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The unified theory of $\Gamma - t$ absolute randomized truth degree in Goguen$_{\Delta,\sim}$ system

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Abstract. In this paper, we mainly carry out quantitative research in $\Gamma - t$ absolute randomized truth degree. Using the randomization method of valuation set, we firstly give the definition of $\Gamma - t$ absolute randomized truth degree of formula relative to local finite theory Γ under the t conjunction in Goguen$_{\Delta,\sim}$ n-valued propositional logic system (t takes Δ, $\sim$), and prove some related properties of $\Gamma - t$ absolute randomized truth degree and some inference rules such as MP, HS, intersection inference and union inference. Secondly, we introduce the concepts of $\Gamma - t$ absolute randomized similarity degree and $\Gamma - t$ absolute randomized pseudo-distance of propositional formulas, and prove some good properties of $\Gamma - t$ absolute randomized similarity degree. We also discuss in $\Gamma - t$ absolute randomized logic metric space $(F(S), \rho_{D,\Gamma})$ the continuity of operators Δ, $\sim$, $\rightarrow$, $\wedge$ and $\vee$ with respect to $\Gamma - t$ absolute randomized pseudo-distance $\rho_{D,\Gamma}$. Finally, we give the concepts of t absolute randomized divergence degree and t absolute randomized consistency degree of arbitrary theory Γ relative to the fixed theory Γ_0. Using the specific property of contradiction, we define non-absolute randomized consistent of arbitrary theory Γ relative to the fixed theory Γ_0, and establish the relationship between them.

1. Introduction

It is well known that mathematical logic is the formalized theory with the character of symbolization, which focuses on formal deduction rather than on numerical calculation. However, numerical calculation pays more attention to solving problems and rarely uses formal deduction methods. In order to establish some connections between the two, Wang Guojun created quantitative logic [19, 21–23], which is a combination of mathematical logic and probability calculation.

The idea of introducing probability methods into mathematical logic has gradually emerged since the 1950s, and a monograph on"probabilistic logic" [1] was published in 1998. Later, many scholars have carried out researches on this basis and have made rich achievements. However, some authors found that two formulas of exactly the same form must have the same truth degree, it contradicts the fact that various

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simple propositions in reality are true with different probabilities , because whether the simple propositions are true is uncertain and random. Some attempts in that direction was carried out in [4, 10–13, 20, 26]. In that work some authors used the randomization method of valuation set to give the randomized truth degree theory of propositional formulas in the logic system and to establish the randomized logic metric space. It realizes the integration of probability logic and quantitative logic.

Currently, some scholars have already felt the difficulties the strong negation in the Gödel system and Goguen system has brought to relevant quantitative researches. In order to overcome it, in [2, 3, 5–7], some authors introduced two basic connectives Δ and $\sim$, and proposed systems BL_{Δ} and $SBL_{\sim}$, as two axiomatic extensions of basic logic system BL. In BL_{Δ} and $SBL_{\sim}$, both Δ deduction theorem and strong completeness theorem have been established, so related researches can be carried out smoothly. In [14], the author realized quantitative research of Δ fuzzy logic system in $SBL_{\sim}$ system. The systems $Gödel_{\Delta,\sim}$ and $Goguen_{\Delta,\sim}$, as two typical representatives of $SBL_{\sim}$ system, have been studied in [8, 9, 18]. Some authors have proposed t truth degree theory, k randomized truth degree theory and $\Gamma - k$ randomized truth degree theory in $Goguen_{\Delta,\sim}$ propositional logic system. In [16], the author evaded infinite product measure in uniformly distributed probability spaces, and introduced concept of absolute truth degree in Lukasiewicz propositional logic. Later, some scholars [15, 17, 24] have carried out researches on this basis. They proposed $\Gamma - t$ absolute truth degree in $Gödel_{\sim,\Delta}$, $\Gamma-$ absolute truth degree in $n-$ valued propositional logic system L_n^* and $\Gamma-$ absolute truth degree in ternary product logic system π_3. We find that most of the research results only consider the truth degree of formulas from the perspective of total assignment, but in real life, we often need to consider the truth degree of an event under certain conditions (i.e., under certain theories). Along this way of thinking, a subsequent question is whether a similar study of Γ absolute randomization can be carried out in $Goguen_{\Delta,\sim}$ propositional logic system, so that the Γ absolute randomized truth degree of any formula can be calculated by computer in a finite number of steps, which makes the algorithm implementation of the method in this paper possible.

The paper is structured as follows: In Section 2, we recall some basic concepts and results concerning BL_{Δ} and $SBL_{\sim}$ logic system employed in this paper. In Section 3, using the randomization method of valuation set, we put forward the definition of $\Gamma - t$ absolute randomized truth degree of formula relative to local finite theory Γ under the t conjunction in $Goguen_{\Delta,\sim}$ n-valued propositional logic system, and prove some inference rules such as MP, HS, intersection inference and union inference of $\Gamma - t$ absolute randomized truth degree. Using concepts and results in Section 3, we introduce the concepts of $\Gamma - t$ absolute randomized similarity degree and $\Gamma - t$ absolute randomized pseudo-distance of propositional formulas in Section 4. In Section 5, we give the concepts of t absolute randomized divergence degree and t absolute randomized consistency degree of arbitrary theory Γ relative to the fixed theory Γ_0, and using specific property of contradiction, we define non-absolute randomized consistent of arbitrary theory Γ relative to the fixed theory Γ_0. Finally, conclusion in Section 6.

The results of this paper generalize the related work in [8, 9, 18] and enrich the quantitative research in $Goguen_{\sim,\Delta}$ propositional logic system. Our work provides the basis for the future study of Γ absolute randomized truth degree in some other propositional logic systems.

2. Preliminaries

In this section, we recall all necessary basic notions. See [2, 3, 14] for more details.

Definition 2.1. ([2]) *The axioms of BL_{Δ} are the following:*

(BL) the axioms of BL.

(Δ1) $\Delta A \vee \neg \Delta A$.

(Δ2) $\Delta(A \vee B) \to (\Delta A \vee \Delta B)$.

(Δ3) $\Delta A \to A$.

(Δ4) $\Delta A \to \Delta\Delta A$.

(Δ5) $\Delta(A \to B) \to (\Delta A \to \Delta B)$.

The inference rules in BL_Δ are MP rule and Δ rule; the MP rule is from A, $A\to B$, inferred B; and the Δ rule is from A inferred ΔA.

If L is an axiomatic extension of BL, then by L_Δ we will denoted the logic resulting from L in the same way as BL_Δ results from BL, and the following Δ deduction theorem holds for the BL_Δ system.

Theorem 2.2. ([5]) *(Δ deduction theorem) Let L be an expansion of BL_Δ. Then for each theory Γ and all formulas A and B, we have:*

$$\Gamma, A \vdash B \ if \ and \ only \ if \ \Gamma \vdash \Delta A \to B.$$

The logic SBL is obtained by adding to BL the axiom $\neg\neg A \vee \neg A$. SBL_Δ is also an axiomatic extension of SBL.

The logic $SBL_\sim$ is obtained by adding to SBL the involutive negating connective $\sim$.

Definition 2.3. ([3]) *The axioms of $SBL_\sim$ are the following:*

(SBL) the axioms of SBL.

($\sim$1) $\sim\sim A\to A$,

($\sim$2) $\Delta(A\to B)\to \Delta(\sim B\to\sim A)$.

($\sim$3) $\neg A\to\sim A$.

Let $\Delta A=\neg \sim A$ in the $SBL_\sim$ system. Then we can establish the relationship between the SBL_Δ system and the $SBL_\sim$ system, i.e., $SBL_\sim$ has the following equivalent axioms:

(SBL_Δ) the axioms of SBL_Δ.

($\sim$1) $\sim\sim A\to A$.

($\sim$ 2) $\Delta(A\to B)\to \Delta(\sim B\to\sim A)$.

The inference rules in $SBL_\sim$ are MP rule and Δ rule. If L is an axiomatic extension of SBL, then by $L_\sim$ we will denote the logic resulting from L in the same way as $SBL_\sim$ results from SBL, and Gödel$_\sim$ and Goguen$_\sim$ are the two basic types of axiomatic extension of $SBL_\sim$. Because $SBL_\sim$ is also an axiomatic extension of BL_Δ, Δ deduction theorem in $SBL_\sim$ also holds.

Theorem 2.4. ([3]) *(strong completeness theorem) Let L an axiomatic extension of $SBL_\sim$. Then for theory Γ and formula A, the following two conditions are equivalent:*

(i) $\Gamma \vdash A$.

(ii) For every L-algebra and every model e of theory Γ, $e(A) = 1$.

Definition 2.5. ([8]) *Let $S = \{p_1, p_2, \ldots\}$ be a countable set, $\sim$ and Δ be two unary operations on S, $\vee$, $\wedge$ and $\to$ be three binary operations on S, and $F(S)$ be the free algebra of type (1, 1, 2, 2, 2) generated by S. Then an element of S is called an atomic formula or atomic proposition, and an element of $F(S)$ is called a formula or proposition.*

Definition 2.6. ([8]) *The Goguen propositional logic system is also called product system, which we shall denote by Π. Let $\Pi_{\Delta,\sim} = \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$. Define on $\Pi_{\Delta,\sim}$ an unary operator and two binary operators as follows: $\sim x = 1-x$, $\Delta x = \{^{1,x=1}_{0,x<1}$, $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$, and $x \to y = \{^{1,x\leq y}_{\frac{y}{x},x>y}$, $x, y \in \Pi_{\Delta,\sim}$. The system Goguen$_{\Delta,\sim}$ is called an expansion of n-valued product propositional logic system. It is abbreviated as $\Pi_{\Delta,\sim}$.*

Definition 2.7. ([8]) *Assume that $A = A(p_1, p_2, \ldots, p_m)$ is a formula generated by atomic formulas $p_1, p_2, \ldots, p_m$ through connectives $\Delta, \sim, \vee, \wedge$ and $\to$. Substitute x_i for p_i in $A(i = 1, \ldots, m)$ and keep the logic connective in A unchanged but explain them as the corresponding operators defined on the valuation lattice $\Pi_{\Delta,\sim}$. Then we get a function $\overline{A} : \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}^m \to [0, 1]$ and call $\overline{A}$ the $n-$ valued $m-$ elements function corresponding to A.*

Definition 2.8. ([11]) *Let $N = (1, 2, \ldots)$, $D = (p_1, p_2, p_3)$ and $0 < p_n < 1$ $(n = 1, 2, \ldots)$. Then D is called a randomized sequence in $(0, 1)$.*

Definition 2.9. ([4]) *Let* $D_0 = (p_{01}, p_{02}, \ldots)$, $D_{\frac{1}{n-1}} = \{p_{\frac{1}{n-1}1}, p_{\frac{1}{n-1}2}, \ldots\}, \ldots$, $D_1 = (p_{11}, p_{12}, \ldots)$ *be an n randomized sequences in* $(0,1)$, *and* $p_{0k} + p_{\frac{1}{n-1}k} + \ldots + p_{1k} = 1$ $(k = 1, 2, \ldots)$. *Then* $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1$ $(n \geq 2)$ *is called an n-valued randomized number sequence in* $(0,1)$.

Definition 2.10. ([4]) *Suppose that* $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1$ $(n \geq 2)$ *be a series of n randomized numbers in* $(0,1)$. *For* $\alpha = (x_1, x_2, \ldots, x_m) \in \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}^m$, *let* $\varphi(\alpha) = Q_1 \times \ldots \times Q_m$, *here for any* $1 \leq k \leq m$, *when* $x_k = 0$, $Q_k = d_{0k}$; *when* $x_k = \frac{i}{n-1}$, $Q_k = d_{\frac{i}{n-1}k}$ $(i = 1, 2, \ldots, n-2)$; *when* $x_k = 1$, $Q_k = d_{1k}$. *Then we get a mapping*

$$\varphi : \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}^m \to [0,1],$$

called the D-randomization map of $\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}^m$.

Proposition 2.11. ([4]) *Let* φ *be a D-randomization map of* $\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}^m$. *Then*

$$\sum \{\varphi(\alpha) : \alpha \in \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}^m\} = 1.$$

3. $\Gamma - t$ absolute randomized truth degree of propositional formula

In this section, we will give the main results of this paper. To this end, we firstly give the definition of $\Gamma - t$ absolute randomized truth degree of propositional formula. Although some results of the $\Gamma - k$ randomized truth degree of propositional formula and Γ absolute truth degree of propositional formula have been given in [9, 15], absolute randomized truth degree of propositional formula have not been discussed so far.

Now we give the definition of $\Gamma - t$ absolute randomized truth degree of formula relative to local finite theory Γ under the t conjunction in Goguen n-valued propositional logic system of adding two operators.

Definition 3.1. *Let* $\Gamma \subseteq F(S)$ *and* $A \in F(S)$. *It is stipulated in* $\Pi_{\Delta,\sim}$: $S_\Gamma = \{p \in S \mid \exists B \in \Gamma$ *s.t. p is the atomic propositional of* $B\}$ *and* $S_A = \{p \in S \mid p$ *appears in* $A\}$. *Then we call* Γ *is theory of* $\Pi_{\Delta,\sim}$ *propositional logic syetem. Especially, when* S_Γ *is finite,* Γ *is called the locally finite theory of* $\Pi_{\Delta,\sim}$ *propositional logic system.*

Definition 3.2. *Let* $\Gamma \subseteq F(S)$, S_Γ *be finite,* $A \in F(S)$, $S = S_\Gamma \sqcup S_A = \{p_1, p_2, \ldots, p_m\}$, *and* $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1$ $(n \geq 2)$ *be an n-valued randomized number sequence in* $(0,1)$. *Define*

$$[tA]_1 = \overline{tA}^{-1}(1),$$

$$\mu([tA]_1) = \sum \{\varphi(\alpha) : \alpha \in \overline{tA}^{-1}(1)\},$$

$$\text{when } \Delta_n(\Gamma, tA) = \emptyset, \tau_{D,\Gamma}(tA) = 1,$$

$$\text{when } \Delta_n(\Gamma, tA) \neq \emptyset, \tau_{D,\Gamma}(tA) = \frac{|\mu([tA]_1)|}{|\sum\{\varphi(\alpha) : \alpha \in \Delta_n(\Gamma, tA)\}|}.$$

Among them, $\Delta_n(\Gamma, tA) = \{\alpha \in \Pi^m_{\Delta,\sim} | \forall B \in \Gamma, \overline{B}(\alpha) = 1\}$, $\overline{tA}^{-1}(1) = \{\alpha \in \Delta_n(\Gamma, tA) \mid \overline{tA}(\alpha) = 1\}$. *Then* $\tau_{D,\Gamma}(tA)$ *is called the* $\Gamma - t$ *absolute randomized truth degree of propositional formula A, where t takes* Δ *and* $\sim$.

Remark 3.3. *Unless there are another instructions in the text, the following points remain unchanged: (i) Discuss in* $\Pi_{\Delta,\sim}$. *(ii) Basic grammar, semantic concepts, etc. are the same as classic proposition logic. (iii)* p, q, r, z, m, l *take* Δ *and* $\sim$.

Remark 3.4. *(i) If* Γ *consists entirely of tautology or* $\Gamma = \emptyset$, $\Delta_n(\Gamma, tA) = \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}^m$, *then* $\Gamma - t$ *absolute randomized truth degree of formula A is t absolute randomized truth degree of A. (ii)* $\forall A \in F(S)$, *we have* $0 \leq \tau_{D,\Gamma}(tA) \leq 1$.

The following theorem gives the relation of $\Gamma-t$ absolute randomized truth degree between two inclusion theories.

Theorem 3.5. *Let $\Gamma_1 \subseteq \Gamma_2 \subseteq F(S)$, $A \in F(S)$, S_Γ be finite, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1$ ($n \geq 2$) be an n-valued randomized number sequence in $(0,1)$. If $\tau_{D,\Gamma_1}(tA) = 1$, then $\tau_{D,\Gamma_2}(tA) = 1$.*

Proof. As $\Gamma_1 \subseteq \Gamma_2$, we have $\Delta_n(\Gamma_2, tA) \subseteq \Delta_n(\Gamma_1, tA)$. When $\Delta_n(\Gamma_2, tA) = \emptyset$, we have $\tau_{D,\Gamma_2}(tA) = 1$ and $\Delta_n(\Gamma_1, tA) \neq \emptyset$. Since $\tau_{D,\Gamma_1}(tA) = 1$, by Definition 3.2, we get that $|\sum\{\varphi(\alpha) : \alpha \in \Delta_n(\Gamma_1, tA)\}| = |\mu([tA]_1)|$. Thus $\forall\alpha \in \Delta_n(\Gamma_1, tA)$, we have $|\sum\{\varphi(\alpha) : \alpha \in \Delta_n(\Gamma_1, tA)\}| = |\mu([tA]_1)|$. Then $\forall\alpha \in \Delta_n(\Gamma_2, tA)$, we have $|\sum\{\varphi(\alpha) : \alpha \in \Delta_n(\Gamma_2, tA)\}| = |\mu([tA]_1)|$. Thus $|\sum\{\varphi(\alpha) : \alpha \in \Delta_n(\Gamma_2, tA)\}| = |\mu([tA]_1)|$, i.e., $\tau_{D,\Gamma_2}(tA) = 1$. □

Now, we discuss the most basic properties of $\Gamma - t$ absolute randomized truth degree.

Theorem 3.6. *Let $\Gamma \subseteq F(S)$, $A, B \in F(S)$, S_Γ be finite, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1$ ($n \geq 2$) be an n-valued randomized number sequence in $(0,1)$. Then*

(i) If $\Gamma \models A$, then $\tau_{D,\Gamma}(\Delta A) = 1$ and $\tau_{D,\Gamma}(\sim A) = 0$.

(ii) If $\Gamma \models\sim A$, then $\tau_{D,\Gamma}(\Delta A) = 0$ and $\tau_{D,\Gamma}(\sim A) = 1$.

(iii) If $A \approx B$, then $\tau_{D,\Gamma}(tA) = \tau_{D,\Gamma}(tB)$.

(iv) If $\vDash pA \to qB$, then $\tau_{D,\Gamma}(pA) \leq \tau_{D,\Gamma}(qB)$.

(v) If $\Delta_n(\Gamma, tA) \neq \emptyset$, then $\tau_{D,\Gamma}(\sim tA) = 1 - \tau_{D,\Gamma}(tA)$.

Proof. (i): Let $S_\Gamma \cup S_A = \{p_1, p_2, \ldots, p_m\}$. Then $\Delta_n(\Gamma, tA) = \{\alpha \in \Pi^m_{\Delta,\sim} \mid \forall B \in \Gamma, \overline{B}(\alpha) = 1\}$. If $\Gamma \models A$, then for any $\alpha \in \Delta_n(\Gamma, tA)$, we have $\overline{A}(\alpha) = 1$. According to the definition of the Δ conjunction, when $\Gamma \models A$, we have $\alpha \in \Delta_n(\Gamma, \Delta A)$ and $\overline{\Delta A}(\alpha) = 1$. It follows from Definition 3.2 that $\tau_{D,\Gamma}(\Delta A) = \dfrac{|\mu([\Delta A]_1)|}{|\sum\{\varphi(\alpha) : \alpha \in \Delta_n(\Gamma, \Delta A)\}|} = 1$. According to the definition of the $\sim$ conjunction, when $\Gamma \models A$, we have $\forall\alpha \in \Delta_n(\Gamma, \sim A)$ and $\overline{\sim A}(\alpha) = 0$. By Definition 3.2, we get that $\tau_{D,\Gamma}(\sim A) = \dfrac{|\mu([\sim A]_1)|}{|\sum\{\varphi(\alpha) : \alpha \in \Delta_n(\Gamma, \sim A)\}|} = 0$.

(ii): Carrying out a proof similar to that of (i), we can get that if $\Gamma \models\sim A$, then $\tau_{D,\Gamma}(\Delta A) = 0$ and $\tau_{D,\Gamma}(\sim A) = 1$.

(iii): Let $S_\Gamma \cup S_A = S_\Gamma \cup S_B = \{p_1, p_2, \ldots, p_m\}$. Then $\Delta_n(\Gamma, pA) = \{\alpha \in \Pi^m_{\Delta,\sim} \mid \forall R \in \Gamma, \overline{R}(\alpha) = 1\} = \Delta_n(\Gamma, qB)$. It follows from $pA \approx qB$ that $\overline{pA}(\alpha) = \overline{qB}(\alpha)$. Thus $|\overline{pA}^{-1}(1)| = |\alpha \in \Delta_n(\Gamma, pA)|\overline{pA}(\alpha) = 1| = |\alpha \in \Delta_n(\Gamma, qB)|\overline{qB}(\alpha) = 1| = |\overline{qB}^{-1}(1)|$. By Definition 3.2, we get that $\tau_{D,\Gamma}(pA) = \dfrac{|\mu([pA]_1)|}{|\sum\{\varphi(\alpha) : \alpha \in \Delta_n(\Gamma, pA)\}|} = \dfrac{|\mu([qB]_1)|}{|\sum\{\varphi(\alpha) : \alpha \in \Delta_n(\Gamma, qB)\}|} = \tau_{D,\Gamma}(qB)$.

(iv): Let $S_\Gamma \cup S_A = S_\Gamma \cup S_B = \{p_1, p_2, \ldots, p_m\}$. Then $\Delta_n(\Gamma, pA) = \{\alpha \in \Pi^m_{\Delta,\sim} \mid \forall R \in \Gamma, \overline{R}(\alpha) = 1\} = \Delta_n(\Gamma, qB)$. It follows from $\models pA \to qB$ that $\overline{pA \to qB}(\alpha) = 1$, and hence $\overline{pA}(\alpha) \leq \overline{qB}(\alpha)$. Thus $|\overline{pA}^{-1}(1)| = |\alpha \in \Delta_n(\Gamma, pA)|\overline{pA}(\alpha) = 1| \leq |\alpha \in \Delta_n(\Gamma, qB)|\overline{qB}(\alpha) = 1| = |\overline{qB}^{-1}(1)|$. By Definition 3.2, we get that $\tau_{D,\Gamma}(pA) = \dfrac{|\mu([pA]_1)|}{|\sum\{\varphi(\alpha) : \alpha \in \Delta_n(\Gamma, pA)\}|} \leq \dfrac{|\mu([qB]_1)|}{|\sum\{\varphi(\alpha) : \alpha \in \Delta_n(\Gamma, qB)\}|} = \tau_{D,\Gamma}(qB)$.

(v): Let $S_\Gamma \cup S_A = S_\Gamma \cup S_{\sim A} = \{p_1, p_2, \ldots, p_m\}$. Then $\Delta_n(\Gamma, tA) = \{\alpha \in \Pi^m_{\Delta,\sim} \mid \forall B \in \Gamma, \overline{B}(\alpha) = 1\} = \Delta_n(\Gamma, \sim tA)$.

It follows from Definition 3.2 that

$$\begin{aligned}\tau_{D,\Gamma}(\sim tA) &= \frac{|\mu([\sim tA]_1)|}{|\sum\{\varphi(\alpha):\alpha\in\Delta_n(\Gamma,\sim tA)\}|}\\ &= \frac{|\mu([\sim tA]_1)|}{|\sum\{\varphi(\alpha):\alpha\in\Delta_n(\Gamma,tA)\}|}\\ &= \frac{|1-\mu([tA]_1)|}{|\sum\{\varphi(\alpha):\alpha\in\Delta_n(\Gamma,tA)\}|}\\ &= 1-\frac{|\mu([tA]_1)|}{|\sum\{\varphi(\alpha):\alpha\in\Delta_n(\Gamma,tA)\}|}\\ &= 1-\tau_{D,\Gamma}(tA).\end{aligned}$$

□

Lemma 3.7. *Let* $a,b\in\Pi_{\Delta,\sim}$. *Then*

(i) $1\to qb=qb$.

(ii) $pa\to qb\geq qb$.

Proof. (i): If $qb=1$, then $1\to qb=1\to 1=1=qb$; if $qb<1$, then $1\to qb=qb$. So $1\to qb=qb$.

(ii): If $pa\leq qb$, then $pa\to qb=1\geq qb$; if $pa>qb$, then $pa\to qb=\frac{qb}{pa}>qb$. So $pa\to qb\geq qb$. □

Theorem 3.8. *Let* $\Gamma\subseteq F(S)$, $A,B\in F(S)$, S_Γ *be finite, and* $D_0,D_{\frac{1}{n-1}},\ldots,D_{\frac{n-2}{n-1}},D_1 (n\geq 2)$ *be an n-valued randomized number sequence in* $(0,1)$. *If* $\Gamma\models pA$, *then*

(i) $\tau_{D,\Gamma}(pA\to qB)=\tau_{D,\Gamma}(pA\wedge qB)=\tau_{D,\Gamma}(qB)$.

(ii) $\tau_{D,\Gamma}(qB\to pA)=1$.

Proof. Let A and B contain the same atomic formulas $p_1,p_2,\ldots,p_m$. If $\Gamma\models pA$, then for any $\alpha\in\Delta_n(\Gamma,pA)$, we have $\overline{pA}(\alpha)=1$.

(i): Let $S_\Gamma\sqcup S_{A\to B}=S_\Gamma\sqcup S_{A\wedge B}=S_\Gamma\sqcup S_B$. Then $\Delta_n(\Gamma,pA\to qB)=\{\alpha\in\Pi^m_{\Delta,\sim}\mid\forall R\in\Gamma,\overline{R}(\alpha)=1\}=\Delta_n(\Gamma,pA\wedge qB)=\Delta_n(\Gamma,qB)$. By Lemma 3.7(i), we have that $\overline{pA\to qB}(\alpha)=(\overline{pA}\to\overline{qB})(\alpha)=\overline{pA}(\alpha)\to\overline{qB}(\alpha)=1\to\overline{qB}(\alpha)=\overline{qB}(\alpha)$ and $\overline{pA\wedge qB}(\alpha)=(\overline{pA}\wedge\overline{qB})(\alpha)=\overline{pA}(\alpha)\wedge\overline{qB}(\alpha)=1\wedge\overline{qB}(\alpha)=\overline{qB}(\alpha)$. Thus $|\overline{pA\to qB}^{-1}(1)|=|\alpha\in\Delta_n(\Gamma,pA\to qB)|\overline{pA\to qB}(\alpha)=1|=|\alpha\in\Delta_n(\Gamma,qB)|\overline{qB}(\alpha)=1|=|\overline{qB}^{-1}(1)|$ and $|\overline{pA\wedge qB}^{-1}(1)|=|\alpha\in\Delta_n(\Gamma,pA\wedge qB)|\overline{pA\wedge qB}(\alpha)=1|=|\alpha\in\Delta_n(\Gamma,qB)|\overline{qB}(\alpha)=1|=|\overline{qB}^{-1}(1)|$. It follows from Definition 3.2 that

$$\begin{aligned}\tau_{D,\Gamma}(pA\to qB) &= \frac{|\mu([pA\to qB]_1)|}{|\sum\{\varphi(\alpha):\alpha\in\Delta_n(\Gamma,pA\to qB)\}|}\\ &= \frac{|\mu([qB]_1)|}{|\sum\{\varphi(\alpha):\alpha\in\Delta_n(\Gamma,qB)\}|}\\ &= \tau_{D,\Gamma}(qB).\end{aligned}$$

$$\begin{aligned}\tau_{D,\Gamma}(pA\wedge qB) &= \frac{|\mu([pA\wedge qB]_1)|}{|\sum\{\varphi(\alpha):\alpha\in\Delta_n(\Gamma,pA\wedge qB)\}|}\\ &= \frac{|\mu([qB]_1)|}{|\sum\{\varphi(\alpha):\alpha\in\Delta_n(\Gamma,qB)\}|}\\ &= \tau_{D,\Gamma}(qB).\end{aligned}$$

Thus $\tau_{D,\Gamma}(pA\to qB)=\tau_{D,\Gamma}(pA\wedge qB)=\tau_{D,\Gamma}(qB)$.

(ii): For any $\alpha \in \Delta_n(\Gamma, qB \to pA)$, by Lemma 3.7(ii), we get that $\overline{qB \to pA}(\alpha) = (\overline{qB} \to \overline{pA})(\alpha) \geq \overline{pA}(\alpha) = 1$. Thus $\forall \alpha \in \Delta_n(\Gamma, qB \to pA)$, we have $\overline{qB \to pA}(\alpha) = 1$. It follows from Definition 3.2 that $\tau_{D,\Gamma}(qB \to pA) = \dfrac{|\mu([qB \to pA]_1)|}{|\sum\{\varphi(\alpha) : \alpha \in \Delta_n(\Gamma, qB \to pA)\}|} = 1$. □

Example 3.9. *Let* $\Gamma = (p_1 \to \Delta p_2) \to\sim p_1$, $A = (\sim p_1 \vee \Delta p_2) \to p_2$, $B = (\sim p_1 \to\sim p_2) \to p_1$, $C = (\Delta p_1 \to\sim p_2) \to\sim p_1$, $E_1 = (\Delta A \wedge \Delta B) \to\sim C$, *and* $D_0 = \{0.1, 0.2\}$, $D_{\frac{1}{3}} = \{0.2, 0.1\}$, $D_{\frac{2}{3}} = \{0.3, 0.4\}$ *and* $D_1 = \{0.4, 0.3\}$ *be a* 4*-valued randomized number sequence in* $(0, 1)$. *Calculate* $\tau_{D,\Gamma}(E_1)$.

Answer. $\overline{A}(x, y) : \{0, \frac{1}{3}, \frac{2}{3}, 1\}^2 \to [0, 1]$, $\overline{A}(x, y) = (\sim x \vee \Delta y) \to y$.

$\overline{B}(x, y) : \{0, \frac{1}{3}, \frac{2}{3}, 1\}^2 \to [0, 1]$, $\overline{B}(x, y) = (\sim x \to\sim y) \to x$.

$\overline{C}(x, y) : \{0, \frac{1}{3}, \frac{2}{3}, 1\}^2 \to [0, 1]$, $\overline{C}(x, y) = (\Delta x \to\sim y) \to\sim x$.

In order to facilitate calculation and understanding, the following chart is made.

x	y	$\overline{A}(x, y)$	$\overline{B}(x, y)$	$\overline{C}(x, y)$	E_1	$\Delta_n(\Gamma, E_1)$
0	0	0	0	1	1	1
0	$\frac{1}{3}$	$\frac{1}{3}$	0	1	1	1
0	$\frac{2}{3}$	$\frac{2}{3}$	0	1	1	1
0	1	1	1	1	0	0
$\frac{1}{3}$	0	0	$\frac{1}{3}$	$\frac{2}{3}$	1	1
$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$	1	1
$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{2}{3}$	$\frac{2}{3}$	1	1
$\frac{1}{3}$	1	1	1	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$
$\frac{2}{3}$	0	0	$\frac{2}{3}$	$\frac{1}{3}$	1	1
$\frac{2}{3}$	$\frac{1}{3}$	1	$\frac{2}{3}$	$\frac{1}{3}$	1	1
$\frac{2}{3}$	$\frac{2}{3}$	1	$\frac{2}{3}$	$\frac{1}{3}$	1	1
$\frac{2}{3}$	1	1	1	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$
1	0	1	1	0	1	1
1	$\frac{1}{3}$	1	1	0	1	1
1	$\frac{2}{3}$	1	1	0	1	1
1	1	1	1	1	0	0

Thus $\Delta_n(\Gamma, E_1) == \{(0,0), (0, \frac{1}{3}), (0, \frac{2}{3}), (0, 1), (\frac{1}{3}, 0), (\frac{1}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3}), (\frac{2}{3}, 0), (\frac{2}{3}, \frac{1}{3}), (\frac{2}{3}, \frac{2}{3}), (1, 0), (1, \frac{1}{3}), (1, \frac{2}{3})\}$. $\overline{E_1}^{-1}(1) = \{x, y \in \Delta_n(\Gamma, E_1) | \overline{E_1}(x, y) = 1\} = \{(0, 0), (0, \frac{1}{3}), (0, \frac{2}{3}), (\frac{1}{3}, 0), (\frac{1}{3}, \frac{1}{3}), (\frac{1}{3} \frac{2}{3}), (\frac{2}{3}, 0), (\frac{2}{3}, \frac{1}{3}), (\frac{2}{3}, \frac{2}{3}), (1, 0), (1, \frac{1}{3}), (1, \frac{2}{3})\}$. Thus $|\sum\{\varphi(\alpha) : \alpha \in \Delta_n(\Gamma, E_1)| = |0.1 \times (0.2 + 0.1 + 0.4 + 0.3) + 0.2 \times (0.2 + 0.1 + 0.4) + 0.3 \times (0.2 + 0.1 + 0.4) + 0.4 \times (0.2 + 0.1 + 0.4)| = 0.73$. $|\mu([E]_1)| = |0.1 \times (0.2 + 0.1 + 0.4) + 0.2 \times (0.2 + 0.1 + 0.4) + 0.3 \times (0.2 + 0.1 + 0.4) + 0.4 \times (0.2 + 0.1 + 0.4)| = 0.7$. It follows from Definition 3.2 that $\tau_{D,\Gamma}(E_1) = \frac{0.7}{0.73} = \frac{70}{73}$.

In [9], the definition of $\overline{tA}^{-1}(1) = \{\alpha \in \Pi^m_{\Delta,\sim} | \overline{tA}(\alpha) = 1\}$ is given. We find that it is different from the one given in this paper. In order to emphasize the difference between the two, we give the following example specifically.

Example 3.10. *Let* $\Gamma = (p_1 \vee \Delta p_2) \to\sim p_1$, $A = (\sim p_1 \vee \Delta p_2) \to p_2$, $B = (\sim p_1 \to\sim p_2) \to p_1$, $C = (\Delta p_1 \to\sim p_2) \to\sim p_1$, $E_2 = (\Delta A \vee \Delta B) \to\sim C$, *and* $D_0 = \{0.2, 0.1\}$, $D_{\frac{1}{3}} = \{0.1, 0.2\}$, $D_{\frac{2}{3}} = \{0.4, 0.3\}$ *and* $D_1 = \{0.3, 0.4\}$ *be a* 4*-valued randomized number sequence in* $(0, 1)$. *Calculate* $\tau_{D,\Gamma}(E_2)$.

Answer. $\overline{A}(x,y):\{0,\frac{1}{3},\frac{2}{3},1\}^2\to[0,1]$, $\overline{A}(x,y)=(\sim x\vee\Delta y)\to y$.

$\overline{B}(x,y):\{0,\frac{1}{3},\frac{2}{3},1\}^2\to[0,1]$, $\overline{B}(x,y)=(\sim x\to\sim y)\to x$.

$\overline{C}(x,y):\{0,\frac{1}{3},\frac{2}{3},1\}^2\to[0,1]$, $\overline{C}(x,y)=(\Delta x\to\sim y)\to\sim x$.

In order to facilitate calculation and understanding, the following chart is made.

x	y	$\overline{A}(x,y)$	$\overline{B}(x,y)$	$\overline{C}(x,y)$	E_2	$\Delta_n(\Gamma,E_2)$
0	0	0	0	1	1	1
0	$\frac{1}{3}$	$\frac{1}{3}$	0	1	1	1
0	$\frac{2}{3}$	$\frac{2}{3}$	0	1	1	1
0	1	1	1	1	0	1
$\frac{1}{3}$	0	0	$\frac{1}{3}$	$\frac{2}{3}$	1	1
$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$	1	1
$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	1
$\frac{1}{3}$	1	1	1	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$
$\frac{2}{3}$	0	0	$\frac{2}{3}$	$\frac{1}{3}$	1	$\frac{1}{2}$
$\frac{2}{3}$	$\frac{1}{3}$	1	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{2}$
$\frac{2}{3}$	$\frac{2}{3}$	1	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{2}$
$\frac{2}{3}$	1	1	1	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$
1	0	1	1	0	1	0
1	$\frac{1}{3}$	1	1	0	1	0
1	$\frac{2}{3}$	1	1	0	1	0
1	1	1	1	1	0	0

Thus $\Delta_n(\Gamma,E_2) == \{(0,0),(0,\frac{1}{3}),(0,\frac{2}{3}),(0,1),(\frac{1}{3},0),(\frac{1}{3},\frac{1}{3}),(\frac{1}{3},\frac{2}{3})\}$. $\overline{E_2}^{-1}(1)=\{x,y\in\Delta_n(\Gamma,E_2)|\overline{E_2}(x,y)=1\}=\{(0,0),(0,\frac{1}{3}),(0,\frac{2}{3}),(\frac{1}{3},0),(\frac{1}{3},\frac{1}{3})\}$.
Thus $|\sum\{\varphi(\alpha):\alpha\in\Delta_n(\Gamma,E_2)|=|0.2\times(0.1+0.2+0.3+0.4)+0.1\times(0.1+0.2+0.3)|=0.26$. $|\mu([E]_2)|=|0.2\times(0.1+0.2+0.3)+0.1\times(0.1+0.2)|=0.15$. It follows from Definition 3.2 that $\tau_{D,\Gamma}(E_2)=\frac{0.15}{0.26}=\frac{15}{26}$.

Lemma 3.11. *Let $a,b\in\Pi_{\Delta,\sim}$. Then $qb\vee pa=qb+pa-(qb\wedge pa)$.*

Proof. Let $\lambda_1=qb\vee pa-qb-pa+(qb\wedge pa)$.

Case 1: $qb\le pa$. Then $\lambda_1=pa-qb-pa+qb=0$, i.e., $qb\vee pa=qb+pa-(qb\wedge pa)$.

Case 2: $qb>pa$. Then $\lambda_1=qb-qb-pa+pa$, i.e., $qb\vee pa=qb+pa-(qb\wedge pa)$.

So to sum up $qb\vee pa=qb+pa-(qb\wedge pa)$. □

Theorem 3.12. *Let $\Gamma\subseteq F(S)$, $A,B\in F(S)$, S_Γ be finite, and $D_0,D_{\frac{1}{n-1}},\dots,D_{\frac{n-2}{n-1}},D_1(n\ge2)$ be an n-valued randomized number sequence in $(0,1)$. Then $\tau_{D,\Gamma}(qB\vee pA)=\tau_{D,\Gamma}(qB)+\tau_{D,\Gamma}(pA)-\tau_{D,\Gamma}(qB\wedge pA)$.*

Proof. Let $S_\Gamma\cup S_A=S_\Gamma\cup S_B=S_\Gamma\cup S_{A\vee B}=S_\Gamma\cup S_{A\wedge B}=\{p_1,p_2,\dots,p_m\}$. Then $\Delta_n(\Gamma,pA\vee qB)=\{\alpha\in\Pi^m_{\Delta,\sim}\mid\forall R\in\Gamma,\overline{R}(\alpha)=1\}=\Delta_n(\Gamma,pA\wedge qB)=\Delta_n(\Gamma,pA)=\Delta_n(\Gamma,qB)=$. By Lemma 3.11, we get that $\overline{qB\vee pA}(\alpha)=\overline{qB}(\alpha)+\overline{pA}(\alpha)-\overline{qB\wedge pA}(\alpha)$. Thus $|\overline{pA\vee qB}^{-1}(1)|=|\alpha\in\Delta_n(\Gamma,pA\vee qB)|\overline{pA\vee qB}(\alpha)=1|=|\alpha\in\Delta_n(\Gamma,pA)|\overline{pA}(\alpha)=1|+|\alpha\in\Delta_n(\Gamma,qB)|\overline{qB}(\alpha)=1|-|\alpha\in\Delta_n(\Gamma,pA\wedge qB)|\overline{pA\wedge qB}(\alpha)=1|=|\overline{pA}^{-1}(1)|+|\overline{qB}^{-1}(1)|-|\overline{pA\wedge qB}^{-1}(1)|$. It follows from Definition 3.2 that

$$\begin{aligned}\tau_{D,\Gamma}(pA\vee qB)&=\frac{|\mu([pA\wedge qB]_1)|}{|\sum\{\varphi(\alpha):\alpha\in\Delta_n(\Gamma,pA\vee qB)\}|}\\&=\frac{|\mu([pA]_1)|}{|\sum\{\varphi(\alpha):\alpha\in\Delta_n(\Gamma,pA)\}|}+\frac{|\mu([qB]_1)|}{|\sum\{\varphi(\alpha):\alpha\in\Delta_n(\Gamma,qB)\}|}-\frac{|\mu([pA\wedge qB]_1)|}{|\sum\{\varphi(\alpha):\alpha\in\Delta_n(\Gamma,pA\wedge qB)\}|}\\&=\tau_{D,\Gamma}(pA)+\tau_{D,\Gamma}(qB)-\tau_{D,\Gamma}(pA\wedge qB).\end{aligned}$$

□

Remark 3.13. *Because p, q take Δ and $\sim$, the conclusion of Theorem 3.12 has specifically the following four forms:*

(i) $\tau_{D,\Gamma}(\Delta B \vee \Delta A) = \tau_{D,\Gamma}(\Delta B) + \tau_{D,\Gamma}(\Delta A) - \tau_{D,\Gamma}(\Delta B \wedge \Delta A)$.

(ii) $\tau_{D,\Gamma}(\Delta B \vee \sim A) = \tau_{D,\Gamma}(\Delta B) + \tau_{D,\Gamma}(\sim A) - \tau_{D,\Gamma}(\Delta B \wedge \sim A)$.

(iii) $\tau_{D,\Gamma}(\sim B \vee \Delta A) = \tau_{D,\Gamma}(\sim B) + \tau_{D,\Gamma}(\Delta A) - \tau_{D,\Gamma}(\sim B \wedge \Delta A)$.

(iv) $\tau_{D,\Gamma}(\sim B \vee \sim A) = \tau_{D,\Gamma}(\sim B) + \tau_{D,\Gamma}(\sim A) - \tau_{D,\Gamma}(\sim B \wedge \sim A)$.

Now, let us consider whether the $\Gamma - t$ absolute randomized truth degree MP rule holds. To do this, the following lemma is required.

Lemma 3.14. *Let $a, b \in \Pi_{\Delta,\sim}$. Then $qb \geq pa + (pa \to qb) - 1$.*

Proof. Let $\lambda_2 = qb - pa - (pa \to qb) - 1$.

Case 1: $pa \leq qb$. Then $\lambda_2 = qb - pa \geq 0$, i.e., $qb \geq pa + (pa \to qb) - 1$.

Case 2: $pa > qb$.Then $\lambda_2 = qb - pa - \frac{qb}{pa} + 1 = \frac{pa(qb-pa)}{pa} - \frac{qb-pa}{pa} = \frac{(qb-pa)(pa-1)}{pa} \geq 0$, i.e., $qb \geq pa + (pa \to qb) - 1$.

So to sum up $qb \geq pa + (pa \to qb) - 1$. □

Next, we give the inference rules of $\Gamma - t$ absolute randomized truth degree.

Theorem 3.15. *($\Gamma - t$ absolute randomized truth degree MP rule) Let $\Gamma \subseteq F(S)$, $A, B \in F(S)$, S_Γ be finite, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1$ $(n \geq 2)$ be an n-valued randomized number sequence in $(0, 1)$. If $\tau_{D,\Gamma}(pA) \geq \alpha$ and $\tau_{D,\Gamma}(pA \to qB) \geq \beta$, then $\tau_{D,\Gamma}(qB) \geq \alpha + \beta - 1$.*

Proof. Let $S_\Gamma \cup S_A = S_\Gamma \cup S_B = S_\Gamma \cup S_{A\to B} = \{p_1, p_2, \ldots, p_m\}$. Then $\Delta_n(\Gamma, pA) = \{\alpha \in \Pi^m_{\Delta,\sim} \mid \forall R \in \Gamma, \overline{R}(\alpha) = 1\} = \Delta_n(\Gamma, qB) = \Delta_n(\Gamma, pA \to qB)$. By Lemma 3.14, we get that $\overline{qB}(\alpha) \geq \overline{pA}(\alpha) + \overline{pA \to qB}(\alpha) - 1$. Thus $|\overline{qB}^{-1}(1)| = |\alpha \in \Delta_n(\Gamma, qB) | \overline{qB}(\alpha) = 1| \geq |\alpha \in \Delta_n(\Gamma, pA) | \overline{pA}(\alpha) = 1| + |\alpha \in \Delta_n(\Gamma, pA \to qB) | \overline{pA \to qB}(\alpha) = 1| - |\Delta_n(\Gamma, qB)| = |\overline{pA}^{-1}(1)| + |\overline{pA \to qB}^{-1}(1)| - |\Delta_n(\Gamma, qB)|$. It follows from Definition 3.2 that

$$\begin{aligned}
\tau_{D,\Gamma}(qB) &= \frac{|\mu([qB]_1)|}{|\sum\{\varphi(\alpha) : \alpha \in \Delta_n(\Gamma, qB)|} \\
&\geq \frac{|\mu([pA]_1)|}{|\sum\{\varphi(\alpha) : \alpha \in \Delta_n(\Gamma, pA)|} + \frac{|\mu([pA \to qB]_1)|}{|\sum\{\varphi(\alpha) : \alpha \in \Delta_n(\Gamma, pA \to qB)|} - \frac{|\sum\{\varphi(\alpha) : \alpha \in \Delta_n(\Gamma, qB)|}{|\sum\{\varphi(\alpha) : \alpha \in \Delta_n(\Gamma, qB)|} \\
&= \tau_{D,\Gamma}(pA) + \tau_{D,\Gamma}(pA \to qB) - 1.
\end{aligned}$$

Thus $\tau_{D,\Gamma}(qB) \geq \alpha + \beta - 1$. □

In particular, when $\tau_{D,\Gamma}(pA) = 1$ and $\tau_{D,\Gamma}(pA \to qB) = 1$, Theorem 3.15 has the following conclusion.

Corollary 3.16. *Let $\Gamma \subseteq F(S)$, $A, B \in F(S)$, $S\Gamma$ be finite, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ be an n-valued randomized number sequence in $(0, 1)$. If $\tau_{D,\Gamma}(pA) = 1$ and $\tau_{D,\Gamma}(pA \to qB) = 1$, then $\tau_{D,\Gamma}(qB) = 1$.*

Lemma 3.17. *Let $a, b, c \in \Pi_{\Delta,\sim}$. Then $(pa \to rc) \geq (pa \to qb) + (qb \to rc) - 1$.*

Proof. Let $\lambda_3 = (pa \to rc) - (pa \to qb) - (qb \to rc) + 1$.

(1) Caes 1: $pa \leq rc$.

(1.1) Case 1.1: $rc \leq qb$. Then $\lambda_3 = 1 - 1 - \frac{rc}{qb} + 1 \geq 0$,
i.e., $(pa \to rc) \geq (pa \to qb) + (qb \to rc) - 1$.

(1.2) Case 1.2: $rc > qb$.

(1.2.1) Case 1.2.1: $pa < qb$. Then $\lambda_3 = 1 - 1 - 1 + 1 = 0$,
i.e., $(pa \to rc) \geq (pa \to qb) + (qb \to rc) - 1$.

(1.2.2) Case 1.2.2: $pa \geq qb$. Then $\lambda_3 = 1 - \frac{qb}{pa} - 1 + 1 \geq 0$, i.e., $(pa \to rc) \geq (pa \to qb) + (qb \to rc) - 1$.

(2) Case 2: $pa > rc$.

(2.1) Case 2.1: $qb \geq pa$. Then $\lambda_3 = \frac{rc}{pa} - 1 - \frac{rc}{qb} + 1 \geq 0$,

i.e., $(pa \to rc) \geq (pa \to qb) + (qb \to rc) - 1$.

(2.2) Case 2.2: $qb < pa$.

(2.2.1) Case 2.2.1: $qb > rc$. Then $\lambda_3 = \frac{rc}{pa} - \frac{qb}{pa} - \frac{rc}{qb} + 1 = \frac{rc-qb}{pa} - \frac{rc}{qb} + 1 \geq 0$,

i.e., $(pa \to rc) \geq (pa \to qb) + (qb \to rc) - 1$.

(2.2.2) Case 2.2.2: $qb \leq rc$. Then $\lambda_3 = \frac{rc}{pa} - \frac{qb}{pa} - 1 + 1 = \frac{rc-qb}{pa} \geq 0$,

i.e., $(pa \to rc) \geq (pa \to qb) + (qb \to rc) - 1$.

So to sum up $(pa \to rc) \geq (pa \to qb) + (qb \to rc) - 1$. □

Theorem 3.18. *($\Gamma - t$ absolute randomized truth degree HS rule) Let $\Gamma \subseteq F(S)$, $A, B, C \in F(S)$, S_Γ be finite, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1$ ($n \geq 2$) be an n-valued randomized number sequence in $(0, 1)$. If $\tau_{D,\Gamma}(pA \to qB) \geq \alpha$ and $\tau_{D,\Gamma}(qB \to rC) \geq \beta$, then $\tau_{D,\Gamma}(pA \to rC) \geq \alpha + \beta - 1$.*

Proof. Let $S_\Gamma \cup S_{A\to B} = S_\Gamma \cup S_{B\to C} = S_\Gamma \cup S_{A\to C} = \{p_1, p_2, \ldots, p_m\}$. Then $\Delta_n(\Gamma, pA \to qB) = \{\alpha \in \Pi^m_{\Delta,\sim} \mid \forall R \in \Gamma, \overline{R}(\alpha) = 1\} = \Delta_n(\Gamma, qB \to rC) = \Delta_n(\Gamma, pA \to rC)$. By Lemma 3.17, we get that $\overline{pA \to rC}(\alpha) \geq \overline{pA \to qB}(\alpha) + \overline{qB \to rC}(\alpha) - 1$. Thus $|\overline{pA \to rC}^{-1}(1)| = |\alpha \in \Delta_n(\Gamma, pA \to rC) | \overline{pA \to rC}(\alpha) = 1| \geq |\alpha \in \Delta_n(\Gamma, pA \to qB) | \overline{pA \to qB}(\alpha) = 1| + |\alpha \in \Delta_n(\Gamma, qB \to rC) | \overline{qB \to rC}(\alpha) = 1| - |\Delta_n(\Gamma, pA \to rC)| = |\overline{pA \to qB}^{-1}(1)| + |\overline{qB \to rC}^{-1}(1)| - |\Delta_n(\Gamma, pA \to rC)|$. It follows from Definition 3.2 that

$$\begin{aligned}
\tau_{D,\Gamma}(pA \to rC) &= \frac{|\mu([pA \to rC]_1)|}{|\sum\{\varphi(\alpha) : \alpha \in \Delta_n(\Gamma, pA \to rC)|} \\
&\geq \frac{|\mu([pA \to qB]_1)|}{|\sum\{\varphi(\alpha) : \alpha \in \Delta_n(\Gamma, pA \to qB)|} + \frac{|\mu([qB \to rC]_1)|}{|\sum\{\varphi(\alpha) : \alpha \in \Delta_n(\Gamma, qB \to rC)|} \\
&- \frac{|\sum\{\varphi(\alpha) : \alpha \in \Delta_n(\Gamma, pA \to rC)|}{|\sum\{\varphi(\alpha) : \alpha \in \Delta_n(\Gamma, pA \to rC)|} \\
&= \tau_{D,\Gamma}(pA \to qB) + \tau_{D,\Gamma}(qB \to rC) - 1.
\end{aligned}$$

Thus $\tau_{D,\Gamma}(pA \to rC) \geq \alpha + \beta - 1$. □

Corollary 3.19. *Let $\Gamma \subseteq F(S)$, $A, B, C \in F(S)$, S_Γ be finite, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1$ ($n \geq 2$) be an n-valued randomized number sequence in $(0, 1)$. If $\tau_{D,\Gamma}(pA \to qB) = 1$ and $\tau_{D,\Gamma}(qB \to rC) = 1$, then $\tau_{D,\Gamma}(pA \to rC) = 1$.*

Lemma 3.20. *Let $a, b, c \in \Pi_{\Delta,\sim}$. Then $pa \to (qb \wedge rc) = (pa \to qb) \wedge (pa \to rc)$.*

Proof. Let $\lambda_4 = (pa \to (qb \wedge rc)) - ((pa \to qb) \wedge (pa \to rc))$.

(1) Case 1: $qb \leq rc$.

(1.1) Case 1.1: $pa \leq qb$. Then $\lambda_4 = (pa \to qb) - (1 \wedge 1) = 1 - 1 = 0$,

i.e., $pa \to (qb \wedge rc) = (pa \to qb) \wedge (pa \to rc)$.

(1.2) Case 1.2: $pa > qb$.

(1.2.1) Case 1.2.1: $pa \geq rc$. Then $\lambda_4 = (pa \to qb) - (\frac{qb}{pa} \wedge \frac{rc}{pa}) = \frac{qb}{pa} - \frac{qb}{pa} = 0$,

i.e., $pa \to (qb \wedge rc) = (pa \to qb) \wedge (pa \to rc)$.

(1.2.2) Case 1.2.2: $pa < rc$. Then $\lambda_4 = (pa \to qb) - (\frac{qb}{pa} \wedge 1) = \frac{qb}{pa} - \frac{qb}{pa} = 0$,

i.e., $pa \to (qb \wedge rc) = (pa \to qb) \wedge (pa \to rc)$.

(2) Case 2: $qb > rc$.

(2.1) Case 2.1: $rc > pa$. Then $\lambda_4 = (pa \to rc) - (1 \wedge 1) = 1 - 1 = 0$,

i.e., $pa \to (qb \wedge rc) = (pa \to qb) \wedge (pa \to rc)$.

(2.2) Case 2.2: $rc \leq pa$.

(2.2.1) Case 2.2.1: $qb \leq pa$. Then $\lambda_4 = (pa \to rc) - (\frac{qb}{pa} \wedge \frac{rc}{pa}) = \frac{rc}{pa} - \frac{rc}{pa} = 0$, i.e., $pa \to (qb \wedge rc) = (pa \to qb) \wedge (pa \to rc)$.

(2.2.2) Case 2.2.2: $qb > pa$. Then $\lambda_4 = (pa \to rc) - (1 \wedge \frac{rc}{pa}) = \frac{rc}{pa} - \frac{rc}{pa} = 0$,

i.e., $pa \to (qb \wedge rc) = (pa \to qb) \wedge (pa \to rc)$.

So to sum up $pa \to (qb \wedge rc) = (pa \to qb) \wedge (pa \to rc)$. □

Theorem 3.21. *Let $\Gamma \subseteq F(S)$, $A, B, C \in F(S)$, S_Γ be finite, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1$ ($n \geq 2$) be an n-valued randomized number sequence in $(0,1)$. Then $\tau_{D,\Gamma}(pA \to (qB \wedge rC)) = \tau_{D,\Gamma}(pA \to qB) + \tau_{D,\Gamma}(pA \to rC) - \tau_{D,\Gamma}((pA \to qB) \vee (pA \to rC))$.*

Proof. Let A, B and C contain the same atomic formulas $p_1, p_2, \ldots, p_m$. Then by Lemma 3.20, we have that $pA \to (qB \wedge rC) \approx (pA \to qB) \wedge (pA \to rC)$. It follows from Theorem 3.6(iii) that $\tau_{D,\Gamma}(pA \to (qB \wedge rC)) = \tau_{D,\Gamma}((pA \to qB) \wedge (pA \to rC))$. By Theorem 3.12, we get that $\tau_{D,\Gamma}(pA \to (qB \wedge rC)) = \tau_{D,\Gamma}(pA \to qB) + \tau_{D,\Gamma}(pA \to rC) - \tau_{D,\Gamma}((pA \to qB) \vee (pA \to rC))$. □

Corollary 3.22. *($\Gamma - t$ absolute randomized truth degree intersection inference rule) Let $\Gamma \subseteq F(S)$, $A, B, C \in F(S)$, S_Γ be finite, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1$ ($n \geq 2$) be an n-valued randomized number sequence in $(0,1)$. If $\tau_{D,\Gamma}(pA \to qB) \geq \alpha$ and $\tau_{D,\Gamma}(pA \to rC) \geq \beta$, then $\tau_{D,\Gamma}(pA \to (qB \wedge rC)) \geq \alpha + \beta - 1$.*

Corollary 3.23. *Let $\Gamma \subseteq F(S)$, $A, B, C \in F(S)$, S_Γ be finite, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1$ ($n \geq 2$) be an n-valued randomized number sequence in $(0,1)$. If $\tau_{D,\Gamma}(pA \to qB) = 1$ and $\tau_{D,\Gamma}(pA \to rC) = 1$, then $\tau_{D,\Gamma}(pA \to (qB \wedge rC)) = 1$.*

Lemma 3.24. *Let $a, b, c \in \Pi_{\Delta,\sim}$. Then $(pa \vee qb) \to rc = (pa \to rc) \wedge (qb \to rc)$.*

Proof. Carrying out a proof similar to that of Lemma 3.20, we can get that $(pa \vee qb) \to rc = (pa \to rc) \wedge (qb \to rc)$. □

Theorem 3.25. *Let $\Gamma \subseteq F(S)$, $A, B, C \in F(S)$, S_Γ be finite, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1$ ($n \geq 2$) be an n-valued randomized number sequence in $(0,1)$. Then $\tau_{D,\Gamma}((pA \vee qB) \to rC) = \tau_{D,\Gamma}(pA \to rC) + \tau_{D,\Gamma}(qB \to rC) - \tau_{D,\Gamma}((pA \to rC) \vee (qB \to rC))$.*

Proof. Let A, B and C contain the same atomic formulas $p_1, p_2, \ldots, p_m$. Then by Lemma 3.24, we get that $(pA \vee qB) \to rC \approx (pA \to rC) \wedge (qB \to rC)$. It follows from Theorem 3.6(iii) that $\tau_{D,\Gamma}((pA \vee qB) \to rC) = \tau_{D,\Gamma}((pA \to rC) \wedge (qB \to rC))$. By Theorem 3.12, we have that $\tau_{D,\Gamma}((pA \vee qB) \to rC) = \tau_{D,\Gamma}(pA \to rC) + \tau_{D,\Gamma}(qB \to rC) - \tau_{D,\Gamma}((pA \to rC) \vee (qB \to rC))$. □

Corollary 3.26. *($\Gamma - t$ absolute randomized truth degree union inference rule) Let $\Gamma \subseteq F(S)$, $A, B, C \in F(S)$, S_Γ be finite, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1$ ($n \geq 2$) be an n-valued randomized number sequence in $(0,1)$. If $\tau_{D,\Gamma}(pA \to rC) \geq \alpha$ and $\tau_{D,\Gamma}(qB \to rC) \geq \beta$, then $\tau_{D,\Gamma}((pA \vee qB) \to rC) \geq \alpha + \beta - 1$.*

Corollary 3.27. *Let $\Gamma \subseteq F(S)$, $A, B, C \in F(S)$, S_Γ be finite, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1$($n \geq 2$) be an n-valued randomized number sequence in $(0,1)$. If $\tau_{D,\Gamma}(pA \to rC) = 1$ and $\tau_{D,\Gamma}(qB \to rC) = 1$, then $\tau_{D,\Gamma}((pA \vee qB) \to rC) = 1$.*

For the rest of this section, we will mainly discuss important properties of Δ conjunction.

Lemma 3.28. *Let $a, b, c \in \Pi_{\Delta,\sim}$. Then*

(i) $\Delta a \to \sim b = \Delta a \wedge \sim b - \Delta a + 1$.

(ii) $\Delta a \to \Delta b = \Delta a \wedge \Delta b - \Delta a + 1$.

(iii) $\Delta a \to b = \Delta a \wedge b - \Delta a + 1$.

Proof. (i): Let $\lambda_5 = \Delta a \to \sim b - \Delta a \wedge \sim b + \Delta a - 1$.

(1) Case 1: $\Delta a \leq \sim b$. Then $\lambda_5 = 1 - \Delta a + \Delta a - 1 = 0$, i.e., $\Delta a \to \sim b = \Delta a \wedge \sim b - \Delta a + 1$.

(2) Case 2: $\Delta a > \sim b$. It follows from definition of the Δ conjunction that $a = 1$, i.e., $\Delta a = 1$. Thus $\lambda_5 = \frac{\sim b}{1} - \sim b + 1 - 1 = 0$, i.e., $\Delta a \to \sim b = \Delta a \wedge \sim b - \Delta a + 1$.

So to sum up $\Delta a \to \sim b = \Delta a \wedge \sim b - \Delta a + 1$.

Carrying out a proof similar to that of (i), we can give the proofs of (ii) and (iii). □

Theorem 3.29. *Let* $\Gamma \subseteq F(S)$, $A, B \in F(S)$, S_Γ *be finite, and* $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1$ $(n \geq 2)$ *be an n-valued randomized number sequence in* $(0, 1)$*. Then*

(i) $\tau_{D,\Gamma}(\Delta A \rightarrow\sim B) = \tau_{D,\Gamma}(\Delta A\wedge \sim B) - \tau_{D,\Gamma}(\Delta A) + 1$.

(ii) $\tau_{D,\Gamma}(\Delta A \rightarrow \Delta B) = \tau_{D,\Gamma}(\Delta A \wedge \Delta B) - \tau_{D,\Gamma}(\Delta A) + 1$.

(iii) $\tau_{D,\Gamma}(\Delta A \rightarrow B) = \tau_{D,\Gamma}(\Delta A \wedge B) - \tau_{D,\Gamma}(\Delta A) + 1$.

Proof. (i): Let $S_\Gamma \cup S_{A\rightarrow B} = S_\Gamma \cup S_{A\wedge B} = S_\Gamma \cup S_A = \{p_1, p_2, \ldots, p_m\}$. Then $\Delta_n(\Gamma, \Delta A \rightarrow\sim B) = \{\alpha \in \Pi^m_{\Delta,\sim} \mid \forall R \in \Gamma, \overline{R}(\alpha) = 1\} = \Delta_n(\Gamma, \Delta A\wedge \sim B) = \Delta_n(\Gamma, \Delta A)$. By Lemma 3.28(i), we have that $\overline{\Delta A \rightarrow\sim B}(\alpha) = \overline{\Delta A\wedge \sim B}(\alpha) - \overline{\Delta A}(\alpha) + 1$. Thus $|\overline{\Delta A \rightarrow\sim B}^{-1}(1)| = |\alpha \in \Delta_n(\Gamma, \Delta A \rightarrow\sim B)|\overline{\Delta A \rightarrow\sim B}(\alpha) = 1| = |\alpha \in \Delta_n(\Gamma, \Delta A\wedge \sim B)|\overline{\Delta A\wedge \sim B}(\alpha) = 1| - |\alpha \in \Delta_n(\Gamma, \Delta A)|\overline{\Delta A}(\alpha) = 1| + |\Delta_n(\Gamma, \Delta A \rightarrow\sim B)| = |\overline{\Delta A\wedge \sim B}^{-1}(1)| + |\overline{\Delta A}^{-1}(1)| - |\Delta_n(\Gamma, \Delta A \rightarrow\sim B)|$. It follows from Definition 3.2 that

$$\begin{aligned}
\tau_{D,\Gamma}(\Delta A \rightarrow\sim B) &= \frac{|\mu([\Delta A \rightarrow\sim B]_1)|}{|\sum\{\varphi(\alpha) : \alpha \in \Delta_n(\Gamma, \Delta A \rightarrow\sim B)|} \\
&\geq \frac{|\mu([\Delta A\wedge \sim B]_1)|}{|\sum\{\varphi(\alpha) : \alpha \in \Delta_n(\Gamma, \Delta A\wedge \sim B)|} - \frac{|\mu([\Delta A]_1)|}{|\sum\{\varphi(\alpha) : \alpha \in \Delta_n(\Gamma, \Delta A)|} \\
&- \frac{|\sum\{\varphi(\alpha) : \alpha \in \Delta_n(\Gamma, \Delta A \rightarrow\sim B)|}{|\sum\{\varphi(\alpha) : \alpha \in \Delta_n(\Gamma, \Delta A \rightarrow\sim B)|} \\
&= \tau_{D,\Gamma}(\Delta A\wedge \sim B) + \tau_{D,\Gamma}(\Delta A) - 1.
\end{aligned}$$

(ii): Carrying out a proof similar to that of (i), we can get that $\tau_{D,\Gamma}(\Delta A \rightarrow \Delta B) = \tau_{D,\Gamma}(\Delta A\wedge\Delta B) - \tau_{D,\Gamma}(\Delta A) + 1$.
(iii): Carrying out a proof similar to that of (i), we can prove $\tau_{D,\Gamma}(\Delta A \rightarrow B) = \tau_{D,\Gamma}(\Delta A\wedge B) - \tau_{D,\Gamma}(\Delta A) + 1$. □

Theorem 3.30. *Let* $\Gamma \subseteq F(S)$, $A, B \in F(S)$, S_Γ *be finite, and* $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1$ $(n \geq 2)$ *be an n-valued randomized number sequence in* $(0, 1)$*. Then*

(i) $\tau_{D,\Gamma}(\Delta A \rightarrow (\sim B\vee \sim C)) + \tau_{D,\Gamma}(\Delta A \rightarrow (\sim B\wedge \sim C)) = \tau_{D,\Gamma}(\Delta A \rightarrow\sim B) + \tau_{D,\Gamma}(\Delta A \rightarrow\sim C)$.

(ii) $\tau_{D,\Gamma}((\Delta A \vee \Delta B) \rightarrow \Delta C) + \tau_{D,\Gamma}((\Delta A \wedge \Delta B) \rightarrow \Delta C) = \tau_{D,\Gamma}(\Delta A \rightarrow \Delta C) + \tau_{D,\Gamma}(\Delta B \rightarrow \Delta C)$.

(iii) $\tau_{D,\Gamma}((\Delta A \vee B) \rightarrow (\Delta A \wedge B)) = \tau_{D,\Gamma}(\Delta A \rightarrow B) + \tau_{D,\Gamma}(B \rightarrow \Delta A)) - 1$.

Proof. Let A, B and C contain the same atomic formulas $p_1, p_2, \ldots, p_m$.

(i): By Theorem 3.12 and Theorem 3.29(i), we have that

$$\begin{aligned}
\tau_{D,\Gamma}(\Delta A \rightarrow (\sim B\vee \sim C)) + \tau_{D,\Gamma}(\Delta A \rightarrow (\sim B\wedge \sim C)) &= \tau_{D,\Gamma}(\Delta A \wedge (\sim B\vee \sim C)) - \tau_{D,\Gamma}(\Delta A) + 1 \\
&+ \tau_{D,\Gamma}(\Delta A\wedge \sim B\wedge \sim C) - \tau_{D,\Gamma}(\Delta A) + 1 \\
&= \tau_{D,\Gamma}((\Delta A\wedge \sim B) \vee (\Delta A\wedge \sim C)) + 2 \\
&+ \tau_{D,\Gamma}((\Delta A\wedge \sim B) \wedge (\Delta A\wedge \sim C)) - 2\tau_{D,\Gamma}(\Delta A) \\
&= \tau_{D,\Gamma}(\Delta A\wedge \sim B) + \tau_{D,\Gamma}(\Delta A\wedge \sim C) - 2\tau_{D,\Gamma}(\Delta A) + 2 \\
&= \tau_{D,\Gamma}(\Delta A \rightarrow\sim B) + \tau_{D,\Gamma}(\Delta A \rightarrow\sim C).
\end{aligned}$$

(ii): By Theorem 3.12 and Theorem 3.29(ii), we get that

$$\begin{aligned}
\tau_{D,\Gamma}((\Delta A \vee \Delta B) \rightarrow \Delta C) + \tau_{D,\Gamma}((\Delta A \wedge \Delta B) \rightarrow \Delta C) &= \tau_{D,\Gamma}((\Delta A \wedge \Delta B) \vee \Delta C) - \tau_{D,\Gamma}(\Delta A \vee \Delta B) + 1 \\
&+ \tau_{D,\Gamma}(\Delta A \wedge \Delta B \wedge \Delta C) - \tau_{D,\Gamma}(\Delta A \wedge \Delta B) + 1 \\
&= \tau_{D,\Gamma}((\Delta A \wedge \Delta C) \vee (\Delta B \wedge \Delta C)) - \tau_{D,\Gamma}(\Delta A \vee \Delta B) + 1 \\
&+ \tau_{D,\Gamma}((\Delta A \wedge \Delta C) \wedge (\Delta B \wedge \Delta C)) - \tau_{D,\Gamma}(\Delta A \wedge \Delta B) + 1 \\
&= \tau_{D,\Gamma}(\Delta A \wedge \Delta C) + \tau_{D,\Gamma}(\Delta B \wedge \Delta C) \\
&- \tau_{D,\Gamma}(\Delta A) - \tau_{D,\Gamma}(\Delta B) + 2 \\
&= \tau_{D,\Gamma}(\Delta A \rightarrow \Delta C) + \tau_{D,\Gamma}(\Delta B \rightarrow \Delta C).
\end{aligned}$$

(iii): It follows from Theorem 3.12 and Theorem 3.29(iii) that

$$\begin{aligned}
\tau_{D,\Gamma}((\Delta A \vee B) \to (\Delta A \wedge B)) &= \tau_{D,\Gamma}((\Delta A \vee B) \wedge (\Delta A \wedge B)) - \tau_{D,\Gamma}(\Delta A \vee B) + 1 \\
&= \tau_{D,\Gamma}(\Delta A \vee B) + \tau_{D,\Gamma}(\Delta A \wedge B) - \tau_{D,\Gamma}((\Delta A \vee B) \vee (\Delta A \wedge B)) - \tau_{D,\Gamma}(\Delta A \vee B) + 1 \\
&= \tau_{D,\Gamma}(\Delta A \wedge B) - \tau_{D,\Gamma}(\Delta A \vee B) + 1 \\
&= \tau_{D,\Gamma}(\Delta A \wedge B) - (\tau_{D,\Gamma}(\Delta A) + \tau_{D,\Gamma}(B) - \tau_{D,\Gamma}(\Delta A \wedge B)) + 1 \\
&= \tau_{D,\Gamma}(\Delta A \wedge B) - \tau_{D,\Gamma}(\Delta A) + 1 + \tau_{D,\Gamma}(B \to \Delta A) - 1 \\
&= \tau_{D,\Gamma}(\Delta A \to B) + \tau_{D,\Gamma}(B \to \Delta A) - 1.
\end{aligned}$$

□

Lemma 3.31. *Let $a, b, c \in \Pi_{\Delta,\sim}$.*

(i) $(\Delta a \to \sim b) \to \sim b = \Delta a \vee \sim b$.

(ii) $(\Delta a \to \Delta b) \to \Delta b = \Delta a \vee \Delta b$.

(iii) $(\Delta a \to b) \to b = \Delta a \vee b$.

Proof. We illustrate this with the case of (i), the other cases being similar.

(i): Let $\lambda_6 = ((\Delta a \to \sim b) \to \sim b) - (\Delta a \vee \sim b)$.

(1) Case 1: $\Delta a \leq \sim b$. Then $\lambda_6 = (1 \to \sim b) - \sim b = 0$, i.e., $(\Delta a \to \sim b) \to \sim b = \Delta a \vee \sim b$.

(2) Case 2: $\Delta a > \sim b$. It follows from definition of the Δ conjunction that $a = 1$, i.e., $\Delta a = 1$. Thus $\lambda_6 = (\frac{\sim b}{1} \to \sim b) - 1 = (\sim b \to \sim b) - 1 = 0$, i.e., $(\Delta a \to \sim b) \to \sim b = \Delta a \vee \sim b$.

So to sum up $(\Delta a \to \sim b) \to \sim b = \Delta a \vee \sim b$. □

Theorem 3.32. *Let $\Gamma \subseteq F(S)$, $A, B \in F(S)$, S_Γ be finite, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1$ $(n \geq 2)$ be an n-valued randomized number sequence in* $(0,1)$. *Then*

(i) $\tau_{D,\Gamma}((\Delta A \to \sim B) \to \sim B) = \tau_{D,\Gamma}(\Delta A \vee \sim B)$.

(ii) $\tau_{D,\Gamma}((\Delta A \to \Delta B) \to \Delta B) = \tau_{D,\Gamma}(\Delta A \vee \Delta B)$.

(iii) $\tau_{D,\Gamma}((\Delta A \to B) \to B) = \tau_{D,\Gamma}(\Delta A \vee B)$.

Proof. Let A and B contain the same atomic formulas $p_1, p_2, \ldots, p_m$.

(i): By Lemma 3.31(i), we get that $(\Delta A \to \sim B) \to \sim B \approx \Delta A \vee \sim B$. It follows from Theorem 3.6(iii) that $\tau_{D,\Gamma}((\Delta A \to \sim B) \to \sim B) = \tau_{D,\Gamma}(\Delta A \vee \sim B)$.

(ii): By Lemma 3.31(ii), we get that $(\Delta A \to \Delta B) \to \Delta B \approx \Delta A \vee \Delta B$. It follows from Theorem 3.6(iii) that $\tau_{D,\Gamma}((\Delta A \to \Delta B) \to \Delta B) = \tau_{D,\Gamma}(\Delta A \vee \Delta B)$.

(iii): By Lemma 3.31(iii), we get that $(\Delta A \to B) \to B \approx \Delta A \vee B$. It follows from Theorem 3.6(iii) that $\tau_{D,\Gamma}((\Delta A \to B) \to B) = \tau_{D,\Gamma}(\Delta A \vee B)$. □

Theorem 3.33. *Let $\Gamma \subseteq F(S)$, $A, B \in F(S)$, S_Γ be finite, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1$ $(n \geq 2)$ be an n-valued randomized number sequence in* $(0,1)$. *Then*

(i) $\tau_{D,\Gamma}((\Delta A \to (\sim B \vee \sim C)) \to (\sim B \vee \sim C)) + \tau_{D,\Gamma}((\Delta A \to (\sim B \wedge \sim C)) \to (\sim B \wedge \sim C)) = \tau_{D,\Gamma}(\Delta A \vee \sim B) + \tau_{D,\Gamma}(\Delta A \vee \sim C)$.

(ii) $\tau_{D,\Gamma}(((\Delta A \vee \Delta B) \to \Delta C) \to \Delta C) + \tau_{D,\Gamma}(((\Delta A \wedge \Delta B) \to \Delta C) \to \Delta C) = \tau_{D,\Gamma}(\Delta A \vee \Delta C) + \tau_{D,\Gamma}(\Delta B \vee \Delta C)$.

(iii) $\tau_{D,\Gamma}(((\Delta A \vee B) \to (\Delta A \wedge B)) \to (\Delta A \wedge B)) = \tau_{D,\Gamma}(\Delta A \vee B)$

Proof. Let A, B and C contain the same atomic formulas $p_1, p_2, \ldots, p_m$.

(i): By Theorem 3.12 and Theorem 32(i), we have that

$$\begin{aligned}\tau_{D,\Gamma}((\Delta A \to (\sim B \vee \sim C)) \to (\sim B \vee \sim C)) &+ \tau_{D,\Gamma}((\Delta A \to (\sim B \wedge \sim C)) \to (\sim B \wedge \sim C))\\ &= \tau_{D,\Gamma}(\Delta A \vee (\sim B \vee \sim C)) + \tau_{D,\Gamma}(\Delta A \vee (\sim B \wedge \sim C))\\ &= \tau_{D,\Gamma}((\Delta A \vee \sim B) \vee (\Delta A \vee \sim C)) + \tau_{D,\Gamma}((\Delta A \vee \sim B) \wedge (\Delta A \vee \sim C))\\ &= \tau_{D,\Gamma}(\Delta A \vee \sim B) + \tau_{D,\Gamma}(\Delta A \vee \sim C)\\ &- \tau_{D,\Gamma}((\Delta A \vee \sim B) \wedge (\Delta A \vee \sim C)) + \tau_{D,\Gamma}((\Delta A \vee \sim B) \wedge (\Delta A \vee \sim C))\\ &= \tau_{D,\Gamma}(\Delta A \vee \sim B) + \tau_{D,\Gamma}(\Delta A \vee \sim C).\end{aligned}$$

(ii): By Theorem 3.12 and Theorem 3.32(ii), we get that

$$\begin{aligned}\tau_{D,\Gamma}(((\Delta A \vee \Delta B) \to \Delta C) \to \Delta C) &+ \tau_{D,\Gamma}(((\Delta A \wedge \Delta B) \to \Delta C) \to \Delta C)\\ &= \tau_{D,\Gamma}((\Delta A \vee \Delta B) \vee \Delta C) + \tau_{D,\Gamma}((\Delta A \wedge \Delta B) \vee \Delta C)\\ &= \tau_{D,\Gamma}((\Delta A \vee \Delta C) \vee (\Delta B \vee \Delta C)) + \tau_{D,\Gamma}((\Delta A \vee \Delta B) \wedge (\Delta B \vee \Delta C))\\ &= \tau_{D,\Gamma}(\Delta A \vee \Delta C) + \tau_{D,\Gamma}(\Delta B \vee \Delta C)\\ &- \tau_{D,\Gamma}((\Delta A \vee \Delta C) \wedge (\Delta B \vee \Delta C)) + \tau_{D,\Gamma}((\Delta A \vee \Delta C) \wedge (\Delta B \vee \Delta C))\\ &= \tau_{D,\Gamma}(\Delta A \vee \Delta C) + \tau_{D,\Gamma}(\Delta B \vee \Delta C).\end{aligned}$$

(iii): By Theorem 3.32(iii), we have that

$$\begin{aligned}\tau_{D,\Gamma}(((\Delta A \vee B) \to (\Delta A \wedge B)) \to (\Delta A \wedge B)) &= \tau_{D,\Gamma}((\Delta A \vee B) \vee (\Delta A \wedge B))\\ &= \tau_{D,\Gamma}(\Delta A \vee B).\end{aligned}$$

□

4. $\Gamma - t$ absolute randomized similarity degree and $\Gamma - t$ absolute randomized pseudo-distance of propositional formulas

In this section, based on the concept and properties of $\Gamma - t$ absolute randomized truth degree in Goguen$_{\Delta,\sim}$, we introduce the $\Gamma - t$ absolute randomized similarity degree and the $\Gamma - t$ absolute randomized pseudo-distance of formulae, study some properties of the $\Gamma - t$ absolute randomized similarity degree, establish the $\Gamma - t$ absolute randomized logic metric space, and give a simple proof of the continuity of logic operations in the $\Gamma - t$ absolute randomized logic metric space.

Definition 4.1. *Let* $\Gamma \subseteq F(S)$, $A, B \in F(S)$, S_Γ *be finite, and* $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1$ $(n \geq 2)$ *be an n-valued randomized number sequence in* $(0, 1)$. *Define*

$$\xi_{D,\Gamma}(pA, qB) = \tau_{D,\Gamma}((pA \to qB) \wedge (qB \to pA)).$$

Then $\xi_{D,\Gamma}(pA, qB)$ *is called the* $\Gamma - t$ *absolute randomized similarity degree of propositional formulas A and B.*

Lemma 4.2. *Let* $a, b \in \Pi_{\Delta,\sim}$. *Then*

(i) $((pa \to qb) \to qb) \wedge (qb \to (pa \to qb)) = pa \vee qb$.

(ii) $((pa \to qb) \to (pa \wedge qb)) \wedge ((pa \wedge qb) \to (pa \to qb)) = pa$.

Proof. (i): Let $\lambda_7 = (((pa \to qb) \to qb) \wedge (qb \to (pa \to qb))) - (pa \vee qb)$.

(1) case 1: $pa \leq qb$. Then $\lambda_7 = (1 \to qb) \wedge (qb \to 1) - qb = qb - qb = 0$,
i.e., $((pa \to qb) \to qb) \wedge (qb \to (pa \to qb)) = pa \vee qb$.

(2) case 2: $pa > qb$. Then $\lambda_7 = (\frac{qb}{pa} \to qb) \wedge (qb \to \frac{qb}{pa}) - pa = (pa \wedge 1) - pa = 0$,
i.e., $((pa \to qb) \to qb) \wedge (qb \to (pa \to qb)) = pa \vee qb$.

So to sum up $((pa \to qb) \to qb) \wedge (qb \to (pa \to qb)) = pa \vee qb$.

(ii): Let $\lambda_8 = (((pa \to qb) \to (pa \wedge qb)) \wedge ((pa \wedge qb) \to (pa \to qb))) - pa$.

(1) case 1: $pa \le qb$. Then $\lambda_8 = (1 \to pa) \wedge (pa \to 1) - pa = pa - pa = 0$,
i.e., $((pa \to qb) \to (pa \wedge qb)) \wedge ((pa \wedge qb) \to (pa \to qb)) = pa$.
(2) case 2: $pa > qb$. Then $\lambda_8 = (\frac{qb}{pa} \to pa) \wedge (qb \to \frac{qb}{pa}) - pa = (pa \wedge 1) - pa = 0$,
i.e., $((pa \to qb) \to (pa \wedge qb)) \wedge ((pa \wedge qb) \to (pa \to qb)) = pa$.
So to sum up $((pa \to qb) \to (pa \wedge qb)) \wedge ((pa \wedge qb) \to (pa \to qb)) = pa$. □

Next, we discuss the most basic properties of $\Gamma - t$ absolute randomized similarity degree.

Theorem 4.3. *Let $\Gamma \subseteq F(S)$, $A, B \in F(S)$, S_Γ be finite, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1$ $(n \ge 2)$ be an n-valued randomized number sequence in $(0,1)$. Then*

(i) If $A \approx B$, then $\xi_{D,\Gamma}(tA, tB) = 1$.
(ii) $\xi_{D,\Gamma}(pA, qB) = \xi_{D,\Gamma}(qB, pA)$.
(iii) $\xi_{D,\Gamma}(pA \vee qB, pA) = \tau_{D,\Gamma}(qB \to pA)$.
(iv) $\xi_{D,\Gamma}(pA \wedge qB, pA) = \tau_{D,\Gamma}(pA \to qB)$.
(v) $\xi_{D,\Gamma}(pA \to qB, qB) = \tau_{D,\Gamma}(pA \vee qB)$.
(vi) $\xi_{D,\Gamma}(pA \to qB, pA \wedge qB) = \tau_{D,\Gamma}(pA)$.

Proof. Let A and B contain the same atomic formulas $p_1, p_2, \ldots, p_m$.

(i): As $A \approx B$, we have $tA \approx tB$. Thus $\models (tA \to tB) \wedge (tB \to tA)$. It follows from Definition 3.2 and Definition 4.1 that $\xi_{D,\Gamma}(tA, tB) = \tau_{D\Gamma}((tA \to tB) \wedge (tB \to tA)) = 1$.

(ii): For any $a, b \in \Pi_{\Delta,\sim}$, we have $(pa \to qb) \wedge (qb \to pa) = (qb \to pa) \wedge (pa \to qb)$. Thus $(pA \to qB) \wedge (qB \to pA) \approx (qB \to pA) \wedge (pA \to qB)$. Then by Theorem 3.6(iii), we get that $\tau_{D,\Gamma}((pA \to qB) \wedge (qB \to pA)) = \tau_{D,\Gamma}((pA \to qB) \wedge (qB \to pA))$. It follows from Definition 4.1 that $\xi_{D,\Gamma}(pA, qB) = \xi_D(qB, pA)$.

(iii): By Lemma 3.24, we have that $(pA \vee qB) \to pA \approx (pA \to pA) \wedge (qB \to pA) = qB \to pA$. It follows from Lemma 3.20 that $pA \to (pA \vee qB) \approx (pA \to pA) \vee (pA \to qB) = pA \to pA$. Then by Definition 4.1, we get that

$$\begin{aligned}
\xi_{D,\Gamma}(pA \vee qB, pA) &= \tau_{D,\Gamma}(((pA \vee qB) \to pA) \wedge (pA \to (pA \vee qB))) \\
&= \tau_{D,\Gamma}(((pA \to pA) \wedge (qB \to pA)) \wedge ((pA \to pA) \vee (pA \to qB))) \\
&= \tau_{D,\Gamma}((qB \to pA) \wedge (pA \to pA)) \\
&= \tau_{D,\Gamma}(qB \to pA).
\end{aligned}$$

(iv): By Lemma 3.24, we have that $(pA \wedge qB) \to pA \approx (pA \to pA) \vee (qB \to pA) = pA \to pA$. It follows from Lemma 3.20 that $pA \to (pA \wedge qB) \approx (pA \to pA) \wedge (pA \to qB) = pA \to qB$. Then by Definition 4.1, we have that

$$\begin{aligned}
\xi_{D,\Gamma}(pA \wedge qB, pA) &= \tau_{D,\Gamma}(((pA \wedge qB) \to pA) \wedge (pA \to (pA \wedge qB))) \\
&= \tau_{D,\Gamma}(((pA \to pA) \vee (qB \to pA)) \wedge ((pA \to pA) \wedge (pA \to qB))) \\
&= \tau_{D,\Gamma}((pA \to pA) \wedge (pA \to qB)) \\
&= \tau_{D,\Gamma}(pA \to qB).
\end{aligned}$$

(v): It follows from Lemma 4.2(i) that $((pA \to qB) \to qB) \wedge (qB \to (pA \to qB)) \approx pA \vee qB$. By Theorem 3.6(iii), we get that $\tau_{D,\Gamma}(((pA \to qB) \to qB) \wedge (qB \to (pA \to qB))) = \tau_{D,\Gamma}(pA \vee qB)$. It follows from Definition 4.1 that $\xi_{D,\Gamma}(pA \to qB, qB) = \tau_{D,\Gamma}(pA \vee qB)$.

(vi): By Lemma 4.2(ii), we have that $((pA \to qB) \to (pA \wedge qB)) \wedge ((pA \wedge qB) \to (pA \to qB)) \approx pA$. It follows from Theorem 3.6(iii) that $\tau_{D,\Gamma}(((pA \to qB) \to (pA \wedge qB)) \wedge ((pA \wedge qB) \to (pA \to qB))) = \tau_{D,\Gamma}(pA)$. By Definition 4.1, we get that $\xi_{D,\Gamma}(pA \to qB, pA \wedge qB) = \tau_{D,\Gamma}(pA)$. □

Corollary 4.4. *Let $\Gamma \subseteq F(S)$, $A, B \in F(S)$, S_Γ be finite, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1$ $(n \ge 2)$ be an n-valued randomized number sequence in $(0,1)$. Then*

(i) $\xi_{D,\Gamma}(pA \wedge qB, qB) = \xi_{D,\Gamma}(pA \vee qB, pA) = \tau_{D,\Gamma}(qB \to pA)$.
(ii) $\xi_{D,\Gamma}(pA \vee qB, qB) = \xi_{D,\Gamma}(pA \wedge qB, pA) = \tau_{D,\Gamma}(pA \to qB)$.

Theorem 4.5. *Let $\Gamma \subseteq F(S)$, $A, B \in F(S)$, S_Γ be finite, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1$ $(n \geq 2)$ be an n-valued randomized number sequence in $(0,1)$. If $\Gamma \models pA$, then*

(i) $\xi_{D,\Gamma}(pA, qB) = \tau_{D,\Gamma}(qB)$.

(ii) $\xi_{D,\Gamma}(pA \vee qB, pA) = 1$.

(iii) $\xi_{D,\Gamma}(pA \wedge qB, pA) = \tau_{D,\Gamma}(qB)$.

Proof. Let A and B contain the same atomic formulas $p_1, p_2, \ldots, p_m$.

(i): By Definition 4.1, we get that $\xi_{D,\Gamma}(pA, qB) = \tau_{D,\Gamma}((pA \to qB) \wedge (qB \to pA))$. It follows from Theorem 3.12 that $\xi_{D,\Gamma}(pA, qB) = \tau_{D,\Gamma}(pA \to qB) + \tau_{D,\Gamma}(qB \to pA) - \tau_{D,\Gamma}((pA \to qB) \vee (qB \to pA))$. Thus $\xi_{D,\Gamma}(pA, qB) = \tau_{D,\Gamma}(pA \to qB) + \tau_{D,\Gamma}(qB \to pA) - 1$. Since $\Gamma \models pA$, by Theorem 3.8, we have that $\xi_{D,\Gamma}(pA, qB) = \tau_{D,\Gamma}(qB) + 1 - 1 = \tau_{D,\Gamma}(qB)$.

(ii): By Theorem 4.3(iii), we get that $\xi_{D,\Gamma}(pA \vee qB, pA) = \tau_{D,\Gamma}(qB \to pA)$. Since $\Gamma \models pA$, it follows from Theorem 3.8(ii) that $\xi_{D,\Gamma}(pA \vee qB, pA) = 1$.

(iii): By Theorem 4.3(iv), we have that $\xi_{D,\Gamma}(pA \wedge qB, pA) = \tau_{D,\Gamma}(pA \to qB)$. Since $\Gamma \models pA$, it follows from Theorem 3.8(i) that $\xi_{D,\Gamma}(pA \wedge qB, pA) = \tau_{D,\Gamma}(qB)$. □

Assume that $\Gamma \models qB$, then the corollary is as follows.

Corollary 4.6. *Let $\Gamma \subseteq F(S)$, $A, B \in F(S)$, S_Γ be finite, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1$ $(n \geq 2)$ be an n-valued randomized number sequence in $(0,1)$. If $\Gamma \models qB$, then*

(i) $\xi_{D,\Gamma}(pA, qB) = \tau_{D,\Gamma}(pA)$.

(ii) $\xi_{D,\Gamma}(pA \vee qB, pA) = \tau_{D,\Gamma}(pA)$.

(iii) $\xi_{D,\Gamma}(pA \wedge qB, pA) = 1$.

Lemma 4.7. *Let $a, b, c \in \Pi_{\Delta,\sim}$. Then $(pa \to qb) \to ((pa \vee rc) \to (qb \vee rc)) = 1$.*

Proof. Let $\lambda_9 = (pa \to qb) \to ((pa \vee rc) \to (qb \vee rc)) - 1$.

(1) Case 1: $pa \leq qb$.

(1.1) Case 1.1: $qb \leq rc$. Then $\lambda_9 = 1 \to (rc \to rc) - 1 = 1 - 1 = 0$,
i.e., $(pa \to qb) \to ((pa \vee rc) \to (qb \vee rc)) = 1$.

(1.2) Case 1.2: $qb > rc$.

(1.2.1) Case 1.2.1: $pa \leq rc$. Then $\lambda_9 = 1 \to (rc \to qb) - 1 = 1 - 1 = 0$,
i.e., $(pa \to qb) \to ((pa \vee rc) \to (qb \vee rc)) = 1$.

(1.2.2) Case 1.2.2: $pa > rc$. Then $\lambda_9 = 1 \to (pa \to qb) - 1 = 1 - 1 = 0$,
i.e., $(pa \to qb) \to ((pa \vee rc) \to (qb \vee rc)) = 1$.

(2) Case 2: $pa > qb$.

(2.1) Case 2.1: $qb > rc$. Then $\lambda_9 = \frac{qb}{pa} \to (pa \to qb) - 1 = (\frac{qb}{pa} \to \frac{qb}{pa}) - 1 = 0$,
i.e., $(pa \to qb) \to ((pa \vee rc) \to (qb \vee rc)) = 1$.

(2.2) Case 2.2: $qb \leq rc$.

(2.2.1) Case 2.2.1: $pa \leq rc$. Then $\lambda_9 = \frac{qb}{pa} \to (rc \to rc) - 1 = (\frac{qb}{pa} \to 1) - 1 = 0$,
i.e., $(pa \to qb) \to ((pa \vee rc) \to (qb \vee rc)) = 1$.

(2.2.2) Case 2.2.2: $pa > rc$. Then $\lambda_9 = \frac{qb}{pa} \to (pa \to rc) - 1 = (\frac{qb}{pa} \to \frac{rc}{pa}) - 1 = 0$,
i.e., $(pa \to qb) \to ((pa \vee rc) \to (qb \vee rc)) = 1$.

So to sum up $(pa \to qb) \to ((pa \vee rc) \to (qb \vee rc)) = 1$. □

Lemma 4.8. *Let $a, b, c \in \Pi_{\Delta,\sim}$. Then $(pa \to qb) \to ((pa \wedge rc) \to (qb \wedge rc)) = 1$.*

Proof. Carrying out a proof similar to that of Lemma 4.7, we can show that $(pa \to qb) \to ((pa \wedge rc) \to (qb \wedge rc)) = 1$. □

Theorem 4.9. *Let* $\Gamma \subseteq F(S)$, $A, B, C \in F(S)$, S_Γ *be finite, and* $D_0, D_{\frac{1}{n-1}}, \dots, D_{\frac{n-2}{n-1}}, D_1$ ($n \geq 2$) *be an n-valued randomized number sequence in* $(0,1)$. *Then*

(i) $\xi_{D,\Gamma}(pA \vee rC, qB \vee rC) \geq \xi_{D,\Gamma}(pA, qB)$.

(ii) $\xi_{D,\Gamma}(pA \wedge rC, qB \wedge rC) \geq \xi_{D,\Gamma}(pA, qB)$.

(iii) $\xi_{D,\Gamma}(pA \to rC, qB \to rC) \geq \xi_{D,\Gamma}(pA, qB)$.

Proof. Let A, B and C contain the same atomic formulas $p_1, p_2, \dots, p_m$.

(i): By Lemma 4.7, we get that $\models (pA \to qB) \to ((pA \vee rC) \to (qB \vee rC))$ and $\models (qB \to pA) \to ((qB \vee rC) \to (pA \vee rC))$. Thus $\models ((pA \to qB) \wedge (qB \to pA)) \to (((pA \vee rC) \to (qB \vee rC)) \wedge ((qB \vee rC) \to (pA \vee rC)))$. Then by Theorem 3.6(iv), we have that $\tau_{D,\Gamma}((pA \to qB) \wedge (qB \to pA)) \leq \tau_{D,\Gamma}(((pA \vee rC) \to (qB \vee rC)) \wedge ((qB \vee rC) \to (pA \vee rC)))$. It follows from Definition 4.1 that

$$\begin{aligned} \xi_{D,\Gamma}(pA \vee rC, qB \vee rC) &= \tau_{D,\Gamma}(((pA \vee rC) \to (qB \vee rC)) \wedge ((qB \vee rC) \to (pA \vee rC))) \\ &\geq \tau_{D,\Gamma}((pA \to qB) \wedge (qB \to pA)) \\ &= \xi_{D,\Gamma}(pA, qB). \end{aligned}$$

Carrying out a proof similar to that of (i), we can get (ii) and (iii). □

The following corollary gives another form of Theorem 4.9.

Corollary 4.10. *Let* $\Gamma \subseteq F(S)$, $A, B, C \in F(S)$, S_Γ *be finite, and* $D_0, D_{\frac{1}{n-1}}, \dots, D_{\frac{n-2}{n-1}}, D_1$ ($n \geq 2$) *be an n-valued randomized number sequence in* $(0,1)$. *Then*

(i) $\xi_{D,\Gamma}(pA \vee qB, pA \vee rC) \geq \xi_{D,\Gamma}(qB, rC)$.

(ii) $\xi_{D,\Gamma}(pA \wedge qB, pA \wedge rC) \geq \xi_{D,\Gamma}(qB, rC)$.

(iii) $\xi_{D,\Gamma}(pA \to qB, pA \to rC) \geq \xi_{D,\Gamma}(qB, rC)$.

Theorem 4.11. *Let* $\Gamma \subseteq F(S)$, $A, B \in F(S)$, S_Γ *be finite, and* $D_0, D_{\frac{1}{n-1}}, \dots, D_{\frac{n-2}{n-1}}, D_1 (n \geq 2)$ *be an n-valued randomized number sequence in* $(0,1)$. *Then* $\xi_{D,\Gamma}(pA, qB) = \tau_{D,\Gamma}(pA \to qB) + \tau_{D,\Gamma}(qB \to pA) - 1 \geq \tau_{D,\Gamma}(pA) + \tau_{D,\Gamma}(qB) - 1$.

Proof. Let A and B contain the same atomic formulas $p_1, p_2, \dots, p_m$. Then by Definition 4.1 and Theorem 3.12, we have that

$$\begin{aligned} \xi_{D,\Gamma}(pA, qB) &= \tau_{D,\Gamma}((pA \to qB) \wedge (qB \to pA)) \\ &= \tau_{D,\Gamma}(pA \to qB) + \tau_{D,\Gamma}(qB \to pA) - \tau_{D,\Gamma}((pA \to qB) \vee (qB \to pA)) \\ &= \tau_{D,\Gamma}(pA \to qB) + \tau_{D,\Gamma}(qB \to pA) - 1. \end{aligned}$$

Since $\models pA \to (qB \to pA)$ and $\models qB \to (pA \to qB)$, it follows from Theorem 3.6(iv) that $\tau_{D,\Gamma}(pA) \leq \tau_{D,\Gamma}(qB \to pA)$ and $\tau_{D,\Gamma}(qB) \leq \tau_{D,\Gamma}(pA \to qB)$. Thus $\xi_{D,\Gamma}(pA, qB) = \tau_{D,\Gamma}(pA \to qB) + \tau_{D,\Gamma}(qB \to pA) - 1 \geq \tau_{D,\Gamma}(pA) + \tau_{D,\Gamma}(qB) - 1$. □

Theorem 4.12. *Let* $\Gamma \subseteq F(S)$, $A, B, C \in F(S)$, S_Γ *be finite, and* $D_0, D_{\frac{1}{n-1}}, \dots, D_{\frac{n-2}{n-1}}, D_1$ ($n \geq 2$) *be an n-valued randomized number sequence in* $(0,1)$. *Then* $\xi_{D,\Gamma}(pA, rC) \geq \xi_{D,\Gamma}(pA, qB) + \xi_{D,\Gamma}(qB, rC) - 1$.

Proof. Let A, B and C contain the same atomic formulas $p_1, p_2, \dots, p_m$. Then by Theorem 4.11, we get that $\xi_{D,\Gamma}(pA, qB) + \xi_{D,\Gamma}(qB, rC) - 1 = [\tau_{D,\Gamma}(pA \to qB) + \tau_{D,\Gamma}(qB \to pA) - 1] + [\tau_{D,\Gamma}(qB \to rC) + \tau_{D,\Gamma}(rC \to qB) - 1] - 1 = [\tau_{D,\Gamma}(pA \to qB) + \tau_{D,\Gamma}(qB \to rC) - 1] + [\tau_{D,\Gamma}(rC \to qB) + \tau_{D,\Gamma}(qB \to pA) - 1] - 1$. It follows from Theorem 3.18 that $\xi_{D,\Gamma}(pA, qB) + \xi_{D,\Gamma}(qB, rC) - 1 \leq \tau_{D,\Gamma}(pA \to rC) + \tau_{D,\Gamma}(rC \to pA) - 1 = \xi_{D,\Gamma}(pA, rC)$. □

Theorem 4.13. *Let* $\Gamma \subseteq F(S)$, $A, B, C \in F(S)$, S_Γ *be finite, and* $D_0, D_{\frac{1}{n-1}}, \dots, D_{\frac{n-2}{n-1}}, D_1$ ($n \geq 2$) *be an n-valued randomized number sequence in* $(0,1)$. *Then*

(i) $\xi_{D,\Gamma}(pA \to rC, qB \to zD) \geq \xi_{D,\Gamma}(pA, qB) + \xi_{D,\Gamma}(rC, zD) - 1$.

(ii) $\xi_{D,\Gamma}(pA \vee rC, qB \vee zD) \geq \xi_{D,\Gamma}(pA, qB) + \xi_{D,\Gamma}(rC, zD) - 1$.

(iii) $\xi_{D,\Gamma}(pA \wedge rC, qB \wedge zD) \geq \xi_{D,\Gamma}(pA, qB) + \xi_{D,\Gamma}(rC, zD) - 1$.

Proof. Let A, B, C and D contain the same atomic formulas $p_1, p_2, \ldots, p_m$.

(i) By Theorem 4.12, we get that $\xi_{D,\Gamma}(pA \to rC, qB \to zD) \geq \xi_{D,\Gamma}(pA \to rC, qB \to rC) + \xi_{D,\Gamma}(qB \to rC, qB \to zD) - 1$. It follows from Theorem 4.9(iii) that $\xi_{D,\Gamma}(pA \to rC, qB \to rC) \geq \xi_{D,\Gamma}(pA, qB)$. By Corollary 4.10(iii), we have that $\xi_{D,\Gamma}(qB \to rC, qB \to zD) \geq \xi_{D,\Gamma}(rC, zD)$. Thus $\xi_{D,\Gamma}(pA \to rC, qB \to zD) \geq \xi_{D,\Gamma}(pA, qB) + \xi_{D,\Gamma}(rC, zD) - 1$.

Carrying out a proof similar to that of (i), we can have (ii) and (iii). □

Definition 4.14. *Let $\Gamma \subseteq F(S)$, $A, B \in F(S)$, S_Γ be finite, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1$ $(n \geq 2)$ be an n-valued randomized number sequence in $(0,1)$. Stipulate that $\rho_{D,\Gamma} : F(S) \times F(S) \to [0,1]$. Define*

$$\rho_{D,\Gamma}(pA, qB) = 1 - \xi_{D,\Gamma}(pA, qB).$$

Then $\rho_{D,\Gamma}$ is called the $\Gamma - t$ absolute randomized pseudo-distance on $F(S)$, and $(F(S), \rho_{D,\Gamma})$ is called the $\Gamma - t$ absolute randomized logic metric space.

Remark 4.15. *Let A, B and C contain the same atomic formulas $p_1, p_2, \ldots, p_m$. Then*

(i) By Definition 4.14 and Theorem 4.3(i), we have that $\rho_{D,\Gamma}(pA, pA) = 1 - \xi_{D,\Gamma}(pA, pA) = 0$.

(ii) It follows from Definition 4.14 and Theorem 4.3(ii) that $\rho_{D,\Gamma}(pA, qB) = \rho_{D,\Gamma}(qB, pA)$.

(iii) By Definition 4.14 and Theorem 4.12, we get that $\rho_{D,\Gamma}(pA, rC) = 1 - \xi_{D,\Gamma}(pA, rC) \leq 1 - [\xi_{D,\Gamma}(pA, qB) + \xi_{D,\Gamma}(qB, rC) - 1] = 1 - \xi_{D,\Gamma}(pA, qB) + 1 - \xi_{D,\Gamma}(qB, rC) = \rho_{D,\Gamma}(pA, qB) + \rho_{D,\Gamma}(qB, rC)$.

Thus $\rho_{D,\Gamma}(pA, qB)$ is called the $\Gamma - t$ absolute randomized pseudo-distance of propositional formulas A and B, i.e., $\Gamma - t$ absolute randomized truth degree can form three properties satisfying $\Gamma - t$ absolute randomized pseudo-distance. Then it can form $\Gamma - t$ absolute randomized logic metric space. So Definition 4.14 is reasonable.

Theorem 4.16. *Let $\Gamma \subseteq F(S)$, $A, B \in F(S)$, S_Γ be finite, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1$ $(n \geq 2)$ be an n-valued randomized number sequence in $(0,1)$. Then*

(i) $\rho_{D,\Gamma}(pA \vee qB, pA) = 1 - \tau_{D,\Gamma}(qB \to pA)$.

(ii) $\rho_{D,\Gamma}(pA \wedge qB, pA) = 1 - \tau_{D,\Gamma}(pA \to qB)$.

(iii) $\rho_{D,\Gamma}(pA \to qB, qB) = 1 - \tau_{D,\Gamma}(pA \vee qB)$.

(iv) $\rho_{D,\Gamma}(pA \to qB, pA \wedge qB) = 1 - \tau_{D,\Gamma}(pA)$.

Proof. It is easy to prove Theorem 4.16 by Definition 4.14 and Theorem 4.3(iii) (iv) (v) (vi). □

Theorem 4.17. *Let $\Gamma \subseteq F(S)$, $A, B \in F(S)$, S_Γ be finite, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1$ $(n \geq 2)$ be an n-valued randomized number sequence in $(0,1)$. If $\Gamma \models pA$, then*

(i) $\rho_{D,\Gamma}(pA, qB) = 1 - \tau_{D,\Gamma}(qB)$.

(ii) $\rho_{D,\Gamma}(pA \vee qB, pA) = 0$.

(iii) $\rho_{D,\Gamma}(pA \wedge qB, pA) = 1 - \tau_{D,\Gamma}(qB)$.

Proof. Theorem 4.17 can be easily proved by Definition 4.14 and Theorem 4.5. □

Corollary 4.18. *Let $\Gamma \subseteq F(S)$, $A, B \in F(S)$, S_Γ be finite, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1$ $(n \geq 2)$ be an n-valued randomized number sequence in $(0,1)$. If $\Gamma \models qB$, then*

(i) $\rho_{D,\Gamma}(pA, qB) = 1 - \tau_{D,\Gamma}(pA)$.

(ii) $\rho_{D,\Gamma}(pA \vee qB, pA) = 1 - \tau_{D,\Gamma}(pA)$.

(iii) $\rho_{D,\Gamma}(pA \wedge qB, pA) = 0$.

Theorem 4.19. *Let $\Gamma \subseteq F(S)$, $A, B, C \in F(S)$, S_Γ be finite, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1$ $(n \geq 2)$ be an n-valued randomized number sequence in $(0,1)$. Then*

(i) $\rho_{D,\Gamma}(pA, qB) \geq \rho_{D,Gamma}(pA \vee rC, qB \vee rC)$.

(ii) $\rho_{D,\Gamma}(pA, qB) \geq \rho_{D,\Gamma}(pA \wedge rC, qB \wedge rC)$.

(iii) $\rho_{D,\Gamma}(pA, qB) \geq \rho_{D,\Gamma}(pA \to rC, qB \to rC)$.

Proof. It is easy to prove Theorem 4.19 by Definition 4.14 and Theorem 4.9. □

Corollary 4.20. *Let* $\Gamma \subseteq F(S)$*,* $A, B, C \in F(S)$*,* S_Γ *be finite, and* $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1$ $(n \geq 2)$ *be an n-valued randomized number sequence in* $(0, 1)$*. Then*

(i) $\rho_{D,\Gamma}(qB, rC) \geq \rho_{D,\Gamma}(pA \vee qB, pA \vee rC)$*.*

(ii) $\rho_{D,\Gamma}(qB, rC) \geq \rho_{D,\Gamma}(pA \wedge qB, pA \wedge rC)$*.*

(iii) $\rho_{D,\Gamma}(qB, rC) \geq \rho_{D,\Gamma}(pA \to qB, pA \to rC)$*.*

Theorem 4.21. *Let* $\Gamma \subseteq F(S)$*,* $A, B, \in F(S)$*,* S_Γ *be finite, and* $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1$ $(n \geq 2)$ *be an n-valued randomized number sequence in* $(0, 1)$*. Then* $\rho_{D,\Gamma}(pA, qB) = 2 - \tau_{D,\Gamma}(pA \to qB) - \tau_{D,\Gamma}(qB \to pA) \leq 2 - \tau_{D,\Gamma}(pA) - \tau_{D,\Gamma}(qB)$*.*

Proof. Theorem 4.21 can be easily proved by Definition 4.14 and Theorem 4.11. □

Corollary 4.22. *If the* $\Gamma - t$ *absolute randomized truth degree of each formula is* 1*, then the* $\Gamma - t$ *absolute randomized pseudo-distance between them is* 0*.*

Theorem 4.23. *Let* $\Gamma \subseteq F(S)$*,* $A, B, C, D \in F(S)$*,* S_Γ *be finite, and* $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}, D_1$ $(n \geq 2)$ *be an n-valued randomized number sequence in* $(0, 1)$*. Then*

(i) $\rho_{D,\Gamma}(pA \to rC, qB \to zD) \geq \rho_{D,\Gamma}(pA, qB) + \rho_{D,\Gamma}(rC, zD)$*.*

(ii) $\rho_{D,\Gamma}(pA \vee rC, qB \vee zD) \geq \rho_{D,\Gamma}(pA, qB) + \rho_{D,\Gamma}(rC, zD)$*.*

(iii) $\rho_{D,\Gamma}(pA \wedge rC, qB \wedge zD) \geq \rho_{D,\Gamma}(pA, qB) + \rho_{D,\Gamma}(rC, zD)$*.*

Proof. It is easy to prove Theorem 4.23 by Definition 4.14 and Theorem 4.13. □

In logic metric spaces, it is a very meaning topic to discuss the continuity of related operators. The following theorem shows the continuity of such operators.

Theorem 4.24. *Let* $(F(S), \rho_{D,\Gamma})$ *be the* $\Gamma - t$ *absolute randomized logic metric space, and* $\rho_{D,\Gamma}$ *be the* $\Gamma - t$ *absolute randomized pseudo-distance on* $F(S)$*. Then*

(i) The binary operator $\to$ *is continuous with respect to the* $\Gamma - t$ *absolute randomized pseudo-distance* $\rho_{D,\Gamma}$ *in the* $\Gamma - t$ *absolute randomized logic metric space* $(F(S), \rho_{D,\Gamma})$*.*

(ii) The binary operator $\vee$ *is continuous with respect to the* $\Gamma - t$ *absolute randomized pseudo-distance* $\rho_{D,\Gamma}$ *in the* $\Gamma - t$ *absolute randomized logic metric space* $(F(S), \rho_{D,\Gamma})$*.*

(iii) The binary operator $\wedge$ *is continuous with respect to the* $\Gamma - t$ *absolute randomized pseudo-distance* $\rho_{D,\Gamma}$ *in the* $\Gamma - t$ *absolute randomized logic metric space* $(F(S), \rho_{D,\Gamma})$*.*

(iv) The unitary operator Δ *is not continuous with respect to the* $\Gamma - t$ *absolute randomized pseudo-distance* $\rho_{D,\Gamma}$ *in the* $\Gamma - t$ *absolute randomized logic metric space* $(F(S), \rho_{D,\Gamma})$*.*

(v) The unitary operator $\sim$ *is not continuous with respect to the* $\Gamma - t$ *absolute randomized pseudo-distance* $\rho_{D,\Gamma}$ *in the* $\Gamma - t$ *absolute randomized logic metric space* $(F(S), \rho_{D,\Gamma})$*.*

Proof. We illustrate this with the case of (i), (iv) and (v), the other cases being similar. Let A, B, C, D, A_n and B_n contain the same atomic formulas $p_1, p_2, \ldots, p_m$.

(i): By Theorem 4.23(i), we get that $\rho_{D,\Gamma}(pA \to rC, qB \to zD) \leq \rho_{D,\Gamma}(pA, qB) + \rho_{D,\Gamma}(rC, zD)$. If $\lim_{n\to\infty} \rho_{D,\Gamma}(mA_n, pA) = 0$ and $\lim_{n\to\infty} \rho_{D,\Gamma}(lB_n, qB) = 0$, then $\lim_{n\to\infty} \rho_{D,\Gamma}(mA_n \to lB_n, pA \to qB) \leq \lim_{n\to\infty} \rho_{D,\Gamma}(mA_n, pA) + \lim_{n\to\infty} \rho_{D,\Gamma}(lB_n, qB) = 0$.

Therefore, the binary operator $\to$ is continuous with respect to the $\Gamma - t$ absolute randomized pseudo-distance $\rho_{D,\Gamma}$ in the $\Gamma - t$ absolute randomized logic metric space $(F(S), \rho_{D,\Gamma})$.

(iv): For any $a, b \in \Pi_{\Delta,\sim}$, when $pa \neq 1$, we have $pa \to qb \leq \Delta pa \to \Delta qb$, when $qb \neq 1$, we have $qb \to pa \leq \Delta qb \to \Delta pa$.

So when $pa \neq 1$ and $qb \neq 1$, we have $\models (pA \to qB) \to (\Delta pA \to \Delta qB)$ and $\models (qB \to pA) \to (\Delta qB \to \Delta pA)$. Then by Theorem 3.6(iv), we get that $\tau_{D,\Gamma}(pA \to qB) \leq \tau_{D,\Gamma}(\Delta pA \to \Delta qB)$ and $\tau_{D,\Gamma}(qB \to pA) \leq \tau_{D,\Gamma}(\Delta qB \to$

ΔpA). It follows from Theorem 4.21 that $\rho_{D,\Gamma}(\Delta pA, \Delta qB) = 2 - \tau_{D,\Gamma}(\Delta pA \to \Delta qB) - \tau_{D,\Gamma}(\Delta qB \to \Delta pA) \leq 2 - \tau_{D,\Gamma}(pA \to qB) - \tau_{D,\Gamma}(qB \to pA) = \rho_{D,\Gamma}(pA, qB)$. If $\lim\limits_{n\to\infty} \rho_{D,\Gamma}(mA_n, pA) = 0$, then $\lim\limits_{n\to\infty} \rho_{D,\Gamma}(\Delta mA_n, \Delta pA) \leq \lim\limits_{n\to\infty} \rho_{D,\Gamma}(mA_n, pA) = 0$.

Thus the unitary operator Δ is continuous with respect to the $\Gamma - t$ absolute randomized pseudo-distance $\rho_{D,\Gamma}$ in the $\Gamma - t$ absolute randomized logic metric space $(F(S), \rho_{D,\Gamma})$ only when $pa \neq 1$ and $qb \neq 1$.

Therefore, the unitary operator Δ is not continuous with respect to the $\Gamma - t$ absolute randomized pseudo-distance $\rho_{D,\Gamma}$ in the $\Gamma - t$ absolute randomized logic metric space $(F(S), \rho_{D,\Gamma})$.

(v): As $(\neg pA \to \neg qB) \approx (qB \to pA)$ and $(\neg qB \to \neg pA) \approx (pA \to qB)$, we have $(\neg pA \to \neg qB) \wedge (\neg qB \to \neg pA) \approx (qB \to pA) \wedge (pA \to qB)$. It follows from Theorem 3.6(iii) that $\tau_{D,\Gamma}((\neg pA \to \neg qB) \wedge (\neg qB \to \neg pA)) = \tau_{D,\Gamma}((qB \to pA) \wedge (pA \to qB))$. Thus $\rho_{D,\Gamma}(\neg pA, \neg qB) = 1 - \xi_{D,\Gamma}(\neg pA, \neg qB) = 1 - \tau_{D,\Gamma}((\neg pA \to \neg qB) \wedge (\neg qB \to \neg pA)) = 1 - \tau_{D,\Gamma}((qB \to pA) \wedge (pA \to qB)) = 1 - \xi_{D,\Gamma}(qB, pA) = \rho_{D,\Gamma}(qB, pA) = \rho_{D,\Gamma}(pA, qB)$. If $\lim\limits_{n\to\infty} \rho_{D,\Gamma}(mA_n, pA) = 0$, then $\lim\limits_{n\to\infty} \rho_{D,\Gamma}(\neg mA_n, \neg pA) = \lim\limits_{n\to\infty} \rho_{D,\Gamma}(mA_n, pA) = 0$.

So the unitary operator $\neg$ is continuous with respect to the $\Gamma - t$ absolute randomized pseudo-distance $\rho_{D,\Gamma}$ in the $\Gamma - t$ absolute randomized logic metric space $(F(S), \rho_{D,\Gamma})$.

Since $\Delta = \neg \sim$, it follows from Theorem 4.24(iv) that the unitary operator $\sim$ is not continuous with respect to the $\Gamma - t$ absolute randomized pseudo-distance $\rho_{D,\Gamma}$ in the $\Gamma - t$ absolute randomized logic metric space $(F(S), \rho_{D,\Gamma})$. □

Remark 4.25. *The above 5 connectives are the most basic connectives in $\Pi_{\Delta,\sim}$, and other connectives can be transformed through these 5 connectives and their continuity can be got accordingly.*

5. *t* absolute randomized divergence degree and *t* absolute randomized consistency degree of arbitrary theory Γ relative to the fixed theory Γ_0

In [25], the author gave the definition of consistency degree in the case of the finite theory of the Lukasiewicz fuzzy propositional logic system. In [27], the author gave the definition of consistency degree in the cases of the the infinite theory of the Lukasiewicz, Gödel, Product, and R_0 fuzzy propositional system. And in [28], the author supplemented the above two definitions of consistency degrees. Along this way of thinking, in this section, we give the concepts of t absolute randomized divergence degree and t absolute randomized consistency degree of arbitrary theory Γ relative to the fixed theory Γ_0. Using the specific property of contradiction, we define non-absolute randomized consistent of arbitrary theory Γ relative to the fixed theory Γ_0, and obtain the relationship between them.

Definition 5.1. *Let $\Gamma_0 \subseteq F(S)$, $A, B \in F(S)$, S_{Γ_0} be finite, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}$, D_1 $(n \geq 2)$ be an n-valued randomized number sequence in $(0, 1)$. For $\Gamma \subseteq F(S)$, define*

$$div_{D,\Gamma_0}(\Gamma) = \sup\{\rho_{D,\Gamma_0}(pA, qB) | pA, qB \in D_{\Gamma_0}(\Gamma)\}.$$

Then $div_{D,\Gamma_0}(\Gamma)$ is called the t absolute randomized divergence degree of arbitrary theory Γ relative to the fixed theory Γ_0. When $div_{D,\Gamma_0}(\Gamma) = 1$, called the theory Γ relative to the fixed theory Γ_0 is absolute randomized fully divergent.

Remark 5.2. *(i) $0 \leq div_{D,\Gamma_0}(\Gamma) \leq 1$. (ii) $div_{D,\Gamma_0}(\Gamma) = \sup\{\rho_{D,\Gamma_0}(pA, qB) | pA, qB \in D(\Gamma \cup \Gamma_0)$. (iii) If Γ_0 consists entirely of theorem or $\Gamma_0 = \varnothing$, then t absolute randomized divergence degree of theory Γ relative to the fixed theory Γ_0 is t absolute randomized divergence degree of theory Γ [9].*

Definition 5.3. *Let $\Gamma_0 \subseteq F(S)$, $A, B \in F(S)$, S_{Γ_0} be finite, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}$, D_1 $(n \geq 2)$ be an n-valued randomized number sequence in $(0, 1)$. For $\Gamma \subseteq F(S)$, define*

$$i_{D,\Gamma_0}(\Gamma) = 1 - \min\{\lceil 1 - \rho_{D,\Gamma_0}(pA, qB) \rceil | pA, qB \in D_{\Gamma_0}(\Gamma)\}.$$

Then $i_{D,\Gamma_0}(\Gamma)$ is called the t absolute randomized polar index of theory Γ relative to the fixed theory Γ_0.

Remark 5.4. *From [1], we know* $\lceil x \rceil = \{^{1,\ 0<x\leq 1}_{0,\ x=0}$. *Then the value range of* $i_{D,\Gamma_0}(\Gamma)$ *is as follows:*

(i) $\lceil 1-\rho_{D,\Gamma_0}(pA,qB)\rceil = \{^{1,\ 0\leq\rho_{D,\Gamma_0}(pA,qB)<1}_{0,\ \rho_{D,\Gamma_0}(pA,qB)=1}$.

(ii) $i_{D,\Gamma_0}(\Gamma)$ *can only take 0 and 1.*

In [9], the author proposed a definition of non-randomized consistent of theory Γ relative to the fixed theory Γ_0. Based on his idea, we will give a definition of non-absolute randomized consistent of theory Γ relative to the fixed theory Γ_0.

Definition 5.5. *Let* $\Gamma_0,\Gamma\subseteq F(S)$, $A\in F(S)$, *and* $D_0, D_{\frac{1}{n-1}},\ldots,D_{\frac{n-2}{n-1}}, D_1$ $(n\geq 2)$ *be an n-valued randomized number sequence in* $(0,1)$. *If* $\overline{0}\in D(\Gamma_0\cup\Gamma)$ *and* $\Delta_n(\Gamma,tA)\neq\emptyset$, *then we say that the theory* Γ *relative to the fixed theory* Γ_0 *is not absolute randomized consistent; otherwise, we say that the theory* Γ *relative to the fixed theory* Γ_0 *is absolute randomized consistent.*

In particular, if $\Gamma\subseteq D(\Gamma_0)$, then we say that the theory Γ relative to the fixed theory Γ_0 is completely absolute randomized consistent.

Theorem 5.6. *Let* $\Gamma_0,\Gamma\subseteq F(S)$, $A,B\in F(S)$, S_{Γ_0} *be finite, and* $D_0, D_{\frac{1}{n-1}},\ldots,D_{\frac{n-2}{n-1}}$, D_1 $(n\geq 2)$ *be an n-valued randomized number sequence in* $(0,1)$. *Then*

(i) Theory Γ *relative to the fixed theory* Γ_0 *is completely absolute randomized consistent if and only if* $div_{D,\Gamma_0}(\Gamma)=0$.

(ii) Theory Γ *relative to the fixed theory* Γ_0 *is not absolute randomized consistent if and only if there exists* $pA,qB\in D(\Gamma_0\cup\Gamma)$ *such that* $\rho_{D,\Gamma_0}(pA,qB)=1$.

(iii) Theory Γ *relative to the fixed theory* Γ_0 *is absolute randomized consistent if and only if* $\rho_{D,\Gamma_0}(pA,qB)<1$, $pA,qB\in D(\Gamma_0\sqcup\Gamma)$.

Proof. Let A and B contain the same atomic formulas $p_1,p_2,\ldots,p_m$.

(i): Assume that $div_{D,\Gamma_0}(\Gamma)=0$. Then from Definition 5.1 we get that $pA\in\Gamma$ and $\rho_{D,\Gamma_0}(pA,\Gamma)=0$. It is obvious that $pA\approx((pA\to\Gamma)\wedge(\Gamma\to pA))$. By Theorem 3.6(iii), we get that $\tau_{D,\Gamma_0}(pA)=\tau_{D,\Gamma_0}((pA\to\Gamma)\wedge(\Gamma\to pA))=1-\rho_{D,\Gamma_0}(pA,\Gamma)=1$. Then $pA\in D(\Gamma_0)$. Thus $\Gamma\subseteq D(\Gamma_0)$, i.e., theory Γ relative to the fixed theory Γ_0 is completely absolute randomized consistent.

Conversely, if theory Γ relative to the fixed theory Γ_0 is completely absolute randomized consistent, then $\Gamma\subseteq D(\Gamma_0)$. Thus $D(\Gamma_0\cup\Gamma)=D(\Gamma_0)$. Thus $\forall pA,qB\in D(\Gamma_0\cup\Gamma)$, we have $pA,qB\in D(\Gamma_0)$. Then $\rho_{D,\Gamma_0}(pA,qB)=1-\tau_{D,\Gamma_0}((pA\to qB)\wedge(qB\to pA))=0$. By Remark 5.2(ii), we have that $div_{D,\Gamma_0}(\Gamma)=0$.

(ii): If there exists $pA,qB\in D(\Gamma_0\cup\Gamma)$ such that $\rho_{D,\Gamma_0}(pA,qB)=1$, then $\tau_{D,\Gamma_0}((pA\to qB)\wedge(qB\to pA))=1-\rho_{D,\Gamma_0}(pA,qB)=0$. By Definition 3.2, we get that $\Delta_n(\Gamma,pA)\neq\emptyset$.

The following is divided into two aspects to prove $\overline{0}\in D(\Gamma_0\cup\Gamma)$.

On the one hand, since $pA,qB\in D(\Gamma_0\cup\Gamma)$, $\vdash pA\to(qB\to pA)$ and $\vdash qB\to(pA\to qB)$, it follows from MP rule that $\Gamma_0\cup\Gamma\vdash(pA\to qB)\wedge(qB\to pA)$.

On the other hand, by Theorem 3.6(v), we get that $\tau_{D,\Gamma_0}(\sim((pA\to qB)\wedge(qB\to pA)))=1-\tau_{D,\Gamma_0}((pA\to qB)\wedge(qB\to pA))=\rho_{D,\Gamma_0}(pA,qB)=1$. Thus $\Gamma_0\cup\Gamma\vdash\sim(pA\to qB)\wedge(qB\to pA)$.

Thus $\Gamma_0\cup\Gamma\vdash((pA\to qB)\wedge(qB\to pA))\wedge(\sim((pA\to qB)\wedge(qB\to pA)))$. As $((pA\to qB)\wedge(qB\to pA))\wedge(\sim((pA\to qB)\wedge(qB\to pA)))\approx 0$, we have $\Gamma_0\cup\Gamma\vdash\overline{0}$, i.e., $\overline{0}\in D(\Gamma_0\cup\Gamma)$.

So to sum up if there exists $pA,qB\in D(\Gamma_0\cup\Gamma)$ such that $\rho_{D,\Gamma_0}(pA,qB)=1$, then theory Γ relative to the fixed theory Γ_0 is not absolute randomized consistent.

Conversely, if theory Γ relative to the fixed theory Γ_0 is not absolute randomized consistent, then $\overline{0}\in D(\Gamma_0\cup\Gamma)$ and $\Delta_n(\Gamma,pA)\neq\emptyset$. It follows from Definition 3.2 and $\Delta_n(\Gamma,pA)\neq\emptyset$ that $\tau_{D,\Gamma_0}(\overline{0})=0$. It is obvious that $(\overline{0}\to\Gamma)\wedge(\Gamma\to\overline{0})\approx\overline{0}$. By Theorem 3.6(iii), we get that $\tau_{D,\Gamma_0}((\overline{0}\to\Gamma)\wedge(\Gamma\to\overline{0}))=\tau_{D,\Gamma_0}(\overline{0})=0$. Therefore, there exists $\overline{0},\Gamma\in D(\Gamma_0\cup\Gamma)$ such that $\rho_{D,\Gamma_0}(\overline{0},\Gamma)=1-\tau_{D,\Gamma_0}((\overline{0}\to\Gamma)\wedge(\Gamma\to\overline{0}))=1$.

(iii): According to the conclusion in (ii), it can be proved that (iii) is true. □

Theorem 5.7. *Let $\Gamma_0, \Gamma \subseteq F(S)$, $A, B \in F(S)$, S_{Γ_0} be finite, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}$, D_1 ($n \geq 2$) be an n-valued randomized number sequence in $(0,1)$. Then*

(i) Theory Γ relative to the fixed theory Γ_0 is not absolute randomized consistent if and only if there exists $pA, qB \in D(\Gamma_0 \cup \Gamma)$ such that $i_{D,\Gamma_0}(\Gamma) = 1$.

(ii) Theory Γ relative to the fixed theory Γ_0 is absolute randomized consistent if and only if $i_{D,\Gamma_0}(\Gamma) = 0$, $\forall pA, qB \in D(\Gamma_0 \cup \Gamma)$.

Proof. Let A and B contain the same atomic formulas $p_1, p_2, \ldots, p_m$.

(i): By Theorem 5.6(ii), Remark 5.4(i)and Definition 5.3, we get that theory Γ relative to the fixed theory Γ_0 is not absolute randomized consistent if and only if there exists $pA, qB \in D(\Gamma_0 \cup \Gamma)$ such that $\rho_{D,\Gamma_0}(pA, qB) = 1$, if and only if there exists $pA, qB \in D(\Gamma_0 \cup \Gamma)$ such that $\lceil 1 - \rho_{D,\Gamma_0}(pA, qB) \rceil = 0$ and if and only if there exists $pA, qB \in D(\Gamma_0 \cup \Gamma)$ such that $i_{D,\Gamma_0}(\Gamma) = 1$.

(ii): By Theorem 5.6(iii), Remark 5.4(i)and Definition 5.3, we get that theory Γ relative to the fixed theory Γ_0 is not absolute randomized consistent if and only if $\rho_{D,\Gamma_0}(pA, qB) < 1$, if and only if $\lceil 1 - \rho_{D,\Gamma_0}(pA, qB) \rceil = 1$ and if and only if $i_{D,\Gamma_0}(\Gamma) = 0$. □

Definition 5.8. *Let $\Gamma_0 \subseteq F(S)$, $A, B \in F(S)$, S_{Γ_0} be finite, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}$, $D_1 (n \geq 2)$ be an n-valued randomized number sequence in $(0,1)$. For $\Gamma \subseteq F(S)$, define*

$$\eta_{D,\Gamma_0}(\Gamma) = 1 - \frac{1}{2} div_{D,\Gamma_0}(\Gamma)(1 + i_{D,\Gamma_0}(\Gamma)).$$

Then $\eta_{D,\Gamma_0}(\Gamma)$ is called the t absolute randomized consistency degree of arbitrary theory Γ relative to the fixed theory Γ_0.

The following theorem gives the relationship between Definition 5.5 and Definition 5.8.

Theorem 5.9. *Let $\Gamma_0, \Gamma \subseteq F(S)$, $A, B \in F(S)$, S_{Γ_0} be finite, and $D_0, D_{\frac{1}{n-1}}, \ldots, D_{\frac{n-2}{n-1}}$, $D_1 (n \geq 2)$ be an n-valued randomized number sequence in $(0,1)$. Then*

(i) Theory Γ relative to the fixed theory Γ_0 is completely absolute randomized consistent if and only if $\eta_{D,\Gamma_0}(\Gamma) = 1$.

(ii) Theory Γ relative to the fixed theory Γ_0 is absolute randomized consistent if and only if $\frac{1}{2} \leq \eta_{D,\Gamma_0}(\Gamma) \leq 1$, $\forall pA, qB \in D(\Gamma_0 \cup \Gamma)$.

(iii) Theory Γ relative to the fixed theory Γ_0 is not absolute randomized consistent if and only if there exists $pA, qB \in D(\Gamma_0 \cup \Gamma)$ such that $\eta_{D,\Gamma_0}(\Gamma) = 0$.

(iv) Theory Γ relative to the fixed theory Γ_0 is absolute randomized consistent and absolute randomized fullly divergent if and only if $\eta_{D,\Gamma_0}(\Gamma) = \frac{1}{2}$, $\forall pA, qB \in D(\Gamma_0 \cup \Gamma)$.

Proof. Let A and B contain the same atomic formulas $p_1, p_2, \ldots, p_m$.

(i): By Theorem 5.6(i) and Definition 5.8, we get that theory Γ relative to the fixed theory Γ_0 is completely absolute randomized consistent if and only if $div_{D,\Gamma_0}(\Gamma) = 0$ and if and only if $\eta_{D,\Gamma_0}(\Gamma) = 1$.

(ii): It follows from Theorem 5.7(ii) that theory Γ relative to the fixed theory Γ_0 is absolute randomized consistent if and only if $i_{D,\Gamma_0}(\Gamma) = 0$. Since $0 \leq div_{D,\Gamma_0}(\Gamma) \leq 1$, By Definition 5.8, we have that theory Γ relative to the fixed theory Γ_0 is absolute randomized consistent if and only if $\frac{1}{2} \leq \eta_{D,\Gamma_0}(\Gamma) \leq 1$.

(iii): By Theorem 5.6(ii) and Definition 5.1, we get that theory Γ relative to the fixed theory Γ_0 is not absolute randomized consistent if and only if there exists $pA, qB \in D(\Gamma_0 \cup \Gamma)$ such that $\rho_{D,\Gamma_0}(pA, qB) = 1$ and if and oniy if there exists $pA, qB \in D(\Gamma_0 \cup \Gamma)$ such that $div_{D,\Gamma_0}(\Gamma) = 1$. It follows from Theorem 5.7(i) and Definition 5.8 that theory Γ relative to the fixed theory Γ_0 is not absolute randomized consistent if and only if there exists $pA, qB \in D(\Gamma_0 \cup \Gamma)$ such that $i_{D,\Gamma_0}(\Gamma) = 1$ and if and oniy if there exists $pA, qB \in D(\Gamma_0 \cup \Gamma)$ such that $\eta_{D,\Gamma_0}(\Gamma) = 0$.

(iv): By Definition 5.1, Theorem 5.7(ii) and Definition 5.8, we have that theory Γ relative to the fixed theory Γ_0 is absolute randomized consistent and absolute randomized fullly divergent if and only if $div_{D,\Gamma_0}(\Gamma) = 1$, if and only if $i_{D,\Gamma_0}(\Gamma) = 0$ and if and only if $\eta_{D,\Gamma_0}(\Gamma) = \frac{1}{2}$. □

6. Conclusions and further work

In this paper, using the randomization method of valuation set, we first put forward the definition of $\Gamma - t$ absolute randomized truth degree of formula relative to local finite theory Γ under the t conjunction in $\Pi_{\Delta,\sim}$ n-valued propositional logic system, and prove some inference rules such as MP, HS, intersection inference and union inference of $\Gamma - t$ absolute randomized truth degree. Then we introduce the concepts of $\Gamma - t$ absolute randomized similarity degree and $\Gamma - t$ absolute randomized pseudo-distance of propositional formulas. We also give the concepts of t absolute randomized divergence degree and t absolute randomized consistency degree of arbitrary theory Γ relative to the fixed theory Γ_0, and using specific property of contradiction, we define non-absolute randomized consistent of arbitrary theory Γ relative to the fixed theory Γ_0.

Based on the work in this paper, the following two problems deserve further research:

(i) What are the properties of approximate reasoning for $\Gamma - t$ absolute randomized truth degree in $\Pi_{\Delta,\sim}$ propositional logic system?

(ii) What are the properties of Γ absolute randomized truth degree in other multivalued propositional logic systems?

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References

[1] E. Adams, A primer of probability logic, Stanford: CSLI Publications. (1998).
[2] M. Baaz, Infinite-valued Gödel logic with 0-1 projections and relativisations, Comput Set Phys Lect Notes Logic. 6 (1996) 23-33.
[3] P. Cintula, E. Klement, R. Mesiar, et al, Fuzzy logics with an additional involutive negation, Fuzzy. Sets. Syst. 161 (2010) 390-411.
[4] M. Cui, Randomized truth degree and approximate reasoning of theory in n-valued propositional logic system Lukasiewicz, Acta Mathematics Applicatae Sinica. 35 (2012) 209-220.
[5] P. Cintula, Weakly implicative (fuzzy) logics I: basic properties, Arch. Math. Logic. 45 (2006) 673-704.
[6] F. Eteva, L. Godo, P. Hajek, et al, Residuated fuzzy logics with an involutive negation, Arch. Math. Logic. 39(2000) 103-124.
[7] T. Flaminio, E. Marchioni, T-norm based logics with an independent an involutive negation, Fuzzy. Sets. Syst. 157 (2006) 3125-3144.
[8] X. Gao, X. Hui, N. Zhu, The theory of k randomized truth degree of axiomatic extension of Goguen propositional logic system, Fuzzy. Syst. Math. 31 (2017) 6-15.
[9] X. Hui, X. Gao, N. Zhu, The theory and properities of $\Gamma - k$ randomized truth degree on axiomatic extension of Goguen propositional logic system, Acta Electronica Sinica. 45 (2017) 2656-2662.
[10] X. Hui, Q. Nan, Q. Xu, Distribution of $p-$ Randomized Divergence Degree of Theory in Propositional Logic System ζ^{*n}, Systems Science and Mathematics. 43 (2023) 2310-2318.
[11] X. Hui, G. Wang, Research on randomization of classical reasoning mode and its application, Science in China: E Series. 37 (2007) 801-812.
[12] X. Hui, G. Wang, Research on randomization of classical reasoning patterns and its application (II), Fuzzy. Syst. Math. 22 (2008) 21-26.
[13] X. Hui, Randomization of three-valued R_0 propositional logic system, Journal of Applied Mathematics. 32 (2009) 19-27.
[14] X. Hui, Quantification of truth-based $SBL_\sim$ axiomatic expansion systems, Science in China: Information Science. 44 (2014) 900-911.
[15] S. Li, Z. Chen, The theory of $\Gamma-$ absolute truth degree of formulas in ternary product logic system π_3, Journal of Yanan University (Natural Science). 42 (2023) 63-66+73.
[16] J. Li, J. Li, Y. Zhou, Absolute truth degree theory of formulas in $n-$valued Lukasiewicz propositional logic system, Lanzhou University of Technology Journal. 34 (2008) 135-139.
[17] S. Li, Z. Wang, The theory of $\Gamma-$ absolute truth degree of formulas in $n-$ valued propositional logic system L_n^*, Journal of Guizhou University (Natural Science). 34 (2017) 4-7.
[18] N. Nan, X. Hui, M. Jin, t truth degree and properties on Goguen n-valed propositional logic system of adding two operators, Fuzzy Syst. Math. 35 (2021) 50-58.
[19] G. Wang, Quantitative logic, Journal of Engineering Mathematics. 23 (2006) 159-168.
[20] G. Wang, X. Hui, Extension of basic theorems of probabilistic logic, Acta Electronica Sinica. 37 (2007) 1333-1340.
[21] G. Wang, Y. Leung, Integrated semantics and logic metric spaces, Fuzzy. Sets. Syst. 136 (2003) 71-91.
[22] G. Wang, B. Li, The truth degree theory and limit theorem of formulas in Lukasiewicz n-valued propositional logic, Science in China: E Series. 35 (2005) 561-569.
[23] G. Wang, W. Li, Logical metric spaces, Acta Mathematica Sinica. 44 (2001) 159-168.
[24] Y. Wang, On truth degree theory in axiomatic extension system of Gödel $n-$ valued propositional logic, Yanan: Yanan University. (2018).

[25] G. Wang, W. Zhang, Consistency degrees of finite theories in Lukasiewicz propositional fuzzy logic, Fuzzy. Sets. Syst. 149 (2005) 275-284.

[26] Q. Xu, X. Hui, Q. Nan, Conditional randomized truth degree of formulas in $n-$ valued R_0 propositional logic system, Journal of Hubei University (Natural Science). 46 (2024) 133-140.

[27] X. Zhou, G. Wang, Consistency degrees of theories in some systems of propositional fuzzy logic, Fuzzy. Sets. Syst. 152 (2005) 321-331.

[28] H. Zhou, G. Wang, A new theory consistency index based on deduction theorems in several logic system, Fuzzy. Sets. Syst. 157 (2006) 327-443.