \to \tilde{\mathtt{R}}_+$ with*

(i) $\max\{\breve{p}, \breve{q}\} \leq \hat{\daleth}(\breve{p}, \breve{q})$; $\breve{p}, \breve{q} \geq 0$,

(ii) $\hat{\daleth}$ *is nondecreasing and continuous,*

(iii) $\hat{\daleth}(\breve{p}_1 + \breve{p}_2, \breve{q}_1 + \breve{q}_2) \leq \hat{\daleth}(\breve{p}_1, \breve{q}_1) + \hat{\daleth}(\breve{p}_2, \breve{q}_2)$.

As an example, we can consider $\hat{\daleth}(\breve{p}, \breve{q}) = \breve{p} + \breve{q}$, for all $\breve{p}, \breve{q} > 0$.

$\bar{\mathcal{E}}_1$ and $\bar{\mathcal{E}}_2$ will be nonempty convex subsets of a Banach space $\hat{\mathtt{M}}$ in this section.

Definition 3.4. *Consider $\left(\bar{\mathcal{E}}_1, \bar{\mathcal{E}}_2\right)$ as a pair of convex and nonempty subsets of a Banach space $\hat{\mathtt{M}}$ equipped with a* MNC $\aleph$. *$\hat{\Lambda}$ is nondecreasing continuous functions, $\hat{\Xi} \in \bar{\mathfrak{R}}$, $\hat{\daleth} \in \bar{\mathtt{H}}$, and for any $0 < \check{\mathtt{e}} < \check{\mathtt{f}} < \infty$ there exists $0 < \hat{\varkappa}\left(\check{\mathtt{e}}, \check{\mathtt{f}}\right) < 1$. A mapping $\hat{\mathcal{G}} : \bar{\mathcal{E}}_1 \cup \bar{\mathcal{E}}_2 \to \bar{\mathcal{E}}_1 \cup \bar{\mathcal{E}}_2$, which is cyclic* (*noncyclic*) *is said to be a* $(\hat{\Xi}\hat{\daleth}\hat{\Lambda})$*-contractive operator such that for any pair of convex, nonempty, proximinal, closed, bounded and $\hat{\mathcal{G}}$-invariant subsets $\left(\check{\mathtt{U}}_1, \check{\mathtt{U}}_2\right)$ of $\left(\bar{\mathcal{E}}_1, \bar{\mathcal{E}}_2\right)$ such that $dist\left(\check{\mathtt{U}}_1, \check{\mathtt{U}}_2\right) = dist\left(\bar{\mathcal{E}}_1, \bar{\mathcal{E}}_2\right)$ with,*

$$\check{\mathtt{e}} \leq \hat{\daleth}\left(\aleph\left(\check{\mathtt{U}}_1 \cup \check{\mathtt{U}}_2\right), \hat{\Lambda}\left(\aleph\left(\check{\mathtt{U}}_1 \cup \check{\mathtt{U}}_2\right)\right)\right) \leq \check{\mathtt{f}} \Longrightarrow$$

$$\hat{\Xi}\left\{\hat{\daleth}\left(\aleph\left(\hat{\mathcal{G}}\left(\check{\mathtt{U}}_1\right) \cup \hat{\mathcal{G}}\left(\check{\mathtt{U}}_2\right)\right), \hat{\Lambda}\left(\aleph\left(\hat{\mathcal{G}}\left(\check{\mathtt{U}}_1\right) \cup \hat{\mathcal{G}}\left(\check{\mathtt{U}}_2\right)\right)\right)\right), \hat{\varkappa}\left(\check{\mathtt{e}}, \check{\mathtt{f}}\right) \hat{\daleth}\left(\aleph\left(\check{\mathtt{U}}_1 \cup \check{\mathtt{U}}_2\right), \hat{\Lambda}\left(\aleph\left(\check{\mathtt{U}}_1 \cup \check{\mathtt{U}}_2\right)\right)\right)\right\} \geq 0. \quad (1)$$

Theorem 3.5. *Consider a relatively nonexpansive, cyclic and* $(\hat{\Xi}\hat{\daleth}\hat{\Lambda})$*-contractive operator* $\hat{\mathcal{G}} : \bar{\mathcal{E}}_1 \cup \bar{\mathcal{E}}_2 \to \bar{\mathcal{E}}_1 \cup \bar{\mathcal{E}}_2$*. Then* $\hat{\mathcal{G}}$ *has a BPP, if* $\bar{\bar{\mathcal{E}}}_1 \neq \emptyset$*.*

Proof. Since $\bar{\bar{\mathcal{E}}}_1 \neq \emptyset, \left(\bar{\bar{\mathcal{E}}}_1, \bar{\bar{\mathcal{E}}}_2\right) \neq \emptyset$. By the given conditions on $\hat{\mathcal{G}}$, clearly $\left(\bar{\bar{\mathcal{E}}}_1, \bar{\bar{\mathcal{E}}}_2\right)$ is a closed, convex, proximinal and $\hat{\mathcal{G}}$-invariant pair. For each $\acute{\sigma} \in \bar{\bar{\mathcal{E}}}_1$, there is a $\bar{\zeta} \in \bar{\bar{\mathcal{E}}}_2$ satisfying $\|\acute{\sigma} - \bar{\zeta}\| = dist\left(\bar{\mathcal{E}}_1, \bar{\mathcal{E}}_2\right)$. Since $\hat{\mathcal{G}}$ is relatively nonexpansive, so we get $\|\hat{\mathcal{G}}\acute{\sigma} - \hat{\mathcal{G}}\bar{\zeta}\| \leq \|\acute{\sigma} - \bar{\zeta}\| = dist\left(\bar{\mathcal{E}}_1, \bar{\mathcal{E}}_2\right)$, which implies $\hat{\mathcal{G}}\acute{\sigma} \in \bar{\bar{\mathcal{E}}}_2$, that is, $\hat{\mathcal{G}}\left(\bar{\bar{\mathcal{E}}}_1\right) \subseteq \bar{\bar{\mathcal{E}}}_2$. Similarly, $\hat{\mathcal{G}}\left(\bar{\bar{\mathcal{E}}}_2\right) \subseteq \bar{\bar{\mathcal{E}}}_1$. Hence, we get $\hat{\mathcal{G}}$ is cyclic on $\bar{\bar{\mathcal{E}}}_1 \cup \bar{\bar{\mathcal{E}}}_2$.

Let us assume that $\hat{X}_0 = \bar{\bar{\mathcal{E}}}_1$ and $\hat{Y}_0 = \bar{\bar{\mathcal{E}}}_2$ and $\left\{\left(\hat{X}_n, \hat{Y}_n\right)\right\}$ be a sequence of pairs with $\hat{X}_n = \overline{conv}\left(\hat{\mathcal{G}}\left(\hat{X}_{n-1}\right)\right)$ and $\hat{Y}_n = \overline{conv}\left(\hat{\mathcal{G}}\left(\hat{Y}_{n-1}\right)\right)$, for all $n \in \mathbb{N}$. Now our claim is, $\hat{X}_{n+1} \subseteq \hat{Y}_n$ and $\hat{Y}_n \subseteq \hat{X}_{n-1}$ for all $n \in \mathbb{N}$. In fact $\hat{Y}_1 = \overline{conv}\left(\hat{\mathcal{G}}\left(\hat{Y}_0\right)\right) = \overline{conv}\left(\hat{\mathcal{G}}\left(\bar{\bar{\mathcal{E}}}_2\right)\right) \subseteq \overline{conv}\left(\bar{\bar{\mathcal{E}}}_1\right) = \bar{\bar{\mathcal{E}}}_1 = \hat{X}_0$. Hence, we can write,

$$\hat{\mathcal{G}}\left(\hat{Y}_1\right) \subseteq \hat{\mathcal{G}}\left(\hat{X}_0\right) \text{ and } \hat{Y}_2 = \overline{conv}\left(\hat{\mathcal{G}}\left(\hat{Y}_1\right)\right) \subseteq \overline{conv}\left(\hat{\mathcal{G}}\left(\hat{X}_0\right)\right) = \hat{X}_1.$$

With the similar argument, we get by using induction that $\hat{Y}_n \subseteq \hat{X}_{n-1}$. Similarly, we get $\hat{X}_{n+1} \subseteq \hat{Y}_n$, for all $n \in \mathbb{N}$. Hence, we can write $\hat{X}_{n+2} \subseteq \hat{Y}_{n+1} \subseteq \hat{X}_n \subseteq \hat{Y}_{n-1}$, for all $n \in \mathbb{N}$. So, in $\bar{\bar{\mathcal{E}}}_1 \times \bar{\bar{\mathcal{E}}}_2$, the decreasing sequence of NBCC pairs is $\left\{\left(\hat{X}_{2n}, \hat{Y}_{2n}\right)\right\}$. Moreover,

$$\hat{\mathcal{G}}\left(\hat{Y}_{2n}\right) \subseteq \hat{\mathcal{G}}\left(\hat{X}_{2n-1}\right) \subseteq \overline{conv}\left(\hat{\mathcal{G}}\left(\hat{X}_{2n-1}\right)\right) = \hat{X}_{2n}, \tag{2}$$

$$\hat{\mathcal{G}}\left(\hat{X}_{2n}\right) \subseteq \hat{\mathcal{G}}\left(\hat{Y}_{2n-1}\right) \subseteq \overline{conv}\left(\hat{\mathcal{G}}\left(\hat{Y}_{2n-1}\right)\right) = \hat{Y}_{2n}. \tag{3}$$

Hence, we get $\left(\hat{X}_{2n}, \hat{Y}_{2n}\right)$ is a $\hat{\mathcal{G}} - invariant$ pair, for all n $\in \mathbb{N}$. Now, if $(\check{\mathfrak{p}}, \check{\mathfrak{q}}) \in \bar{\bar{\mathcal{E}}}_1 \times \bar{\bar{\mathcal{E}}}_2$ which is proximinal, we have,

$$dist\left(\hat{X}_{2n}, \hat{Y}_{2n}\right) \leq \|\hat{\mathcal{G}}^{2n}\check{\mathfrak{p}} - \hat{\mathcal{G}}^{2n}\check{\mathfrak{q}}\| \leq \|\check{\mathfrak{p}} - \check{\mathfrak{q}}\| = dist\left(\bar{\mathcal{E}}_1, \bar{\mathcal{E}}_2\right).$$

Now, we are to show that, for all $n \in \mathbb{N}$, the pair $\left(\hat{X}_n, \hat{Y}_n\right)$ is proximinal. For n=0, we have $\left(\hat{X}_0, \hat{Y}_0\right)$ is a proximinal pair. Let us assume that $\left(\hat{X}_k, \hat{Y}_k\right)$ is proximinal and there is an arbitrary $\hat{\mathfrak{a}}$ such that $\hat{\mathfrak{a}} \in \hat{X}_{k+1} = \overline{conv}\left(\hat{\mathcal{G}}\left(\hat{X}_k\right)\right)$. So $\hat{\mathfrak{a}} = \sum_{\acute{j}=1}^{\acute{x}} \mathcal{D}_{\acute{j}}\hat{\mathcal{G}}\left(\bar{\omega}_{\acute{j}}\right)$ with $\bar{\omega}_{\acute{j}} \in \hat{X}_k$, $\acute{x} \in [1, \infty)$, $\mathcal{D}_{\acute{j}} \geq 0$ and $\sum_{\acute{j}=1}^{\acute{x}} \mathcal{D}_{\acute{j}} = 1$. By assumption, we have $\left(\hat{X}_k, \hat{Y}_k\right)$ is a proximinal pair, so there exists $\widetilde{K}_{\acute{j}} \in \hat{Y}_k$ $(1 \leq \acute{j} \leq \acute{x})$ such that $\|\bar{\omega}_{\acute{j}} - \widetilde{K}_{\acute{j}}\| = dist\left(\hat{X}_k, \hat{Y}_k\right) = dist\left(\bar{\mathcal{E}}_1, \bar{\mathcal{E}}_2\right)$. Consider $\hat{\mathfrak{b}} = \sum_{\acute{j}=1}^{\acute{x}} \mathcal{D}_{\acute{j}}\hat{\mathcal{G}}\left(\widetilde{K}_{\acute{j}}\right)$, then $\hat{\mathfrak{b}} \in \overline{conv}\left(\hat{\mathcal{G}}\left(\hat{Y}_k\right)\right) = \hat{Y}_{k+1}$, and

$$\|\hat{\mathfrak{a}} - \hat{\mathfrak{b}}\| = \|\sum_{\acute{j}=1}^{\acute{x}} \mathcal{D}_{\acute{j}}\hat{\mathcal{G}}\left(\bar{\omega}_{\acute{j}}\right) - \sum_{\acute{j}=1}^{\acute{x}} \mathcal{D}_{\acute{j}}\hat{\mathcal{G}}\left(\widetilde{K}_{\acute{j}}\right)\| \leq \sum_{\acute{j}=1}^{\acute{x}} \mathcal{D}_{\acute{j}}\|\bar{\omega}_{\acute{j}} - \widetilde{K}_{\acute{j}}\| = dist\left(\bar{\mathcal{E}}_1, \bar{\mathcal{E}}_2\right).$$

Hence, $\left(\hat{X}_{k+1}, \hat{Y}_{k+1}\right)$ is a proximinal pair, and by induction hypothesis our claim is proved.

Now, if $\aleph\left(\hat{X}_{2n} \cup \hat{Y}_{2n}\right) = 0$ for some $n \in [1, \infty) \cup \{0\}$, then we have $\hat{\mathcal{G}} : \hat{X}_{2n} \cup \hat{Y}_{2n} \to \hat{X}_{2n} \cup \hat{Y}_{2n}$ is compact. Thus from corollary (2.5), $\hat{\mathcal{G}}$ has a *BPP*. Hence, we consider that $\aleph\left(\hat{X}_{2n} \cup \hat{Y}_{2n}\right) > 0$, for all $n \in [1, \infty)$. Then $\hat{\daleth}\left(\aleph\left(\hat{X}_{2n} \cup \hat{Y}_{2n}\right), \hat{\Lambda}\left(\aleph\left(\hat{X}_{2n} \cup \hat{Y}_{2n}\right)\right)\right) > 0$ for all $n \in [1, \infty)$. For $\check{\mathfrak{e}} \leq \hat{\daleth}\left(\aleph\left(\hat{X}_{2n} \cup \hat{Y}_{2n}\right), \hat{\Lambda}\left(\aleph\left(\hat{X}_{2n} \cup \hat{Y}_{2n}\right)\right)\right) \leq \check{\mathfrak{f}}$,

$$\begin{aligned}
0 &\leq \hat{\Xi}\Big\{\hat{\daleth}\left(\aleph\left(\hat{\mathcal{G}}\left(\hat{X}_{2n}\right) \cup \hat{\mathcal{G}}\left(\hat{Y}_{2n}\right)\right), \hat{\Lambda}\left(\aleph\left(\hat{\mathcal{G}}\left(\hat{X}_{2n}\right) \cup \hat{\mathcal{G}}\left(\hat{Y}_{2n}\right)\right)\right)\right), \hat{\varkappa}\left(\check{\mathfrak{e}}, \check{\mathfrak{f}}\right)\hat{\daleth}\left(\aleph\left(\hat{X}_{2n} \cup \hat{Y}_{2n}\right), \hat{\Lambda}\left(\aleph\left(\hat{X}_{2n} \cup \hat{Y}_{2n}\right)\right)\right)\Big\} \\
&= \hat{\Xi}\Big\{\hat{\daleth}\left(\max\left\{\aleph\left(\hat{\mathcal{G}}\left(\hat{X}_{2n}\right)\right), \aleph\left(\hat{\mathcal{G}}\left(\hat{Y}_{2n}\right)\right)\right\}, \hat{\Lambda}\left(\max\left\{\aleph\left(\hat{\mathcal{G}}\left(\hat{X}_{2n}\right)\right), \aleph\left(\hat{\mathcal{G}}\left(\hat{Y}_{2n}\right)\right)\right\}\right)\right), \\
&\hat{\varkappa}\left(\check{\mathfrak{e}}, \check{\mathfrak{f}}\right)\hat{\daleth}\left(\max\left\{\aleph\left(\hat{X}_{2n}\right), \aleph\left(\hat{Y}_{2n}\right)\right\}, \hat{\Lambda}\left(\max\left\{\aleph\left(\hat{X}_{2n}\right), \aleph\left(\hat{Y}_{2n}\right)\right\}\right)\right)\Big\} \\
&= \hat{\Xi}\Big\{\hat{\daleth}\Big(\max\left\{\aleph\left(\overline{conv}\left(\hat{\mathcal{G}}\left(\hat{X}_{2n}\right)\right)\right), \aleph\left(\overline{conv}\left(\hat{\mathcal{G}}\left(\hat{Y}_{2n}\right)\right)\right)\right\},
\end{aligned}$$

$$\hat{\Lambda}\left(\max\left\{\aleph\left(\overline{conv}\left(\hat{\mathcal{G}}\left(\hat{X}_{2n}\right)\right)\right),\aleph\left(\overline{conv}\left(\hat{\mathcal{G}}\left(\hat{X}_{2n}\right)\right)\right)\right\}\right)\right),$$
$$\hat{\varkappa}\left(\check{e},\check{f}\right)\hat{\daleth}\left(\max\left\{\aleph\left(\hat{X}_{2n}\right),\aleph\left(\hat{Y}_{2n}\right)\right\},\hat{\Lambda}\left(\max\left\{\aleph\left(\hat{X}_{2n}\right),\aleph\left(\hat{Y}_{2n}\right)\right\}\right)\right)\Big\}$$
$$=\hat{\Xi}\Big\{\hat{\daleth}\left(\max\left\{\aleph\left(\hat{X}_{2n+1}\right),\aleph\left(\hat{Y}_{2n+1}\right)\right\},\hat{\Lambda}\left(\max\left\{\aleph\left(\hat{X}_{2n+1}\right),\aleph\left(\hat{Y}_{2n+1}\right)\right\}\right)\right),$$
$$\hat{\varkappa}\left(\check{e},\check{f}\right)\hat{\daleth}\left(\max\left\{\aleph\left(\hat{X}_{2n}\right),\aleph\left(\hat{Y}_{2n}\right)\right\},\hat{\Lambda}\left(\max\left\{\aleph\left(\hat{X}_{2n}\right),\aleph\left(\hat{Y}_{2n}\right)\right\}\right)\right)\Big\}$$
$$=\hat{\Xi}\Big\{\hat{\daleth}\left(\aleph\left(\hat{X}_{2n+1}\cup\hat{Y}_{2n+1}\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_{2n+1}\cup\hat{Y}_{2n+1}\right)\right)\right),\hat{\varkappa}\left(\check{e},\check{f}\right)\hat{\daleth}\left(\aleph\left(\hat{X}_{2n}\cup\hat{Y}_{2n}\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_{2n}\cup\hat{Y}_{2n}\right)\right)\right)\Big\}$$
$$\leq\hat{\varkappa}\left(\check{e},\check{f}\right)\hat{\daleth}\left(\aleph\left(\hat{X}_{2n}\cup\hat{Y}_{2n}\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_{2n}\cup\hat{Y}_{2n}\right)\right)\right)-\hat{\daleth}\left(\aleph\left(\hat{X}_{2n+1}\cup\hat{Y}_{2n+1}\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_{2n+1}\cup\hat{Y}_{2n+1}\right)\right)\right)$$

i.e

$$\hat{\varkappa}\left(\check{e},\check{f}\right)\geq\frac{\hat{\daleth}\left(\aleph\left(\hat{X}_{2n+1}\cup\hat{Y}_{2n+1}\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_{2n+1}\cup\hat{Y}_{2n+1}\right)\right)\right)}{\hat{\daleth}\left(\aleph\left(\hat{X}_{2n}\cup\hat{Y}_{2n}\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_{2n}\cup\hat{Y}_{2n}\right)\right)\right)}$$

If

$$\hat{\daleth}\left(\aleph\left(\hat{X}_{2n+1}\cup\hat{Y}_{2n+1}\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_{2n+1}\cup\hat{Y}_{2n+1}\right)\right)\right)\geq\hat{\daleth}\left(\aleph\left(\hat{X}_{2n}\cup\hat{Y}_{2n}\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_{2n}\cup\hat{Y}_{2n}\right)\right)\right)$$

Then $\hat{\varkappa}\left(\check{e},\check{f}\right)>1$, which is a contradiction.
Hence, for all $n\in[1,\infty)$,

$$\hat{\daleth}\left(\aleph\left(\hat{X}_{2n+1}\cup\hat{Y}_{2n+1}\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_{2n+1}\cup\hat{Y}_{2n+1}\right)\right)\right)<\hat{\daleth}\left(\aleph\left(\hat{X}_{2n}\cup\hat{Y}_{2n}\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_{2n}\cup\hat{Y}_{2n}\right)\right)\right)$$

Thus $\left\{\hat{\daleth}\left(\aleph\left(\hat{X}_{2n}\cup\hat{Y}_{2n}\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_{2n}\cup\hat{Y}_{2n}\right)\right)\right)\right\}$ is a non negative decreasing sequence. So there exists $\tilde{\lambda}\geq 0$ such that

$$\lim_{n\to\infty}\hat{\daleth}\left(\aleph\left(\hat{X}_{2n}\cup\hat{Y}_{2n}\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_{2n}\cup\hat{Y}_{2n}\right)\right)\right)=\tilde{\lambda}.$$

Suppose $\tilde{\lambda}>0$, Then for all $n\geq 0$

$$0<\tilde{\lambda}=\check{e}\leq\hat{\daleth}\left(\aleph\left(\hat{X}_{2n}\cup\hat{Y}_{2n}\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_{2n}\cup\hat{Y}_{2n}\right)\right)\right)$$
$$\leq\hat{\daleth}\left(\aleph\left(\hat{X}_{0}\cup\hat{Y}_{0}\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_{0}\cup\hat{Y}_{0}\right)\right)\right)=\check{f}$$

Again, we have for the convex, nonempty, proximal, closed, bounded and $\hat{\mathcal{G}}$-invariant subsets $\left(\hat{X}_{2n},\hat{Y}_{2n}\right)$ with dist $\left(\hat{X}_{2n},\hat{Y}_{2n}\right)=\text{dist}\left(\bar{\mathcal{E}}_1,\bar{\mathcal{E}}_2\right)$, there exists $0<\hat{\varkappa}\left(\check{e},\check{f}\right)<1$ such that

$$0\leq\hat{\Xi}\Big\{\hat{\daleth}\left(\aleph\left(\hat{\mathcal{G}}\left(\hat{X}_{2n}\right)\cup\hat{\mathcal{G}}\left(\hat{Y}_{2n}\right)\right),\hat{\Lambda}\left(\aleph\left(\hat{\mathcal{G}}\left(\hat{X}_{2n}\right)\cup\hat{\mathcal{G}}\left(\hat{Y}_{2n}\right)\right)\right)\right),\hat{\varkappa}\left(\check{e},\check{f}\right)\hat{\daleth}\left(\aleph\left(\hat{X}_{2n}\cup\hat{Y}_{2n}\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_{2n}\cup\hat{Y}_{2n}\right)\right)\right)\Big\}$$
$$=\hat{\Xi}\Big\{\hat{\daleth}\left(\aleph\left(\hat{X}_{2n+1}\cup\hat{Y}_{2n+1}\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_{2n+1}\cup\hat{Y}_{2n+1}\right)\right)\right),\hat{\varkappa}\left(\check{e},\check{f}\right)\hat{\daleth}\left(\aleph\left(\hat{X}_{2n}\cup\hat{Y}_{2n}\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_{2n}\cup\hat{Y}_{2n}\right)\right)\right)\Big\}$$

Consider

$$\hat{\daleth}\left(\aleph\left(\hat{X}_{2n+1}\cup\hat{Y}_{2n+1}\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_{2n+1}\cup\hat{Y}_{2n+1}\right)\right)\right)=\check{x}_{2n}$$
$$\hat{\varkappa}\left(\check{e},\check{f}\right)\hat{\daleth}\left(\aleph\left(\hat{X}_{2n}\cup\hat{Y}_{2n}\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_{2n}\cup\hat{Y}_{2n}\right)\right)\right)=\check{y}_{2n}$$

Since for all $n \geq 0$,
$\check{x}_{2n} < \check{y}_{2n}$ and $\lim_{n\to\infty} \check{x}_{2n} = \tilde{\lambda}$, $\lim_{n\to\infty} \check{y}_{2n} = \tilde{\lambda}\hat{\varkappa}\left(\check{e}, \check{f}\right)$.

Thus

$$\lim_{n\to\infty} \sup \hat{\Xi}\left\{\hat{\daleth}\left(\aleph\left(\hat{\mathcal{G}}\left(\hat{X}_{2n}\right) \cup \hat{\mathcal{G}}\left(\hat{Y}_{2n}\right)\right), \hat{\Lambda}\left(\aleph\left(\hat{\mathcal{G}}\left(\hat{X}_{2n}\right) \cup \hat{\mathcal{G}}\left(\hat{Y}_{2n}\right)\right)\right)\right), \hat{\varkappa}\left(\check{e}, \check{f}\right)\hat{\daleth}\left(\aleph\left(\hat{X}_{2n} \cup \hat{Y}_{2n}\right), \hat{\Lambda}\left(\aleph\left(\hat{X}_{2n} \cup \hat{Y}_{2n}\right)\right)\right)\right\}$$
$$\leq \tilde{\lambda}\hat{\varkappa}\left(\check{e}, \check{f}\right) - \tilde{\lambda} < 0,$$

which is a contradiction. Hence $\tilde{\lambda} = 0$

Therefore $\lim_{n\to\infty} \hat{\daleth}\left(\aleph\left(\hat{X}_{2n} \cup \hat{Y}_{2n}\right), \hat{\Lambda}\left(\aleph\left(\hat{X}_{2n} \cup \hat{Y}_{2n}\right)\right)\right) = 0$.

Thus we get
$\lim_{n\to\infty} \aleph\left(\hat{X}_{2n} \cup \hat{Y}_{2n}\right) = 0$, and $\lim_{n\to\infty} \hat{\Lambda}\left(\aleph\left(\hat{X}_{2n} \cup \hat{Y}_{2n}\right)\right) = 0$.

Let $\hat{X}_\infty = \cap_{n=0}^{\infty} \hat{X}_{2n}$ and $\hat{Y}_\infty = \cap_{n=0}^{\infty} \hat{Y}_{2n}$, so, we get a nonempty, compact, convex pair $\left(\hat{X}_\infty, \hat{Y}_\infty\right)$ which is $\hat{\mathcal{G}}$-invariant with $\text{dist}\left(\hat{X}_\infty, \hat{Y}_\infty\right) = \text{dist}\left(\bar{\mathcal{E}}_1, \bar{\mathcal{E}}_2\right)$. Hence from corollary 2.5, $\hat{\mathcal{G}}$ has a $\mathcal{BPP}$. □

Theorem 3.6. *Consider a relatively nonexpansive, noncyclic and* $(\hat{\Xi}\hat{\daleth}\hat{\Lambda})$*-contractive operator* $\hat{\mathcal{G}} : \bar{\mathcal{E}}_1 \cup \bar{\mathcal{E}}_2 \to \bar{\mathcal{E}}_1 \cup \bar{\mathcal{E}}_2$ *on a strictly convex Banach space* $\hat{M}$. *Then* $\hat{\mathcal{G}}$ *has a best proximity pair, if* $\bar{\bar{\mathcal{E}}}_1 \neq \emptyset$.

Proof. Following the proof of the Theorem 3.5, define a pair $\left(\hat{X}_n, \hat{Y}_n\right)$ as $\hat{X}_n = \overline{conv}\left(\hat{\mathcal{G}}\left(\hat{X}_{n-1}\right)\right)$ and $\hat{Y}_n = \overline{conv}\left(\hat{\mathcal{G}}\left(\hat{Y}_{n-1}\right)\right)$, $n \in \mathbb{N}$ with $\hat{X}_0 = \bar{\bar{\mathcal{E}}}_1$ and $\hat{Y}_0 = \bar{\bar{\mathcal{E}}}_2$, we get NBCC and decreasing sequence of pairs $\left\{\left(\hat{X}_n, \hat{Y}_n\right)\right\}$ in $\bar{\bar{\mathcal{E}}}_1 \times \bar{\bar{\mathcal{E}}}_2$. Also,

$$\hat{\mathcal{G}}\left(\hat{X}_n\right) \subseteq \hat{\mathcal{G}}\left(\hat{X}_{n-1}\right) \subseteq \overline{conv}\left(\hat{\mathcal{G}}\left(\hat{X}_{n-1}\right)\right) = \hat{X}_n, \tag{4}$$
$$\hat{\mathcal{G}}\left(\hat{Y}_n\right) \subseteq \hat{\mathcal{G}}\left(\hat{Y}_{n-1}\right) \subseteq \overline{conv}\left(\hat{\mathcal{G}}\left(\hat{Y}_{n-1}\right)\right) = \hat{Y}_n. \tag{5}$$

Thus, the pair $\left(\hat{X}_n, \hat{Y}_n\right)$ is $\hat{\mathcal{G}}$-invariant, for all n ≥ 1. Following the proof of the Theorem 3.5, we obtain a proximinal pair $\left(\hat{X}_n, \hat{Y}_n\right)$ for all non negative integer n such that $\text{dist}\left(\hat{X}_n, \hat{Y}_n\right) = \text{dist}\left(\bar{\mathcal{E}}_1, \bar{\mathcal{E}}_2\right)$. Now, if $\aleph\left(\hat{X}_n, \hat{Y}_n\right) = 0$, for some positive integer n, then $\hat{\mathcal{G}} : \hat{X}_n \cup \hat{Y}_n \to \hat{X}_n \cup \hat{Y}_n$ is compact. Hence, from corollary (2.6), we get the desired result. Thus, we consider that $\aleph\left(\hat{X}_n, \hat{Y}_n\right) > 0$. Then $\hat{\daleth}\left(\aleph\left(\hat{X}_n \cup \hat{Y}_n\right), \hat{\Lambda}\left(\aleph\left(\hat{X}_n \cup \hat{Y}_n\right)\right)\right) > 0$ for all $n \in [1, \infty)$. We have, for $\check{e} \leq \hat{\daleth}\left(\aleph\left(\hat{X}_n \cup \hat{Y}_n\right), \hat{\Lambda}\left(\aleph\left(\hat{X}_n \cup \hat{Y}_n\right)\right)\right) \leq \check{f}$,

$$0 \leq \hat{\Xi}\left\{\hat{\daleth}\left(\aleph\left(\hat{\mathcal{G}}\left(\hat{X}_n\right) \cup \hat{\mathcal{G}}\left(\hat{Y}_n\right)\right), \hat{\Lambda}\left(\aleph\left(\hat{\mathcal{G}}\left(\hat{X}_n\right) \cup \hat{\mathcal{G}}\left(\hat{Y}_n\right)\right)\right)\right), \hat{\varkappa}\left(\check{e}, \check{f}\right)\hat{\daleth}\left(\aleph\left(\hat{X}_n \cup \hat{Y}_n\right), \hat{\Lambda}\left(\aleph\left(\hat{X}_n \cup \hat{Y}_n\right)\right)\right)\right\}$$
$$= \hat{\Xi}\left\{\hat{\daleth}\left(\max\left\{\aleph\left(\hat{\mathcal{G}}\left(\hat{X}_n\right)\right), \aleph\left(\hat{\mathcal{G}}\left(\hat{Y}_n\right)\right)\right\}, \hat{\Lambda}\left(\max\left\{\aleph\left(\hat{\mathcal{G}}\left(\hat{X}_n\right)\right), \aleph\left(\hat{\mathcal{G}}\left(\hat{Y}_n\right)\right)\right\}\right)\right),\right.$$
$$\left.\hat{\varkappa}\left(\check{e}, \check{f}\right)\hat{\daleth}\left(\max\left\{\aleph\left(\hat{X}_n\right), \aleph\left(\hat{Y}_n\right)\right\}, \hat{\Lambda}\left(\max\left\{\aleph\left(\hat{X}_n\right), \aleph\left(\hat{Y}_n\right)\right\}\right)\right)\right\}$$
$$= \hat{\Xi}\left\{\hat{\daleth}\left(\max\left\{\aleph\left(\overline{conv}\left(\hat{\mathcal{G}}\left(\hat{X}_n\right)\right)\right), \aleph\left(\overline{conv}\left(\hat{\mathcal{G}}\left(\hat{Y}_n\right)\right)\right)\right\},\right.\right.$$
$$\hat{\Lambda}\left(\max\left\{\aleph\left(\overline{conv}\left(\hat{\mathcal{G}}\left(\hat{X}_n\right)\right)\right), \aleph\left(\overline{conv}\left(\hat{\mathcal{G}}\left(\hat{X}_n\right)\right)\right)\right\}\right),$$

$$\hat{\varkappa}\left(\check{e},\check{f}\right)\hat{\daleth}\left(\max\left\{\aleph\left(\hat{X}_n\right),\aleph\left(\hat{Y}_n\right)\right\},\hat{\Lambda}\left(\max\left\{\aleph\left(\hat{X}_n\right),\aleph\left(\hat{Y}_n\right)\right\}\right)\right)\Big\}$$
$$=\hat{\Xi}\Big\{\hat{\daleth}\left(\max\left\{\aleph\left(\hat{X}_{n+1}\right),\aleph\left(\hat{Y}_{n+1}\right)\right\},\hat{\Lambda}\left(\max\left\{\aleph\left(\hat{X}_{n+1}\right),\aleph\left(\hat{Y}_{n+1}\right)\right\}\right)\right),$$
$$\hat{\varkappa}\left(\check{e},\check{f}\right)\hat{\daleth}\left(\max\left\{\aleph\left(\hat{X}_n\right),\aleph\left(\hat{Y}_n\right)\right\},\hat{\Lambda}\left(\max\left\{\aleph\left(\hat{X}_n\right),\aleph\left(\hat{Y}_n\right)\right\}\right)\right)\Big\}$$
$$=\hat{\Xi}\Big\{\hat{\daleth}\left(\aleph\left(\hat{X}_{n+1}\cup\hat{Y}_{n+1}\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_{n+1}\cup\hat{Y}_{n+1}\right)\right)\right),\hat{\varkappa}\left(\check{e},\check{f}\right)\hat{\daleth}\left(\aleph\left(\hat{X}_n\cup\hat{Y}_n\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_n\cup\hat{Y}_n\right)\right)\right)\Big\}$$
$$\leq\hat{\varkappa}\left(\check{e},\check{f}\right)\hat{\daleth}\left(\aleph\left(\hat{X}_n\cup\hat{Y}_n\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_n\cup\hat{Y}_n\right)\right)\right)-\hat{\daleth}\left(\aleph\left(\hat{X}_{n+1}\cup\hat{Y}_{n+1}\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_{n+1}\cup\hat{Y}_{n+1}\right)\right)\right)$$

i.e

$$\hat{\varkappa}\left(\check{e},\check{f}\right)\geq\frac{\hat{\daleth}\left(\aleph\left(\hat{X}_{n+1}\cup\hat{Y}_{n+1}\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_{n+1}\cup\hat{Y}_{n+1}\right)\right)\right)}{\hat{\daleth}\left(\aleph\left(\hat{X}_n\cup\hat{Y}_n\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_n\cup\hat{Y}_n\right)\right)\right)}$$

If

$$\hat{\daleth}\left(\aleph\left(\hat{X}_{n+1}\cup\hat{Y}_{n+1}\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_{n+1}\cup\hat{Y}_{n+1}\right)\right)\right)\geq\hat{\daleth}\left(\aleph\left(\hat{X}_n\cup\hat{Y}_n\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_n\cup\hat{Y}_n\right)\right)\right)$$

Then $\hat{\varkappa}\left(\check{e},\check{f}\right)>1$, which is a contradiction.
Hence, for all $n\in[1,\infty)$,

$$\hat{\daleth}\left(\aleph\left(\hat{X}_{n+1}\cup\hat{Y}_{n+1}\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_{n+1}\cup\hat{Y}_{n+1}\right)\right)\right)<\hat{\daleth}\left(\aleph\left(\hat{X}_n\cup\hat{Y}_n\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_n\cup\hat{Y}_n\right)\right)\right)$$

Thus $\left\{\hat{\daleth}\left(\aleph\left(\hat{X}_n\cup\hat{Y}_n\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_n\cup\hat{Y}_n\right)\right)\right)\right\}$ is a nonnegative decreasing sequence. So there exists $\tilde{\lambda}\geq 0$ such that

$$\lim_{n\to\infty}\hat{\daleth}\left(\aleph\left(\hat{X}_n\cup\hat{Y}_n\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_n\cup\hat{Y}_n\right)\right)\right)=\tilde{\lambda}.$$

Suppose $\tilde{\lambda}>0$, Then for all $n\geq 0$

$$0<\tilde{\lambda}=\check{e}\leq\hat{\daleth}\left(\aleph\left(\hat{X}_n\cup\hat{Y}_n\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_n\cup\hat{Y}_n\right)\right)\right)$$
$$\leq\hat{\daleth}\left(\aleph\left(\hat{X}_0\cup\hat{Y}_0\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_0\cup\hat{Y}_0\right)\right)\right)=\check{f}$$

Again we have for the convex, nonempty, proximal, closed, bounded and $\hat{\mathcal{G}}$-invariant subsets $\left(\hat{X}_n,\hat{Y}_n\right)$ with $\mathrm{dist}\left(\hat{X}_n,\hat{Y}_n\right)=\mathrm{dist}\left(\bar{\mathcal{E}}_1,\bar{\mathcal{E}}_2\right)$, there exists $0<\hat{\varkappa}\left(\check{e},\check{f}\right)<1$ such that

$$0\leq\hat{\Xi}\Big\{\hat{\daleth}\left(\aleph\left(\hat{\mathcal{G}}\left(\hat{X}_n\right)\cup\hat{\mathcal{G}}\left(\hat{Y}_n\right)\right),\hat{\Lambda}\left(\aleph\left(\hat{\mathcal{G}}\left(\hat{X}_n\right)\cup\hat{\mathcal{G}}\left(\hat{Y}_n\right)\right)\right)\right),\hat{\varkappa}\left(\check{e},\check{f}\right)\hat{\daleth}\left(\aleph\left(\hat{X}_n\cup\hat{Y}_n\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_n\cup\hat{Y}_n\right)\right)\right)\Big\}$$
$$=\hat{\Xi}\Big\{\hat{\daleth}\left(\aleph\left(\hat{X}_{n+1}\cup\hat{Y}_{n+1}\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_{n+1}\cup\hat{Y}_{n+1}\right)\right)\right),\hat{\varkappa}\left(\check{e},\check{f}\right)\hat{\daleth}\left(\aleph\left(\hat{X}_n\cup\hat{Y}_n\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_n\cup\hat{Y}_n\right)\right)\right)\Big\}$$

Consider

$$\hat{\daleth}\left(\aleph\left(\hat{X}_{n+1}\cup\hat{Y}_{n+1}\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_{n+1}\cup\hat{Y}_{n+1}\right)\right)\right)=\check{x}_n$$
$$\hat{\varkappa}\left(\check{e},\check{f}\right)\hat{\daleth}\left(\aleph\left(\hat{X}_n\cup\hat{Y}_n\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_n\cup\hat{Y}_n\right)\right)\right)=\check{y}_n$$

Since for all $n\geq 0$,
$\check{x}_n<\check{y}_n$ and $\lim_{n\to\infty}\check{x}_n=\tilde{\lambda}$, $\lim_{n\to\infty}\check{y}_n=\tilde{\lambda}\hat{\varkappa}\left(\check{e},\check{f}\right)$.

Thus

$$\lim_{n\to\infty} \sup \hat{\Xi}\left\{\hat{ך}\left(\aleph\left(\hat{\mathcal{G}}\left(\hat{X}_n\right)\cup\hat{\mathcal{G}}\left(\hat{Y}_n\right)\right),\hat{\Lambda}\left(\aleph\left(\hat{\mathcal{G}}\left(\hat{X}_n\right)\cup\hat{\mathcal{G}}\left(\hat{Y}_n\right)\right)\right)\right),\hat{\varkappa}\left(\check{e},\check{f}\right)\hat{ך}\left(\aleph\left(\hat{X}_n\cup\hat{Y}_n\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_n\cup\hat{Y}_n\right)\right)\right)\right\}$$
$$\leq \tilde{\lambda}\hat{\varkappa}\left(\check{e},\check{f}\right)-\tilde{\lambda}<0,$$

which is a contradiction. Hence $\tilde{\lambda}=0$

Therefore $\lim_{n\to\infty} \hat{ך}\left(\aleph\left(\hat{X}_n\cup\hat{Y}_n\right),\hat{\Lambda}\left(\aleph\left(\hat{X}_n\cup\hat{Y}_n\right)\right)\right)=0.$

Thus we get
$\lim_{n\to\infty}\aleph\left(\hat{X}_n\cup\hat{Y}_n\right)=0$, and $\lim_{n\to\infty}\hat{\Lambda}\left(\aleph\left(\hat{X}_n\cup\hat{Y}_n\right)\right)=0$.

Let $\hat{X}_\infty=\cap_{n=0}^{\infty}\hat{X}_n$ and $\hat{Y}_\infty=\cap_{n=0}^{\infty}\hat{Y}_n$, so, we get a nonempty, compact, convex pair $\left(\hat{X}_\infty,\hat{Y}_\infty\right)$ which is $\hat{\mathcal{G}}$-invariant with $\mathrm{dist}\left(\hat{X}_\infty,\hat{Y}_\infty\right)=\mathrm{dist}\left(\bar{\mathcal{E}}_1,\bar{\mathcal{E}}_2\right)$. Hence from corollary 2.6, $\hat{\mathcal{G}}$ has a best proximity pair. □

The next results are special cases of Theorem 3.5. The noncyclic version of the following corollaries are satisfied in strictly convex Banach spaces.

Corollary 3.7. *Consider* $\left(\bar{\mathcal{E}}_1,\bar{\mathcal{E}}_2\right)$ *be a pair of convex and nonempty subsets of a Banach space* $\hat{\mathtt{M}}$ *equipped with a* MNC $\aleph$ *and a relatively nonexpansive, cyclic operator* $\hat{\mathcal{G}}:\bar{\mathcal{E}}_1\cup\bar{\mathcal{E}}_2\to\bar{\mathcal{E}}_1\cup\bar{\mathcal{E}}_2$ *such that,*

$$\check{e}\leq\aleph\left(\check{U}_1\cup\check{U}_2\right)+\hat{\Lambda}\left(\aleph\left(\check{U}_1\cup\check{U}_2\right)\right)\leq\check{f}\Longrightarrow$$
$$\hat{\Xi}\left\{\aleph\left(\hat{\mathcal{G}}\left(\check{U}_1\right)\cup\hat{\mathcal{G}}\left(\check{U}_2\right)\right)+\hat{\Lambda}\left(\aleph\left(\hat{\mathcal{G}}\left(\check{U}_1\right)\cup\hat{\mathcal{G}}\left(\check{U}_2\right)\right)\right),\hat{\varkappa}\left(\check{e},\check{f}\right)\left(\aleph\left(\check{U}_1\cup\check{U}_2\right)+\hat{\Lambda}\left(\aleph\left(\check{U}_1\cup\check{U}_2\right)\right)\right)\right\}\geq 0. \tag{6}$$

where $\left(\check{U}_1,\check{U}_2\right)$ *is a pair of convex, nonempty, proximinal, closed, bounded and* $\hat{\mathcal{G}}$*-invariant subset of* $\left(\bar{\mathcal{E}}_1,\bar{\mathcal{E}}_2\right)$ *with* $dist\left(\check{U}_1,\check{U}_2\right)=dist\left(\bar{\mathcal{E}}_1,\bar{\mathcal{E}}_2\right)$, $\hat{\Lambda}$ *is nondecreasing continuous functions,* $\hat{\Xi}\in\bar{\mathfrak{R}}$, *and for any* $0<\check{e}<\check{f}<\infty$ *there exists* $0<\hat{\varkappa}\left(\check{e},\check{f}\right)<1$. *Then* $\hat{\mathcal{G}}$ *has a BPP, if* $\bar{\bar{\mathcal{E}}}_1\neq\emptyset$.

Proof. Putting $\hat{ך}(\check{p},\check{q})=\check{p}+\check{q}$ for all $\check{p},\check{q}>0$ in equation (1) of definition (3.4) and using Theorem 3.5, we obtain the result shown above. □

Corollary 3.8. *Consider* $\left(\bar{\mathcal{E}}_1,\bar{\mathcal{E}}_2\right)$ *be a pair of convex and nonempty subsets of a Banach space* $\hat{\mathtt{M}}$ *equipped with a* MNC $\aleph$ *and a relatively nonexpansive, cyclic operator* $\hat{\mathcal{G}}:\bar{\mathcal{E}}_1\cup\bar{\mathcal{E}}_2\to\bar{\mathcal{E}}_1\cup\bar{\mathcal{E}}_2$ *such that,*

$$\check{e}\leq\aleph\left(\check{U}_1\cup\check{U}_2\right)\leq\check{f}\Longrightarrow\hat{\Xi}\left\{\aleph\left(\hat{\mathcal{G}}\left(\check{U}_1\right)\cup\hat{\mathcal{G}}\left(\check{U}_2\right)\right),\hat{\varkappa}\left(\check{e},\check{f}\right)\left(\aleph\left(\check{U}_1\cup\check{U}_2\right)\right)\right\}\geq 0. \tag{7}$$

where $\left(\check{U}_1,\check{U}_2\right)$ *is a pair of convex, nonempty, proximinal, closed, bounded and* $\hat{\mathcal{G}}$*-invariant subset of* $\left(\bar{\mathcal{E}}_1,\bar{\mathcal{E}}_2\right)$ *with* $dist\left(\check{U}_1,\check{U}_2\right)=dist\left(\bar{\mathcal{E}}_1,\bar{\mathcal{E}}_2\right)$, $\hat{\Xi}\in\bar{\mathfrak{R}}$, *and for any* $0<\check{e}<\check{f}<\infty$ *there exists* $0<\hat{\varkappa}\left(\check{e},\check{f}\right)<1$. *Then* $\hat{\mathcal{G}}$ *has a BPP, if* $\bar{\bar{\mathcal{E}}}_1\neq\emptyset$.

Proof. Putting $\hat{\Lambda}\equiv 0$ in equation (6) of corollary (3.7) and using Theorem 3.5, we obtain the result shown above. □

Corollary 3.9. *Consider* $\left(\bar{\mathcal{E}}_1,\bar{\mathcal{E}}_2\right)$ *be a pair of convex and nonempty subsets of a Banach space* $\hat{\mathtt{M}}$ *equipped with a* MNC $\aleph$ *and a relatively nonexpansive, cyclic operator* $\hat{\mathcal{G}}:\bar{\mathcal{E}}_1\cup\bar{\mathcal{E}}_2\to\bar{\mathcal{E}}_1\cup\bar{\mathcal{E}}_2$ *such that,*

$$\check{e}\leq\aleph\left(\check{U}_1\cup\check{U}_2\right)\leq\check{f}\Longrightarrow\tilde{\mathfrak{G}}_2\left(\aleph\left(\hat{\mathcal{G}}\left(\check{U}_1\right)\cup\hat{\mathcal{G}}\left(\check{U}_2\right)\right)\right)\leq\tilde{\mathfrak{G}}_1\left(\hat{\varkappa}\left(\check{e},\check{f}\right)\left(\aleph\left(\check{U}_1\cup\check{U}_2\right)\right)\right). \tag{8}$$

where $\left(\check{\mathtt{U}}_1,\check{\mathtt{U}}_2\right)$ *is a pair of convex, nonempty, proximinal, closed, bounded and* $\hat{\mathcal{G}}$*-invariant subset of* $\left(\bar{\mathcal{E}}_1,\bar{\mathcal{E}}_2\right)$ *with* $dist\left(\check{\mathtt{U}}_1,\check{\mathtt{U}}_2\right)=dist\left(\bar{\mathcal{E}}_1,\bar{\mathcal{E}}_2\right)$, $\tilde{\mathfrak{G}}_1,\tilde{\mathfrak{G}}_2$ *are two alternating distance functions, and for any* $0<\check{\mathtt{e}}<\check{\mathtt{f}}<\infty$ *there exists* $0<\hat{\varkappa}\left(\check{\mathtt{e}},\check{\mathtt{f}}\right)<1$. *Then* $\hat{\mathcal{G}}$ *has a BPP, if* $\bar{\bar{\mathcal{E}}}_1\neq\emptyset$.

Proof. Putting $\hat{\Xi}\left(\check{p},\check{q}\right)=\tilde{\mathfrak{G}}_1\left(\check{q}\right)-\tilde{\mathfrak{G}}_2\left(\check{p}\right)$ for all $\check{p},\check{q}>0$ in equation (7) of corollary (3.8) and using Theorem 3.5, we obtain the result shown above. □

Corollary 3.10. *Consider* $\left(\bar{\mathcal{E}}_1,\bar{\mathcal{E}}_2\right)$ *be a pair of convex and nonempty subsets of a Banach space* $\hat{\mathtt{M}}$ *equipped with a* MNC $\aleph$ *and a relatively nonexpansive, cyclic operator* $\hat{\mathcal{G}}:\bar{\mathcal{E}}_1\cup\bar{\mathcal{E}}_2\to\bar{\mathcal{E}}_1\cup\bar{\mathcal{E}}_2$ *such that,*

$$\check{\mathtt{e}}\leq\aleph\left(\check{\mathtt{U}}_1\cup\check{\mathtt{U}}_2\right)\leq\check{\mathtt{f}}\Longrightarrow\aleph\left(\hat{\mathcal{G}}\left(\check{\mathtt{U}}_1\right)\cup\hat{\mathcal{G}}\left(\check{\mathtt{U}}_2\right)\right)\leq\aleph\left(\check{\mathtt{U}}_1\cup\check{\mathtt{U}}_2\right). \tag{9}$$

where $\left(\check{\mathtt{U}}_1,\check{\mathtt{U}}_2\right)$ *is a pair of convex, nonempty, proximinal, closed, bounded and* $\hat{\mathcal{G}}$*-invariant subset of* $\left(\bar{\mathcal{E}}_1,\bar{\mathcal{E}}_2\right)$ *with* $dist\left(\check{\mathtt{U}}_1,\check{\mathtt{U}}_2\right)=dist\left(\bar{\mathcal{E}}_1,\bar{\mathcal{E}}_2\right)$, *and* $0<\check{\mathtt{e}}<\check{\mathtt{f}}<\infty$. *Then* $\hat{\mathcal{G}}$ *has a BPP, if* $\bar{\bar{\mathcal{E}}}_1\neq\emptyset$.

Proof. Putting $\hat{\Xi}\left(\check{p},\check{q}\right)=\check{q}-\check{p}$ for all $\check{p},\check{q}>0$ in equation (7) of corollary (3.8) and using Theorem 3.5, we obtain the result shown above. □

Example 3.11. *For a Banach space* $\hat{\mathtt{M}}$, *consider a convex, nonempty, compact, proximinal and* $\hat{\mathcal{G}}$*-invariant pair* $\left(\bar{\mathcal{E}}_1,\bar{\mathcal{E}}_2\right)$ *equipped with a* MNC $\aleph$. *Then every relatively nonexpansive, cyclic and* $(\hat{\Xi}\hat{\urcorner}\hat{\Lambda})$*-contractive mapping* $\hat{\mathcal{G}}:\bar{\mathcal{E}}_1\cup\bar{\mathcal{E}}_2\to\bar{\mathcal{E}}_1\cup\bar{\mathcal{E}}_2$ *has a BPP.*

Proof. Let $\left(\hat{X}_1,\hat{Y}_1\right)\subseteq\left(\bar{\mathcal{E}}_1,\bar{\mathcal{E}}_2\right)$ be a convex, nonempty, bounded, proximinal, closed, and $\hat{\mathcal{G}}$-invariant pair, with $\mathrm{dist}\left(\hat{X}_1,\hat{Y}_1\right)=\mathrm{dist}\left(\bar{\mathcal{E}}_1,\bar{\mathcal{E}}_2\right)$. Now we are show that $\left(\hat{\mathcal{G}}(\hat{X}_1),\hat{\mathcal{G}}(\hat{Y}_1)\right)$ is a relatively compact pair. Consider $\{p_n\}$ be a sequence in $\hat{X}_1$. Since $\left(\hat{X}_1,\hat{Y}_1\right)$ is proximinal, there exists a sequence $\{q_n\}$ in $\hat{Y}_1$ such that $\|p_n-q_n\|=\mathrm{dist}\left(\bar{\mathcal{E}}_1,\bar{\mathcal{E}}_2\right)$, for all n ≥ 1. Thus $\|\hat{\mathcal{G}}p_n-\hat{\mathcal{G}}q_n\|\leq\|p_n-q_n\|=\mathrm{dist}\left(\bar{\mathcal{E}}_1,\bar{\mathcal{E}}_2\right)$, for all n ≥ 1. Since $\left(\bar{\mathcal{E}}_1,\bar{\mathcal{E}}_2\right)$ is compact, and proximinal pair, thus the sequence $\{(\hat{\mathcal{G}}p_n,\hat{\mathcal{G}}q_n)\}$ has a convergent subsequence which implies that $\left(\hat{\mathcal{G}}(\hat{X}_1),\hat{\mathcal{G}}(\hat{Y}_1)\right)$ is relatively compact. Hence $\aleph(\hat{\mathcal{G}}(\hat{X}_1),\hat{\mathcal{G}}(\hat{Y}_1))=0$. By putting $\hat{\urcorner}\left(\check{p},\check{q}\right)=\check{p}+\check{q}$, $\hat{\Lambda}=0$, and $\hat{\Xi}\left(\check{p},\check{q}\right)=\check{q}-\check{p}$, for all $\check{p},\check{q}>0$ in equation (1) of definition (3.4), we get a $(\hat{\Xi}\hat{\urcorner}\hat{\Lambda})$-contractive mapping as equation (9) of corollary (3.10). Thus $\aleph(\hat{\mathcal{G}}(\hat{X}_1),\hat{\mathcal{G}}(\hat{Y}_1))=0$ implies that $\hat{\mathcal{G}}$ is a $(\hat{\Xi}\hat{\urcorner}\hat{\Lambda})$-contractive mapping. Therefore $\hat{\mathcal{G}}$ has a *BPP*. □

4. Applications

We apply our findings to investigate the optimum solution the following system of integro differential equations:

$$\begin{cases}\hat{\Phi}\left(\hat{\mathtt{s}}\right)=\hat{\Omega}\left(\hat{\mathtt{s}},\hat{\Phi}\left(\hat{\mathtt{s}}\right)\right)\left(\dfrac{\hat{a}_0}{\hat{\Omega}\left(\hat{\mathtt{s}}_0,\hat{a}_0\right)}+\dfrac{1}{\bar{\Gamma}\left(\hat{\omega}\right)}\displaystyle\int_0^{\hat{\mathtt{s}}}\left(\hat{\mathtt{s}}-\hat{\delta}\right)^{\hat{\omega}-1}\hat{\Psi}\left(\hat{\delta},\hat{\Phi}\left(\hat{\delta}\right)\right)d\hat{\delta}\right)+\dfrac{1}{\bar{\Gamma}\left(\hat{\gamma}\right)}\displaystyle\int_0^{\hat{\mathtt{s}}}\left(\hat{\mathtt{s}}-\hat{\delta}\right)^{\hat{\gamma}-1}\hat{\Theta}_1\left(\hat{\delta}\right)d\hat{\delta},\\ \hat{\Upsilon}\left(\hat{\mathtt{s}}\right)=\hat{\Omega}\left(\hat{\mathtt{s}},\hat{\Upsilon}\left(\hat{\mathtt{s}}\right)\right)\left(\dfrac{\hat{b}_0}{\hat{\Omega}\left(\hat{\mathtt{s}}_0,\hat{b}_0\right)}+\dfrac{1}{\bar{\Gamma}\left(\hat{\omega}\right)}\displaystyle\int_0^{\hat{\mathtt{s}}}\left(\hat{\mathtt{s}}-\hat{\delta}\right)^{\hat{\omega}-1}\hat{\Pi}\left(\hat{\delta},\hat{\Upsilon}\left(\hat{\delta}\right)\right)d\hat{\delta}\right)+\dfrac{1}{\bar{\Gamma}\left(\hat{\gamma}\right)}\displaystyle\int_0^{\hat{\mathtt{s}}}\left(\hat{\mathtt{s}}-\hat{\delta}\right)^{\hat{\gamma}-1}\hat{\Theta}_2\left(\hat{\delta}\right)d\hat{\delta},\end{cases} \tag{10}$$

for $\hat{\mathtt{s}},\hat{\mathtt{s}}_0,\hat{\delta}\in[0,1]=\bar{\mathcal{E}}$, $\hat{\omega},\hat{\gamma}>0$, and $\hat{a}_0,\hat{b}_0\geq0$. Also, assume that $(\tilde{\mathtt{R}},\|.\|)$ be a Banach space and two closed ball $\check{\mathcal{N}}_1=\tilde{\mathcal{D}}\left(\hat{a}_0,\vec{d}\right)$ and $\check{\mathcal{N}}_2=\tilde{\mathcal{D}}\left(\hat{b}_0,\vec{d}\right)$ in $\hat{\mathtt{M}}$ with $\hat{a}_0,\hat{b}_0\in\hat{\mathtt{M}}$. And $\hat{\Omega}:\check{\mathtt{J}}\times\tilde{\mathtt{R}}\longrightarrow\tilde{\mathtt{R}}\setminus\{0\}$, and $\hat{\Psi},\hat{\Pi}:\check{\mathtt{J}}\times\tilde{\mathtt{R}}\longrightarrow\tilde{\mathtt{R}}$; $\hat{\Theta}_1,\hat{\Theta}_2:\tilde{\mathtt{R}}\longrightarrow\tilde{\mathtt{R}}$ are continuous map. Consider a standard Banach space $\mathbf{W}=C\left(\check{\mathtt{J}},\hat{\mathtt{M}}\right)$ of continuous function with supremum norm for $\check{\mathtt{J}}=[0,1)\subseteq\bar{\mathcal{E}}$. Also $\bar{\Gamma}(,)$ denotes Euler's gamma function. Let:

$\mathbf{W}_1 = C\left(\check{\mathrm{J}}, \check{\mathcal{N}}_1\right) = \{\acute{\Phi} : \check{\mathrm{J}} \to \check{\mathcal{N}}_1 : \acute{\Phi} \in \mathbf{W}\}$,
$\mathbf{W}_2 = C\left(\check{\mathrm{J}}, \check{\mathcal{N}}_2\right) = \{\acute{\Upsilon} : \check{\mathrm{J}} \to \check{\mathcal{N}}_2 : \acute{\Upsilon} \in \mathbf{W}\}$.
Then $(\mathbf{W}_1, \mathbf{W}_2)$ is a NBCC pair in $\mathbf{W}$. Now, for every $\acute{\Phi} \in \mathbf{W}_1$ and $\acute{\Upsilon} \in \mathbf{W}_2$, $\|\acute{\Phi} - \acute{\Upsilon}\| = \sup_{\hat{\mathrm{s}} \in \check{\mathrm{J}}} \|\acute{\Phi}(\hat{\mathrm{s}}) - \acute{\Upsilon}(\hat{\mathrm{s}})\| \geq \|\hat{a}_0 - \hat{b}_0\|$. Thus, dist$(\mathbf{W}_1, \mathbf{W}_2) = \|\hat{a}_0 - \hat{b}_0\|$. Now, we define $\hat{\mathcal{G}} : \mathbf{W}_1 \cup \mathbf{W}_2 \to \mathbf{W}$ such that

$$\hat{\mathcal{G}}\left(\acute{\Phi}(\hat{\mathrm{s}})\right) = \begin{cases} \hat{\Omega}\left(\hat{\mathrm{s}}, \acute{\Phi}(\hat{\mathrm{s}})\right)\left(\dfrac{\hat{b}_0}{\hat{\Omega}\left(\hat{\mathrm{s}}_0, \hat{b}_0\right)} + \dfrac{1}{\bar{\Gamma}(\hat{\omega})}\int_0^{\hat{\mathrm{s}}}\left(\hat{\mathrm{s}} - \hat{\delta}\right)^{\hat{\omega}-1}\hat{\Pi}\left(\hat{\delta}, \acute{\Phi}\left(\hat{\delta}\right)\right) d\hat{\delta}\right) + \dfrac{1}{\bar{\Gamma}(\hat{\gamma})}\int_0^{\hat{\mathrm{s}}}\left(\hat{\mathrm{s}} - \hat{\delta}\right)^{\hat{\gamma}-1}\hat{\Theta}_2\left(\hat{\delta}\right) d\hat{\delta}, \acute{\Phi} \in \mathbf{W}_1, \\ \hat{\Omega}\left(\hat{\mathrm{s}}, \acute{\Phi}(\hat{\mathrm{s}})\right)\left(\dfrac{\hat{a}_0}{\hat{\Omega}\left(\hat{\mathrm{s}}_0, \hat{a}_0\right)} + \dfrac{1}{\bar{\Gamma}(\hat{\omega})}\int_0^{\hat{\mathrm{s}}}\left(\hat{\mathrm{s}} - \hat{\delta}\right)^{\hat{\omega}-1}\hat{\Psi}\left(\hat{\delta}, \acute{\Phi}\left(\hat{\delta}\right)\right) d\hat{\delta}\right) + \dfrac{1}{\bar{\Gamma}(\hat{\gamma})}\int_0^{\hat{\mathrm{s}}}\left(\hat{\mathrm{s}} - \hat{\delta}\right)^{\hat{\gamma}-1}\hat{\Theta}_1\left(\hat{\delta}\right) d\hat{\delta}, \acute{\Phi} \in \mathbf{W}_2. \end{cases}$$

Clearly, $\hat{\mathcal{G}}$ is cyclic, and if $\|\check{s} - \hat{\mathcal{G}}(\check{s})\| = dist(\mathbf{W}_1, \mathbf{W}_2)$, for $\check{s} \in \mathbf{W}_1 \cup \mathbf{W}_2$, then $\check{s}$ is a *BPP* for the operator $\hat{\mathcal{G}}$ and it is equivalent to that $\check{s}$ is an optimum solution of the system (10).

Theorem 4.1. *[9] Consider $\acute{\mathrm{V}}_1 \in C[\check{r}_1, \check{r}_2]$ with $\check{r}_1 < \check{r}_2$. Let $\acute{\mathrm{V}}_2$ is Lebesgue integrable on $[\check{r}_1, \check{r}_2]$ and $\acute{\mathrm{V}}_2$ does not change its sign in $[\check{r}_1, \check{r}_2]$. Then the generalized mean value theorem of integral calculus gives,*

$$\int_{\check{r}_1}^{\check{r}_2} \acute{\mathrm{V}}_1(\hat{\mathrm{s}})\acute{\mathrm{V}}_2(\hat{\mathrm{s}})\, d\hat{\mathrm{s}} = \acute{\mathrm{V}}_1(\hat{g})\int_{\check{r}_1}^{\check{r}_2} \acute{\mathrm{V}}_2(\hat{\mathrm{s}})\, d\hat{\mathrm{s}}, \tag{11}$$

for some $\hat{g} \in (\check{r}_1, \check{r}_2)$.

Theorem 4.2. *Assume that $\aleph_0$ be a* MNC *on* $\mathbf{W}$. *For the continuous functions $\hat{\Omega}, \hat{\Psi}, \hat{\Pi}, \hat{\Theta}_1, \hat{\Theta}_2$, there exists $\mathtt{D}, \mathtt{D}_1, \mathtt{D}_2, Ł_1, Ł_2 > o$ such that $\|\hat{\Omega}\left(., \acute{\Phi}(.)\right)\|, \|\hat{\Omega}\left(., \acute{\Upsilon}(.)\right)\| \leq \mathtt{D}$, $\|\hat{\Psi}\left(\hat{\delta}, \acute{\Phi}\left(\hat{\delta}\right)\right)\| \leq \mathtt{D}_1$, $\|\hat{\Pi}\left(., \acute{\Phi}(.)\right)\|, \|\hat{\Pi}\left(., \acute{\Upsilon}(.)\right)\| \leq \mathtt{D}_2$, $\|\hat{\Theta}_1\left(\hat{\delta}\right)\| \leq Ł_1$, and $\|\hat{\Theta}_2\left(\hat{\delta}\right)\| \leq Ł_2$. Also there exists $\tilde{\mathcal{K}}_0, \tilde{\mathcal{K}}_1, \tilde{\mathcal{K}}_2 > 0$, such that $\dfrac{|\hat{b}_0|\mathtt{D}}{\left|\hat{\Omega}\left(\hat{\mathrm{s}}_0, \hat{b}_0\right)\right|} \leq \tilde{\mathcal{K}}_0$, $\tilde{\mathcal{K}}_0 + |\hat{b}_0| \leq \tilde{\mathcal{K}}_1$, and $\tilde{\mathcal{K}}_1 + \dfrac{\mathtt{D}\mathtt{D}_2}{\bar{\Gamma}(\hat{\omega}+1)} + \dfrac{Ł_2}{\bar{\Gamma}(\hat{\gamma}+1)} \leq \tilde{\mathcal{K}}_2$. Then an optimal solution exists for the system of integro differential equations (10) if:*

(i) For any bounded pair $(\mathtt{J}_1, \mathtt{J}_2) \subseteq (\mathbf{W}_1, \mathbf{W}_2)$, $\aleph_0\left(\left\{\hat{\Pi}(\check{\mathrm{J}} \times \mathtt{J}_1) + \hat{\Theta}_1(\check{\mathrm{J}})\right\}\right), \aleph_0\left(\left\{\hat{\Psi}(\check{\mathrm{J}} \times \mathtt{J}_2) + \hat{\Theta}_2(\check{\mathrm{J}})\right\}\right) > 0$ implies that,

$$\aleph_0\left(\left\{\hat{\Pi}(\check{\mathrm{J}} \times \mathtt{J}_1) + \hat{\Theta}_1(\check{\mathrm{J}})\right\} \cup \left\{\hat{\Psi}(\check{\mathrm{J}} \times \mathtt{J}_2) + \hat{\Theta}_2(\check{\mathrm{J}})\right\}\right) \leq \aleph_0(\mathtt{J}_1 \cup \mathtt{J}_2).$$

(ii) $\hat{\Omega} : \check{\mathrm{J}} \times \tilde{\mathrm{R}} \longrightarrow \tilde{\mathrm{R}} \setminus \{0\}$ be continuous map satisfying

$$\left|\hat{\Omega}\left(\hat{\mathrm{s}}, \acute{\Phi}(\hat{\mathrm{s}})\right)\left(\frac{\hat{b}_0}{\hat{\Omega}\left(\hat{\mathrm{s}}_0, \hat{b}_0\right)}\right) - \hat{\Omega}\left(\hat{\mathrm{s}}, \acute{\Upsilon}(\hat{\mathrm{s}})\right)\left(\frac{\hat{a}_0}{\hat{\Omega}\left(\hat{\mathrm{s}}_0, \hat{a}_0\right)}\right)\right| \leq \frac{1}{2}\left|\hat{a}_0 - \hat{b}_0\right|$$

for $\acute{\Phi} \in \mathbf{W}_1$, $\acute{\Upsilon} \in \mathbf{W}_2$, $\hat{\mathrm{s}}, \hat{\mathrm{s}}_0 \in \bar{\mathcal{E}}$, and $\hat{a}_0, \hat{b}_0 \geq 0$.

(iii) $\hat{\Theta}_1, \hat{\Theta}_2 : \tilde{\mathrm{R}} \longrightarrow \tilde{\mathrm{R}}$ are continuous map satisfying

$$\left|\hat{\Theta}_2\left(\hat{\delta}\right) - \hat{\Theta}_1\left(\hat{\delta}\right)\right| \leq \left(\frac{\bar{\Gamma}(\hat{\gamma}+1)}{2}\left|\left(\hat{a}_0 - \hat{b}_0\right)\right|\right)$$

for $\hat{\delta} \in \bar{\mathcal{E}}$, $\hat{\gamma} > 0$, and $\hat{a}_0, \hat{b}_0 \geq 0$.

(iv) $\hat{\Omega} : \check{\mathrm{J}} \times \tilde{\mathrm{R}} \longrightarrow \tilde{\mathrm{R}} \setminus \{0\}$ be continuous map satisfying

$$\left|\hat{\Omega}\left(\hat{\mathrm{s}}, \acute{\Phi}(\hat{\mathrm{s}})\right) - \hat{\Omega}\left(\hat{\mathrm{s}}, \acute{\Upsilon}(\hat{\mathrm{s}})\right)\right| \leq \frac{\bar{\Gamma}(\hat{\omega}+1)}{2\mathtt{D}_2}|\acute{\Phi} - \acute{\Upsilon}|.$$

for $\acute{\Phi} \in \mathbf{W}_1$, $\acute{\Upsilon} \in \mathbf{W}_2$, $\hat{\mathtt{s}} \in \bar{\mathcal{E}}$, $\hat{\omega}, \mathtt{D}_2 > 0$.

(v) $\hat{\Psi}, \hat{\Pi} : \breve{\mathtt{J}} \times \tilde{\mathtt{R}} \longrightarrow \tilde{\mathtt{R}}$ *are continuous map satisfying*

$$\left|\hat{\Pi}\left(\hat{\delta}, \acute{\Phi}\left(\hat{\delta}\right)\right) - \hat{\Psi}\left(\hat{\delta}, \acute{\Upsilon}\left(\hat{\delta}\right)\right)\right| \leq \frac{\bar{\Gamma}\left(\hat{\omega}+1\right)}{2\mathtt{D}}\left(|\acute{\Phi} - \acute{\Upsilon}| - 2|\hat{a}_0 - \hat{b}_0|\right)$$

for $\acute{\Phi} \in \mathbf{W}_1$, $\acute{\Upsilon} \in \mathbf{W}_2$, $\hat{\delta} \in \bar{\mathcal{E}}$, *and* $\hat{a}_0, \hat{b}_0 \geq 0$.

(vi) $\hat{\Omega} : \breve{\mathtt{J}} \times \tilde{\mathtt{R}} \longrightarrow \tilde{\mathtt{R}} \setminus \{0\}$ *be continuous and there exists* $\tilde{\mathtt{m}} > 0$ *satisfying*

$$\left|\hat{\Omega}\left(\hat{\mathtt{s}}, \acute{\Phi}\left(\hat{\mathtt{s}}\right)\right) - \hat{\Omega}\left(\bar{\hat{\mathtt{s}}}, \acute{\Phi}\left(\bar{\hat{\mathtt{s}}}\right)\right)\right| \leq \tilde{\mathtt{m}}\left|\hat{\mathtt{s}} - \bar{\hat{\mathtt{s}}}\right|$$

for $\acute{\Phi} \in \mathbf{W}_1$, $\hat{\mathtt{s}}, \bar{\hat{\mathtt{s}}} \in \bar{\mathcal{E}}$.

Proof. First we show that the operator $\hat{\mathcal{G}}$ is cyclic.
For $\acute{\Phi} \in \mathbf{W}_1$,

$$\begin{aligned}
&\|\hat{\mathcal{G}}\left(\acute{\Phi}\left(\hat{\mathtt{s}}\right)\right) - \hat{b}_0\| \\
&= \left\|\hat{\Omega}\left(\hat{\mathtt{s}}, \acute{\Phi}\left(\hat{\mathtt{s}}\right)\right)\left(\frac{\hat{b}_0}{\hat{\Omega}\left(\hat{\mathtt{s}}_0, \hat{b}_0\right)} + \frac{1}{\bar{\Gamma}\left(\hat{\omega}\right)}\int_0^{\hat{\mathtt{s}}}\left(\hat{\mathtt{s}} - \hat{\delta}\right)^{\hat{\omega}-1}\hat{\Pi}\left(\hat{\delta}, \acute{\Phi}\left(\hat{\delta}\right)\right)d\hat{\delta}\right) + \frac{1}{\bar{\Gamma}\left(\hat{\gamma}\right)}\int_0^{\hat{\mathtt{s}}}\left(\hat{\mathtt{s}} - \hat{\delta}\right)^{\hat{\gamma}-1}\hat{\Theta}_2\left(\hat{\delta}\right)d\hat{\delta} - \hat{b}_0\right\| \\
&\leq \left|\hat{b}_0\left(\frac{\hat{\Omega}\left(\hat{\mathtt{s}}, \acute{\Phi}\left(\hat{\mathtt{s}}\right)\right)}{\hat{\Omega}\left(\hat{\mathtt{s}}_0, \hat{b}_0\right)} - 1\right)\right| + \left|\frac{\hat{\Omega}\left(\hat{\mathtt{s}}, \acute{\Phi}\left(\hat{\mathtt{s}}\right)\right)}{\bar{\Gamma}\left(\hat{\omega}\right)}\int_0^{\hat{\mathtt{s}}}\left(\hat{\mathtt{s}} - \hat{\delta}\right)^{\hat{\omega}-1}\hat{\Pi}\left(\hat{\delta}, \acute{\Phi}\left(\hat{\delta}\right)\right)d\hat{\delta}\right| + \left|\frac{1}{\bar{\Gamma}\left(\hat{\gamma}\right)}\int_0^{\hat{\mathtt{s}}}\left(\hat{\mathtt{s}} - \hat{\delta}\right)^{\hat{\gamma}-1}\hat{\Theta}_2\left(\hat{\delta}\right)d\hat{\delta}\right| \\
&\leq \left|\hat{b}_0\right|\left|\left(\frac{\hat{\Omega}\left(\hat{\mathtt{s}}, \acute{\Phi}\left(\hat{\mathtt{s}}\right)\right)}{\hat{\Omega}\left(\hat{\mathtt{s}}_0, \hat{b}_0\right)} - 1\right)\right| + \left|\frac{\hat{\Omega}\left(\hat{\mathtt{s}}, \acute{\Phi}\left(\hat{\mathtt{s}}\right)\right)}{\bar{\Gamma}\left(\hat{\omega}\right)}\int_0^{\hat{\mathtt{s}}}\left(\hat{\mathtt{s}} - \hat{\delta}\right)^{\hat{\omega}-1}\hat{\Pi}\left(\hat{\delta}, \acute{\Phi}\left(\hat{\delta}\right)\right)d\hat{\delta}\right| + \left|\frac{1}{\bar{\Gamma}\left(\hat{\gamma}\right)}\int_0^{\hat{\mathtt{s}}}\left(\hat{\mathtt{s}} - \hat{\delta}\right)^{\hat{\gamma}-1}\hat{\Theta}_2\left(\hat{\delta}\right)d\hat{\delta}\right| \\
&\leq \left|\hat{b}_0\right|\left|\left(\frac{\mathtt{D}}{\hat{\Omega}\left(\hat{\mathtt{s}}_0, \hat{b}_0\right)} - 1\right)\right| + \frac{\mathtt{D}\mathtt{D}_2}{\bar{\Gamma}\left(\hat{\omega}\right)}\left|\int_0^{\hat{\mathtt{s}}}\left(\hat{\mathtt{s}} - \hat{\delta}\right)^{\hat{\omega}-1}d\hat{\delta}\right| + \frac{\text{Ł}_2}{\bar{\Gamma}\left(\hat{\gamma}\right)}\left|\int_0^{\hat{\mathtt{s}}}\left(\hat{\mathtt{s}} - \hat{\delta}\right)^{\hat{\gamma}-1}d\hat{\delta}\right| \\
&\leq \left(\frac{|\hat{b}_0|\mathtt{D}}{\left|\hat{\Omega}\left(\hat{\mathtt{s}}_0, \hat{b}_0\right)\right|} + |\hat{b}_0|\right) + \frac{\mathtt{D}\mathtt{D}_2}{\bar{\Gamma}\left(\hat{\omega}+1\right)}|\left(\hat{\mathtt{s}}\right)^{\hat{\omega}}| + \frac{\text{Ł}_2}{\bar{\Gamma}\left(\hat{\gamma}+1\right)}|\left(\hat{\mathtt{s}}\right)^{\hat{\gamma}}| \\
&\leq \left(\tilde{\mathcal{K}}_0 + |\hat{b}_0|\right) + \frac{\mathtt{D}\mathtt{D}_2}{\bar{\Gamma}\left(\hat{\omega}+1\right)} + \frac{\text{Ł}_2}{\bar{\Gamma}\left(\hat{\gamma}+1\right)} \\
&\leq \tilde{\mathcal{K}}_1 + \frac{\mathtt{D}\mathtt{D}_2}{\bar{\Gamma}\left(\hat{\omega}+1\right)} + \frac{\text{Ł}_2}{\bar{\Gamma}\left(\hat{\gamma}+1\right)} \\
&\leq \tilde{\mathcal{K}}_2.
\end{aligned}$$

Hence, $\hat{\mathcal{G}}\left(\acute{\Phi}\left(\hat{\mathtt{s}}\right)\right) \in \mathbf{W}_2$. Using a similar method, we can show that $\hat{\mathcal{G}}\left(\acute{\Phi}\left(\hat{\mathtt{s}}\right)\right) \in \mathbf{W}_1$ for $\acute{\Phi} \in \mathbf{W}_2$. Thus, $\hat{\mathcal{G}}$ is cyclic. We now show that $\hat{\mathcal{G}}\left(\mathbf{W}_1\right)$ is an equicontinuous and bounded subset of $\mathbf{W}_2$. Consider $\acute{\Phi} \in \mathbf{W}_1$ and $\hat{\mathtt{s}} \in \bar{\mathcal{E}}$, we have,

$$\begin{aligned}
&\|\hat{\mathcal{G}}\left(\acute{\Phi}\left(\hat{\mathtt{s}}\right)\right)\| \\
&= \left\|\hat{\Omega}\left(\hat{\mathtt{s}}, \acute{\Phi}\left(\hat{\mathtt{s}}\right)\right)\left(\frac{\hat{b}_0}{\hat{\Omega}\left(\hat{\mathtt{s}}_0, \hat{b}_0\right)} + \frac{1}{\bar{\Gamma}\left(\hat{\omega}\right)}\int_0^{\hat{\mathtt{s}}}\left(\hat{\mathtt{s}} - \hat{\delta}\right)^{\hat{\omega}-1}\hat{\Pi}\left(\hat{\delta}, \acute{\Phi}\left(\hat{\delta}\right)\right)d\hat{\delta}\right) + \frac{1}{\bar{\Gamma}\left(\hat{\gamma}\right)}\int_0^{\hat{\mathtt{s}}}\left(\hat{\mathtt{s}} - \hat{\delta}\right)^{\hat{\gamma}-1}\hat{\Theta}_2\left(\hat{\delta}\right)d\hat{\delta}\right\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \left|\hat{\Omega}\left(\hat{\mathsf{s}},\acute{\Phi}(\hat{\mathsf{s}})\right)\frac{\hat{b}_0}{\hat{\Omega}\left(\hat{\mathsf{s}}_0,\hat{b}_0\right)}\right| + \left|\frac{\hat{\Omega}\left(\hat{\mathsf{s}},\acute{\Phi}(\hat{\mathsf{s}})\right)}{\bar{\Gamma}(\hat{\omega})}\int_0^{\hat{\mathsf{s}}}\left(\hat{\mathsf{s}}-\hat{\delta}\right)^{\hat{\omega}-1}\hat{\Pi}\left(\hat{\delta},\acute{\Phi}\left(\hat{\delta}\right)\right)d\hat{\delta}\right| + \left|\frac{1}{\bar{\Gamma}(\hat{\gamma})}\int_0^{\hat{\mathsf{s}}}\left(\hat{\mathsf{s}}-\hat{\delta}\right)^{\hat{\gamma}-1}\hat{\Theta}_2\left(\hat{\delta}\right)d\hat{\delta}\right| \\
&\leq \mathtt{D}\frac{|\hat{b}_0|}{\left|\hat{\Omega}\left(\hat{\mathsf{s}}_0,\hat{b}_0\right)\right|} + \frac{\mathtt{DD}_2}{\bar{\Gamma}(\hat{\omega}+1)}|(\hat{\mathsf{s}})^{\hat{\omega}}| + \frac{\text{Ł}_2}{\bar{\Gamma}(\hat{\gamma}+1)}|(\hat{\mathsf{s}})^{\hat{\gamma}}| \\
&\leq \tilde{\mathcal{K}}_0 + \frac{\mathtt{DD}_2}{\bar{\Gamma}(\hat{\omega}+1)} + \frac{\text{Ł}_2}{\bar{\Gamma}(\hat{\gamma}+1)} \\
&\leq \tilde{\mathcal{K}}_2.
\end{aligned}
$$

Thus, $\hat{\mathcal{G}}(\mathbf{W}_1)$ is bounded.

Suppose that $\hat{\mathsf{s}}, \bar{\hat{\mathsf{s}}} \in \bar{\mathcal{E}}$, $\hat{\mathsf{s}} > \bar{\hat{\mathsf{s}}}$ and $\acute{\Phi} \in \mathbf{W}_1$. Then

$$
\begin{aligned}
&\left\|\hat{\mathcal{G}}\left(\acute{\Phi}(\hat{\mathsf{s}})\right)-\hat{\mathcal{G}}\left(\acute{\Phi}(\bar{\hat{\mathsf{s}}})\right)\right\| \\
&= \Bigg\|\hat{\Omega}\left(\hat{\mathsf{s}},\acute{\Phi}(\hat{\mathsf{s}})\right)\left(\frac{\hat{b}_0}{\hat{\Omega}\left(\hat{\mathsf{s}}_0,\hat{b}_0\right)}+\frac{1}{\bar{\Gamma}(\hat{\omega})}\int_0^{\hat{\mathsf{s}}}\left(\hat{\mathsf{s}}-\hat{\delta}\right)^{\hat{\omega}-1}\hat{\Pi}\left(\hat{\delta},\acute{\Phi}\left(\hat{\delta}\right)\right)d\hat{\delta}\right)+\frac{1}{\bar{\Gamma}(\hat{\gamma})}\int_0^{\hat{\mathsf{s}}}\left(\hat{\mathsf{s}}-\hat{\delta}\right)^{\hat{\gamma}-1}\hat{\Theta}_2\left(\hat{\delta}\right)d\hat{\delta}- \\
&\hat{\Omega}\left(\bar{\hat{\mathsf{s}}},\acute{\Phi}(\bar{\hat{\mathsf{s}}})\right)\left(\frac{\hat{b}_0}{\hat{\Omega}\left(\hat{\mathsf{s}}_0,\hat{b}_0\right)}+\frac{1}{\bar{\Gamma}(\hat{\omega})}\int_0^{\bar{\hat{\mathsf{s}}}}\left(\bar{\hat{\mathsf{s}}}-\hat{\delta}\right)^{\hat{\omega}-1}\hat{\Pi}\left(\hat{\delta},\acute{\Phi}\left(\hat{\delta}\right)\right)d\hat{\delta}\right)-\frac{1}{\bar{\Gamma}(\hat{\gamma})}\int_0^{\bar{\hat{\mathsf{s}}}}\left(\bar{\hat{\mathsf{s}}}-\hat{\delta}\right)^{\hat{\gamma}-1}\hat{\Theta}_2\left(\hat{\delta}\right)d\hat{\delta}\Bigg\| \\
&\leq \left|\frac{\hat{b}_0}{\hat{\Omega}\left(\hat{\mathsf{s}}_0,\hat{b}_0\right)}\right|\left\|\hat{\Omega}\left(\hat{\mathsf{s}},\acute{\Phi}(\hat{\mathsf{s}})\right)-\hat{\Omega}\left(\bar{\hat{\mathsf{s}}},\acute{\Phi}(\bar{\hat{\mathsf{s}}})\right)\right|+\frac{\left|\hat{\Omega}\left(\hat{\mathsf{s}},\acute{\Phi}(\hat{\mathsf{s}})\right)-\hat{\Omega}\left(\bar{\hat{\mathsf{s}}},\acute{\Phi}(\bar{\hat{\mathsf{s}}})\right)\right|}{\bar{\Gamma}(\hat{\omega})}\int_0^{\hat{\mathsf{s}}}\left(\hat{\mathsf{s}}-\hat{\delta}\right)^{\hat{\omega}-1}\left|\hat{\Pi}\left(\hat{\delta},\acute{\Phi}\left(\hat{\delta}\right)\right)\right|d\hat{\delta}+ \\
&\frac{\left|\hat{\Omega}\left(\bar{\hat{\mathsf{s}}},\acute{\Phi}(\bar{\hat{\mathsf{s}}})\right)\right|}{\bar{\Gamma}(\hat{\omega})}\left(\left|\int_0^{\hat{\mathsf{s}}}\left(\hat{\mathsf{s}}-\hat{\delta}\right)^{\hat{\omega}-1}\hat{\Pi}\left(\hat{\delta},\acute{\Phi}\left(\hat{\delta}\right)\right)d\hat{\delta}-\int_0^{\bar{\hat{\mathsf{s}}}}\left(\bar{\hat{\mathsf{s}}}-\hat{\delta}\right)^{\hat{\omega}-1}\hat{\Pi}\left(\hat{\delta},\acute{\Phi}\left(\hat{\delta}\right)\right)d\hat{\delta}\right|\right)+\frac{\text{Ł}_2}{\bar{\Gamma}(\hat{\gamma})}\left|\int_0^{\hat{\mathsf{s}}}\left(\hat{\mathsf{s}}-\hat{\delta}\right)^{\hat{\gamma}-1}d\hat{\delta}-\int_0^{\bar{\hat{\mathsf{s}}}}\left(\bar{\hat{\mathsf{s}}}-\hat{\delta}\right)^{\hat{\gamma}-1}d\hat{\delta}\right| \\
&\leq \tilde{\mathtt{m}}|\hat{\mathsf{s}}-\bar{\hat{\mathsf{s}}}|\left|\frac{\hat{b}_0}{\hat{\Omega}\left(\hat{\mathsf{s}}_0,\hat{b}_0\right)}\right|+\frac{\left|\hat{\Omega}\left(\hat{\mathsf{s}},\acute{\Phi}(\hat{\mathsf{s}})\right)-\hat{\Omega}\left(\bar{\hat{\mathsf{s}}},\acute{\Phi}(\bar{\hat{\mathsf{s}}})\right)\right|}{\bar{\Gamma}(\hat{\omega}+1)}\mathtt{D}_2|(\hat{\mathsf{s}})^{\hat{\omega}}|+\frac{\mathtt{D}}{\bar{\Gamma}(\hat{\omega})}\left(\left|\int_0^{\bar{\hat{\mathsf{s}}}}\left(\hat{\mathsf{s}}-\hat{\delta}\right)^{\hat{\omega}-1}\hat{\Pi}\left(\hat{\delta},\acute{\Phi}\left(\hat{\delta}\right)\right)d\hat{\delta}\right.\right. \\
&\quad\left.\left.+\int_{\bar{\hat{\mathsf{s}}}}^{\hat{\mathsf{s}}}\left(\hat{\mathsf{s}}-\hat{\delta}\right)^{\hat{\omega}-1}\hat{\Pi}\left(\hat{\delta},\acute{\Phi}\left(\hat{\delta}\right)\right)d\hat{\delta}-\int_0^{\bar{\hat{\mathsf{s}}}}\left(\bar{\hat{\mathsf{s}}}-\hat{\delta}\right)^{\hat{\omega}-1}\hat{\Pi}\left(\hat{\delta},\acute{\Phi}\left(\hat{\delta}\right)\right)d\hat{\delta}\right|\right)+\frac{\text{Ł}_2}{\bar{\Gamma}(\hat{\gamma}+1)}\left|(\hat{\mathsf{s}})^{\hat{\gamma}}-(\bar{\hat{\mathsf{s}}})^{\hat{\gamma}}\right| \\
&\leq \tilde{\mathtt{m}}|\hat{\mathsf{s}}-\bar{\hat{\mathsf{s}}}|\left|\frac{\hat{b}_0}{\hat{\Omega}\left(\hat{\mathsf{s}}_0,\hat{b}_0\right)}\right|+\frac{\mathtt{D}_2}{\bar{\Gamma}(\hat{\omega}+1)}\tilde{\mathtt{m}}|\hat{\mathsf{s}}-\bar{\hat{\mathsf{s}}}|+\frac{\mathtt{DD}_2}{\bar{\Gamma}(\hat{\omega})}\left|\int_0^{\bar{\hat{\mathsf{s}}}}\left(\left(\hat{\mathsf{s}}-\hat{\delta}\right)^{\hat{\omega}-1}-\left(\bar{\hat{\mathsf{s}}}-\hat{\delta}\right)^{\hat{\omega}-1}\right)d\hat{\delta}\right|+\frac{\mathtt{DD}_2}{\bar{\Gamma}(\hat{\omega})}\left|\int_{\bar{\hat{\mathsf{s}}}}^{\hat{\mathsf{s}}}\left(\hat{\mathsf{s}}-\hat{\delta}\right)^{\hat{\omega}-1}d\hat{\delta}\right|+\frac{\text{Ł}_2}{\bar{\Gamma}(\hat{\gamma}+1)}\left|(\hat{\mathsf{s}})^{\hat{\gamma}}-(\bar{\hat{\mathsf{s}}})^{\hat{\gamma}}\right| \\
&\leq \tilde{\mathtt{m}}|\hat{\mathsf{s}}-\bar{\hat{\mathsf{s}}}|\left|\frac{\hat{b}_0}{\hat{\Omega}\left(\hat{\mathsf{s}}_0,\hat{b}_0\right)}\right|+\tilde{\mathtt{m}}|\hat{\mathsf{s}}-\bar{\hat{\mathsf{s}}}|\frac{\mathtt{D}_2}{\bar{\Gamma}(\hat{\omega}+1)}+\frac{\mathtt{DD}_2}{\bar{\Gamma}(\hat{\omega}+1)}\left|-(\hat{\mathsf{s}}-\bar{\hat{\mathsf{s}}})^{\hat{\omega}}+(\hat{\mathsf{s}})^{\hat{\omega}}-(\bar{\hat{\mathsf{s}}})^{\hat{\omega}}\right|+\frac{\mathtt{DD}_2}{\bar{\Gamma}(\hat{\omega}+1)}\left|(\hat{\mathsf{s}}-\bar{\hat{\mathsf{s}}})^{\hat{\omega}}\right|+\frac{\text{Ł}_2}{\bar{\Gamma}(\hat{\gamma}+1)}\left|(\hat{\mathsf{s}})^{\hat{\gamma}}-(\bar{\hat{\mathsf{s}}})^{\hat{\gamma}}\right|
\end{aligned}
$$

As $\bar{\hat{\mathsf{s}}} \to \hat{\mathsf{s}}$, we have, $\|\hat{\mathcal{G}}\left(\acute{\Phi}(\hat{\mathsf{s}})\right)-\hat{\mathcal{G}}\left(\acute{\Phi}(\bar{\hat{\mathsf{s}}})\right)\| \to 0$,

i.e. $\hat{\mathcal{G}}(\mathbf{W}_1)$ is equicontinuous. We can show that $\hat{\mathcal{G}}(\mathbf{W}_2)$ is equicontinuous and bounded with the similar manner. Thus, by Arzela-Ascoli theorem we conclude that $(\mathbf{W}_1, \mathbf{W}_2)$ is a relatively compact pair. Now, for each $\left(\acute{\Phi}, \acute{\Upsilon}\right) \in \mathbf{W}_1 \times \mathbf{W}_2$, we have,

$$
\begin{aligned}
&\left\|\hat{\mathcal{G}}\left(\acute{\Phi}(\hat{\mathsf{s}})\right)-\hat{\mathcal{G}}\left(\acute{\Upsilon}(\hat{\mathsf{s}})\right)\right\| \\
&= \Bigg\|\hat{\Omega}\left(\hat{\mathsf{s}},\acute{\Phi}(\hat{\mathsf{s}})\right)\left(\frac{\hat{b}_0}{\hat{\Omega}\left(\hat{\mathsf{s}}_0,\hat{b}_0\right)}+\frac{1}{\bar{\Gamma}(\hat{\omega})}\int_0^{\hat{\mathsf{s}}}\left(\hat{\mathsf{s}}-\hat{\delta}\right)^{\hat{\omega}-1}\hat{\Pi}\left(\hat{\delta},\acute{\Phi}\left(\hat{\delta}\right)\right)d\hat{\delta}\right)+\frac{1}{\bar{\Gamma}(\hat{\gamma})}\int_0^{\hat{\mathsf{s}}}\left(\hat{\mathsf{s}}-\hat{\delta}\right)^{\hat{\gamma}-1}\hat{\Theta}_2\left(\hat{\delta}\right)d\hat{\delta}- \\
&\hat{\Omega}\left(\hat{\mathsf{s}},\acute{\Upsilon}(\hat{\mathsf{s}})\right)\left(\frac{\hat{a}_0}{\hat{\Omega}(\hat{\mathsf{s}}_0,\hat{a}_0)}+\frac{1}{\bar{\Gamma}(\hat{\omega})}\int_0^{\hat{\mathsf{s}}}\left(\hat{\mathsf{s}}-\hat{\delta}\right)^{\hat{\omega}-1}\hat{\Psi}\left(\hat{\delta},\acute{\Upsilon}\left(\hat{\delta}\right)\right)d\hat{\delta}\right)-\frac{1}{\bar{\Gamma}(\hat{\gamma})}\int_0^{\hat{\mathsf{s}}}\left(\hat{\mathsf{s}}-\hat{\delta}\right)^{\hat{\gamma}-1}\hat{\Theta}_1\left(\hat{\delta}\right)d\hat{\delta}\Bigg\| \\
&\leq \left|\hat{\Omega}\left(\hat{\mathsf{s}},\acute{\Phi}(\hat{\mathsf{s}})\right)\left(\frac{\hat{b}_0}{\hat{\Omega}\left(\hat{\mathsf{s}}_0,\hat{b}_0\right)}\right)-\hat{\Omega}\left(\hat{\mathsf{s}},\acute{\Upsilon}(\hat{\mathsf{s}})\right)\left(\frac{\hat{a}_0}{\hat{\Omega}(\hat{\mathsf{s}}_0,\hat{a}_0)}\right)\right|+\frac{1}{\bar{\Gamma}(\hat{\omega})}\left|\hat{\Omega}\left(\hat{\mathsf{s}},\acute{\Phi}(\hat{\mathsf{s}})\right)\int_0^{\hat{\mathsf{s}}}\left(\hat{\mathsf{s}}-\hat{\delta}\right)^{\hat{\omega}-1}\hat{\Pi}\left(\hat{\delta},\acute{\Phi}\left(\hat{\delta}\right)\right)d\hat{\delta}-\right. \\
&\left.\hat{\Omega}\left(\hat{\mathsf{s}},\acute{\Upsilon}(\hat{\mathsf{s}})\right)\int_0^{\hat{\mathsf{s}}}\left(\hat{\mathsf{s}}-\hat{\delta}\right)^{\hat{\omega}-1}\hat{\Psi}\left(\hat{\delta},\acute{\Upsilon}\left(\hat{\delta}\right)\right)d\hat{\delta}\right|+\frac{1}{\bar{\Gamma}(\hat{\gamma})}\left|\int_0^{\hat{\mathsf{s}}}\left(\hat{\mathsf{s}}-\hat{\delta}\right)^{\hat{\gamma}-1}\left(\hat{\Theta}_2\left(\hat{\delta}\right)-\hat{\Theta}_1\left(\hat{\delta}\right)\right)d\hat{\delta}\right| \\
&\leq \frac{1}{2}\left|\hat{a}_0-\hat{b}_0\right|+\frac{1}{\bar{\Gamma}(\hat{\omega})}\left|\hat{\Omega}\left(\hat{\mathsf{s}},\acute{\Phi}(\hat{\mathsf{s}})\right)\int_0^{\hat{\mathsf{s}}}\left(\hat{\mathsf{s}}-\hat{\delta}\right)^{\hat{\omega}-1}\hat{\Pi}\left(\hat{\delta},\acute{\Phi}\left(\hat{\delta}\right)\right)d\hat{\delta}-\hat{\Omega}\left(\hat{\mathsf{s}},\acute{\Upsilon}(\hat{\mathsf{s}})\right)\int_0^{\hat{\mathsf{s}}}\left(\hat{\mathsf{s}}-\hat{\delta}\right)^{\hat{\omega}-1}\hat{\Pi}\left(\hat{\delta},\acute{\Phi}\left(\hat{\delta}\right)\right)d\hat{\delta}+\right. \\
&\left.\hat{\Omega}\left(\hat{\mathsf{s}},\acute{\Upsilon}(\hat{\mathsf{s}})\right)\int_0^{\hat{\mathsf{s}}}\left(\hat{\mathsf{s}}-\hat{\delta}\right)^{\hat{\omega}-1}\hat{\Pi}\left(\hat{\delta},\acute{\Phi}\left(\hat{\delta}\right)\right)d\hat{\delta}-\hat{\Omega}\left(\hat{\mathsf{s}},\acute{\Upsilon}(\hat{\mathsf{s}})\right)\int_0^{\hat{\mathsf{s}}}\left(\hat{\mathsf{s}}-\hat{\delta}\right)^{\hat{\omega}-1}\hat{\Psi}\left(\hat{\delta},\acute{\Upsilon}\left(\hat{\delta}\right)\right)d\hat{\delta}\right|+\frac{1}{\bar{\Gamma}(\hat{\gamma})}\left|\int_0^{\hat{\mathsf{s}}}\left(\hat{\mathsf{s}}-\hat{\delta}\right)^{\hat{\gamma}-1}\left(\frac{\bar{\Gamma}(\hat{\gamma}+1)}{2}|\left(\hat{a}_0-\hat{b}_0\right)|\right)d\hat{\delta}\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{2}\left|\hat{a}_0-\hat{b}_0\right|+\frac{1}{\bar{\Gamma}(\hat{\omega})}\left|\int_0^{\hat{s}}\left(\hat{s}-\hat{\delta}\right)^{\hat{\omega}-1}\hat{\Pi}\left(\hat{\delta},\acute{\Phi}\left(\hat{\delta}\right)\right)\left(\hat{\Omega}\left(\hat{s},\acute{\Phi}(\hat{s})\right)-\hat{\Omega}\left(\hat{s},\acute{\Upsilon}(\hat{s})\right)\right)d\hat{\delta}\right|+\frac{1}{\bar{\Gamma}(\hat{\omega})}\left|\int_0^{\hat{s}}\hat{\Omega}\left(\hat{s},\acute{\Upsilon}(\hat{s})\right)\left(\hat{s}-\hat{\delta}\right)^{\hat{\omega}-1}\left(\hat{\Pi}\left(\hat{\delta},\acute{\Phi}\left(\hat{\delta}\right)\right)-\hat{\Psi}\left(\hat{\delta},\acute{\Upsilon}\left(\hat{\delta}\right)\right)\right)d\hat{\delta}\right|\\
&\quad+\frac{1}{\bar{\Gamma}(\hat{\gamma}+1)}\frac{\bar{\Gamma}(\hat{\gamma}+1)}{2}\left|\hat{a}_0-\hat{b}_0\right|(\hat{s})^{\hat{\gamma}}\\
&\leq \frac{1}{2}\left|\hat{a}_0-\hat{b}_0\right|+\frac{D_2}{\bar{\Gamma}(\hat{\omega})}\left|\int_0^{\hat{s}}\left(\hat{s}-\hat{\delta}\right)^{\hat{\omega}-1}\left(\frac{\bar{\Gamma}(\hat{\omega}+1)}{2D_2}|\acute{\Phi}-\acute{\Upsilon}|\right)d\hat{\delta}\right|+\frac{D}{\bar{\Gamma}(\hat{\omega})}\left|\int_0^{\hat{s}}\left(\hat{s}-\hat{\delta}\right)^{\hat{\omega}-1}\frac{\bar{\Gamma}(\hat{\omega}+1)}{2D}\left(|\acute{\Phi}-\acute{\Upsilon}|-2|\hat{a}_0-\hat{b}_0|\right)d\hat{\delta}\right|+\frac{|\hat{a}_0-\hat{b}_0|}{2}\\
&\leq \frac{1}{2}\left|\hat{a}_0-\hat{b}_0\right|+\frac{D_2}{\bar{\Gamma}(\hat{\omega}+1)}(\hat{s})^{\hat{\omega}}\frac{\bar{\Gamma}(\hat{\omega}+1)}{2D_2}|\acute{\Phi}-\acute{\Upsilon}|+\frac{D}{\bar{\Gamma}(\hat{\omega}+1)}(\hat{s})^{\hat{\omega}}\frac{\bar{\Gamma}(\hat{\omega}+1)}{2D}\left(|\acute{\Phi}-\acute{\Upsilon}|-2|\hat{a}_0-\hat{b}_0|\right)+\frac{|\hat{a}_0-\hat{b}_0|}{2}\\
&\leq \frac{1}{2}\left|\hat{a}_0-\hat{b}_0\right|+\frac{|\acute{\Phi}-\acute{\Upsilon}|}{2}+\frac{|\acute{\Phi}-\acute{\Upsilon}|}{2}-|\hat{a}_0-\hat{b}_0|+\frac{|\hat{a}_0-\hat{b}_0|}{2}\\
&=|\acute{\Phi}-\acute{\Upsilon}|\\
&\leq \|\acute{\Phi}-\acute{\Upsilon}\|.
\end{aligned}
$$

Thus $\hat{\mathcal{G}}$ is relatively nonexpansive.
Now, we assume that the pair $(\gimel_1,\gimel_2)\subseteq(\mathbf{W}_1,\mathbf{W}_2)$ is a NBCC, $\hat{\mathcal{G}}$-invariant, proximinal pair and $dist(\gimel_1,\gimel_2)=dist(\mathbf{W}_1,\mathbf{W}_2)$. Using assumption (1) of theorem 4.2 and theorem 4.1, we get,

$$
\begin{aligned}
&\aleph_0(\hat{\mathcal{G}}(\gimel_1)\cup\hat{\mathcal{G}}(\gimel_2))\\
&=\max\{\aleph_0(\hat{\mathcal{G}}(\gimel_1)),\aleph_0(\hat{\mathcal{G}}(\gimel_2))\}\\
&\leq \max\{\sup_{\hat{s}\in\breve{J}}\{\aleph_0(\{\hat{\mathcal{G}}\acute{\Phi}(\hat{s}):\acute{\Phi}\in\gimel_1\})\},\sup_{\hat{s}\in\breve{J}}\{\aleph_0(\{\hat{\mathcal{G}}\acute{\Upsilon}(\hat{s}):\acute{\Upsilon}\in\gimel_2\})\}\}\\
&=\max\left\{\sup_{\hat{s}\in\breve{J}}\aleph_0\left(\left\{\hat{\Omega}\left(\hat{s},\acute{\Phi}(\hat{s})\right)\left(\frac{\hat{b}_0}{\hat{\Omega}\left(\hat{s}_0,\hat{b}_0\right)}+\frac{1}{\bar{\Gamma}(\hat{\omega})}\int_0^{\hat{s}}\left(\hat{s}-\hat{\delta}\right)^{\hat{\omega}-1}\hat{\Pi}\left(\hat{\delta},\acute{\Phi}\left(\hat{\delta}\right)\right)d\hat{\delta}\right)+\frac{1}{\bar{\Gamma}(\hat{\gamma})}\int_0^{\hat{s}}\left(\hat{s}-\hat{\delta}\right)^{\hat{\gamma}-1}\hat{\Theta}_2\left(\hat{\delta}\right)d\hat{\delta}\right\}\right),\right.\\
&\left.\sup_{\hat{s}\in\breve{J}}\left\{\aleph_0\left(\left\{\hat{\Omega}\left(\hat{s},\acute{\Upsilon}(\hat{s})\right)\left(\frac{\hat{a}_0}{\hat{\Omega}(\hat{s}_0,\hat{a}_0)}+\frac{1}{\bar{\Gamma}(\hat{\omega})}\int_0^{\hat{s}}\left(\hat{s}-\hat{\delta}\right)^{\hat{\omega}-1}\hat{\Psi}\left(\hat{\delta},\acute{\Upsilon}\left(\hat{\delta}\right)\right)d\hat{\delta}\right)+\frac{1}{\bar{\Gamma}(\hat{\gamma})}\int_0^{\hat{s}}\left(\hat{s}-\hat{\delta}\right)^{\hat{\gamma}-1}\hat{\Theta}_1\left(\hat{\delta}\right)d\hat{\delta}\right\}\right)\right\}\right\}\\
&\leq \max\left\{\sup_{\hat{s}\in\breve{J}}\aleph_0\left(\left\{\frac{D}{\bar{\Gamma}(\hat{\omega})}\hat{\Pi}\left(\hat{g},\acute{\Phi}(\hat{g})\right)\int_0^{\hat{s}}\left(\hat{s}-\hat{\delta}\right)^{\hat{\omega}-1}d\hat{\delta}+\frac{1}{\bar{\Gamma}(\hat{\gamma})}\hat{\Theta}_2(\hat{g})\int_0^{\hat{s}}\left(\hat{s}-\hat{\delta}\right)^{\hat{\gamma}-1}d\hat{\delta}:\hat{g}\in(0,\hat{s})\right\}\right),\right.\\
&\qquad\left.\sup_{\hat{s}\in\breve{J}}\left\{\aleph_0\left(\left\{\frac{D}{\bar{\Gamma}(\hat{\omega})}\hat{\Psi}\left(\hat{g},\acute{\Upsilon}(\hat{g})\right)\int_0^{\hat{s}}\left(\hat{s}-\hat{\delta}\right)^{\hat{\omega}-1}d\hat{\delta}+\frac{1}{\bar{\Gamma}(\hat{\gamma})}\hat{\Theta}_1(\hat{g})\int_0^{\hat{s}}\left(\hat{s}-\hat{\delta}\right)^{\hat{\gamma}-1}d\hat{\delta}:\hat{g}\in(0,\hat{s})\right\}\right)\right\}\right\}\\
&\leq \max\left\{\sup_{\hat{s}\in\breve{J}}\aleph_0\left(\left\{\frac{D}{\bar{\Gamma}(\hat{\omega}+1)}\hat{\Pi}\left(\hat{g},\acute{\Phi}(\hat{g})\right)(\hat{s})^{\hat{\omega}}+\frac{1}{\bar{\Gamma}(\hat{\gamma}+1)}\hat{\Theta}_2(\hat{g})(\hat{s})^{\hat{\gamma}}:\hat{g}\in(0,\hat{s})\right\}\right),\sup_{\hat{s}\in\breve{J}}\left\{\aleph_0\left(\left\{\frac{D}{\bar{\Gamma}(\hat{\omega}+1)}\hat{\Psi}\left(\hat{g},\acute{\Upsilon}(\hat{g})\right)(\hat{s})^{\hat{\omega}}+\frac{1}{\bar{\Gamma}(\hat{\gamma}+1)}\hat{\Theta}_1(\hat{g})(\hat{s})^{\hat{\gamma}}:\hat{g}\in(0,\hat{s})\right\}\right)\right\}\right\}\\
&\leq \aleph_0\left(\left\{\hat{\Pi}(\breve{J}\times\gimel_1)+\hat{\Theta}_1(\breve{J})\right\}\cup\left\{\hat{\Psi}(\breve{J}\times\gimel_2)+\hat{\Theta}_2(\breve{J})\right\}\right)\\
&\leq \aleph_0(\gimel_1\cup\gimel_2).
\end{aligned}
$$

Thus, by corollary (3.10), $\hat{\mathcal{G}}$ has a *BPP*. Hence, we conclude that the system of equations (10) has $s\in\mathbf{W}_1\cup\mathbf{W}_2$ as an optimal solution. □

Example 4.3. *Assume the following system of integro differential equations as*

$$
\begin{aligned}
\acute{\Phi}(\hat{s})&=\hat{s}\acute{\Phi}\left(\frac{1}{\bar{\Gamma}(4)}\int_0^{\hat{s}}\left(\hat{s}-\hat{\delta}\right)^3\frac{\hat{\delta}\acute{\Phi}}{2}d\hat{\delta}\right)+\frac{1}{\bar{\Gamma}(3)}\int_0^{\hat{s}}\left(\hat{s}-\hat{\delta}\right)^2\hat{\delta}\,d\hat{\delta},\\
\acute{\Upsilon}(\hat{s})&=\hat{s}\acute{\Upsilon}\left(\frac{1}{8}+\frac{1}{\bar{\Gamma}(4)}\int_0^{\hat{s}}\left(\hat{s}-\hat{\delta}\right)^3\frac{\hat{\delta}\acute{\Upsilon}}{2}d\hat{\delta}\right)+\frac{1}{\bar{\Gamma}(3)}\int_0^{\hat{s}}\left(\hat{s}-\hat{\delta}\right)^2\hat{\delta}\,d\hat{\delta},
\end{aligned}
$$

with $\frac{17}{5}\leq\acute{\Phi}\leq 4$, $\acute{\Upsilon}\leq 1$, *and* $\left|\acute{\Phi}-\acute{\Upsilon}\right|\geq 2.4$. *Consider* $\mathbf{W}_1=\{\hat{s}\}$ *and* $\mathbf{W}_2=\{\hat{s}+1\}$ *on* $\breve{J}=[0,1)$.
Define an operator $\hat{\mathcal{G}}:\mathbf{W}_1\cup\mathbf{W}_2\to\mathbf{W}$ *such that,*

$$
\hat{\mathcal{G}}\left(\acute{\Phi}(\hat{s})\right)=\begin{cases}\hat{s}\acute{\Phi}\left(\frac{1}{8}+\frac{1}{\bar{\Gamma}(4)}\int_0^{\hat{s}}\left(\hat{s}-\hat{\delta}\right)^3\frac{\hat{\delta}\acute{\Phi}}{2}d\hat{\delta}\right)+\frac{1}{\bar{\Gamma}(3)}\int_0^{\hat{s}}\left(\hat{s}-\hat{\delta}\right)^2\hat{\delta}\,d\hat{\delta}, & \acute{\Phi}\in\mathbf{W}_1,\\[2ex] \hat{s}\acute{\Phi}\left(\frac{1}{\bar{\Gamma}(4)}\int_0^{\hat{s}}\left(\hat{s}-\hat{\delta}\right)^3\frac{\hat{\delta}\acute{\Phi}}{2}d\hat{\delta}\right)+\frac{1}{\bar{\Gamma}(3)}\int_0^{\hat{s}}\left(\hat{s}-\hat{\delta}\right)^2\hat{\delta}\,d\hat{\delta}, & \acute{\Phi}\in\mathbf{W}_2.\end{cases}\tag{12}
$$

Here $\hat{\Psi} : \check{\mathtt{J}} \times \mathtt{J}_2 \longrightarrow \mathtt{J}_2$, and $\hat{\Pi} : \check{\mathtt{J}} \times \mathtt{J}_1 \longrightarrow \mathtt{J}_1$ with $\mathtt{J}_1 = \{\hat{\mathtt{s}}\}$, $\mathtt{J}_2 = \{\hat{\mathtt{s}} + 1\}$ and $(\mathtt{J}_1, \mathtt{J}_2) \subseteq (\mathbf{W}_1, \mathbf{W}_2)$ on $\check{\mathtt{J}} = [0, 1)$, $\aleph_0\Big(\Big\{\hat{\Pi}(\check{\mathtt{J}} \times \mathtt{J}_1) + \hat{\Theta}_1(\check{\mathtt{J}})\Big\}\Big), \aleph_0\Big(\Big\{\hat{\Psi}(\check{\mathtt{J}} \times \mathtt{J}_2) + \hat{\Theta}_2(\check{\mathtt{J}})\Big\}\Big) > 0$.
Now

$$\begin{aligned}
&\aleph_0\Big(\Big\{\hat{\Pi}(\check{\mathtt{J}} \times \mathtt{J}_1) + \hat{\Theta}_1(\check{\mathtt{J}})\Big\} \cup \Big\{\hat{\Psi}(\check{\mathtt{J}} \times \mathtt{J}_2) + \hat{\Theta}_2(\check{\mathtt{J}})\Big\}\Big)\\
&= \aleph_0\left(\left\{\hat{\delta}\left(\frac{\acute{\Phi}}{2} + 1\right) : \acute{\Phi} \in \mathtt{J}_1, \hat{\delta} \in \check{\mathtt{J}}\right\} \cup \left\{\hat{\delta}\left(\frac{\acute{\Upsilon}}{2} + 1\right) : \acute{\Upsilon} \in \mathtt{J}_2, \hat{\delta} \in \check{\mathtt{J}}\right\}\right)\\
&= max\left\{\aleph_0\left(\left\{\hat{\delta}\left(\frac{\acute{\Phi}}{2} + 1\right) : \acute{\Phi} \in \mathtt{J}_1, \hat{\delta} \in \check{\mathtt{J}}\right\}\right), \aleph_0\left(\left\{\hat{\delta}\left(\frac{\acute{\Upsilon}}{2} + 1\right) : \acute{\Upsilon} \in \mathtt{J}_2, \hat{\delta} \in \check{\mathtt{J}}\right\}\right)\right\}\\
&\le max\left\{\aleph_0\left(\left\{\left(\frac{\acute{\Phi}}{2} + 1\right) : \acute{\Phi} \in \mathtt{J}_1\right\}\right), \aleph_0\left(\left\{\left(\frac{\acute{\Upsilon}}{2} + 1\right) : \acute{\Upsilon} \in \mathtt{J}_2\right\}\right)\right\}\\
&\le max\{\aleph_0(\{\mathtt{J}_1\}), \aleph_0(\{\mathtt{J}_2\})\}\\
&= \aleph_0(\mathtt{J}_1 \cup \mathtt{J}_2).
\end{aligned}$$

For $\acute{\Phi} \in \mathbf{W}_1$, $\acute{\Upsilon} \in \mathbf{W}_2$

$$\begin{aligned}
&\left|\hat{\Omega}\left(\hat{\mathtt{s}}, \acute{\Phi}(\hat{\mathtt{s}})\right)\left(\frac{\hat{b}_0}{\hat{\Omega}\left(\hat{\mathtt{s}}_0, \hat{b}_0\right)}\right) - \hat{\Omega}\left(\hat{\mathtt{s}}, \acute{\Upsilon}(\hat{\mathtt{s}})\right)\left(\frac{\hat{a}_0}{\hat{\Omega}(\hat{\mathtt{s}}_0, \hat{a}_0)}\right)\right|\\
&= \left|\hat{\mathtt{s}}\acute{\Phi}\left(\frac{1}{8}\right)\right|\\
&\le \frac{|\acute{\Phi}|}{8} \le \frac{4}{8} = \frac{1}{2}.
\end{aligned}$$

For the continuous maps $\hat{\Theta}_1, \hat{\Theta}_2$ and $\hat{\delta} \in \bar{\mathcal{E}}$, $\hat{\gamma} = 3$,

$$\left|\hat{\Theta}_2\left(\hat{\delta}\right) - \hat{\Theta}_1\left(\hat{\delta}\right)\right| = \left|\hat{\delta} - \hat{\delta}\right| \le \frac{\bar{\Gamma}(4)}{2}.$$

For the continuous map $\hat{\Omega}$ with $\acute{\Phi} \in \mathbf{W}_1$, $\hat{\mathtt{s}} \in \bar{\mathcal{E}}$, $\hat{\omega} = 4, \mathtt{D}_2 = 3$, we have

$$\left|\hat{\Omega}\left(\hat{\mathtt{s}}, \acute{\Phi}(\hat{\mathtt{s}})\right) - \hat{\Omega}\left(\hat{\mathtt{s}}, \acute{\Upsilon}(\hat{\mathtt{s}})\right)\right| = \left|\hat{\mathtt{s}}\acute{\Phi} - \hat{\mathtt{s}}\acute{\Upsilon}\right| \le \left|\acute{\Phi} - \acute{\Upsilon}\right| \le 4\left|\acute{\Phi} - \acute{\Upsilon}\right|.$$

For $\acute{\Phi} \in \mathbf{W}_1$, $\acute{\Upsilon} \in \mathbf{W}_2$, $\hat{\delta} \in \bar{\mathcal{E}}$, $\mathtt{D} = 2$, and the continuous maps $\hat{\Psi}, \hat{\Pi}$, we have

$$\left|\hat{\Pi}\left(\hat{\delta}, \acute{\Phi}\left(\hat{\delta}\right)\right) - \hat{\Psi}\left(\hat{\delta}, \acute{\Upsilon}\left(\hat{\delta}\right)\right)\right| = \left|\frac{\hat{\delta}\acute{\Phi}}{2} - \frac{\hat{\delta}\acute{\Upsilon}}{2}\right| \le \left|\acute{\Phi} - \acute{\Upsilon}\right| \le 6\Big(\left|\acute{\Phi} - \acute{\Upsilon}\right| - 2\Big).$$

For $\acute{\Phi} \in \mathbf{W}_1$, $\hat{\mathtt{s}}, \bar{\hat{\mathtt{s}}} \in \bar{\mathcal{E}}$, $\tilde{\mathtt{m}} = 4$ and the continuous map $\hat{\Omega}$, we have

$$\left|\hat{\Omega}\left(\hat{\mathtt{s}}, \acute{\Phi}(\hat{\mathtt{s}})\right) - \hat{\Omega}\left(\bar{\hat{\mathtt{s}}}, \acute{\Phi}(\bar{\hat{\mathtt{s}}})\right)\right| = \left|\hat{\mathtt{s}}\acute{\Phi} - \bar{\hat{\mathtt{s}}}\acute{\Phi}\right| \le 4\left|\hat{\mathtt{s}} - \bar{\hat{\mathtt{s}}}\right|.$$

Since the above system of equation satisfies all the conditions of Theorem 4.2. Therefore an optimal solution exists for the above system of equation.

Example 4.4. *Assume the following system of integro differential equations as*

$$\acute{\Phi}(\hat{s}) = \hat{s}^2 \sin\acute{\Phi}\left(\frac{1}{\bar{\Gamma}(5)}\int_0^{\hat{s}}\left(\hat{s}-\hat{\delta}\right)^4 \sin\acute{\Phi}\, d\hat{\delta}\right),$$

$$\acute{\Upsilon}(\hat{s}) = \hat{s}^2 \sin\acute{\Upsilon}\left(\frac{1}{6}+\frac{1}{\bar{\Gamma}(5)}\int_0^{\hat{s}}\left(\hat{s}-\hat{\delta}\right)^4 \sin\acute{\Upsilon}\, d\hat{\delta}\right),$$

with $\acute{\Phi} \le 3$, $\acute{\Upsilon} \le 1$ *and* $\left|\acute{\Phi}-\acute{\Upsilon}\right| > 4$. *Consider* $\mathbf{W}_1 = \{\hat{s}\}$ *and* $\mathbf{W}_2 = \{\hat{s}+1\}$ *on* $\check{J} = [0,1)$.
Define an operator $\hat{\mathcal{G}} : \mathbf{W}_1 \cup \mathbf{W}_2 \to \mathbf{W}$ *such that,*

$$\hat{\mathcal{G}}\left(\acute{\Phi}(\hat{s})\right) = \begin{cases} \hat{s}^2 \sin\acute{\Phi}\left(\frac{1}{6}+\frac{1}{\bar{\Gamma}(5)}\int_0^{\hat{s}}\left(\hat{s}-\hat{\delta}\right)^4 \sin\acute{\Phi}\, d\hat{\delta}\right), & \acute{\Phi} \in \mathbf{W}_1, \\ \hat{s}^2 \sin\acute{\Phi}\left(\frac{1}{\bar{\Gamma}(5)}\int_0^{\hat{s}}\left(\hat{s}-\hat{\delta}\right)^4 \sin\acute{\Phi}\, d\hat{\delta}\right), & \acute{\Phi} \in \mathbf{W}_2. \end{cases} \tag{13}$$

Here $\hat{\Psi} : \check{J} \times \beth_2 \longrightarrow \beth_2$, and $\hat{\Pi} : \check{J} \times \beth_1 \longrightarrow \beth_1$ with $\beth_1 = \{\hat{s}\}$, $\beth_2 = \{\hat{s}+1\}$ and $(\beth_1, \beth_2) \subseteq (\mathbf{W}_1, \mathbf{W}_2)$ on $\check{J} = [0,1)$, $\aleph_0\left(\left\{\hat{\Pi}(\check{J}\times\beth_1)+\hat{\Theta}_1(\check{J})\right\}\right), \aleph_0\left(\left\{\hat{\Psi}(\check{J}\times\beth_2)+\hat{\Theta}_2(\check{J})\right\}\right) > 0$.
Now

$$\begin{aligned}
&\aleph_0\left(\left\{\hat{\Pi}(\check{J}\times\beth_1)+\hat{\Theta}_1(\check{J})\right\} \cup \left\{\hat{\Psi}(\check{J}\times\beth_2)+\hat{\Theta}_2(\check{J})\right\}\right) \\
&= \aleph_0\left(\left\{\left(\sin\acute{\Phi}\right) : \acute{\Phi} \in \beth_1\right\} \cup \left\{\left(\sin\acute{\Upsilon}\right) : \acute{\Upsilon} \in \beth_2\right\}\right) \\
&= max\left\{\aleph_0\left(\left\{\sin\acute{\Phi} : \acute{\Phi} \in \beth_1\right\}\right), \aleph_0\left(\left\{\sin\acute{\Upsilon} : \acute{\Upsilon} \in \beth_2\right\}\right)\right\} \\
&\le max\{\aleph_0(\{\beth_1\}), \aleph_0(\{\beth_2\})\} \\
&= \aleph_0(\beth_1 \cup \beth_2).
\end{aligned}$$

For $\acute{\Phi} \in \mathbf{W}_1$, $\acute{\Upsilon} \in \mathbf{W}_2$

$$\begin{aligned}
&\left|\hat{\Omega}\left(\hat{s}, \acute{\Phi}(\hat{s})\right)\left(\frac{\hat{b}_0}{\hat{\Omega}\left(\hat{s}_0, \hat{b}_0\right)}\right) - \hat{\Omega}\left(\hat{s}, \acute{\Upsilon}(\hat{s})\right)\left(\frac{\hat{a}_0}{\hat{\Omega}\left(\hat{s}_0, \hat{a}_0\right)}\right)\right| \\
&= \left|\hat{s}^2 \sin\acute{\Phi}\left(\frac{1}{6}\right)\right| \\
&\le \frac{|\acute{\Phi}|}{6} \le \frac{3}{6} = \frac{1}{2}.
\end{aligned}$$

For the continuous maps $\hat{\Theta}_1, \hat{\Theta}_2$ and $\hat{\delta} \in \bar{\mathcal{E}}$, $\hat{\gamma} = 3$,

$$\left|\hat{\Theta}_2\left(\hat{\delta}\right) - \hat{\Theta}_1\left(\hat{\delta}\right)\right| \le \frac{\bar{\Gamma}(4)}{2}.$$

For the continuous map $\hat{\Omega}$ with $\acute{\Phi} \in \mathbf{W}_1$, $\hat{s} \in \bar{\mathcal{E}}$, $\hat{\omega} = 5$, $\daleth_2 = 3$, we have

$$\begin{aligned}
\left|\hat{\Omega}\left(\hat{s}, \acute{\Phi}(\hat{s})\right) - \hat{\Omega}\left(\hat{s}, \acute{\Upsilon}(\hat{s})\right)\right| = \left|\hat{s}^2 \sin\acute{\Phi} - \hat{s}^2 \sin\acute{\Upsilon}\right| &\le \left|\sin\acute{\Phi} - \sin\acute{\Upsilon}\right| \\
&\le \left|2\cos\left(\frac{\acute{\Phi}+\acute{\Upsilon}}{2}\right)\sin\left(\frac{\acute{\Phi}-\acute{\Upsilon}}{2}\right)\right|
\end{aligned}$$

$$\leq 2\left|\frac{\acute{\Phi}-\acute{\Upsilon}}{2}\right|$$
$$\leq 4\left|\acute{\Phi}-\acute{\Upsilon}\right|.$$

For $\acute{\Phi}\in\mathbf{W}_1$, $\acute{\Upsilon}\in\mathbf{W}_2$, $\hat{\delta}\in\bar{\mathcal{E}}$, $\mathtt{D}=6$, and the continuous maps $\hat{\Psi},\hat{\Pi}$, we have

$$\begin{aligned}\left|\hat{\Pi}\left(\hat{\delta},\acute{\Phi}\left(\hat{\delta}\right)\right)-\hat{\Psi}\left(\hat{\delta},\acute{\Upsilon}\left(\hat{\delta}\right)\right)\right| &= \left|\sin\acute{\Phi}-\sin\acute{\Upsilon}\right|\\ &\leq \left|2\cos\left(\frac{\acute{\Phi}+\acute{\Upsilon}}{2}\right)\sin\left(\frac{\acute{\Phi}-\acute{\Upsilon}}{2}\right)\right|\\ &\leq 2\left|\frac{\acute{\Phi}-\acute{\Upsilon}}{2}\right|\\ &\leq \left|\acute{\Phi}-\acute{\Upsilon}\right|\\ &\leq 2\left|\acute{\Phi}-\acute{\Upsilon}\right|-4.\end{aligned}$$

For $\acute{\Phi}\in\mathbf{W}_1$, $\hat{\mathtt{s}},\bar{\hat{\mathtt{s}}}\in\bar{\mathcal{E}}$, $\tilde{\mathtt{m}}=5$ and the continuous map $\hat{\Omega}$, we have

$$\begin{aligned}\left|\hat{\Omega}\left(\hat{\mathtt{s}},\acute{\Phi}\left(\hat{\mathtt{s}}\right)\right)-\hat{\Omega}\left(\bar{\hat{\mathtt{s}}},\acute{\Phi}\left(\bar{\hat{\mathtt{s}}}\right)\right)\right| &= \left|\hat{\mathtt{s}}^2\sin\acute{\Phi}-\bar{\hat{\mathtt{s}}}^2\sin\acute{\Phi}\right|\\ &\leq |\sin\acute{\Phi}|\left|\hat{\mathtt{s}}^2-\bar{\hat{\mathtt{s}}}^2\right|\\ &\leq \acute{\Phi}\left|\hat{\mathtt{s}}+\bar{\hat{\mathtt{s}}}\right|\left|\hat{\mathtt{s}}-\bar{\hat{\mathtt{s}}}\right|\\ &\leq 5\left|\hat{\mathtt{s}}-\bar{\hat{\mathtt{s}}}\right|.\end{aligned}$$

Since the above system of equation satisfies all the conditions of Theorem 4.2. Therefore an optimal solution exists for the above system of equation.

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