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Amenability and inner amenability of transformation groups

Ali Ebadian[a], **Ali Jabbari**[b,*]

[a]*Department of Mathematics, Faculty of Science, Urmia University, Urmia, Iran*
[b]*Kyrgyz-Turkish Manas University, Bishkek, Kyrgyzstan*

Abstract. In this paper, we investigate the amenability of transformation groups constructed by the semidirect product of groups. We introduce the inner amenability of transformation groups and characterize this property. As an interesting result, we show that every inner amenable transformation group has property (W).

Let G be a locally compact group and X be a locally compact space. Then X is called a transformation group if it is a left G-space, i.e., $(x, s) \longmapsto s \cdot x$ is a continuous left action from $X \times G \longrightarrow X$. Note that X is a unitary G-space, i.e., for any $x \in X$, we have $e_G \cdot x = x$. Following [1], if $g \in C_C(X \times G)$, then g^x will be the map $t \longmapsto g^x(t) = g(x, t)$, and $g(t)$ will be the map $x \longmapsto g(t)(x) = g(x, t)$.

The transformation group (X, G) (or the G-action on X, or the G-space X) is amenable if there is a net $(m_i)_{i\in I}$ of continuous maps $x \longmapsto m_i^x$ from X into the space $\mathrm{Prob}(G)$ (the set of probability measures on G, equipped with the weak*-topology) such that

$$\lim_i \|sm_i^x - m_i^{sx}\|_1 = 0,$$

uniformly on compact subsets of $X \times G$. Such a net $(m_i)_{i\in I}$ will be called an approximate invariant continuous mean (a.i.c.m. for short).

The amenability of transformation groups and semigroups are investigated in many literatures that we refer to [1, 2, 11, 16, 17], for more details. In [1, 2], Anantharaman-Delaroche by characterizing amenability of transformation groups gave some applications related to amenability of C^*-dynamical systems, nuclearity of the corresponding crossed products and operator algebras.

A locally compact group G is amenable if there is a left invariant mean on $L^\infty(G)$, where $L^\infty(G)$ denotes the Banach space of complex-valued essentially bounded functions on G with respect to the Haar measure λ_G and equipped with the essential supremum norm. A left invariant mean is a functional m in $L^\infty(G)^*$ such that $\|m\| = 1$ and for all $\phi \in L^\infty(G)$, $m(l_s\phi) = m(\phi)$ for every $s \in G$, where $l_s\phi(t) = \phi(s^{-1}t)$, for all $s, t \in G$. The left and right actions of G on elements of $L^1(G)$ that they are called the left and right translations are defined as follows:

$$l_s f(t) = f(s^{-1}t) =_s f(t), \quad r_s f(t) = f(ts) = f_s(t),$$

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* Corresponding author: Ali Jabbari
Email addresses: ebadian.ali@gmail.com (Ali Ebadian), ali.jabbari@manas.edu.kg, jabbari_al@yahoo.com (Ali Jabbari)
ORCID iDs: https://orcid.org/0000-0003-4067-6729 (Ali Ebadian), https://orcid.org/0000-0003-4273-1998 (Ali Jabbari)

for all $f \in L^1(G)$ and $s, t \in G$. For more about amenability of locally compact groups and equivalence relations to the amenability of them, we refer to [14].

A locally compact group G is called inner amenable if there is a linear functional m on $L^\infty(G)$ such that $m({}_{s^{-1}}f_s) = m(f)$, for all $f \in L^\infty(G)$ and $s \in G$. The class of inner amenable locally compact groups includes all amenable and [IN]-groups. See [5–7, 12, 13, 18, 19, 21, 22], for inner amenable locally compact groups and its applications. Moreover, a new version of inner amenability of semigroups namely *character inner amenability of semigroups* is defined in [4]. Let S be a semigroup, $\Delta(S)$ be the set of characters on the semigroup S and $\varphi \in \Delta(S)$. Then S is called *φ-inner amenable* if there is a linear functional $m \in \ell^\infty(S)^*$ such that $m(s \cdot f) = m(f \cdot s)$ and $m(\varphi) = 1$, for every $s \in S_\varphi$ and $f \in \ell^\infty(S)$, where $S_\varphi = \{s \in S : \varphi(s) = 1\}$. If, for every $\varphi \in \Delta(S)$, S is φ-inner amenable, then we say it is character inner amenable. We call a linear functional $m \in \ell^\infty(S)^*$ which satisfies in the above conditions, *φ-inner mean* on $\ell^\infty(S)$.

In this work, in the next Section, we study on the amenability of transformation groups constructed by semidirect product of locally compact groups. In Section 3, we define inner amenability of transformation groups and characterize this new notion which our aim for this definition is giving an answer to the following question:

Which transformation groups have property (W)?

1. Amenability of Transformation Groups

We recall the following result from [1]:

Proposition 1.1. *The following conditions are equivalent:*

1. (X, G) *is an amenable transformation group.*
2. *There is a net* $(g_i)i \in I$ *of nonnegative continuous functions on* $X \times G$ *such that*
   (a) *for every* $i \in I$ *and* $x \in X$, $\int_G g_i^x(t)\, d\lambda_G(t) = 1$;
   (b) $\lim_i \int_G \left| g_i^{sx}(st) - g_i^x(t) \right| \, d\lambda_G(t) = 0$ *uniformly on compact subsets of* $X \times G$.
3. *There is a net* $(g_i)i \in I$ *in* $C_C(X \times G)^+$ *such that*
   (a) $\lim_i \int_G g_i^x(t)\, d\lambda_G(t) = 1$ *uniformly on compact subsets of* X;
   (b) $\lim_i \int_G \left| g_i^{sx}(st) - g_i^x(t) \right| \, d\lambda_G(t) = 0$ *uniformly on compact subsets of* $X \times G$.

Let N and H be two locally compact groups with H acting on N; i.e., there is a group homomorphism τ from H to $\mathrm{Aut}(N)$, such that $(n, h) \longrightarrow \tau_h(n)$ is continuous with respect to the product topology on $N \times H$, where $\mathrm{Aut}(N)$ is the group of continuous group automorphisms of N.

We say that $G := N \rtimes_\tau H$ is the *semidirect product* of N and H with respect to τ if G is the group consisting of elements of the form (n, h), where $n \in N$ and $h \in H$, equipped with multiplication given by:

$$(n_1, h_1) * (n_2, h_2) = (n_1 \tau_{h_1}(n_2), h_1 h_2).$$

If G is equipped with the product topology, then G is a locally compact group. Let λ_N and λ_H be two left Haar measures on N and H, respectively, then the left Haar measure on G is as follows:

$$d\lambda_G(n, h) = \sigma(h) d\lambda_N(n) d\lambda_H(h),$$

where $\sigma(h) = \frac{\lambda_N(A)}{\lambda_N(\tau_h(A))}$, for each measurable $A \subset H$ and σ is a continuous group homomorphism from H into $[0, \infty)$, see [8], for more details.

The action related to τ defines a linear isometry and convolution preserving map $T : H \longrightarrow B(L^1(N))$ by $h \longmapsto T_h$ such that $T_h(f)(n) = f(\tau_{h^{-1}}(n))\sigma(h)$, for all $f \in L^1(N)$, $n \in N$ and $h \in H$. Amenability and the topics equivalent to it such as constructions of Følner and Rieter nets on the semidirect product of groups and semigroups are studied in [9, 10, 20].

Let H, N be two locally compact groups and X be a locally compact space such that X is both a left H-space and a left N-space such that X is a trivial left N-space, i.e., $nx = x$, for all $n \in N$ and $x \in X$. Suppose that $G := N \rtimes_\tau H$ is the semidirect product of N and H with respect to τ. Then $X \times X$ by the following action becomes a G-space

$$(n, h) \cdot (x, y) = (x, hy),$$

for all $x, y \in X$ and $(n, h) \in G$.

If (X, N) and (X, H) are amenable transformation groups, then Proposition 1.1(2) implies that there are nets $(f_i)_{i\in I}$ and $(g_j)_{j\in J}$ of nonnegative continuous functions on $X \times N$ and $X \times H$, respectively, such that $\int_N f_i^x(n) d\lambda_N(n) = 1$, $\int_H g_j^x(h) d\lambda_H(h) = 1$, for all $i \in I$, $j \in J$, $x \in X$, and $\lim_i \int_N |f_i^{sx}(sn) - f_i^x(n)| d\lambda_N = 0$, $\lim_i \int_H |g_j^{tx}(th) - g_j^x(h)| d\lambda_H = 0$ uniformly on compact subsets of $X \times N$ and $X \times H$, respectively.

By keeping in mind the above discussion, we have the following result.

Theorem 1.2. *Let (X, N) and (X, H) be amenable transformation groups such that $nx = x$, for all $n \in N$ and $x \in X$. Consider the following assertions:*

(i) There is a net $(e_{i,j})_{i\in I, j\in J}$ for all $i \in I$ and $j \in J$ on $X \times X \times G$ is as $e_{i,j}(x, y, n, h) = f_i(x, n) g_j(y, h) \sigma^{-1}(h)$ for every $x, y \in X$, $n \in N$ and $h \in H$ such that

(a) $\int_G e_{i,j}^{(x,y)}(r)\, d\lambda_G(r) = 1$ for every $i \in I$, $j \in J$, $x, y \in X$ and $r \in G$.

(b) For every $(s, t), a \in G$ and $x, y \in X$,

$$\lim_{i,j} \int_G \left| e_{i,j}^{(s,t)\cdot(x,y)}((s, t) * a) - e_{i,j}^{(x,y)}(a) \right| \, d\lambda_G(a) = 0.$$

(ii) $\|T_t f_i^x - f_i^x\|_{L^1(N)} \longrightarrow 0$ uniformly in t on compact subsets of H.

(iii) $(X \times X, G)$ is an amenable transformation group.

Then (i) $\longleftrightarrow$ (ii) $\longleftrightarrow$ (iii).

Proof. (ii)$\longrightarrow$(i) According to the definition of $(e_{i,j})_{i\in I, j\in J}$, it is a net of nonnegative continuous functions on $X \times G$ such that

$$\begin{aligned} \int_G e_{i,j}^{(x,y)}(r)\, d\lambda_G(r) &= \int_N \int_H f_i^x(n) g_j^y(h)\, d\lambda_H(h)\, d\lambda_N(n) \\ &= \int_N f_i^x(n) \left(\int_H g_j^y(h)\, d\lambda_H(h) \right) d\lambda_N(n) \\ &= 1, \end{aligned} \tag{1}$$

and

$$\begin{aligned} \int_G \left| e_{i,j}^{(s,t)\cdot(x,y)}((s, t) * a) - e_{i,j}^{(x,y)}(a) \right| \, d\lambda_G(a) &= \int_G \left| e_{i,j}^{(x,ty)}((s\tau_t(n), th)) - e_{i,j}^{(x,y)}(a) \right| \, d\lambda_G(n, h) \\ &= \int_N \int_H \left| l_{s^{-1}} T_{t^{-1}} f_i^x(n) g_j^{ty}(th) - f_i^x(n) g_j^y(h) \right| d\lambda_H(h) d\lambda_N(n) \\ &\le \int_N \int_H \left| l_{s^{-1}} T_{t^{-1}} f_i^x(n) g_j^{ty}(th) - l_{s^{-1}} f_i^x(n) g_j^{ty}(th) \right| d\lambda_H(h) d\lambda_N(n) \\ &\quad + \int_N \int_H \left| l_{s^{-1}} f_i^x(n) g_j^{ty}(th) - f_i^x(n) g_j^{ty}(th) \right| d\lambda_H(h) d\lambda_N(n) \\ &\quad + \int_N \int_H \left| f_i^x(n) g_j^{ty}(th) - f_i^x(n) g_j^y(h) \right| d\lambda_H(h) d\lambda_N(n) \end{aligned}$$

$$
\begin{aligned}
&= \int_N \int_H \left| l_{s^{-1}} \left(T_{t^{-1}} f_i^x(n) - f_i^x(n) \right) g_j^{ty}(th) \right| d\lambda_H(h) d\lambda_N(n) \\
&\quad + \int_N \int_H \left| \left(f_i^x(sn) - f_i^x(n) \right) g_j^{ty}(th) \right| d\lambda_H(h) d\lambda_N(n) \\
&\quad + \int_N \int_H \left| f_i^x(n) \left(g_j^{ty}(th) - g_j^y(h) \right) \right| d\lambda_H(h) d\lambda_N(n) \\
&\leq \| l_{s^{-1}} \left(T_{t^{-1}} f_i^x - f_i^{sx} \right) \|_{L^1(N)} \| g_j^{ty} \|_{L^1(H)} \\
&\quad + \int_N \left| f_i^{sx}(sn) - f_i^x(n) \right| d\lambda_N(n) \| g_j^{ty} \|_{L^1(H)} \\
&\quad + \| f_i^x \|_{L^1(N)} \int_H \left| g_j^{ty}(th) - g_j^y(h \right| d\lambda_H(h).
\end{aligned}
$$

This implies that

$$
\lim_{i,j} \int_G \left| e_{i,j}^{(s,t)\cdot(x,y)}((s,t) * a) - e_{i,j}^{(x,y)}(a) \right| \, d\lambda_G(a) = 0. \tag{2}
$$

Thus, (1) and (2) together with Proposition 1.1(2) imply that $(X \times X, G)$ is an amenable transformation group, this means that (i)⟶(iii) holds.

(i)→(ii) Assume that $(e_{i,j})_{i\in I, j\in J}$, $(f_i)_{i\in I}$ and $(g_j)_{i\in J}$ are as given. From $\int_H g_j^x(h) d\lambda_H(h) = 1$, for all $j \in J$ and $x \in X$,

$$
\begin{aligned}
\| T_t f_i^x - f_i^x \|_{L^1(N)} &= \int_N \left| T_t f_i^x(n) - f_i^x(n) \right| \, d\lambda_N(n) \\
&= \int_N \left| T_t f_i^x(n) - f_i^x(n) \right| \left(\int_H g_j^x(h) d\lambda_H(h) \right) \, d\lambda_N(n) \\
&= \int_N \int_H \left| T_t f_i^x(n) g_j^x(h) - f_i^x(n) g_j^x(h) \right| d\lambda_H(h) \, d\lambda_N(n) \\
&\leq \int_N \int_H \left| T_t f_i^x(n) g_j^x(h) - T_t f_i^x(n) g_j^{tx}(th) \right| d\lambda_H(h) \, d\lambda_N(n) \\
&\quad + \int_N \int_H \left| T_t f_i^x(n) g_j^{tx}(th) - f_i^x(n) g_j^x(h) \right| d\lambda_H(h) \, d\lambda_N(n) \\
&\leq \| T_t f_i^x \|_{L^1(N)} \int_H \left| g_j^x(h) - g_j^{tx}(h) \right| d\lambda_H(h) \, d\lambda_N(n) \\
&\quad + \int_G \left| e_{i,j}^{(e_N,t)\cdot(x,y)}((e_N,t) * (n,h)) - e_{i,j}^{(x,y)}(n,h) \right| \, d\lambda_G(n,h).
\end{aligned}
$$

Thus, the proof completes.

(iii)⟶(ii) Assume that $(X \times X, G)$ is an amenable transformation group, then there is a net $(E_i)_{i\in I}$ such that

$$
\int_G E_i^{(x,y)}(a) \, d\lambda_G(a) = 1, \tag{3}
$$

and

$$
\lim_i \int_G \left| E_i^{(s,t)\cdot(x,y)}((s,t) * a) - E_i^{(x,y)}(a) \right| \, d\lambda_G(a) = 0, \tag{4}
$$

for all $i \in I$, $x, y \in X$ and $a, (s,t) \in G$. For every $i \in I$, define

$$
f_i(x,n) = \int_H E_i^{(x,y)}(n,h) \sigma(h) \, d\lambda_H(h),
$$

for all $x \in X$ and $n \in N$. Clearly, $(f_i)_{i\in I}$ is a net of nonnegative continuous functions on $X \times N$. Moreover

$$\begin{aligned}\int_N f_i^x(n)\, d\lambda_N(n) &= \int_N \int_H E_i^{(x,y)}(n,h)\sigma(h)\, d\lambda_H(h)\, d\lambda_N(n) \\ &= \int_G E_i^{(x,y)}(n,h)\, d\lambda_G(n,h) = 1,\end{aligned} \tag{5}$$

and

$$\begin{aligned}\int_N \left|f_i^{sx}(sn) - f_i^x(n)\right|\, d\lambda_N(n) &= \int_N \left|\int_H \left(E_i^{(s,e_H)\cdot(x,y)}(sn,h) - E_i^{(x,y)}(n,h)\right)\sigma(h)\, d\lambda_H(h)\right|\, d\lambda_N(n) \\ &\leq \int_G \left|E_i^{(s,e_H)\cdot(x,y)}(sn,h) - E_i^{(x,y)}(n,h)\right| d\lambda_G(n,h).\end{aligned} \tag{6}$$

From (4) we have

$$\lim_i \int_N \left|f_i^{sx}(sn) - f_i^x(n)\right|\, d\lambda_N(n) = 0. \tag{7}$$

For every $t \in H$,

$$\begin{aligned}\|T_t f_i^x - f_i^x\|_{L^1(N)} &= \int_N \left|T_t f_i^x(n) - f_i^x(n)\right|\, d\lambda_N(n) \\ &= \int_N \left|f_i^x(\tau_{t^{-1}}(n))\sigma(t) - f_i^x(n)\right|\, d\lambda_N(n) \\ &= \int_N \left|\int_H \left(E_i^{(x,y)}(\tau_{t^{-1}}(n),h)\sigma(t) - E_i^{(x,y)}(n,h)\right)\sigma(h)d\lambda_H(h)\right| d\lambda_N(n) \\ &\leq \int_N \int_H \left|E_i^{(x,y)}(\tau_{t^{-1}}(n),h)\sigma(t) - E_i^{(x,y)}(n,h)\right| \sigma(h)\, d\lambda_H(h)d\lambda_N(n) \\ &= \int_N \int_H \left|E_i^{(x,y)}(\tau_{t^{-1}}(n),t^{-1}h) - E_i^{(x,y)}(n,h)\right| \sigma(h)\, d\lambda_H(h)\, d\lambda_N(n) \\ &= \int_G \left|E_i^{(x,y)}(\tau_{t^{-1}}(n),t^{-1}h) - E_i^{(x,y)}(n,h)\right|\, d\lambda_G(n,h) \\ &= \int_G \left|E_i^{(e_N,t^{-1})\cdot(x,y)}((e_N,t^{-1}) * (n,h)) - E_i^{(x,y)}(n,h)\right|\, d\lambda_G(n,h).\end{aligned}$$

Again by (4) we have

$$\lim_i \|T_t f_i^x - f_i^x\|_{L^1(N)} = 0,$$

uniformly in t^{-1} on compact subsets of H. This completes the proof. □

Let $C_0(X)$ be the space of all bounded functions on X vanishes at infinity and (X,G) be a transformation group. The space $C_0(X)$ is a well-known C^*-algebra that is a G-C^*-algebra with the following action: $s \cdot f(x) = f(s^{-1}x)$, for every $s \in G$, $x \in X$ and $f \in C_0(X)$. The corresponding crossed products are denoted by $C(X \rtimes G)$ and $C_r(X \rtimes G)$ respectively. Note that, when X is a point, then the above crossed products are the full and reduced C^*-algebras of G, denoted by $C^*(G)$ and $C_r^*(G)$, respectively.

Let G be a locally compact group, N be a closed normal subgroup of G and H be a closed subgroup such that $N \cap H = \{e_G\}$ and $G = NH$. We now, define $\tau : H \longrightarrow \mathrm{Aut}(N)$ by $\tau(h)(n) = hnh^{-1}$, for all $h \in H$ and $n \in N$. Also, there is a map such as $\varphi : N \rtimes_\tau H \longrightarrow G$ defined by $(n,h) \mapsto nh$ which is an isomorphism between locally compact groups. This means that we can suppose that $G = N \rtimes_\tau H$. Let λ_N and λ_H be the Haar measures on N and H, respectively. Then a Haar measure on G is given by

$$\int_H \int_N f(nh)\sigma_H(h)^{-1} d\lambda_N(n) d\lambda_H(h) = \int_H \int_N f(hn) d\lambda_N(n) d\lambda_H(h).$$

As a result of [1, Theorem 5.3], we have the following result:

Proposition 1.3. *Let G, N and H be as the above. Let (X, N) and (X, H) be two transformation groups. If $(X \times X, G)$ is amenable transformation group, then $C_r^*((X \times X) \rtimes G)$ is nuclear.*

2. Inner Amenability of Transformation Groups

In this section, by a transformation group (X, G), we mean the both left and right actions, i.e., $(x, s) \longmapsto s \cdot x$ and $(x, s) \longmapsto x \cdot s$ are continuous. We define the inner amenability of transformation group (X, G) and characterize the inner amenability of locally compact groups that is different from [3]. As we mentioned in the first section, our aim in this section is to show that the inner amenable transformation groups have the property (W).

Definition 2.1. *Let (X, G) be a transformation group. We say that (X, G) is inner amenable if there is a net $(m_\alpha)_{\alpha \in I}$ of continuous maps $x \longmapsto m_\alpha^x$ from X into the space $\mathrm{Prob}(G)$ such that*

$$\lim_\alpha \|m_\alpha^{sx} s^{-1} - s^{-1} m_\alpha^{xs}\|_1 = 0, \tag{8}$$

uniformly on compact subsets of $X \times G$.

Similar to Proposition 1.1 and its proof we have the following result.

Proposition 2.2. *The following conditions are equivalent:*

(i) (X, G) *is an inner amenable transformation group.*

(ii) There is a net $(f_\alpha)_{\alpha \in I}$ of nonnegative continuous functions on $X \times G$ such that

- *(a) for all $\alpha \in I$ and $x \in X$, $\int_G f_\alpha^x(t)\, d\lambda_G(t) = 1$;*
- *(b)* $\lim_\alpha \int_G \left| f_\alpha^{sx}(s^{-1}t) - f_\alpha^{xs}(ts^{-1}) \right| d\lambda_G(t) = 0$ *uniformly on compact subsets of $X \times G$.*

(iii) There is a net $(f_\alpha)_{\alpha \in I}$ in $C_C(X \times G)^+$ such that

- *(a)* $\lim_\alpha \int_G f_\alpha^x(t)\, d\lambda_G(t) = 1$ *uniformly on compact subsets of X;*
- *(b)* $\lim_\alpha \int_G \left| f_\alpha^{sx}(s^{-1}t) - f_\alpha^{xs}(ts^{-1}) \right| d\lambda_G(t) = 0$ *uniformly on compact subsets of $X \times G$.*

Proof. (i)⟶(ii) Let $(m_\alpha)_{\alpha \in I}$ be a net of continuous maps $x \longmapsto m_i^x$ from X into the space Prob(G) such that

$$\lim_\alpha \|m_\alpha^{sx} s^{-1} - s^{-1} m_\alpha^{xs}\|_1 = 0 \tag{9}$$

uniformly on compact subsets of $X \times G$. Pick $f \in C_C(G)^+$ such that $\int_G f(s)\, d\lambda_G(s) = 1$. Define

$$f_\alpha(x, s) = \int_G f(t^{-1}s)\, dm_\alpha^x(t),$$

for every $(x, s) \in (X, G)$. Then, for all $x \in X$ and $\alpha \in I$,

$$\begin{aligned} \int_G f_\alpha^x(s)\, d\lambda_G(s) &= \int_G \int_G f(t^{-1}s)\, dm_\alpha^x(t)\, d\lambda_G(s) \\ &= \int_G \int_G f(t^{-1}s)\, d\lambda_G(s)\, dm_\alpha^x(t) \\ &= 1. \end{aligned} \tag{10}$$

Thus (10) implies (a). For any $s \in G$ and $x \in X$,

$$\begin{aligned}\int_G \left|f_\alpha^{sx}(s^{-1}t) - f_\alpha^{xs}(ts^{-1})\right| \, d\lambda_G(t) &= \int_G \left|\int_G f(u^{-1}s^{-1}t)\, dm_\alpha^{sx}(u) - \int_G f(u^{-1}ts^{-1})\, dm_\alpha^{xs}(u)\right| \, d\lambda_G(t)\\ &= \int_G \left|\int_G f(u^{-1}s^{-1}ts^{-1})\, d(m_\alpha^{sx}s^{-1})(u) - \int_G f(u^{-1}s^{-1}ts^{-1})\, d(s^{-1}m_\alpha^{xs})(u)\right| \, d\lambda_G(t)\\ &\leq \int_G \int_G f(u^{-1}s^{-1}ts^{-1})\, d\left|m_\alpha^{sx}s^{-1} - s^{-1}m_\alpha^{xs}\right|(u)\, d\lambda_G(t)\\ &= \int_G \left(\int_G f(u^{-1}s^{-1}ts^{-1})\, d\lambda_G(t)\right) d\left|m_\alpha^{sx}s^{-1} - s^{-1}m_\alpha^{xs}\right|(u)\\ &\leq \|m_\alpha^{sx}s^{-1} - s^{-1}m_\alpha^{xs}\|_1. \end{aligned} \tag{11}$$

Then by (9) and (11) we obtain (b).

(ii)⟶(i) Let $(f_\alpha)_{i\in I}$ be as (ii). Define $m_\alpha^x = f_\alpha^x \lambda_G$, for all $\alpha \in I$ and $x \in X$. Then $dm_\alpha^x(t) = f_\alpha^x(t)\, d\lambda_G(t)$, for all $\alpha \in I$, $t \in G$ and $x \in X$. Applying (a) shows that m_α^x is a probability measure on G, for all $\alpha \in I$ and $x \in X$. Moreover, (b) implies that

$$\lim_\alpha \|m_\alpha^{sx}s^{-1} - s^{-1}m_\alpha^{xs}\|_1 = 0$$

uniformly on compact subsets of $X \times G$.

(ii)⟶(iii) Since compactly supported functions are dense in continuous functions, the proof is clear.

(iii)⟶(ii) Similar to the proof of Proposition 1.1, define

$$f_{\alpha,n} = \frac{f_\alpha(x,s) + \frac{1}{n}f(s)}{\int_G f_\alpha(x,t)\, d\lambda_G(t) + \frac{1}{n}},$$

where $(f_\alpha)_{\alpha\in I}$ is a net that satisfies (iii) and $f \in C_C(G)^+$. □

Remark 2.3. 1. *Let (X, G) be an inner amenable transformation group such that X is a point. Proposition 2.2 implies that there is a net $(f_\alpha)_{\alpha\in I}$ such that $\int_G f_\alpha(t)\, d\lambda_G(t) = 1$ and $\int_G \left|f_\alpha(st) - f_\alpha(ts)\right| d\lambda_G(t) \longrightarrow 0$ uniformly on compact subsets of $X \times G$. This implies that*

$$\begin{aligned}\int_G \left|f_\alpha(s^{-1}t) - f_\alpha(ts^{-1})\right| \, d\lambda_G(t) &= \int_G \left|l_s f_\alpha(t) - r_s f_\alpha(t)\right| \, d\lambda_G(t)\\ &= \int_G \left|(\delta_s * f_\alpha)(t) - (f_\alpha * \delta_s)(t)\right| \, d\lambda_G(t)\\ &= \|\delta_s * f_\alpha - f_\alpha * \delta_s\|_1.\end{aligned}$$

This shows that $\|\delta_s * f_\alpha - f_\alpha * \delta_s\|_1$ tends to 0 uniformly on compact subsets of G. Thus G is inner amenable [13, Proposition 1].

2. *Let G be an inner amenable locally compact group and X be a locally compact G-space. Inner amenability of G implies that there is a net $(f_\alpha)_{\alpha\in I}$ of probability measures such that $\|\delta_s * f_\alpha - f_\alpha * \delta_s\|_1 \longrightarrow 0$ for every $s \in G$. Now, we define m_α on $X \times G$ such that its value on X is a constant value and on G is equal to $f_\alpha\, d\lambda_G$. Thus, (m_α) satisfies in (8). This means that every transformation group (X, G) is inner amenable whenever G is inner amenable.*

Example 2.4. 1. *We give an example of transformation groups which show that the converse of 2.3(2), in general, is not true. Let $\mathbb{F}_2$ be the free group with two generators a and b and $\partial\mathbb{F}_2$ be the boundary of $\mathbb{F}_2$ that is the set of all infinite reduced words $\omega = a_1a_2\cdots a_n\cdots$ in the alphabet $S = \{a, a^{-1}, b, b^{-1}\}$. Suppose that $X = \partial\mathbb{F}_2$, then $(X, \mathbb{F}_2)$ becomes a transformation group. By [1, Example 2.7(4)], $(X, \mathbb{F}_2)$ is inner amenable. But $\mathbb{F}_2$ is not inner amenable [14, Proposition 22.38].*

2. *Let $N \in \mathbb{N}$ such that $N \geq 2$. The generalized Thompson group $F(N)$ is the set of piecewise linear homeomorphisms from the closed unit interval $[0,1]$ to itself that are differentiable except at finitely many N-adic rationals and such that at intervals of differentiability the derivatives are powers of N. This group is inner amenable [15, Corollary 3.7]. Thus, by Remark 2.3(2), for any $F(N)$-space X, the transformation group $(X, F(N))$ is inner amenable.*

Corollary 2.5. *Let G be a unimodular locally compact group. Then the transformation group (X,G) is inner amenable if and only if there is a net $(\xi_\alpha)_{\alpha\in I}$ in $C_C(X\times G)^+$ such that the following statements hold:*

(i) $\lim_\alpha \int_G |\xi_\alpha(x,t)|^2\, d\lambda_G(t) = 1$ *uniformly on compact subsets of X;*

(ii) $\lim_\alpha \int_G \left|\xi_\alpha(sx,s^{-1}t) - \xi_\alpha(xs,ts^{-1})\right|^2 d\lambda_G(t) = 0$ *uniformly on compact subsets of $X\times G$.*

Proof. Suppose that (X,G) is inner amenable. Then by Proposition 2.2(iii) there is a net $(f_\alpha)_{\alpha\in I}$ such that satisfies the implications (a) and (b). Set $\xi_\alpha = \sqrt{f_\alpha}$, for any $\alpha \in I$ and define

$$h_\alpha(x,s) = \int_G \overline{\xi_\alpha(sx,s^{-1}t)}\xi_\alpha(xs,ts^{-1})d\lambda_G(t),$$

for all $(x,s)\in X\times G$. Then, for any $x\in X$ and $s\in G$, we have,

$$h_\alpha(sx,e_G) = \int_G \overline{\xi_\alpha(sx,t)}\xi_\alpha(sx,t)d\lambda_G(t) = \int_G \overline{\xi_\alpha(sx,s^{-1}t)}\xi_\alpha(sx,s^{-1}t)d\lambda_G(t)$$

and

$$h_\alpha(xs,e_G) = \int_G \overline{\xi_\alpha(xs,t)}\xi_\alpha(xs,t)d\lambda_G(t) = \int_G \overline{\xi_\alpha(xs,ts^{-1})}\xi_\alpha(xs,ts^{-1})d\lambda_G(t).$$

Then

$$h_\alpha(sx,e_G) + h_\alpha(xs,e_G) - 2h_\alpha(x,s) = \int_G \left|\xi_\alpha(sx,s^{-1}t) - \xi_\alpha(xs,xs^{-1})\right|^2 d\lambda_G(t).$$

By the above equality we get

$$\begin{aligned}\int_G \left|\xi_\alpha(sx,s^{-1}t) - \xi_\alpha(xs,xs^{-1})\right|^2 d\lambda_G(t) &\leq \int_G \left|\xi_\alpha(sx,s^{-1}t)^2 - \xi_\alpha(xs,xs^{-1})^2\right| d\lambda_G(t)\\ &= \int_G \left|f_\alpha(sx,s^{-1}t) - f_\alpha(xs,xs^{-1})\right| d\lambda_G(t)\\ &= \int_G \left|f_\alpha^{sx}(s^{-1}t) - f_\alpha^{xs}(xs^{-1})\right| d\lambda_G(t).\end{aligned}$$

Then by the phrase (b) of Proposition 2.2(iii), the statement (ii) holds. We, also for every $x\in X$, have

$$\begin{aligned}h_\alpha(x,e_G) &= \int_G \overline{\xi_\alpha(x,t)}\xi_\alpha(x,t)d\lambda_G(t) = \int_G |\xi_\alpha(x,t)|^2\, d\lambda_G(t)\\ &= \int_G f_\alpha(x,t)d\lambda_G(t) = \int_G f_\alpha^x(t)d\lambda_G(t).\end{aligned}$$

By the phrase (a) of Proposition 2.2(iii), the statement (i) holds.

Conversely, suppose that there is a net $(\xi_\alpha)_{\alpha\in I}$ in $C_C(X\times G)$ such that satisfies (i) and (ii). Set $f_\alpha = |\xi_\alpha|^2$ for any $\alpha\in I$. Then, the statement (a) of Proposition 2.2 holds. By the Cauchy-Schwarz inequality, we have

$$\int_G \left|f_\alpha^{sx}(s^{-1}t) - f_\alpha^{xs}(xs^{-1})\right| d\lambda_G(t) = \int_G \left|f_\alpha(sx,s^{-1}t) - f_\alpha(xs,xs^{-1})\right| d\lambda_G(t)$$

$$\begin{aligned} &= \int_G \left| \left|\xi_\alpha(sx, s^{-1}t)\right|^2 - \left|\xi_\alpha(xs, xs^{-1})\right|^2 \right| d\lambda_G(t) \\ &\leq \left(\int_G \left(\left|\xi_\alpha(sx, s^{-1}t)\right| + \left|\xi_\alpha(xs, xs^{-1})\right| \right)^2 d\lambda_G(t) \right)^{\frac{1}{2}} \\ &\quad \left(\int_G \left|\xi_\alpha(sx, s^{-1}t) - \xi_\alpha(xs, xs^{-1})\right|^2 d\lambda_G(t) \right)^{\frac{1}{2}}. \end{aligned}$$

Then by the statement (ii), on compact subsets, we have,

$$\lim_\alpha \int_G \left| f_\alpha^{sx}(s^{-1}t) - f_\alpha^{xs}(xs^{-1}) \right| d\lambda_G(t) = 0.$$

Thus, the statement (iii) of Proposition 2.2 holds. Hence, (X, G) is inner amenable. □

Following [1], we call a complex-valued function h defined on the transformation group $X \times G$ a positive type function if, for every $x \in X$, $n \in \mathbb{N}$, $t_1, \dots, t_n \in G$ and $\alpha_1, \dots, \alpha_n \in \mathbb{C}$, we have

$$\sum_{i,j} h(t_i^{-1}x, t_i^{-1}t_j)\overline{\alpha_i}\alpha_j \geq 0.$$

Let (X, G) be a transformation group. Then with the following action the product space $X \times X$ becomes a $G \times G$-space:

$$(s, t) \cdot (x, y) = (sx, ty),$$

for every $(x, y) \in X \times X$ and $(s, t) \in G \times G$. Then, according to the definition of positive type function on $X \times G$, a complex-valued function h defined on $X \times X \times G \times G$ is called positive type function if, for every $(x, y) \in X \times X$, $n \in \mathbb{N}$ and $(s_1, t_1), \dots, (s_n, t_n) \in G \times G$, the matrix $\left[h(s_i^{-1}x, t_i^{-1}y, s_i^{-1}s_j, t_i^{-1}t_j)\right]$ is positive. A closed subset B of $(X \times G) \times (X \times G)$ is called proper with respect to projections or π-proper if for every compact subset K of $x \times G$, the sets $[K \times (X \times G)] \cap B$ and $[(X \times G) \times K] \cap B$ are compact [1]. A continuous function h on $(X \times G) \times (X \times G)$ is called properly supported if its support is π-proper. The property (W) is introduced in [1, Definition 4.3]; a transformation group (X, G) has property (W) if, for every compact subset K of $X \times G$ and every $\varepsilon > 0$ there is a continuous bounded positive type, properly supported, function h on $(X \times G) \times (X \times G)$ such that $|h(x, t, x, t) - 1| \leq \varepsilon$, for all $(x, t) \in K$.

Corollary 2.6. *Every inner amenable transformation group has property* (W).

Proof. Let (X, G) be an inner amenable transformation group. Given $\varepsilon > 0$ and compact subset K of $X \times G$, in light of Proposition 2.2(iii), there exists $f \in C_C(X \times G)^+$ such that $|f(x, t) - 1| \leq \varepsilon$, for every $(x, t) \in K$. Similar to the proof of Corollary 2.5, define $\xi = \sqrt{f}$. We now define h on $(X \times G) \times (X \times G)$ by $h(x, s, y, t) = \xi(x, s)\xi(y, t)$, for every $(x, s, y, t) \in (X \times G) \times (X \times G)$. Then h is a continuous of positive type function and compactly supported on $(X \times G) \times (X \times G)$ and from Corollary 2.5, we have

$$|h(x, t, x, t) - 1| = |\xi(x, t)^2 - 1| = |f(x, t) - 1| \leq \varepsilon,$$

for every $(x, t) \in X \times G$. Thus, (X, G) has property (W). □

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