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Characterization of some non-additive mixed biskew Jordan type derivations on $*$-rings

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Abstract. Let $\mathcal{F}$ be a 2-torsion free $*$-ring having a unital element and a non-trivial symmetric idempotent. In the present paper, we demonstrate that, under certain mild conditions, if a map $\Psi : \mathcal{F} \to \mathcal{F}$ (not necessarily additive) satisfies

$$\Psi(\mathcal{U}\Diamond\mathcal{K} \circ \mathcal{L}) = \Psi(\mathcal{U})\Diamond\mathcal{K} \circ \mathcal{L} + \mathcal{U}\Diamond\Psi(\mathcal{K}) \circ \mathcal{L} + \mathcal{U}\Diamond\mathcal{K} \circ \Psi(\mathcal{L})$$

for all $\mathcal{U}, \mathcal{K}, \mathcal{L} \in \mathcal{F}$, then Ψ is an additive $*$-derivation. Particularly, we apply our main result to certain special classes of $*$-algebras such as prime $*$-algebra, von Neumann algebras with no central summands of type I_1 and standard operator algebras.

1. Introduction

Over the entire article, $\mathcal{F}$ denotes an associative ring with centre $\mathcal{Z}(\mathcal{F})$. A map $* : \mathcal{F} \to \mathcal{F}$ is called an involution on $\mathcal{F}$ if it satisfies the following properties: (i) $(\mathcal{U}\mathcal{K})^* = \mathcal{K}^*\mathcal{U}^*$; (ii) $(\mathcal{U} + \mathcal{K})^* = \mathcal{U}^* + \mathcal{K}^*$ and $(\mathcal{U}^*)^* = \mathcal{U}$ for all $\mathcal{U}, \mathcal{K} \in \mathcal{F}$. $\mathcal{F}$ is called 2-torsion free if $2\mathcal{U} = 0 \Rightarrow \mathcal{U} = 0$, for all $\mathcal{U} \in \mathcal{F}$. If $\mathcal{F}$ admits an involution '$*$', then $\mathcal{F}$ is called a $*$-ring. Suppose that $\mathcal{F}$ is a $*$-ring, then an element $\mathcal{E} \in \mathcal{F}$ with $\mathcal{E}^2 = \mathcal{E}^* = \mathcal{E}$ is termed as a symmetric idempotent. Furthermore, if $\mathcal{E} \notin \{0, I\}$, then $\mathcal{E}$ is known as a non-trivial symmetric idempotent. An additive map $\Psi : \mathcal{F} \to \mathcal{F}$ is termed as an additive derivation if $\Psi(\mathcal{U}\mathcal{K}) = \Psi(\mathcal{U})\mathcal{K} + \mathcal{U}\Psi(\mathcal{K})$ for any $\mathcal{U}, \mathcal{K} \in \mathcal{F}$. Furthermore, if $\mathcal{F}$ has an involution '$*$' and the additive derivation Ψ also satisfies $\Psi(\mathcal{U}^*) = \Psi(\mathcal{U})^*$ for every $\mathcal{U} \in \mathcal{F}$, then Ψ is referred to as an additive $*$-derivation. Let $\Psi : \mathcal{F} \to \mathcal{F}$ be a map (not necessarily additive). Then Ψ is termed as a Jordan $*$-derivation if $\Psi(\mathcal{U} \bullet \mathcal{K}) = \Psi(\mathcal{U}) \bullet \mathcal{K} + \mathcal{U} \bullet \Psi(\mathcal{K})$ holds for all $\mathcal{U}, \mathcal{K} \in \mathcal{F}$ and is termed as a bi-skew Jordan derivation if $\Psi(\mathcal{U} \diamond \mathcal{K}) = \Psi(\mathcal{U}) \diamond \mathcal{K} + \mathcal{U} \diamond \Psi(\mathcal{K})$ for any $\mathcal{U}, \mathcal{K} \in \mathcal{F}$ respectively. A map $\Psi : \mathcal{F} \to \mathcal{F}$ (without additivity

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assumption) is known as a skew Lie derivation if $\Psi([\mathcal{U},\mathcal{K}]_*) = [\Psi(\mathcal{U}),\mathcal{K}]_* + [\mathcal{U},\Psi(\mathcal{K})]_*$ for all $\mathcal{U},\mathcal{K} \in \mathcal{F}$ and a bi-skew Lie derivation if $\Psi([\mathcal{U},\mathcal{K}]_\diamond) = [\Psi(\mathcal{U}),\mathcal{K}]_\diamond + [\mathcal{U},\Psi(\mathcal{K})]_\diamond$ for all $\mathcal{U},\mathcal{K} \in \mathcal{F}$. Let $\mathcal{A}$ be an algebra, then a map Ψ (not necessarily additive) on $\mathcal{A}$ is coined as a nonlinear mixed Lie triple derivation, if

$$\Psi([[\mathcal{U},\mathcal{K}]_*,\mathcal{L}]) = [[\Psi(\mathcal{U}),\mathcal{K}]_*,\mathcal{L}] + [[\mathcal{U},\ \Psi(\mathcal{K})]_*,\mathcal{L}] + [[\mathcal{U},\mathcal{K}]_*,\Psi(\mathcal{L})]$$

for all $\mathcal{U},\mathcal{K},\mathcal{L} \in \mathcal{A}$ (see [17]). In the present article, we define a map Ψ (not necessarily additive) on $\mathcal{F}$ and we call it a mixed biskew Jordan triple derivation, i.e., a map $\Psi : \mathcal{F} \to \mathcal{F}$ such that Ψ satisfies $\Psi(\mathcal{U}\diamond\mathcal{K}\circ\mathcal{L}) = \Psi(\mathcal{U})\diamond\mathcal{K}\circ\mathcal{L}+\mathcal{U}\diamond\Psi(\mathcal{K})\circ\mathcal{L}+\mathcal{U}\diamond\mathcal{K}\circ\Psi(\mathcal{L})$ for all $\mathcal{U},\mathcal{K},\mathcal{L} \in \mathcal{F}$, where, $(\mathcal{U}\diamond\mathcal{K}\circ\mathcal{L})$ is defined as $(\mathcal{U}\diamond\mathcal{K})\circ\mathcal{L}$.

In the last few years, there has been a significant interest from algebraists in exploring the constraints under which a map becomes an additive $*$-derivation in the setting of rings and algebras. For example Yu et al. [29], showed that on a factor von Neumann algebra $\mathcal{N}$ over H, where H is a Hilbert space with dim(H) ≥ 2 that if $\Psi : \mathcal{N} \to \mathcal{N}$ is a skew Lie derivation, then Ψ is an additive $*$-derivation. Lately, Kong and Zhang [13], extended this result to prime $*$-rings and demonstrated that if $\mathcal{A}$ is a 2-torsion free unital prime $*$-ring with a non-trivial symmetric idempotent, then a map $\Psi : \mathcal{A} \to \mathcal{A}$ is a skew Lie derivation if and only if Ψ is an additive $*$-derivation. Liang and Zhang[17] proved that on a factor von Neumann algebra $\mathcal{N}$, every nonlinear mixed Lie triple derivable mapping is an additive $*$-derivation. Moreover, Zhou et al. [30] discussed that on prime $*$-algebra, every nonlinear mixed Lie triple derivation is an additive $*$-derivation. Li et al.[15] discussed that any nonlinear $*$-Jordan derivation is an additive $*$-derivation. Very recently, Kong and Li [12], showed that on a 2-torsion free unital prime $*$-ring with a symmetric idempotent, every mixed Lie triple derivation is an additive $*$-derivation. Also, Siddeeque and Shikeh [26], established that every skew Jordan $*$-derivation on a $*$-ring is an additive $*$-derivation.

In recent years, many authors have studied these types of maps in nonassociative structures such as alternative algebras and alternative rings. For instance, Pieren et al. [19] showed that if $\mathcal{A}$ is a unital alternative $*$-algebra with a distinguished idempotent element, then a map $\Psi : \mathcal{A} \to \mathcal{A}$ is a nonlinear mixed $*$-Jordan type derivation on $\mathcal{A}$ if and only if Ψ is an additive $*$-derivation. Ferreira et al. [10] investigated the structural properties of reverse multiplicative $*$-Lie n-maps between two $*$-Jordan algebras. Andrade et al. [1] study the characterization of multiplicative $*$-Lie-type maps and as application, they obtained the result on alternative W^*-algebras. Moreover, Ferreira et al. [8] investigated that on an alternative ring $\mathcal{R}$, under some conditions, if Ψ on $\mathcal{R}$ satisfies $\Psi(\mathcal{U}.\mathcal{V}\mathcal{U}) = \Psi(\mathcal{U}).\mathcal{V}\mathcal{U} + \mathcal{U}.\Psi(\mathcal{V})\mathcal{U} + \mathcal{U}.\mathcal{V}\Psi(\mathcal{U})$ for all $\mathcal{U},\mathcal{V} \in \mathcal{R}$, then Ψ is additive. For other works see [9, 11, 20]

Inspired by the aforementioned findings, in the present article, we discuss the mixed biskew Jordan triple product on $*$-rings and in particular, we investigate the structure of mixed biskew Jordan triple derivations in $*$-rings. Mainly, we show under certain mild assumptions, every mixed biskew Jordan triple derivation on a $*$-ring is an additive $*$-derivation.

Before discussing our main result, it is vital to provide an example of a non-trivial map Ψ on $\mathcal{F}$ that satisfies

$$\Psi(\mathcal{U}\diamond\mathcal{K}\circ\mathcal{L}) = \Psi(\mathcal{U})\diamond\mathcal{K}\circ\mathcal{L}+\mathcal{U}\diamond\Psi(\mathcal{K})\circ\mathcal{L}+\mathcal{U}\diamond\mathcal{K}\circ\Psi(\mathcal{L})$$

for all $\mathcal{U},\mathcal{K},\mathcal{L} \in \mathcal{F}$.

Example 1.1. *Consider $\mathcal{F} = \mathcal{M}_2(\mathbb{R})$, the 2-torsion free unital ring of all 2×2 matrices over real field. Define the maps $*, \Psi : \mathcal{F} \to \mathcal{F}$ as*

$$*\left\{\begin{bmatrix}\eta_1 & \eta_2\\ \eta_3 & \eta_4\end{bmatrix}\right\} = \begin{bmatrix}\eta_1 & \eta_3\\ \eta_2 & \eta_4\end{bmatrix} \text{and } \Psi\left\{\begin{bmatrix}\eta_1 & \eta_2\\ \eta_3 & \eta_4\end{bmatrix}\right\} = \begin{bmatrix}\eta_2+\eta_3 & \eta_4-\eta_1\\ \eta_4-\eta_1 & -\eta_2-\eta_3\end{bmatrix}.$$

Then it can be easily verified that '$*$' is an involution and Ψ satisfies

$$\Psi(\mathcal{U}\diamond\mathcal{K}\circ\mathcal{L}) = \Psi(\mathcal{U})\diamond\mathcal{K}\circ\mathcal{L}+\mathcal{U}\diamond\Psi(\mathcal{K})\circ\mathcal{L}+\mathcal{U}\diamond\mathcal{K}\circ\Psi(\mathcal{L}) \quad (1)$$

for all $\mathcal{U},\mathcal{K},\mathcal{L} \in \mathcal{F}$. In order to prove (1), first of all we prove that Ψ satisfies the following:

(i). Ψ is additive, i.e., $\Psi(\mathcal{U}+\mathcal{K}) = \Psi(\mathcal{U}) + \Psi(\mathcal{K})$ for any $\mathcal{U},\mathcal{K} \in \mathcal{F}$.
Let $\mathcal{U} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\mathcal{K} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$. Then

$$\Psi(\mathcal{U}+\mathcal{K}) = \begin{bmatrix} b+c+y+z & d+w-a-x \\ d+w-a-x & -b-c-y-z \end{bmatrix} = \Psi(\mathcal{U}) + \Psi(\mathcal{K}).$$

Hence Ψ is additive.
(ii). Ψ preserves $*$, i.e., $\Psi(\mathcal{U}^*) = \Psi(\mathcal{U})^*$ for all $\mathcal{U} \in \mathcal{F}$.
Let $\mathcal{U} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$\Psi(\mathcal{U}^*) = \begin{bmatrix} b+c & d-a \\ d-a & -b-c \end{bmatrix} = \Psi(\mathcal{U})^*.$$

Hence Ψ preserves $*$.
(iii) $\Psi(\mathcal{U}\mathcal{K}) = \Psi(\mathcal{U})\mathcal{K} + \mathcal{U}\Psi(\mathcal{K})$ for any $\mathcal{U},\mathcal{K} \in \mathcal{F}$.
Let $\mathcal{U} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\mathcal{K} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$. Then

$$\Psi(\mathcal{U}\mathcal{K}) = \begin{bmatrix} by+bw+cx+dz & cy+dw-cx-dz \\ cy+dw-cx-dz & -by-bw-cx-dz \end{bmatrix} = \Psi(\mathcal{U})\mathcal{K} + \mathcal{U}\Psi(\mathcal{K}).$$

Now using (i), (ii) and (iii) in (1), we have

$$\begin{aligned}
\Psi(\mathcal{U}\lozenge\mathcal{K}\circ\mathcal{L}) &= \Psi((\mathcal{U}\mathcal{K}^* + \mathcal{K}\mathcal{U}^*)\circ\mathcal{L}) \\
&= \Psi((\mathcal{U}\mathcal{K}^* + \mathcal{K}\mathcal{U}^*)\mathcal{L} + \mathcal{L}(\mathcal{U}\mathcal{K}^* + \mathcal{K}\mathcal{U}^*)) \\
&= \Psi(\mathcal{U}\mathcal{K}^*\mathcal{L} + \mathcal{K}\mathcal{U}^*\mathcal{L} + \mathcal{L}\mathcal{U}\mathcal{K}^* + \mathcal{L}\mathcal{K}\mathcal{U}^*) \\
&= \Psi(\mathcal{U}\mathcal{K}^*\mathcal{L}) + \Psi(\mathcal{K}\mathcal{U}^*\mathcal{L}) + \Psi(\mathcal{L}\mathcal{U}\mathcal{K}^*) + \Psi(\mathcal{L}\mathcal{K}\mathcal{U}^*) \\
&= \Psi(\mathcal{U})\Psi(\mathcal{K})^*\Psi(\mathcal{L}) + \Psi(\mathcal{K})\Psi(\mathcal{U})^*\Psi(\mathcal{L}) \\
&+ \Psi(\mathcal{L})\Psi(\mathcal{U})\Psi(\mathcal{L})^* + \Psi(\mathcal{L})\Psi(\mathcal{K})\Psi(\mathcal{U})^* \\
&= \Psi(\mathcal{U})\lozenge\mathcal{K}\circ\mathcal{L} + \mathcal{U}\lozenge\Psi(\mathcal{K})\circ\mathcal{L} + \mathcal{U}\lozenge\mathcal{K}\circ\Psi(\mathcal{L}).
\end{aligned}$$

Thus Ψ satisfies (1). Moreover $\mathcal{F}$ contains a non-trivial symmetric idempotent element $P = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and Ψ is a non-trivial map.

In various research fields of algebra, the products listed in the table below are gaining importance, leading to increased attention and study from numerous researchers, see ([4, 5, 7, 14, 16, 21–28]) and their bibliographic content.

Name	Notation	Definitions
Jordan Product	$\mathcal{U}\circ\mathcal{K}$	$\mathcal{U}\mathcal{K}+\mathcal{K}\mathcal{U}$
Lie product	$[\mathcal{U},\mathcal{K}]$	$\mathcal{U}\mathcal{K}-\mathcal{K}\mathcal{U}$
Jordan $*$-product	$\mathcal{U}\bullet\mathcal{K}$	$\mathcal{U}\mathcal{K}+\mathcal{K}\mathcal{U}^*$
Skew Lie product	$[\mathcal{U},\mathcal{K}]_*$	$\mathcal{U}\mathcal{K}-\mathcal{K}\mathcal{U}^*$
Bi-skew Lie product	$[\mathcal{U},\mathcal{K}]_\diamond$	$\mathcal{U}^*\mathcal{K}-\mathcal{K}^*\mathcal{U}$
Left bi-skew Jordan Product	$\mathcal{U}\diamond\mathcal{K}$	$\mathcal{U}^*\mathcal{K}+\mathcal{K}^*\mathcal{U}$
Right bi-skew Jordan Product	$\mathcal{U}\lozenge\mathcal{K}$	$\mathcal{U}\mathcal{K}^*+\mathcal{K}\mathcal{U}^*$

2. Results

Theorem 2.1. *Consider a 2-torsion free $*$-ring $\mathcal{F}$ with unity I and a symmetric idempotent $\mathcal{E}_1 \neq 0, I$ such that*

$$\mathcal{U}\mathcal{F}\mathcal{E}_t = 0 \quad \Rightarrow \quad \mathcal{U} = 0, \ t \in \{1, 2\} \tag{2}$$

where $\mathcal{E}_2 = I - \mathcal{E}_1, \mathcal{U} \in \mathcal{F}$. Suppose that a map $\Psi : \mathcal{F} \to \mathcal{F}$ (not necessarily additive) satisfies

$$\Psi(\mathcal{U} \lozenge \mathcal{K} \circ \mathcal{L}) = \Psi(\mathcal{U}) \lozenge \mathcal{K} \circ \mathcal{L} + \mathcal{U} \lozenge \Psi(\mathcal{K}) \circ \mathcal{L} + \mathcal{U} \lozenge \mathcal{K} \circ \Psi(\mathcal{L})$$

for all $\mathcal{U}, \mathcal{K}, \mathcal{L} \in \mathcal{F}$. Then Ψ is an additive map. Furthermore, if $\Psi(\mathcal{E}_i), i \in \{1, 2\}$ is self adjoint, then Ψ is a $*$-derivation.

Proof. Let $\mathcal{F}_{ij} = \mathcal{E}_i \mathcal{F} \mathcal{E}_j$ for $i, j = 1, 2$. Through Peirce decomposition of $\mathcal{F}$ related to $\mathcal{E}_1$ and $\mathcal{E}_2$, we have $\mathcal{F} = \mathcal{F}_{11} \oplus \mathcal{F}_{12} \oplus \mathcal{F}_{21} \oplus \mathcal{F}_{22}$. Observe that every $\mathcal{U} \in \mathcal{F}$ can be represented as $\mathcal{U} = \mathcal{U}_{11} + \mathcal{U}_{12} + \mathcal{U}_{21} + \mathcal{U}_{22}$, where $\mathcal{U}_{ij} \in \mathcal{F}_{ij}$ for $i, j = 1, 2$. Now to prove the additivity of Ψ on $\mathcal{F}$, we use the above partition of $\mathcal{F}$ and prove certain lemmas that will show that Ψ is additive on each $\mathcal{F}_{ij}$ for $i, j = 1, 2$. □

Thus the proof is completed by the following subsequent lemmas.

Lemma 2.2. $\Psi(0) = 0$.

Proof. It is trivial to see that for any $\mathcal{U} \in \mathcal{F}$, we have

$$\begin{aligned} \Psi(0) &= \Psi(0 \lozenge 0 \circ \mathcal{U}) \\ &= \Psi(0) \lozenge 0 \circ \mathcal{U} + 0 \lozenge \Psi(0) \circ \mathcal{U} + 0 \lozenge 0 \circ \Psi(\mathcal{U}) \\ &= 0. \end{aligned}$$

□

Lemma 2.3. *Let $\mathcal{U}_{12} \in \mathcal{F}_{12}$ and $\mathcal{U}_{21} \in \mathcal{F}_{21}$. We have $\Psi(\mathcal{U}_{12} + \mathcal{U}_{21}) = \Psi(\mathcal{U}_{12}) + \Psi(\mathcal{U}_{21})$.*

Proof. Let $\mathcal{M} = \Psi(\mathcal{U}_{12} + \mathcal{U}_{21}) - \Psi(\mathcal{U}_{12}) - \Psi(\mathcal{U}_{21})$. Since $I \lozenge (\mathcal{E}_2 - \mathcal{E}_1) \circ \mathcal{U}_{12} = I \lozenge (\mathcal{E}_2 - \mathcal{E}_1) \circ \mathcal{U}_{21} = 0$, utilizing Lemma 2.2, we can see

$$\begin{aligned} \Psi(I) \lozenge (\mathcal{E}_2 - \mathcal{E}_1) \circ (\mathcal{U}_{12} + \mathcal{U}_{21}) \quad + \quad & I \lozenge \Psi(\mathcal{E}_2 - \mathcal{E}_1) \circ (\mathcal{U}_{12} + \mathcal{U}_{21}) \\ & + I \lozenge (\mathcal{E}_2 - \mathcal{E}_1) \circ \Psi(\mathcal{U}_{12} + \mathcal{U}_{21}) \\ = \quad & \Psi(I \lozenge (\mathcal{E}_2 - \mathcal{E}_1) \circ (\mathcal{U}_{12} + \mathcal{U}_{21})) \\ = \quad & \Psi(I \lozenge (\mathcal{E}_2 - \mathcal{E}_1) \circ \mathcal{U}_{12}) + \Psi(I \lozenge (\mathcal{E}_2 - \mathcal{E}_1) \circ \mathcal{U}_{21}) \\ = \quad & \Psi(I) \lozenge (\mathcal{E}_2 - \mathcal{E}_1) \circ (\mathcal{U}_{12} + \mathcal{U}_{21}) \\ & + I \lozenge \Psi(\mathcal{E}_2 - \mathcal{E}_1) \circ (\mathcal{U}_{12} + \mathcal{U}_{21}) \\ & + I \lozenge (\mathcal{E}_2 - \mathcal{E}_1) \circ (\Psi(\mathcal{U}_{12}) + \Psi(\mathcal{U}_{21})). \end{aligned}$$

From this, we have

$$I \lozenge (\mathcal{E}_2 - \mathcal{E}_1) \circ \mathcal{M} = 0,$$

which implies that $2\mathcal{E}_2\mathcal{M} - 2\mathcal{E}_1\mathcal{M} + 2\mathcal{M}\mathcal{E}_2 - 2\mathcal{M}\mathcal{E}_1 = 0$. Multiplying both sides by $\mathcal{E}_1$, we get $4\mathcal{E}_1\mathcal{M}\mathcal{E}_1 = 0$. As $\mathcal{F}$ is 2-torsion free, we get $\mathcal{E}_1\mathcal{M}\mathcal{E}_1 = 0$. Similarly, multiplying both sides by $\mathcal{E}_2$ and using 2-torsion freeness of $\mathcal{F}$, we obtain $\mathcal{E}_2\mathcal{M}\mathcal{E}_2 = 0$.

Now, for any $\mathcal{V}_{11} \in \mathcal{F}_{11}$, we have $\mathcal{U}_{12} \lozenge \mathcal{E}_1 \circ \mathcal{V}_{11} = 0$. Utilizing Lemma 2.2, we observe

$$\begin{aligned} \Psi(\mathcal{U}_{12} + \mathcal{U}_{21}) \lozenge \mathcal{E}_1 \circ \mathcal{V}_{11} \quad + \quad & (\mathcal{U}_{12} + \mathcal{U}_{21}) \lozenge \Psi(\mathcal{E}_1) \circ \mathcal{V}_{11} + (\mathcal{U}_{12} + \mathcal{U}_{21}) \lozenge \mathcal{E}_1 \circ \Psi(\mathcal{V}_{11}) \\ = \quad & \Psi((\mathcal{U}_{12} + \mathcal{U}_{21}) \lozenge \mathcal{E}_1 \circ \mathcal{V}_{11}) \\ = \quad & \Psi(\mathcal{U}_{12} \lozenge \mathcal{E}_1 \circ \mathcal{V}_{11}) + \Psi(\mathcal{U}_{21} \lozenge \mathcal{E}_1 \circ \mathcal{V}_{11}) \end{aligned}$$

$$= (\Psi(\mathcal{U}_{12}) + \Psi(\mathcal{U}_{21}))\Diamond\mathcal{E}_1 \circ \mathcal{V}_{11} + (\mathcal{U}_{12} + \mathcal{U}_{21})\Diamond\Psi(\mathcal{E}_1) \circ \mathcal{V}_{11} + (\mathcal{U}_{12} + \mathcal{U}_{21})\Diamond\mathcal{E}_1 \circ \Psi(\mathcal{V}_{11}).$$

Hence, we get $\mathcal{M}\Diamond\mathcal{E}_1 \circ \mathcal{V}_{11} = 0$, i.e., $\mathcal{M}\mathcal{V}_{11} + \mathcal{E}_1\mathcal{M}^*\mathcal{V}_{11} + \mathcal{V}_{11}\mathcal{M}\mathcal{E}_1 + \mathcal{V}_{11}\mathcal{M}^* = 0$. Multiplying by $\mathcal{E}_2$ from left, we get $\mathcal{E}_2\mathcal{M}\mathcal{V}_{11} = 0$. Using (2), we get $\mathcal{M}_{21} = 0$. Similarly, we can show that $\mathcal{M}_{12} = 0$. Hence, $\mathcal{M} = 0$, i.e., $\Psi(\mathcal{U}_{12} + \mathcal{U}_{21}) = \Psi(\mathcal{U}_{12}) + \Psi(\mathcal{U}_{21})$. □

Lemma 2.4. *For every $\mathcal{U}_{11} \in \mathcal{F}_{11}, \mathcal{U}_{12} \in \mathcal{F}_{12}, \mathcal{U}_{21} \in \mathcal{F}_{21}$ and $\mathcal{U}_{22} \in \mathcal{F}_{22}$, we have*
(*i*) $\Psi(\mathcal{U}_{12} + \mathcal{U}_{21} + \mathcal{U}_{22}) = \Psi(\mathcal{U}_{12}) + \Psi(\mathcal{U}_{21}) + \Psi(\mathcal{U}_{22})$;
(*ii*) $\Psi(\mathcal{U}_{11} + \mathcal{U}_{12} + \mathcal{U}_{21}) = \Psi(\mathcal{U}_{11}) + \Psi(\mathcal{U}_{12}) + \Psi(\mathcal{U}_{21})$.

Proof. (*i*) Let $\mathcal{M} = \Psi(\mathcal{U}_{12} + \mathcal{U}_{21} + \mathcal{U}_{22}) - \Psi(\mathcal{U}_{12}) - \Psi(\mathcal{U}_{21}) - \Psi(\mathcal{U}_{22})$. Since $I\Diamond\mathcal{E}_1 \circ \mathcal{U}_{22} = 0$, using Lemma 2.3, we have

$$\begin{aligned}
\Psi(I)\Diamond\mathcal{E}_1 \circ (\mathcal{U}_{12} + \mathcal{U}_{21} + \mathcal{U}_{22}) &+ I\Diamond\Psi(\mathcal{E}_1) \circ (\mathcal{U}_{12} + \mathcal{U}_{21} + \mathcal{U}_{22}) + I\Diamond\mathcal{E}_1 \circ \Psi(\mathcal{U}_{12} + \mathcal{U}_{21} + \mathcal{U}_{22}) \\
&= \Psi(I\Diamond\mathcal{E}_1 \circ (\mathcal{U}_{12} + \mathcal{U}_{21} + \mathcal{U}_{22})) \\
&= \Psi(I\Diamond\mathcal{E}_1 \circ \mathcal{U}_{22}) + \Psi(I\Diamond\mathcal{E}_1 \circ (\mathcal{U}_{12} + \mathcal{U}_{21})) \\
&= \Psi(I\Diamond\mathcal{E}_1 \circ \mathcal{U}_{12}) + \Psi(I\Diamond\mathcal{E}_1 \circ \mathcal{U}_{21}) + \Psi(I\Diamond\mathcal{E}_1 \circ \mathcal{U}_{22}) \\
&= \Psi(I)\Diamond\mathcal{E}_1 \circ (\mathcal{U}_{12} + \mathcal{U}_{21} + \mathcal{U}_{22}) + I\Diamond\Psi(\mathcal{E}_1) \circ (\mathcal{U}_{12} \\
&\quad + \mathcal{U}_{21} + \mathcal{U}_{22}) + I\Diamond\mathcal{E}_1 \circ (\Psi(\mathcal{U}_{12}) + \Psi(\mathcal{U}_{21}) + \Psi(\mathcal{U}_{22})).
\end{aligned}$$

Hence, we get $I\Diamond\mathcal{E}_1 \circ \mathcal{M} = 0$. Thus, we yield $2\mathcal{E}_1\mathcal{M} + 2\mathcal{M}\mathcal{E}_1 = 0$. Solving it and invoking 2-torsion freeness of $\mathcal{F}$, we obtain $\mathcal{E}_1 M\mathcal{E}_1 = \mathcal{E}_2 M\mathcal{E}_1 = \mathcal{E}_1 M\mathcal{E}_2 = 0$.
Also, we have

$$I\Diamond(\mathcal{E}_2 - \mathcal{E}_1) \circ \mathcal{U}_{12} = I\Diamond(\mathcal{E}_2 - \mathcal{E}_1) \circ \mathcal{U}_{21} = 0.$$

Applying Lemma 2.2, it follows that

$$\begin{aligned}
\Psi(I)\Diamond(\mathcal{E}_2 - \mathcal{E}_1) \circ (\mathcal{U}_{12} + \mathcal{U}_{21} + \mathcal{U}_{22}) &+ I\Diamond\Psi(\mathcal{E}_2 - \mathcal{E}_1) \circ (\mathcal{U}_{12} + \mathcal{U}_{21} + \mathcal{U}_{22}) \\
+ I\Diamond(\mathcal{E}_2 - \mathcal{E}_1) \circ \Psi(\mathcal{U}_{12} + \mathcal{U}_{21} + \mathcal{U}_{22}) &= \Psi(I\Diamond(\mathcal{E}_2 - \mathcal{E}_1) \circ (\mathcal{U}_{12} + \mathcal{U}_{21} + \mathcal{U}_{22})) \\
&= \Psi(I\Diamond(\mathcal{E}_2 - \mathcal{E}_1) \circ \mathcal{U}_{12}) + \Psi(I\Diamond(\mathcal{E}_2 - \mathcal{E}_1) \circ \mathcal{U}_{21}) \\
&\quad + \Psi(I\Diamond(\mathcal{E}_2 - \mathcal{E}_1) \circ \mathcal{U}_{22}) \\
&= \Psi(I)\Diamond(\mathcal{E}_2 - \mathcal{E}_1) \circ (\mathcal{U}_{12} + \mathcal{U}_{21} + \mathcal{U}_{22}) \\
&\quad + I\Diamond\Psi(\mathcal{E}_2 - \mathcal{E}_1) \circ (\mathcal{U}_{12} + \mathcal{U}_{21} + \mathcal{U}_{22}) \\
&\quad + I\Diamond(\mathcal{E}_2 - \mathcal{E}_1) \circ (\Psi(\mathcal{U}_{12}) + \Psi(\mathcal{U}_{21}) + \Psi(\mathcal{U}_{22})).
\end{aligned}$$

Solving this, we get $I\Diamond(\mathcal{E}_2 - \mathcal{E}_1) \circ \mathcal{M} = 0$, which implies that $2\mathcal{E}_2\mathcal{M} - 2\mathcal{E}_1\mathcal{M} + 2\mathcal{M}\mathcal{E}_2 - 2\mathcal{M}\mathcal{E}_1 = 0$. Multiplying both sides by $\mathcal{E}_2$ and using 2-torsion freeness of $\mathcal{F}$, we get $\mathcal{E}_2\mathcal{M}\mathcal{E}_2 = 0$. Thus $\mathcal{M} = 0$. Analogously, we can verify the other part also. □

Lemma 2.5. *Let $\mathcal{U}_{ij} \in \mathcal{F}_{ij}$, $i, j \in \{1, 2\}$. Then*

$$\Psi(\sum_{i,j=1}^{2} \mathcal{U}_{ij}) = \sum_{i,j=1}^{2} \Psi(\mathcal{U}_{ij}).$$

Proof. Let $\mathcal{M} = \Psi(\mathcal{U}_{11} + \mathcal{U}_{12} + \mathcal{U}_{21} + \mathcal{U}_{22}) - \Psi(\mathcal{U}_{11}) - \Psi(\mathcal{U}_{12}) - \Psi(\mathcal{U}_{21}) - \Psi(\mathcal{U}_{22})$. Since $\mathcal{E}_2\Diamond I \circ \mathcal{U}_{11} = 0$, invoking Lemma 2.4, we have

$$\begin{aligned}
\Psi(\mathcal{E}_2)\Diamond I \circ (\mathcal{U}_{11} + \mathcal{U}_{12} + \mathcal{U}_{21} + \mathcal{U}_{22}) &+ \mathcal{E}_2\Diamond\Psi(I) \circ (\mathcal{U}_{11} + \mathcal{U}_{12} + \mathcal{U}_{21} + \mathcal{U}_{22}) \\
&+ \mathcal{E}_2\Diamond I \circ \Psi(\mathcal{U}_{11} + \mathcal{U}_{12} + \mathcal{U}_{21} + \mathcal{U}_{22}) \\
&= \Psi(\mathcal{E}_2\Diamond I \circ (\mathcal{U}_{11} + \mathcal{U}_{12} + \mathcal{U}_{21} + \mathcal{U}_{22}))
\end{aligned}$$

$$\begin{aligned} &= \Psi(\mathcal{E}_2\Diamond I \circ (\mathcal{U}_{12} + \mathcal{U}_{21} + \mathcal{U}_{22})) + \Psi(\mathcal{E}_2\Diamond I \circ \mathcal{U}_{11}) \\ &= \Psi(\mathcal{E}_2\Diamond I \circ \mathcal{U}_{11}) + \Psi(\mathcal{E}_2\Diamond I \circ \mathcal{U}_{12}) \\ &\quad + \Psi(\mathcal{E}_2\Diamond I \circ \mathcal{U}_{21}) + \Psi(\mathcal{E}_2\Diamond I \circ \mathcal{U}_{22}) \\ &= \Psi(\mathcal{E}_2)\Diamond I \circ (\mathcal{U}_{11} + \mathcal{U}_{12} + \mathcal{U}_{21} + \mathcal{U}_{22}) \\ &\quad + \mathcal{E}_2\Diamond\Psi(I) \circ (\mathcal{U}_{11} + \mathcal{U}_{12} + \mathcal{U}_{21} + \mathcal{U}_{22}) \\ &\quad + \mathcal{E}_2\Diamond I \circ (\Psi(\mathcal{U}_{11}) + \Psi(\mathcal{U}_{12}) \\ &\quad + \Psi(\mathcal{U}_{21}) + \Psi(\mathcal{U}_{22})). \end{aligned}$$

Hence, $\mathcal{E}_2\Diamond I \circ \mathcal{M} = 0$, from which we obtain $\mathcal{E}_1\mathcal{M}\mathcal{E}_2 = \mathcal{E}_2\mathcal{M}\mathcal{E}_1 = \mathcal{E}_2\mathcal{M}\mathcal{E}_2 = 0$. Similarly, we can show that $\mathcal{E}_1\mathcal{M}\mathcal{E}_1 = 0$. Thus $\mathcal{M} = 0$, i.e.,

$$\Psi(\sum_{i,j=1}^{2} \mathcal{U}_{ij}) = \sum_{i,j=1}^{2} \Psi(\mathcal{U}_{ij}).$$

□

Lemma 2.6. *For any $\mathcal{U}_{ij}, \mathcal{K}_{ij} \in \mathcal{F}_{ij}$ with $i \neq j$, $\Psi(\mathcal{U}_{ij} + \mathcal{K}_{ij}) = \Psi(\mathcal{U}_{ij}) + \Psi(\mathcal{K}_{ij})$.*

Proof. For simplicity, suppose $i = 1, j = 2$. Analogous is the case $i = 2, j = 1$. Let $\mathcal{N} = \Psi(\mathcal{U}_{12} + \mathcal{K}_{12}) - \Psi(\mathcal{U}_{12}) - \Psi(\mathcal{K}_{12})$. Since $(\mathcal{E}_2 - \mathcal{E}_1)\Diamond I \circ \mathcal{U}_{12} = 0 = (\mathcal{E}_2 - \mathcal{E}_1)\Diamond I \circ \mathcal{K}_{12}$, applying Lemma 2.2, we have

$$\begin{aligned} \Psi(\mathcal{E}_2 - \mathcal{E}_1)\Diamond I \circ (\mathcal{U}_{12} + \mathcal{K}_{12}) &+ (\mathcal{E}_2 - \mathcal{E}_1)\Diamond\Psi(I) \circ (\mathcal{U}_{12} + \mathcal{K}_{12}) + (\mathcal{E}_2 - \mathcal{E}_1)\Diamond I \circ \Psi(\mathcal{U}_{12} + \mathcal{K}_{12}) \\ &= \Psi((\mathcal{E}_2 - \mathcal{E}_1)\Diamond I \circ (\mathcal{U}_{12} + \mathcal{K}_{12})) \\ &= \Psi((\mathcal{E}_2 - \mathcal{E}_1)\Diamond I \circ \mathcal{U}_{12}) + \Psi((\mathcal{E}_2 - \mathcal{E}_1)\Diamond I \circ \mathcal{K}_{12}) \\ &= \Psi(\mathcal{E}_2 - \mathcal{E}_1)\Diamond I \circ (\mathcal{U}_{12} + \mathcal{U}_{21}) + (\mathcal{E}_2 - \mathcal{E}_1)\Diamond\Psi(I) \circ (\mathcal{U}_{12} + \mathcal{U}_{21}) \\ &\quad + (\mathcal{E}_2 - \mathcal{E}_1)\Diamond I \circ (\Psi(\mathcal{U}_{12}) + \Psi(\mathcal{U}_{21})). \end{aligned}$$

Thus we arrive at

$$(\mathcal{E}_2 - \mathcal{E}_1)\Diamond I \circ \mathcal{N} = 0,$$

which implies that $2\mathcal{E}_2\mathcal{N} - 2\mathcal{E}_1\mathcal{N} + 2\mathcal{N}\mathcal{E}_2 - 2\mathcal{N}\mathcal{E}_1 = 0$. Multiplying both sides by $\mathcal{E}_1$, we get $4\mathcal{E}_1\mathcal{N}\mathcal{E}_1 = 0$. Using 2-torsion freeness of $\mathcal{F}$, we get $\mathcal{E}_1\mathcal{N}\mathcal{E}_1 = 0$. Similarly, multiplying both sides by $\mathcal{E}_2$ and applying 2-torsion freeness of $\mathcal{F}$, we conclude that $\mathcal{E}_2\mathcal{N}\mathcal{E}_2 = 0$.

Noticing that $\mathcal{U}_{12}\Diamond\mathcal{E}_1 \circ \mathcal{E}_1 = 0$ for any $\mathcal{U}_{12} \in \mathcal{F}_{12}$, we can easily obtain that

$$\begin{aligned} \Psi(\mathcal{U}_{12} + \mathcal{K}_{12})\Diamond\mathcal{E}_1 \circ \mathcal{E}_1 &+ (\mathcal{U}_{12} + \mathcal{K}_{12})\Diamond\Psi(\mathcal{E}_1) \circ \mathcal{E}_1 + (\mathcal{U}_{12} + \mathcal{K}_{12})\Diamond\mathcal{E}_1 \circ \Psi(\mathcal{E}_1) \\ &= \Psi((\mathcal{U}_{12} + \mathcal{K}_{12})\Diamond\mathcal{E}_1 \circ \mathcal{E}_1) \\ &= \Psi(\mathcal{U}_{12}\Diamond\mathcal{E}_1 \circ \mathcal{E}_1) + \Psi(\mathcal{K}_{12}\Diamond\mathcal{E}_1 \circ \mathcal{E}_1) \\ &= (\Psi(\mathcal{U}_{12}) + \Psi(\mathcal{K}_{12}))\Diamond\mathcal{E}_1 \circ \mathcal{E}_1 + (\mathcal{U}_{12} + \mathcal{K}_{12})\Diamond\Psi(\mathcal{E}_1) \circ \mathcal{E}_1 \\ &\quad + (\mathcal{U}_{12} + \mathcal{K}_{12})\Diamond\mathcal{E}_1 \circ \Psi(\mathcal{E}_1). \end{aligned}$$

Thus we get, $\mathcal{N}\Diamond\mathcal{E}_1 \circ \mathcal{E}_1 = 0$, which implies that $\mathcal{E}_2\mathcal{N}\mathcal{E}_1 = 0$, i.e., $\mathcal{N}_{21} = 0$. Similarly, we can show that $\mathcal{E}_1\mathcal{N}\mathcal{E}_2 = 0$. Thus, $\Psi(\mathcal{U}_{12} + \mathcal{K}_{12}) = \Psi(\mathcal{U}_{12}) + \Psi(\mathcal{K}_{12})$.

Lemma 2.7. *Let $\mathcal{U}_{ii}, \mathcal{K}_{ii} \in \mathcal{F}_{ii}$, $(i = 1, 2)$. Then $\Psi(\mathcal{U}_{ii} + \mathcal{K}_{ii}) = \Psi(\mathcal{U}_{ii}) + \Psi(\mathcal{K}_{ii})$.*

Proof. Let $\mathcal{Z} = \Psi(\mathcal{U}_{ii} + \mathcal{K}_{ii}) - \Psi(\mathcal{U}_{ii}) - \Psi(\mathcal{K}_{ii})$. Since $I\Diamond\mathcal{E}_j \circ \mathcal{U}_{ii} = I\Diamond\mathcal{E}_j \circ \mathcal{K}_{ii} = 0$, it follows that

$$\begin{aligned} \Psi(I)\Diamond\mathcal{E}_j \circ (\mathcal{U}_{ii} + \mathcal{K}_{ii}) &+ I\Diamond\Psi(\mathcal{E}_j) \circ (\mathcal{U}_{ii} + \mathcal{K}_{ii}) + I\Diamond\mathcal{E}_j \circ \Psi(\mathcal{U}_{ii} + \mathcal{K}_{ii}) \\ &= \Psi(I\Diamond\mathcal{E}_j \circ (\mathcal{U}_{ii} + \mathcal{K}_{ii})) \\ &= \Psi(I\Diamond\mathcal{E}_j \circ \mathcal{U}_{ii}) + \Psi(I\Diamond\mathcal{E}_j \circ \mathcal{K}_{ii}) \end{aligned}$$

$$\begin{aligned} &= \Psi(I)\lozenge\mathcal{E}_j \circ (\mathcal{U}_{ii} + \mathcal{K}_{ii}) + I\lozenge\Psi(\mathcal{E}_j) \circ (\mathcal{U}_{ii} + \mathcal{K}_{ii}) \\ &\quad + I\lozenge \mathcal{E}_j \circ (\Psi(\mathcal{U}_{ii}) + \Psi(\mathcal{K}_{ii})). \end{aligned}$$

From this, we get

$$I\lozenge\mathcal{E}_j \circ \mathcal{Z} = 0.$$

Now solving this, we get $2\mathcal{E}_j\mathcal{Z} + 2\mathcal{Z}\mathcal{E}_j = 0$, which implies that $\mathcal{E}_i\mathcal{Z}\mathcal{E}_j = \mathcal{E}_j\mathcal{Z}\mathcal{E}_i = \mathcal{E}_j\mathcal{Z}\mathcal{E}_j = 0$. Thus, upto this stage, we arrive at $\mathcal{Z} = \mathcal{Z}_{ii}$.

Now, for any $\mathcal{V}_{ij} \in \mathcal{F}_{ij}$ with $i \neq j$ and applying Lemmas 2.5 and 2.6, we have

$$\begin{aligned} \Psi(\mathcal{V}_{ij})\lozenge\mathcal{E}_j \circ (\mathcal{U}_{ii} + \mathcal{K}_{ii}) &+ \mathcal{V}_{ij}\lozenge\Psi(\mathcal{E}_j) \circ (\mathcal{U}_{ii} + \mathcal{K}_{ii}) + \mathcal{V}_{ij}\lozenge\mathcal{E}_j \circ \Psi(\mathcal{U}_{ii} + \mathcal{K}_{ii}) \\ &= \Psi(\mathcal{V}_{ij}\lozenge\mathcal{E}_j \circ (\mathcal{U}_{ii} + \mathcal{K}_{ii})) \\ &= \Psi((\mathcal{V}_{ij} + \mathcal{V}_{ij}^*)(\mathcal{U}_{ii} + \mathcal{K}_{ii}) + (\mathcal{U}_{ii} + \mathcal{K}_{ii})(\mathcal{V}_{ij} + \mathcal{V}_{ij}^*)) \\ &= \Psi(\mathcal{V}_{ij}\lozenge\mathcal{E}_j \circ \mathcal{U}_{ii}) + \Psi(\mathcal{V}_{ij}\lozenge\mathcal{E}_j \circ \mathcal{K}_{ii}) \\ &= \Psi(\mathcal{V}_{ij})\lozenge\mathcal{E}_j \circ (\mathcal{U}_{ii} + \mathcal{K}_{ii}) + \mathcal{V}_{ij}\lozenge\Psi(\mathcal{E}_j) \circ (\mathcal{U}_{ii} + \mathcal{K}_{ii}) \\ &\quad + \mathcal{V}_{ij}\lozenge\mathcal{E}_j \circ (\Psi(\mathcal{U}_{ii}) + \Psi(\mathcal{K}_{ii})). \end{aligned}$$

Hence $\mathcal{V}_{ij}\lozenge\mathcal{E}_j \circ \mathcal{Z} = 0$. Solving this relation, we get $\mathcal{V}_{ij}\mathcal{Z} + \mathcal{V}_{ij}^*\mathcal{Z} + \mathcal{Z}\mathcal{V}_{ij} + \mathcal{Z}\mathcal{V}_{ij}^* = 0$. Finally, we have $\mathcal{Z}_{ii}\mathcal{V}_{ij} = 0$. It follows from (2) that $\mathcal{E}_i\mathcal{Z}\mathcal{E}_i = 0$. Thus, $\mathcal{Z} = 0$. □

Lemma 2.8. Ψ *is additive.*

Proof. For every $\mathcal{U}, \mathcal{K} \in \mathcal{F}$, we have $\mathcal{U} = \sum_{i,j=1}^2 \mathcal{U}_{ij}$ and $\mathcal{K} = \sum_{i,j=1}^2 \mathcal{K}_{ij}$. By using Lemmas 2.5 - 2.7, we get

$$\begin{aligned} \Psi(\mathcal{U} + \mathcal{K}) &= \Psi(\sum_{i,j=1}^2 \mathcal{U}_{ij} + \sum_{i,j=1}^2 \mathcal{K}_{ij}) \\ &= \Psi(\sum_{i,j=1}^2 (\mathcal{U}_{ij} + \mathcal{K}_{ij})) \\ &= \sum_{i,j=1}^2 \Psi(\mathcal{U}_{ij} + \mathcal{K}_{ij}) \\ &= \sum_{i,j=1}^2 \Psi(\mathcal{U}_{ij}) + \sum_{i,j=1}^2 \Psi(\mathcal{K}_{ij}) \\ &= \Psi(\sum_{i,j=1}^2 \mathcal{U}_{ij}) + \Psi(\sum_{i,j=1}^2 \mathcal{K}_{ij}) \\ &= \Psi(\mathcal{U}) + \Psi(\mathcal{K}). \end{aligned}$$

□

Lemma 2.9. (*i*) $\mathcal{E}_1\Psi(\mathcal{E}_1)\mathcal{E}_2 = -\mathcal{E}_1\Psi(\mathcal{E}_2)\mathcal{E}_2$;
(*ii*) $\mathcal{E}_2\Psi(\mathcal{E}_1)\mathcal{E}_1 = -\mathcal{E}_2\Psi(\mathcal{E}_2)\mathcal{E}_1$;
(*iii*) $\mathcal{E}_1\Psi(\mathcal{E}_2)\mathcal{E}_1 = \mathcal{E}_2\Psi(\mathcal{E}_1)\mathcal{E}_2 = 0$.

Proof. (*i*) From $\mathcal{E}_2\lozenge\mathcal{E}_2 \circ \mathcal{E}_1 = 0$ and Lemmas 2.2 and 2.8, it follows that

$$\begin{aligned} 0 &= \Psi(\mathcal{E}_2\lozenge\mathcal{E}_2 \circ \mathcal{E}_1) \\ &= \Psi(\mathcal{E}_2)\lozenge\mathcal{E}_2 \circ \mathcal{E}_1 + \mathcal{E}_2\lozenge\Psi(\mathcal{E}_2) \circ \mathcal{E}_1 + \mathcal{E}_2\lozenge\mathcal{E}_2 \circ \Psi(\mathcal{E}_1) \\ &= 2\mathcal{E}_2\Psi(\mathcal{E}_2)^*\mathcal{E}_1 + 2\mathcal{E}_1\Psi(\mathcal{E}_2)\mathcal{E}_2 + 2\mathcal{E}_2\Psi(\mathcal{E}_1) + 2\Psi(\mathcal{E}_1)\mathcal{E}_2. \end{aligned}$$

By multiplying both sides on the left by $\mathcal{E}_1$ and on the right by $\mathcal{E}_2$, we obtain

$$\mathcal{E}_1\Psi(\mathcal{E}_1)\mathcal{E}_2 = -\mathcal{E}_1\Psi(\mathcal{E}_2)\mathcal{E}_2.$$

(*ii*) Since $\mathcal{E}_1\Diamond\mathcal{E}_1 \circ \mathcal{E}_2 = 0$, using Lemmas 2.2 and 2.8, we have

$$\begin{aligned} 0 &= \Psi(\mathcal{E}_1\Diamond\mathcal{E}_1 \circ \mathcal{E}_2) \\ &= \Psi(\mathcal{E}_1)\Diamond\mathcal{E}_1 \circ \mathcal{E}_2 + \mathcal{E}_1\Diamond\Psi(\mathcal{E}_1) \circ \mathcal{E}_2 + \mathcal{E}_1\Diamond\mathcal{E}_1 \circ \Psi(\mathcal{E}_2) \\ &= 2\mathcal{E}_1\Psi(\mathcal{E}_1)^*\mathcal{E}_2 + 2\mathcal{E}_2\Psi(\mathcal{E}_1)\mathcal{E}_1 + 2\mathcal{E}_1\Psi(\mathcal{E}_2) + 2\Psi(\mathcal{E}_2)\mathcal{E}_1. \end{aligned}$$

By multiplying both sides on the left by $\mathcal{E}_2$ and on the right by $\mathcal{E}_1$, we arrive at

$$\mathcal{E}_2\Psi(\mathcal{E}_2)\mathcal{E}_1 = -\mathcal{E}_2\Psi(\mathcal{E}_1)\mathcal{E}_1.$$

(*iii*) From (*i*) just above, we have

$$0 = 2\mathcal{E}_2\Psi(\mathcal{E}_2)^*\mathcal{E}_1 + 2\mathcal{E}_1\Psi(\mathcal{E}_2)\mathcal{E}_2 + 2\mathcal{E}_2\Psi(\mathcal{E}_1) + 2\Psi(\mathcal{E}_1)\mathcal{E}_2. \tag{3}$$

Multiplying (3) by $\mathcal{E}_2$ from both left and right, we get

$$4\mathcal{E}_2\Psi(\mathcal{E}_1)\mathcal{E}_2 = 0.$$

Using $\mathcal{F}$ as a 2-torsion free in last relation, we get $\mathcal{E}_2\Psi(\mathcal{E}_1)\mathcal{E}_2 = 0$. Multiplying (3) by $\mathcal{E}_2$ from both left and right and invoking 2-torsion freeness of $\mathcal{F}$, we get $\mathcal{E}_2\Psi(\mathcal{E}_1)\mathcal{E}_2 = 0$. Similarly, from (*ii*) just above, we have

$$0 = 2\mathcal{E}_1\Psi(\mathcal{E}_1)^*\mathcal{E}_2 + 2\mathcal{E}_2\Psi(\mathcal{E}_1)\mathcal{E}_1 + 2\mathcal{E}_1\Psi(\mathcal{E}_2) + 2\Psi(\mathcal{E}_2)\mathcal{E}_1. \tag{4}$$

Multiplying (4) by $\mathcal{E}_1$ from both left and right, we get

$$4\mathcal{E}_1\Psi(\mathcal{E}_2)\mathcal{E}_1 = 0.$$

Using $\mathcal{F}$ as a 2-torsion free in last relation, we get $\mathcal{E}_1\Psi(\mathcal{E}_2)\mathcal{E}_1 = 0$. Multiplying (4) by $\mathcal{E}_1$ from both right and left, we get $\mathcal{E}_1\Psi(\mathcal{E}_2)\mathcal{E}_1 = 0$. □

Lemma 2.10. $\mathcal{E}_1\Psi(\mathcal{E}_1)\mathcal{E}_1 = \mathcal{E}_2\Psi(\mathcal{E}_2)\mathcal{E}_2 = 0.$

Proof. For every $\mathcal{U}_{12} \in \mathcal{F}_{12}$, it follows from Lemma 2.8 that

$$\Psi(\mathcal{E}_1\Diamond\mathcal{E}_1 \circ \mathcal{U}_{12}) = 2\Psi(\mathcal{U}_{12}).$$

On the other hand, we have

$$\begin{aligned} \Psi(\mathcal{E}_1\Diamond\mathcal{E}_1 \circ \mathcal{U}_{12}) &= \Psi(\mathcal{E}_1)\Diamond\mathcal{E}_1 \circ \mathcal{U}_{12} + \mathcal{E}_1\Diamond\Psi(\mathcal{E}_1) \circ \mathcal{U}_{12} + \mathcal{E}_1\Diamond\mathcal{E}_1 \circ \Psi(\mathcal{U}_{12}) \\ &= 2\Psi(\mathcal{E}_1)\mathcal{U}_{12} + 2\mathcal{E}_1\Psi(\mathcal{E}_1)^*\mathcal{U}_{12} + 2\mathcal{U}_{12}\Psi(\mathcal{E}_1)\mathcal{E}_1 + 2\mathcal{E}_1\Psi(\mathcal{U}_{12}) + 2\Psi(\mathcal{U}_{12})\mathcal{E}_1. \end{aligned}$$

By comparing the above two relations, we get

$$\begin{aligned} 2\Psi(\mathcal{E}_1)\mathcal{U}_{12} &+ 2\mathcal{E}_1\Psi(\mathcal{E}_1)^*\mathcal{U}_{12} + 2\mathcal{U}_{12}\Psi(\mathcal{E}_1)\mathcal{E}_1 \\ &+ 2\mathcal{E}_1\Psi(\mathcal{U}_{12}) + 2\Psi(\mathcal{U}_{12})\mathcal{E}_1 - 2\Psi(\mathcal{U}_{12}) = 0. \end{aligned}$$

Multiplying the above relation on the left by $\mathcal{E}_1$ and on the right by $\mathcal{E}_2$ and using given hypothesis, we get $\mathcal{E}_1\Psi(\mathcal{E}_1)\mathcal{U}_{12} = 0$, i.e., $\mathcal{E}_1\Psi(\mathcal{E}_1)\mathcal{E}_1\mathcal{U}\mathcal{E}_2 = 0$ for all $\mathcal{U} \in \mathcal{F}$. It follows from (2) that $\mathcal{E}_1\Psi(\mathcal{E}_1)\mathcal{E}_1 = 0$. Analogously, we can verify that $\mathcal{E}_2\Psi(\mathcal{E}_2)\mathcal{E}_2 = 0$. □

Lemma 2.11. (*i*) $\Psi(\mathcal{E}_1) = \mathcal{E}_1\Psi(\mathcal{E}_1)\mathcal{E}_2 + \mathcal{E}_2\Psi(\mathcal{E}_1)\mathcal{E}_1$ *and*
$\Psi(\mathcal{E}_2) = \mathcal{E}_1\Psi(\mathcal{E}_2)\mathcal{E}_2 + \mathcal{E}_2\Psi(\mathcal{E}_2)\mathcal{E}_1$;
(*ii*) $\Psi(I) = 0$ *and* $\Psi(I)^* = \Psi(I)$.

Proof. (i) By Peirce decomposition, we have

$$\Psi(\mathcal{E}_1) = \sum_{i,j=1}^{2} \mathcal{E}_i \Psi(\mathcal{E}_1)\mathcal{E}_j.$$

Now, using Lemmas 2.9 - 2.10, we get $\Psi(\mathcal{E}_1) = \mathcal{E}_1\Psi(\mathcal{E}_1)\mathcal{E}_2 + \mathcal{E}_2\Psi(\mathcal{E}_1)\mathcal{E}_1$. Analogously, we can show that $\Psi(\mathcal{E}_2) = \mathcal{E}_1\Psi(\mathcal{E}_2)\mathcal{E}_2 + \mathcal{E}_2\Psi(\mathcal{E}_2)\mathcal{E}_1$.
(ii) Clearly from Lemmas 2.8-2.10 , we have

$$\begin{aligned} \Psi(I) &= \Psi(\mathcal{E}_1 + \mathcal{E}_2) \\ &= \Psi(\mathcal{E}_1) + \Psi(\mathcal{E}_2) \\ &= \mathcal{E}_1\Psi(\mathcal{E}_1)\mathcal{E}_2 + \mathcal{E}_2\Psi(\mathcal{E}_1)\mathcal{E}_1 + \mathcal{E}_1\Psi(\mathcal{E}_2)\mathcal{E}_2 + \mathcal{E}_2\Psi(\mathcal{E}_2)\mathcal{E}_1 \\ &= 0. \end{aligned}$$

Hence $\Psi(I)^* = \Psi(I)$ □

Lemma 2.12. Ψ *preserves '$*$', i.e.,* $\Psi(\mathcal{U}^*) = \Psi(\mathcal{U})^*$ *for every* $\mathcal{U} \in \mathcal{F}$.

Proof. Invoking Lemma 2.8, we can see

$$\Psi(\mathcal{U}\Diamond I \circ I) = 2\Psi(\mathcal{U} + \mathcal{U}^*) = 2\Psi(\mathcal{U}) + 2\Psi(\mathcal{U}^*).$$

On the other way, using Lemmas 2.8 and 2.11, we observe

$$\begin{aligned} \Psi(\mathcal{U}\Diamond I \circ I) &= \Psi(\mathcal{U})\Diamond I \circ I \\ &= 2\Psi(\mathcal{U}) + 2\Psi(\mathcal{U})^*. \end{aligned}$$

By analyzing the preceding two relations, we obtain $\Psi(\mathcal{U}^*) = \Psi(\mathcal{U})^*$ for all $\mathcal{U} \in \mathcal{F}$. □

Now, set $\mathcal{D} = \mathcal{E}_1\Psi(\mathcal{E}_1)\mathcal{E}_2 - \mathcal{E}_2\Psi(\mathcal{E}_1)\mathcal{E}_1$. Then $\mathcal{D} = -\mathcal{D}^*$. Consider a map $\mathcal{T} : \mathcal{F} \to \mathcal{F}$ defined by $\mathcal{T}(\mathcal{Y}) = \Psi(\mathcal{Y}) - (\mathcal{Y}\mathcal{D} - \mathcal{D}\mathcal{Y})$ for all $\mathcal{Y} \in \mathcal{F}$. It is simple to check that $\mathcal{T}$ is an additive map and also fulfills the condition $\mathcal{T}(\mathcal{E}_1) = \mathcal{T}(\mathcal{E}_2) = \mathcal{T}(I) = 0$, $\mathcal{T}(\mathcal{Y}^*) = \mathcal{T}(\mathcal{Y})^*$ for all $\mathcal{Y} \in \mathcal{F}$ and $\mathcal{T}(\mathcal{U}\Diamond\mathcal{K} \circ \mathcal{L}) = \mathcal{T}(\mathcal{U})\Diamond\mathcal{K} \circ \mathcal{L} + \mathcal{U}\Diamond\mathcal{T}(\mathcal{K}) \circ \mathcal{L} + \mathcal{U}\Diamond\mathcal{K} \circ \mathcal{T}(\mathcal{L})$ for all $\mathcal{U}, \mathcal{K}, \mathcal{L} \in \mathcal{F}$. Moreover, $\mathcal{T}$ is a $*$-derivation if and only if Ψ is a $*$-derivation.

Lemma 2.13. *For any* $R_{ij} \in \mathcal{F}_{ij}$*, we have* $\mathcal{T}(R_{ij}) \in \mathcal{F}_{ij}$, $i, j \in \{1, 2\}$.

Proof. For any $R_{ij} \in \mathcal{F}_{ij}$ $(1 \le i \ne j \le 2)$, since $\mathcal{E}_i\Diamond I \circ R_{ij} = 2R_{ij}, \mathcal{T}(\mathcal{E}_i) = \mathcal{T}(I) = 0$ and using additivity, we have

$$\begin{aligned} 2\mathcal{T}(R_{ij}) &= \mathcal{T}(\mathcal{E}_i\Diamond I \circ R_{ij}) \\ &= \mathcal{E}_i\Diamond I \circ \mathcal{T}(R_{ij}) \\ &= 2\mathcal{E}_i\mathcal{T}(R_{ij}) + 2\mathcal{T}(R_{ij})\mathcal{E}_i. \end{aligned}$$

Thus, multiplying by $\mathcal{E}_i$ on bothsides of the previous relation and using 2-torsion freeness, we get $\mathcal{E}_i\mathcal{T}(R_{ij})\mathcal{E}_i = 0$. Similarly from $2R_{ij} = \mathcal{E}_j\Diamond I \circ R_{ij}$, we conclude $\mathcal{E}_j\mathcal{T}(R_{ij})\mathcal{E}_j = 0$.
As $R_{ij}\Diamond\mathcal{E}_i \circ \mathcal{E}_i = 0$ and using $\mathcal{T}(\mathcal{E}_i) = 0$, we have

$$\begin{aligned} 0 &= \mathcal{T}(R_{ij}\Diamond\mathcal{E}_i \circ \mathcal{E}_i) \\ &= \mathcal{T}(R_{ij})\Diamond\mathcal{E}_i \circ \mathcal{E}_i \\ &= \mathcal{T}(R_{ij})\mathcal{E}_i + \mathcal{E}_i\mathcal{T}(R_{ij})^*\mathcal{E}_i + \mathcal{E}_i\mathcal{T}(R_{ij})\mathcal{E}_i + \mathcal{E}_i\mathcal{T}(R_{ij})^*. \end{aligned}$$

Multiplying left by $\mathcal{E}_j$ in previous relation, we obtain $\mathcal{E}_j\mathcal{T}(R_{ij})\mathcal{E}_i = 0$. Hence $\mathcal{T}(R_{ij}) = \mathcal{E}_i\mathcal{T}(R_{ij})\mathcal{E}_j \in \mathcal{F}_{ij}$ for $i \ne j$.
Now let $R_{ii} \in \mathcal{F}_{ii}, i = 1, 2$. Then for $i \ne j \in \{1, 2\}$ with $\mathcal{E}_j\Diamond I \circ R_{ii} = 0$ and $\mathcal{T}(\mathcal{E}_j) = \mathcal{T}(I) = 0$, we have

$$\begin{aligned} 0 &= \mathcal{T}(\mathcal{E}_j\Diamond I \circ R_{ii}) \\ &= \mathcal{E}_j\Diamond I \circ \mathcal{T}(R_{ii}) \end{aligned}$$

$$= \quad 2\mathcal{E}_j\mathcal{T}(\mathrm{R}_{ii}) + 2\mathcal{T}(\mathrm{R}_{ii})\mathcal{E}_j.$$

Applying 2-torsion freeness of $\mathcal{F}$, we deduce that $\mathcal{E}_j\mathcal{T}(\mathrm{R}_{ii})\mathcal{E}_j = \mathcal{E}_j\mathcal{T}(\mathrm{R}_{ii})\mathcal{E}_i = \mathcal{E}_i\mathcal{T}(\mathrm{R}_{ii})\mathcal{E}_j = 0$ and hence $\mathcal{T}(\mathrm{R}_{ii}) = \mathcal{E}_i\mathcal{T}(\mathrm{R}_{ii})\mathcal{E}_i \in \mathcal{F}_{ii}$ for $i = 1, 2$. □

Lemma 2.14. *For any* $\mathrm{R}_{ii}, \mathrm{S}_{ii} \in \mathcal{F}_{ii}$, $\mathrm{R}_{ij}, \mathrm{S}_{ij} \in \mathcal{F}_{ij}$, $\mathrm{R}_{ji}, \mathrm{S}_{ji} \in \mathcal{F}_{ji}$, $\mathrm{R}_{jj}, \mathrm{S}_{jj} \in \mathcal{F}_{jj} (1 \leq i \neq j \leq 2)$, *we have*

(*i*) $\mathcal{T}(\mathrm{R}_{ii}\mathrm{S}_{ij}) = \mathcal{T}(\mathrm{R}_{ii})\mathrm{S}_{ij} + \mathrm{R}_{ii}\mathcal{T}(\mathrm{S}_{ij})$;
(*ii*) $\mathcal{T}(\mathrm{R}_{ij}\mathrm{S}_{ji}) = \mathcal{T}(\mathrm{R}_{ij})\mathrm{S}_{ji} + \mathrm{R}_{ij}\mathcal{T}(\mathrm{S}_{ji})$;
(*iii*) $\mathcal{T}(\mathrm{R}_{ij}\mathrm{S}_{jj}) = \mathcal{T}(\mathrm{R}_{ij})\mathrm{S}_{jj} + \mathrm{R}_{ij}\mathcal{T}(\mathrm{S}_{jj})$;
(*iv*) $\mathcal{T}(\mathrm{R}_{ii}\mathrm{S}_{ii}) = \mathcal{T}(\mathrm{R}_{ii})\mathrm{S}_{ii} + \mathrm{R}_{ii}\mathcal{T}(\mathrm{S}_{ii})$.

Proof. (*i*) On the one way, using additivity of $\mathcal{T}$, we have

$$\mathcal{T}(I\lozenge\mathrm{S}_{ij} \circ \mathrm{R}_{ii}) = \mathcal{T}(\mathrm{S}_{ij}^*\mathrm{R}_{ii}) + \mathcal{T}(\mathrm{R}_{ii}\mathrm{S}_{ij}).$$

On the other way, using Lemma 2.13 and $\mathcal{T}(I) = 0$, we have

$$\begin{aligned}\mathcal{T}(I\lozenge\mathrm{S}_{ij} \circ \mathrm{R}_{ii}) &= \mathcal{T}(I)\lozenge\mathrm{S}_{ij} \circ \mathrm{R}_{ii} + I\lozenge\mathcal{T}(\mathrm{S}_{ij}) \circ \mathrm{R}_{ii} + I\lozenge\mathrm{S}_{ij} \circ \mathcal{T}(\mathrm{R}_{ii}) \\ &= I\lozenge\mathcal{T}(\mathrm{S}_{ij}) \circ \mathrm{R}_{ii} + I\lozenge\mathrm{S}_{ij} \circ \mathcal{T}(\mathrm{R}_{ii}) \\ &= \mathcal{T}(\mathrm{S}_{ij})^*\mathrm{R}_{ii} + \mathrm{R}_{ii}\mathcal{T}(\mathrm{S}_{ij}) + \mathrm{S}_{ij}^*\mathcal{T}(\mathrm{R}_{ii}) + \mathcal{T}(\mathrm{R}_{ii})\mathrm{S}_{ij}.\end{aligned}$$

By examining the preceding two relations, we obtain $\mathcal{T}(\mathrm{R}_{ii}\mathrm{S}_{ij}) = \mathcal{T}(\mathrm{R}_{ii})\mathrm{S}_{ij} + \mathrm{R}_{ii}\mathcal{T}(\mathrm{S}_{ij})$.

(*ii*) On the one way, additivity of $\mathcal{T}$ gives

$$\mathcal{T}(I\lozenge\mathrm{R}_{ij} \circ \mathrm{S}_{ji}) = \mathcal{T}(\mathrm{R}_{ij}\mathrm{S}_{ji}) + \mathcal{T}(\mathrm{S}_{ji}\mathrm{R}_{ij}).$$

On the other way, using Lemma 2.13 and $\mathcal{T}(I) = 0$, we arrive at

$$\begin{aligned}\mathcal{T}(I\lozenge\mathrm{R}_{ij} \circ \mathrm{S}_{ji}) &= \mathcal{T}(I)\lozenge\mathrm{R}_{ij} \circ \mathrm{S}_{ji} + I\lozenge\mathcal{T}(\mathrm{R}_{ij}) \circ \mathrm{S}_{ji} + I\lozenge\mathrm{R}_{ij} \circ \mathcal{T}(\mathrm{S}_{ji}) \\ &= I\lozenge\mathcal{T}(\mathrm{R}_{ij}) \circ \mathrm{S}_{ji} + I\lozenge\mathrm{R}_{ij} \circ \mathcal{T}(\mathrm{S}_{ji}) \\ &= \mathcal{T}(\mathrm{R}_{ij})\mathrm{S}_{ji} + \mathrm{S}_{ji}\mathcal{T}(\mathrm{R}_{ij}) + \mathrm{R}_{ij}\mathcal{T}(\mathrm{S}_{ji}) + \mathcal{T}(\mathrm{S}_{ji})\mathrm{R}_{ij}.\end{aligned}$$

From the preceding two relations and using 2-torsion freeness of $\mathcal{F}$, we obtain
$\mathcal{T}(\mathrm{R}_{ij}\mathrm{S}_{ji}) = \mathcal{T}(\mathrm{R}_{ij})\mathrm{S}_{ji} + \mathrm{R}_{ij}\mathcal{T}(\mathrm{S}_{ji})$.

(*iii*) As $(\mathrm{R}_{ij}\lozenge\mathcal{E}_j \circ \mathrm{S}_{jj}) = \mathrm{R}_{ij}\mathrm{S}_{jj} + \mathrm{S}_{jj}\mathrm{R}_{ij}^*$, using additivity of $\mathcal{T}$, Lemmas 2.13 - 2.14 (*i*), we have

$$\begin{aligned}\mathcal{T}(\mathrm{R}_{ij}\lozenge\mathcal{E}_j \circ \mathrm{S}_{jj}) &= \mathcal{T}(\mathrm{R}_{ij}\mathrm{S}_{jj} + \mathrm{S}_{jj}\mathrm{R}_{ij}^*) \\ &= \mathcal{T}(\mathrm{R}_{ij}\mathrm{S}_{jj}) + \mathcal{T}(\mathrm{S}_{jj}\mathrm{R}_{ij}^*) \\ &= \mathcal{T}(\mathrm{R}_{ij}\mathrm{S}_{jj}) + \mathcal{T}(\mathrm{S}_{jj})\mathrm{R}_{ij}^* + \mathrm{S}_{jj}\mathcal{T}(\mathrm{R}_{ij})^*.\end{aligned}$$

On the other way, from $\mathcal{T}(\mathcal{E}_j) = 0$ $(j = 1, 2)$, we have

$$\begin{aligned}\mathcal{T}(\mathrm{R}_{ij}\lozenge\mathcal{E}_j \circ \mathrm{S}_{jj}) &= \mathcal{T}(\mathrm{R}_{ij})\lozenge\mathcal{E}_j \circ \mathrm{S}_{jj} + \mathrm{R}_{ij}\lozenge\mathcal{T}(\mathcal{E}_j) \circ \mathrm{S}_{jj} + \mathrm{R}_{ij}\lozenge\mathcal{E}_j \circ \mathcal{T}(\mathrm{S}_{jj}) \\ &= \mathcal{T}(\mathrm{R}_{ij})\mathrm{S}_{jj} + \mathrm{S}_{jj}\mathcal{T}(\mathrm{R}_{ij})^* + \mathrm{R}_{ij}\mathcal{T}(\mathrm{S}_{jj}) + \mathcal{T}(\mathrm{S}_{jj})\mathrm{R}_{ij}^*.\end{aligned}$$

From the aforementioned two relations, we derive $\mathcal{T}(\mathrm{R}_{ij}\mathrm{S}_{jj}) = \mathcal{T}(\mathrm{R}_{ij})\mathrm{S}_{jj} + \mathrm{R}_{ij}\mathcal{T}(\mathrm{S}_{jj})$.

(*iv*) For any $X_{ij} \in \mathcal{F}_{ij}$, it emerges from the Claim 2.14 (*i*) that

$$\begin{aligned}\mathcal{T}(\mathrm{R}_{ii}\mathrm{S}_{ii})X_{ij} + \mathrm{R}_{ii}\mathrm{S}_{ii}\mathcal{T}(X_{ij}) &= \mathcal{T}(\mathrm{R}_{ii}\mathrm{S}_{ii}X_{ij}) \\ &= \mathcal{T}(\mathrm{R}_{ii})\mathrm{S}_{ii}X_{ij} + \mathrm{R}_{ii}\mathcal{T}(\mathrm{S}_{ii}X_{ij}) \\ &= \mathcal{T}(\mathrm{R}_{ii})\mathrm{S}_{ii}X_{ij} + \mathrm{R}_{ii}\mathcal{T}(\mathrm{S}_{ii})X_{ij} + \mathrm{R}_{ii}\mathrm{S}_{ii}\mathcal{T}(X_{ij}).\end{aligned}$$

Hence $(\mathcal{T}(\mathrm{R}_{ii}\mathrm{S}_{ii}) - \mathcal{T}(\mathrm{R}_{ii})\mathrm{S}_{ii} - \mathrm{R}_{ii}\mathcal{T}(\mathrm{S}_{ii}))X_{ij} = 0$. In view of (2), it follows that $\mathcal{T}(\mathrm{R}_{ii}\mathrm{S}_{ii}) = \mathcal{T}(\mathrm{R}_{ii})\mathrm{S}_{ii} + \mathrm{R}_{ii}\mathcal{T}(\mathrm{S}_{ii})$.
□

Lemma 2.15. $\mathcal{T}(\mathrm{RS}) = \mathcal{T}(\mathrm{R})\mathrm{S} + \mathrm{R}\mathcal{T}(\mathrm{S})$ *for any* $\mathrm{R}, \mathrm{S} \in \mathcal{F}$.

Proof. Let $\mathrm{R}, \mathrm{S} \in \mathcal{F}$. Then we can write $\mathrm{R} = \sum_{i,j=1}^{2} \mathrm{R}_{ij}$ and $\mathrm{S} = \sum_{i,j=1}^{2} \mathrm{S}_{ij}$. From additivity of $\mathcal{T}$ and Lemma 2.14, we find that

$$\begin{aligned}
\mathcal{T}(\mathrm{RS}) &= \mathcal{T}(\sum_{i,j=1}^{2} \mathrm{R}_{ij}\mathrm{S}_{ij}) \\
&= \sum_{i,j=1}^{2} \mathcal{T}(\mathrm{R}_{ij}\mathrm{S}_{ij}) \\
&= \mathcal{T}(\sum_{i,j=1}^{2} \mathrm{R}_{ij})(\sum_{i,j=1}^{2} \mathrm{S}_{ij}) + (\sum_{i,j=1}^{2} \mathrm{R}_{ij})\, \mathcal{T}(\sum_{i,j=1}^{2} \mathrm{S}_{ij}) \\
&= \mathcal{T}(\mathrm{R})\mathrm{S} + \mathrm{R}\mathcal{T}(\mathrm{S}).
\end{aligned}$$

□

Hence, $\mathcal{T}$ is an additive $*$-derivation. Consequently, by Lemmas 2.8, 2.12 and 2.15, we have proved that Ψ is an additive $*$-derivation. Hence this concludes the proof of Theorem 2.1.

□

Lastly, we have provided an example that demonstrates the necessity of the 2-torsion freeness condition for $\mathcal{F}$ in Theorem 2.1.

Example 2.16. *Let* $\mathcal{F} = \mathcal{M}_2(\mathbb{K})$ *be the ring of all* 2×2 *matrices over any field* $\mathbb{K}$ *of characteristic 2 equipped with the standard transpose involution* $'*'$. *Then* $\mathcal{F}$ *is a unital* $*$*-ring. Define the map* $\Psi : \mathcal{F} \to \mathcal{F}$ *by*

$$\Psi\left\{\begin{bmatrix} \eta_1 & \eta_2 \\ \eta_3 & \eta_4 \end{bmatrix}\right\} = \begin{bmatrix} \eta_1 + \eta_4 & \eta_2 + \eta_3 \\ \eta_2 + \eta_3 & \eta_1 + \eta_4 \end{bmatrix}.$$

Then it is straightforward to verify that Ψ *satisfies*

$$\Psi(\mathcal{U} \Diamond \mathcal{K} \circ \mathcal{L}) = \Psi(\mathcal{U}) \Diamond \mathcal{K} \circ \mathcal{L} + \mathcal{U} \Diamond \Psi(\mathcal{K}) \circ \mathcal{L} + \mathcal{U} \Diamond \mathcal{K} \circ \Psi(\mathcal{L})$$

for all $\mathcal{U}, \mathcal{K}, \mathcal{L} \in \mathcal{F}$ *but* Ψ *is not an additive* $*$*-derivation.*

3. Corollaries

Let $\mathcal{F}$ be a ring. Then $\mathcal{F}$ is prime if for any $\mathcal{U}, \mathcal{K} \in \mathcal{F}$, whenever $\mathcal{U}\mathcal{F}\mathcal{K} = 0$ implies that either $\mathcal{U} = 0$ or $\mathcal{K} = 0$. If $\mathcal{F}$ is a prime ring, then its maximal symmetric ring of quotients exists and is denoted by $Q_{ms}(\mathcal{F})$. Also $Q_{ms}(\mathcal{F})$ is itself prime with $\mathcal{F} \subseteq Q_{ms}(\mathcal{F})$ and '$*$' can be uniquely extended to $Q_{ms}(\mathcal{F})$ (see [3]). Therefore, if $\mathcal{F}$ is unital and includes a non-trivial symmetric idempotent, then (2) holds automatically.

Theorem 3.1. *Consider* $\mathcal{F}$, *a 2-torsion free prime* $*$*-ring with unity and including a non-trivial symmetric idempotent. Suppose that a map* $\Psi : \mathcal{F} \to Q_{ms}(\mathcal{F})$ *satisfies*

$$\Psi(\mathcal{U} \Diamond \mathcal{K} \circ \mathcal{L}) = \Psi(\mathcal{U}) \Diamond \mathcal{K} \circ \mathcal{L} + \mathcal{U} \Diamond \Psi(\mathcal{K}) \circ \mathcal{L} + \mathcal{U} \Diamond \mathcal{K} \circ \Psi(\mathcal{L})$$

for all $\mathcal{U}, \mathcal{K}, \mathcal{L} \in \mathcal{F}$. *Then* Ψ *is an additive map. Furthermore, if* $\Psi(\mathcal{E}_i), i \in \{1, 2\}$ *is self adjoint, then* Ψ *is a* $*$*-derivation.*

Proof. It is clear from the definition of a prime ring that $\mathcal{F}$ satisfies condition (2). Thus by Theorem 2.1, Ψ is an additive $*$-derivation. □

Let $\mathcal{H}$ be a complex Hilbert space. Suppose F($\mathcal{H}$) and $\mathcal{Z}$ represent the standard operator algebra and von Neumann algebra respectively. Since every standard operator algebra is prime and it is also well known that if a von Neumann algebra has no central summands of type I_1, then $\mathcal{Z}$ fulfills the condition outlined in (2) of Theorem 2.1. Consequently, we have the following immediate corollaries.

Corollary 3.2. *Consider $\mathcal{F}$ as a standard operator algebra over $\mathcal{H}$, an infinite dimensional complex Hilbert space such that $\mathcal{F}$ possesses the identity and is closed under adjoint operation. Suppose that a map $\Psi : \mathcal{F} \to \mathcal{F}$ satisfies*

$$\Psi(\mathcal{U}\Diamond\mathcal{K} \circ \mathcal{L}) = \Psi(\mathcal{U})\Diamond\mathcal{K} \circ \mathcal{L} + \mathcal{U}\Diamond\Psi(\mathcal{K}) \circ \mathcal{L} + \mathcal{U}\Diamond\mathcal{K} \circ \Psi(\mathcal{L})$$

for all $\mathcal{U}, \mathcal{K}, \mathcal{L} \in \mathcal{F}$. Then Ψ is an additive map. Furthermore, if $\Psi(\mathcal{E}_i), i \in \{1, 2\}$ is self adjoint, then Ψ is a $*$-derivation.

Proof. A standard operator algebra, denoted as $\mathcal{F}$, which is a conventional operator algebra is a prime algebra that follows directly from the Hahn-Banach theorem. Since this algebra $\mathcal{F}$ inherently satisfies the condition (2). Therefore, we deduce, by Theorem 2.1 that Ψ is an additive $*$-derivation. □

Corollary 3.3. *Consider a von Neumann algebra $\mathcal{Z}$ having no central summands of type I_1. Suppose that a map $\Psi : \mathcal{Z} \to \mathcal{Z}$ satisfies*

$$\Psi(\mathcal{U}\Diamond\mathcal{K} \circ \mathcal{L}) = \Psi(\mathcal{U})\Diamond\mathcal{K} \circ \mathcal{L} + \mathcal{U}\Diamond\Psi(\mathcal{K}) \circ \mathcal{L} + \mathcal{U}\Diamond\mathcal{K} \circ \Psi(\mathcal{L})$$

for all $\mathcal{U}, \mathcal{K}, \mathcal{L} \in \mathcal{Z}$. Then Ψ is an additive map. Furthermore, if $\Psi(\mathcal{E}_i), i \in \{1, 2\}$ is self adjoint, then Ψ is a $*$-derivation.

Proof. Let $\mathcal{Z}$ be a von Neumann algebra. As shown in [2, 18], any von Neumann algebra $\mathcal{Z}$ without central summands of type I_1 satisfies condition (2). Consequently, by Theorem 2.1, it follows that the previously defined map is an additive $*$-derivation. □

Open Problem

A natural direction for future research is to examine the extent to which the principal results of this study (Theorem 2.1 and the related Lemmas) may be generalized to wider categories of algebraic structures, with particular attention to non-associative rings and algebras, including alternative rings, alternative algebras, W^*-algebras.

In the context of alternative rings, Ferreira and Ferreira established the following characterization of prime rings [8, Theorem 1.1],

Theorem 3.4. *Let $\mathcal{R}$ be a 3-torsion-free alternative ring. Then $\mathcal{R}$ is a prime ring if and only if $\mathcal{U}\mathcal{R}.\mathcal{V} = 0$ or $\mathcal{U}.\mathcal{R}\mathcal{V} = 0$ implies $\mathcal{U} = 0$ or $\mathcal{V} = 0$, $\forall\ \mathcal{U}, \mathcal{V} \in \mathcal{R}$.*

It is well known that the 3-torsion-free condition is unnecessary in the case of associative rings. This raises an interesting open question:
Can the main results of our work be extended to non-associative settings, particularly to alternative rings and other structured algebras, without additional torsion-free assumptions? or $\Psi(\mathcal{E}_i), i \in \{1, 2\}$ is self adjoint?
Investigating this problem could lead to new insights into the structural properties of non-associative algebras and their prime ideals, potentially uncovering deeper connections between associative and non-associative algebraic systems. So we have mentioned a conjecture for future work in this direction as given below.
Conjecture 1 (Extension to alternative $*$-rings). Consider a 2-torsion free alternative $*$-ring $\mathcal{F}$ with unity I and a symmetric idempotent $\mathcal{E}_1 \neq 0, I$ such that

$$\mathcal{U}\mathcal{F}\mathcal{E}_t = 0 \ \Rightarrow\ \mathcal{U} = 0,\ t \in \{1, 2\} \tag{5}$$

where $\mathcal{E}_2 = I - \mathcal{E}_1, \mathcal{U} \in \mathcal{F}$. Suppose that a map $\Psi : \mathcal{F} \to \mathcal{F}$ (not necessarily additive) satisfies

$$\Psi(\mathcal{U}\Diamond\mathcal{K} \circ \mathcal{L}) = \Psi(\mathcal{U})\Diamond\mathcal{K} \circ \mathcal{L} + \mathcal{U}\Diamond\Psi(\mathcal{K}) \circ \mathcal{L} + \mathcal{U}\Diamond\mathcal{K} \circ \Psi(\mathcal{L})$$

for all $\mathcal{U}, \mathcal{K}, \mathcal{L} \in \mathcal{F}$. Then Ψ is an additive map. Furthermore, if $\Psi(\mathcal{E}_i), i \in \{1, 2\}$ is self adjoint, then Ψ is a $*$-derivation.

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