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Abstract non-piecewise syndetic sets

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Abstract. We introduce an abstract scheme $(\mathcal{F}_i)_{i\in I}$ which can be considered as a framework for defining all possible versions of ideals of non-piecewise syndetic sets in a common language. We prove that the family $\nabla((\mathcal{F}_i)_{i\in I}) = \{Z\colon \forall_{i\in I}\exists_{j\in I}\forall_{F\in\mathcal{F}_j}\exists_{G\in\mathcal{F}_i} G \subseteq F \wedge G \cap Z = \emptyset\}$ always forms an ideal. It is shown that, under a certain additional condition, this ideal is equal to the collection of non-piecewise syndetic sets relative to $(\mathcal{F}_i)_{i\in I}$. This correspondence holds in many important cases, including the integer lattice. Moreover, we prove that any F_σ ideal has such representation with finite sets. Finally, we show that the Marczewski-Burstin representation is a particular case of our general abstract scheme.

1. Introduction

1.1. Motivation

The notion of Marczewski-Burstin representation was developed to give an abstract framework for commonly known ideals, σ-ideals, fields and σ-fields. For further research, see: [2], [8], [7], [4], [3], [5], [1], [6].

Recall that the Marczewski-Burstin representation is the family $\mathcal{F}$ of nonempty subsets of the space X. For such a family, we define:

Definition 1.1. For any nonempty family $\mathcal{F} \subseteq P(X) \setminus \{\emptyset\}$ let us define $S(\mathcal{F}) = \{Y \subseteq X\colon \forall_{P\in\mathcal{F}}\exists_{Q\in\mathcal{F}} Q \subseteq P \cap Y \vee Q \subseteq P \setminus Y\}$ and $S^0(\mathcal{F}) = \{Y \subseteq X\colon \forall_{P\in\mathcal{F}}\exists_{Q\in\mathcal{F}} Q \subseteq P \setminus Y\}$.

It turns out that many ideals or σ-ideals can be represented in this way (for example, the σ-ideal of Marczewski null sets (s_0) – see [14] and [15], the σ-ideal of Completely Ramsey null sets, even the classical Lebesgue measure zero sets or the meager sets – see [9]).

The applications of the classical Marczewski-Burstin scheme for the analysis of the classical ideals on the set of natural numbers are discussed in the paper [11].

The aim of this paper is to create an abstract scheme that can support the investigation of certain ideals, especially those listed below, using a unified language.

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(i) The classical ideal of non-piecewise syndetic sets: nPS;

(ii) The ideal on subsets of the integer lattice points: $\mathrm{nPS}_{\square}$ where the base sets are the "integer lattice squares";

(iii) The ideal on subsets of the integer lattice nPS_2 where the base set are the "integer lattice rectangles".

Definitions of the above three ideals can be presented through a single abstract framework. More precisely, we have extended the known characterization ([10]) of an element of *nPS* to an abstract case. The condition ∇, inspired by this characterization, unfortunately is not a characterization of the abstract *nPS* (∇ is a stronger property). However, we have introduced an additional condition, denoted as $\Game$, to ensure that condition ∇ is equivalent to the abstract *nPS*. We find that the condition $\Game$is satisfied for all three families considered in (i), (ii), (iii). Note that condition ∇ for *nPS* and $\mathrm{nPS}_{\square}$ also appeared in [10].

We refer to our scheme as the *PS-scheme*, by analogy to the classical MB-scheme.

1.2. Preliminary definitions

By ω and ω_+ we denote the set of all nonnegative and positive integers, respectively. Throughout, an "integer interval" will refer to a subset of the natural numbers of the form $[a, b] \cap \omega$, where $a \leq b$ are integers.

Definition 1.2. An ideal is a nonempty family $\mathcal{I} \subseteq 2^{\omega}$ closed under taking subsets and finite unions, i.e.

1. $\forall_{A,B}\ A \in \mathcal{I} \wedge B \subset A \implies B \in \mathcal{I}$
2. $\forall_{A,B}\ A \in \mathcal{I} \wedge B \in \mathcal{I} \implies A \cup B \in \mathcal{I}$

An ideal is proper if $\omega \notin \mathcal{I}$, or, equivalently, $\mathcal{I} \neq 2^{\omega}$. We always assume also that all singletons belong to $\mathcal{I}$.

Let us now recall the following definitions:

Definition 1.3. A set S of natural numbers is syndetic if the gaps between its consecutive elements are bounded, that is if $n_0^S < n_1^S < n_2^S, \ldots$ is an increasing enumeration of elements of the set S then there exists a natural number K such that $\forall_{j \in \omega} n_{j+1} - n_j \leq K$. Equivalently, there exists a natural number K such that each integer interval of length K has nonempty intersection with the set S: $\exists_{K \in \omega} \forall_{n \in \omega} \exists_{j<K} n + j \in S$.

Definition 1.4. A set of natural numbers T is said to be thick if it contains integer intervals of arbitrary length, i.e. $\forall_{K \in \omega} \exists_{n \in \omega} \forall_{j<K} n + j \in T$

Definition 1.5. A set A of natural numbers is said to be piecewise syndetic (PS) if it can be represented in the form $A = S \cap T$, where S is a syndetic set and T is a thick set.

The following fact is well known (see for example [10], Fact 1):

Theorem 1.6. *The collection of all subsets of the natural numbers which are not piecewise syndetic forms an ideal.*

We denote this ideal as nPS (and, sometimes this ideal is called the Brown ideal).

2. The system $(\mathcal{F}_i)_{i \in I}$

For any $A \subseteq X$ and $\mathcal{F} \subseteq P(X)$ let us define the *Star* of the set A: $Star(A, \mathcal{F}) = \bigcup\{F \in \mathcal{F} : F \cap A \neq \emptyset\}$.

Definition 2.1. Suppose that X is any set and assume that we have a fixed indexed family of subsets of X, $(\mathcal{F}_i)_{i \in I}$ where I is a set of indices. We also assume that for all $i \in I$, $\mathcal{F}_i$ is nonempty and $\emptyset \notin \mathcal{F}_i$. Hence $\mathcal{F}_i \subseteq P(X) \setminus \{\emptyset\}$.

We now define an abstract version of the classical notions of Syndetic and Thick sets:

Definition 2.2. A set $S \subseteq X$ is called $(\mathcal{F}_i)_{i\in I}$-syndetic if $\exists_{j\in I}\forall_{F\in\mathcal{F}_j} S \cap F \neq \emptyset$. The family of $(\mathcal{F}_i)_{i\in I}$-syndetic sets we denote by $Synd((\mathcal{F}_i)_{i\in I})$.

Definition 2.3. A set $T \subseteq X$ is called $(\mathcal{F}_i)_{i\in I}$-thick if $\forall_{j\in I}\exists_{F\in\mathcal{F}_j} F \subseteq T$. The family of $(\mathcal{F}_i)_{i\in I}$-thick sets we denote by $Thick((\mathcal{F}_i)_{i\in I})$.

It follows immediately from definitions that $S \in Synd((\mathcal{F}_i)_{i\in I})$ if and only if $X \setminus S \notin Thick((\mathcal{F}_i)_{i\in I})$ and $T \in Thick((\mathcal{F}_i)_{i\in I})$ if and only if f $X \setminus T \notin Synd((\mathcal{F}_i)_{i\in I})$.

Definition 2.4. A set $Z \subseteq X$ is called $(\mathcal{F}_i)_{i\in I}$-piecewise syndetic (PS$((\mathcal{F}_i)_{i\in I})$) if there exist $A \in Synd((\mathcal{F}_i)_{i\in I})$ and $B \in Thick((\mathcal{F}_i)_{i\in I})$ such that $Z = A \cap B$.

The following definition represents the negation of the above property:

Definition 2.5. A set $Z \subseteq X$ is called $(\mathcal{F}_i)_{i\in I}$-non-piecewise syndetic (nPS$((\mathcal{F}_i)_{i\in I})$) if there is no $A \in Synd((\mathcal{F}_i)_{i\in I})$ and $B \in Thick((\mathcal{F}_i)_{i\in I})$ such that $Z = A \cap B$.

We now define the crucial notion of this paper.

Definition 2.6. The ∇ condition: A subset $Z \subseteq X$ fulfills the condition $\nabla((\mathcal{F}_i)_{i\in I})$ if

$$\forall_{i\in I}\exists_{j\in I}\forall_{F\in\mathcal{F}_j}\exists_{G\in\mathcal{F}_i} G \subseteq F \wedge G \cap Z = \emptyset.$$

By a slight abuse of notation, we use $\nabla((\mathcal{F}_i)_{i\in I})$ to denote both the condition $\nabla((\mathcal{F}_i)_{i\in I})$ itself and the family of sets satisfying it.

3. Results

3.1. *Relationship between* ∇ *and* nPS$((\mathcal{F}_i)_{i\in I})$

Example 3.1. Let $X = \omega$, and for each $j \in \omega$, let $\mathcal{F}_j$ be the family of all integer intervals of length j, or more precisely:

$$\mathcal{F}_j = \{[k, k + j - 1] \cap \omega : k \in \omega\}.$$

It is easy to see that in this case the family nPS$((\mathcal{F}_i)_{i\in I})$ coincides with the standard definition of nPS sets. The condition $\nabla((\mathcal{F}_i)_{i\in I})$ in this case has the form:

$$\forall_{N\in\omega}\exists_{M\in\omega}\forall_{\substack{\text{interval } I\\ |I|=M}}\exists_{\substack{\text{interval } J\\ |J|=N}} J \subseteq I \wedge J \cap Z = \emptyset.$$

which is the known condition for the nPS sets (see [10], page 3).

We have:

Theorem 3.2. *The family* nPS$((\mathcal{F}_i)_{i\in I})$ *is an ideal.*

Proof. It follows immediately from the definition that PS$((\mathcal{F}_i)_{i\in I}) = \{S\setminus N : S \in Synd((\mathcal{F}_i)_{i\in I}), N \notin Synd((\mathcal{F}_i)_{i\in I})\}$. Therefore, it suffices to prove the following lemma:

Lemma 3.3. *Let* $\emptyset \neq \mathcal{A} \subseteq P(X)$ *be a family such that whenever* $A \in \mathcal{A}$ *and* $B \supseteq A$*, then* $B \in \mathcal{A}$*. Then the family* $\mathcal{I} = P(X) \setminus \{S \setminus N : S \in \mathcal{A}, N \notin \mathcal{A}\}$ *is an ideal.*

Proof. [Proof of the Lemma] Clearly $\emptyset \in \mathcal{I}$.
If $S \setminus N \subseteq Z$, with $S \in \mathcal{A}$ and $N \notin \mathcal{A}$ then $(S \cup Z) \setminus (N \setminus Z) = Z$ and $S \cup Z \in \mathcal{A}$, $N \setminus Z \notin \mathcal{A}$.
It remains to prove that if $A \cup B = S \setminus N$, $S \in \mathcal{A}$, $N \notin \mathcal{A}$, then either A or B is also of this form.
If $A \cup N \in \mathcal{A}$ then $A = (A \cup N) \setminus (N \setminus A)$, $A \cup N \in \mathcal{A}$, and $N \setminus A \notin \mathcal{A}$.
Hence $A \cup N \notin \mathcal{A}$. We have $B = S \setminus \big((A \cup N) \setminus (A \cap B)\big)$, $S \in \mathcal{A}$, $(A \cup N) \setminus (A \cap B) \notin \mathcal{A}$. □

□

Theorem 3.4. *The collection of set fulfilling the condition $\nabla((\mathcal{F}_i)_{i\in I})$ is an ideal.*

Proof. Suppose that Y, Z are any sets with the property $\nabla((\mathcal{F}_i)_{i\in I})$. Let us choose $i \in I$. By virtue of the condition $\nabla((\mathcal{F}_i)_{i\in I})$ applied to the set Y we conclude that there exists $j \in I$ such that $\forall_{F\in\mathcal{F}_j}\exists_{G\in\mathcal{F}_i} G \subseteq F \wedge G \cap Y = \emptyset$. For any fixed $i \in I$ let $j(i) \in I$ be such an index which exists by virtue of the second quantifier from the above sentence. So, let us fix a function $j\colon I \to I$ such that $\forall_{i\in I}\forall_{F\in\mathcal{F}_{j(i)}}\exists_{G\in\mathcal{F}_i} G \subseteq F \wedge G \cap Y = \emptyset$. Analogously, applying the condition $\nabla((\mathcal{F}_i)_{i\in I})$ to the set Z we conclude that there is a function $k\colon I \to I$ such that $\forall_{j\in I}\forall_{F\in\mathcal{F}_{k(j)}}\exists_{G\in\mathcal{F}_j} G \subseteq F \wedge G \cap Z = \emptyset$.

Let us choose any $i \in I$. Further, choose any $F \in \mathcal{F}_{k(j(i))}$. Then there exists $G \in \mathcal{F}_{j(i)}$ such that $G \subseteq F$ and $G \cap Z = \emptyset$. For such G let us find $H \in \mathcal{F}_i$ such that $H \subseteq G$ and $H \cap Y = \emptyset$. Then $H \cap Z \subseteq G \cap Z = \emptyset$. Hence $H \cap (Y \cup Z) = \emptyset$. We have proved that the following condition holds:

$$\forall_{i\in I}\forall_{F\in\mathcal{F}_{k(j(i))}}\exists_{H\in\mathcal{F}_i} H \subseteq F \wedge H \cap (Y \cup Z) = \emptyset,$$

hence

$$\forall_{i\in I}\exists_{k\in I}\forall_{F\in\mathcal{F}_k}\exists_{H\in\mathcal{F}_i} H \subseteq F \wedge H \cap (Y \cup Z) = \emptyset.$$

This proves that $Y \cup Z \in \nabla((\mathcal{F}_i)_{i\in I})$.

It suffices to observe that if $Z \subseteq Y \in \nabla((\mathcal{F}_i)_{i\in I})$ then also $Z \in \nabla((\mathcal{F}_i)_{i\in I})$. The same holds for $Z = \emptyset$: the condition $\nabla((\mathcal{F}_i)_{i\in I})$ is true, since we can take $j := i$ and $G := F$.

So, indeed, the family $\nabla((\mathcal{F}_i)_{i\in I})$ is an ideal. □

Next, we will show:

Theorem 3.5. *Each set which fulfills the condition $\nabla((\mathcal{F}_i)_{i\in I})$ belongs to the ideal* $\mathrm{nPS}((\mathcal{F}_i)_{i\in I})$.

Proof. It suffices to show that if $Z = A \cap B$, $A \in Synd((\mathcal{F}_i)_{i\in I})$, $B \in Thick((\mathcal{F}_i)_{i\in I})$ then $Z \notin \nabla((\mathcal{F}_i)_{i\in I})$.

Let us write down the negation of the condtition $\nabla((\mathcal{F}_i)_{i\in I})$, which we are going to prove:

$$\exists_{i\in I}\forall_{j\in I}\exists_{F\in\mathcal{F}_j}\forall_{G\in\mathcal{F}_i} G \subseteq F \implies G \cap Z \neq \emptyset.$$

Since $A \in Synd((\mathcal{F}_i)_{i\in I}))$ there exists $i_0 \in I$ such that $\forall_{F\in\mathcal{F}_{i_0}} F \cap A \neq \emptyset$. Let us choose any $j \in I$. Since $B \in Thick((\mathcal{F}_i)_{i\in I})$ then there is $F \in \mathcal{F}_j$ such that $F \subseteq B$. Suppose that $G \in \mathcal{F}_{i_0}$, $G \subseteq F$. Then $G \subseteq B$ and $G \cap A \neq \emptyset$ thus $G \cap (A \cap B) \neq \emptyset$. □

Unfortunately, we are unable to provide a counterexample to the reverse inclusion, and therefore we state the following open question:

Problem 3.6. *Is it true that $\nabla((\mathcal{F}_i)_{i\in I}) = \mathrm{nPS}((\mathcal{F}_i)_{i\in I})$?*

Let us define the following property of the family $(\mathcal{F}_i)_{i\in I}$:

Definition 3.7. Assume that for each $i \in I$ we have chosen $G(i) \in \mathcal{F}_i$. Let us also fix a $j \in I$. Then we can fix for each $k \in I$ a set $H(k) \in \mathcal{F}_k$ and there exists a function $i\colon I \to I$ such that

1. $\forall_{k\in I} H(k) \subseteq G(i(k))$;
2. For each $F \in \mathcal{F}_j$ we have $|\{k \in I\colon H(k) \cap F \neq \emptyset\}| \leq 1$.

Let us denote this property as $\supset$.

The reason of introducing this property is the following theorem:

Theorem 3.8. *Assume that the family $(\mathcal{F}_i)_{i\in I}$ fulfills the condition $\supset$. Then a set $A \subseteq X$ has the property $\nabla((\mathcal{F}_i)_{i\in I})$ iff A is* $\mathrm{nPS}((\mathcal{F}_i)_{i\in I})$.

Proof. By virtue of Theorem 3.5 we know that the inclusion from the left to the right is true. Hence it suffices to prove that if $Z \subseteq X$ does not fulfill the condition $\nabla((\mathcal{F}_i)_{i\in I})$ then Z is $\mathrm{PS}((\mathcal{F}_i)_{i\in I})$.

Let us formulate the negation of the condition $\nabla((\mathcal{F}_i)_{i\in I})$:

$$\exists_{i_0\in I}\forall_{j\in I}\exists_{G\in\mathcal{F}_j}\forall_{F\in\mathcal{F}_{i_0}} F \subseteq G \implies F \cap Z \neq \emptyset. \tag{1}$$

For each $j \in I$ choose $G(j) \in \mathcal{F}_j$ which exists by virtue of the third existential quantifier in the condition (1). Thus we have

$$\forall_{F\in\mathcal{F}_{i_0}} F \subseteq G(j) \implies F \cap Z \neq \emptyset. \tag{2}$$

Now using the condition $\supset$ for $G(j)$ and for $j := i_0$ and we obtain that there exists for each $k \in I$ an element $H(k) \in \mathcal{F}_k$ and a function $j\colon I \to I$ such that $\forall_{k\in I} H(k) \subseteq G(j(k))$ and $\forall_{F\in F_{i_0}} |\{k \in I\colon H(k) \cap F \neq \emptyset\}| \leq 1$. Let us define:

$$B := Z \cup \bigcup_{k\in I} H(k);$$

and then

$$A := (X \setminus B) \cup Z.$$

Clearly, $B \in \mathit{Thick}((\mathcal{F}_i)_{i\in I})$.

We now check that $A \in \mathit{Synd}((\mathcal{F}_i)_{i\in I})$. Let $F \in \mathcal{F}_{i_0}$. We know that the set

$$\{k \in I\colon H(k) \cap F \neq \emptyset\}$$

is a singleton or an empty set. Observe that

$$A = \Big[X \setminus (Z \cup \bigcup_{k\in I} H(k))\Big] \cup Z = \Big[(X \setminus Z) \cap X \setminus \bigcup_{k\in I} H(k)\Big] \cup Z = (X \setminus \bigcup_{k\in I} H(k)) \cup Z,$$

hence $X \setminus \bigcup_{k\in I} H(k) \subseteq A$. It suffices just to show that

$$(X \setminus \bigcup_{k\in I} H(k)) \cap F \neq \emptyset. \tag{3}$$

Suppose that $F \subseteq \bigcup_{k\in I} H(k)$. Since the set

$$\{k \in I\colon H(k) \cap F \neq \emptyset\}$$

contains at most one element, so there exists $k_0 \in I$ such that $F \subseteq H(k_0)$. Hence, for $j(k_0)$ we have that $F \subseteq H(k_0) \subseteq G(j(k_0))$. But from the condition (2) applied to F we know that $F \cap Z \neq \emptyset$. Thus $F \cap A \neq \emptyset$, therefore $A \in \mathit{Synd}((\mathcal{F}_i)_{i\in I})$. It suffices to observe that:

$$A \cap B = [(X \setminus B) \cup Z] \cap B = Z \cap B = Z \cap \Big(Z \cup \bigcup_{k\in I} H(k)\Big) = Z.$$

□

It is an important remark that the point (2) in fact means that the corresponding sets $H(k)$ are "far apart" from each other in the sense that there is no $F \in \mathcal{F}_j$ such that it has a non-empty intersection with at least two of them.

Example 3.9. The following systems of families of sets have the property $\supset$:

(i) The standard "lattice intervals" system, i.e:

$$\mathcal{F}_j = \{[k, k+j-1] \cap \omega : k \in \omega\}.$$

for each $j \in \omega$.

The appropriate ideal on the set ω is the classical ideal of non-piecewise syndetic sets (i.e. nPS) and let us denote this nPS-system by $\mathcal{F}(\mathrm{nPS})$.

(ii) The "square" system on the integer lattice:

$$\mathcal{F}_{j^d} = \{(\times_{r=0}^{d-1}[k_r, k_r+j-1] \cap \mathbb{Z}^d) : (k_0, \ldots, k_{d-1}) \in \mathbb{Z}^d\}.$$

for each $j \in \omega_+$

Let us denote by $\mathrm{nPS}_\square$ the appropriate ideal on the integer lattice $\mathbb{Z}^d$ and let us denote this nPS-system by $\mathcal{F}(\mathrm{nPS}_\square)$.

(iii) The "rectangle" system on the integer lattice:

$$\mathcal{F}_j = \{(\times_{r=0}^{d-1}[k_r, k_r+j_r-1]) \cap \mathbb{Z}^d) : (k_0, \ldots, k_{d-1}) \in \mathbb{Z}^d; (j_0, \ldots, j_{d-1}) \in \omega_+^d, \prod_{r=0}^{d-1} j_r = j\}.$$

for each $j \in \omega_+$

Let us denote by nPS_2 the appropriate ideal on the integer lattice $\mathbb{Z}^d$ and let us denote this nPS-system by $\mathcal{F}(\mathrm{nPS}_2)$.

(iv) The "fixed size rectangles" system on the integer lattice $\mathbb{Z}^d$

$$\mathcal{F}_{(j_r)} = \{(\times_{r=0}^{d-1}[k_r, k_r+j_r-1] \cap \mathbb{Z}^2) : (k_0, \ldots, k_{d-1}) \in \mathbb{Z}^d\}.$$

for each $(j_0, \ldots, j_{d-1}) \in \omega_+^d$ (note that in this case the index set I is equal to the product ω_+^d).

The appropriate ideal on the integer lattice $\mathbb{Z}^d$ let us denote by $\mathrm{nPS}_{n\times m}$ and let us denote this nPS-system by $\mathcal{F}(\mathrm{nPS}_{n\times m})$.

Proof. Let us show that all those systems of families of sets have the property $\supset$. At first let us formulate an abstract condition:

Lemma 3.10. *Suppose that $I = \omega$ and assume that the system $(\mathcal{F}_i)_{i\in I}$ has the following properties:*

1. *Every element of each $\mathcal{F}_j$ is a finite set.*
2. $\forall_{j\in I}\forall_{a\in X} Star(\{a\}, \mathcal{F}_j) \in \mathrm{Fin}$;
3. $\forall_{a\in X}\{a\} \in \nabla((\mathcal{F}_i)_{i\in I})$.

Then the family $\nabla((\mathcal{F}_i)_{i\in I})$ fulfills the condition $\supset$.

Proof. Assume that for each $i \in I$ we have chosen $G(i) \in \mathcal{F}_i$. Let us also fix a $j \in I$. We are going to construct an appropriate $H(k) \in \mathcal{F}_k$. We proceed by induction: Assume that we have defined $H(0), \ldots, H(k-1)$. By our assumptions the set $W_k = Star(\bigcup_{l=0}^{k-1} H(l), \mathcal{F}_j)$ is a finite set. Again, by our assumption, this set W_k is in $\nabla((\mathcal{F}_i)_{i\in I})$. Hence there exists $i(k) \in I = \omega$ such that for each $G \in \mathcal{F}_{i(k)}$ there exists $H \in \mathcal{F}_k$ such that $H \subseteq G$ and $H \cap W_k = \emptyset$. So fix any such H for $G := G(i(k))$ and define $H(k) := H$. This finishes the construction of the sequence $(H(k))_{k\in\omega}$. It is easy to see that for each $F \in F_j$ we have $|\{k \in I : H(k) \cap F \neq \emptyset\}| \leq 1$. Indeed, suppose that $H(k_1) \cap F \neq \emptyset$ and $H(k_2) \cap F \neq \emptyset$ for $k_1 < k_2$. Then $Star(\bigcup_{l=0}^{k_2-1} H(l), \mathcal{F}_j) \cap H(k_2) \neq \emptyset$ which is impossible. □

It is easy to observe that all three systems from the Example 3.9 have these properties. Therefore all those systems fulfill the condition $\Game$. □

It turned out, however, that the ideal $\mathrm{nPS}_{n\times m}$ is the same like the ideal $\mathrm{nPS}_{\square}$, which we can prove by establishing the following general theorem:

Theorem 3.11. *Suppose that $I \subseteq J$ and $(\mathcal{F}_j)_{j\in J}$ is an abstract nPS-system. Consider the "restriction" of this system to the set I of indices: $(\mathcal{F}_i)_{i\in I}$. If the following "cofinal" condition holds: There exists a function $i\colon J \to I$ such that*

$$\forall_{j\in J}\forall_{F\in\mathcal{F}_{i(j)}}\exists_{G\in\mathcal{F}_j} G \subseteq F, \tag{4}$$

then $\nabla((\mathcal{F}_i)_{i\in I}) = \nabla((\mathcal{F}_j)_{j\in J})$.

Proof. "⇒": Suppose that $A \in \nabla((\mathcal{F}_i)_{i\in I})$ and $j_0 \in J$. Then by our assumption (4) we know that $\forall_{F\in\mathcal{F}_{i(j_0)}}\exists_{G\in\mathcal{F}_{j_0}} G \subseteq F$. Next, apply the condition $\nabla((\mathcal{F}_i)_{i\in I})$ to the family $\mathcal{F}_{i(j_0)}$ and we obtain that there exists $i_1 \in I \subseteq J$ such that

$$\forall_{H\in\mathcal{F}_{i_1}}\exists_{F\in\mathcal{F}_{i(j_0)}} F \subseteq H \wedge F \cap A = \emptyset.$$

But since $\exists_{G\in\mathcal{F}_{j_0}} G \subseteq F$, finally we obtain that

$$\forall_{H\in\mathcal{F}_{i_1}}\exists_{G\in\mathcal{F}_{j_0}} G \subseteq H \wedge G \cap A = \emptyset.$$

"⇐": On the other hand, assume that $A \in \nabla((\mathcal{F}_j)_{j\in J})$. Let us fix $i_0 \in I$ and by virtue of $\nabla((\mathcal{F}_j)_{j\in J})$ we obtain that there exists $j_1 \in J$ such that

$$\forall_{F\in\mathcal{F}_{j_1}}\exists_{G\in\mathcal{F}_{i_0}} G \subseteq F \wedge G \cap A = \emptyset. \tag{5}$$

For j_1 let us apply the condition (4) and we obtain that

$$\forall_{H\in\mathcal{F}_{i(j_1)}}\exists_{F\in\mathcal{F}_{j_1}} F \subseteq H. \tag{6}$$

By merging formulas (5) and (6) we obtain that

$$\forall_{H\in\mathcal{F}_{i(j_1)}}\exists_{G\in\mathcal{F}_{i_0}} G \subseteq H \wedge G \cap A = \emptyset.$$

□

Corollary 3.12. *The ideals $\mathrm{nPS}_{n\times m}$ and $\mathrm{nPS}_{\square}$ are identical.*

Proof. It suffices to observe by virtue of Theorem 3.11 that if we consider the following sets of indices: $J = \{(k_1,\ldots,k_d)\colon k_1,\ldots,k_d \in \omega^+\}$ and $I = \{\underbrace{(k,\ldots,k)}_{d\times}\colon k \in \omega^+\}$ then we see that the set J is the same set as in the system $\mathcal{F}(\mathrm{nPS}_{n\times m})$ and the subsystem $I \subseteq J$ is isomorphic to the system $\mathcal{F}(\mathrm{nPS}_{\square})$. For any $s \in J$ we define $\mathcal{F}_s$ similar like in the case of $\mathcal{F}(\mathrm{nPS}_{n\times m})$. It is now easy to check that in this case the condition (4) from Theorem 3.11 is fulfilled, therefore the ideals $\mathrm{nPS}_{n\times m}$ and $\mathrm{nPS}_{\square}$ are equal. □

3.2. A special case

In the case when I consists of only one element our scheme reduces to the Marczewski-Burstin case. Namely we have:

Proposition 3.13. *Suppose that $I = \{a\}$. Then:*

(1) The family $\nabla((\mathcal{F}_i)_{i\in I})$ is equivalent to the classical $S^0(\mathcal{F}_a)$ sets;

(2) The condition $Thick((\mathcal{F}_i)_{i\in I})$ in our case is equivalent to the notion that a set A includes at least one set from the family $\mathcal{F}_a$;

(3) The condition $Synd((\mathcal{F}_i)_{i\in I})$ is equivalent to the requirement that a set A is a $\mathcal{F}_a$ dense set, i.e. for each $F \in \mathcal{F}_a$, $F \cap A \neq \emptyset$.

(4) Since the family $(\mathcal{F}_i)_{i\in I}$ is our case has the $\Game$property (the selection $(G(i))_{i\in I}$ has one element only, we can simply define $i = id$ and $H(a) = G(a)$) we deduce that a set A is a $S^0(\mathcal{F}_a)$ set if and only if A is not an intersection of sets B, C such that B includes at least one set from $\mathcal{F}_a$ and C is a $\mathcal{F}_a$ - dense set (this statement can also be proved directly).

Proof. We omit the obvious proofs. □

3.3. Operations on the systems $(\mathcal{F}_i)_{i\in I}$

In analogy to the analogous operation on the Marczewski-Burstin system, let us define the product of two families $(\mathcal{F}_i)_{i\in I}$:

Definition 3.14. Suppose that we have two nPS-systems: $(\mathcal{F}_i)_{i\in I}$ and $(\mathcal{G}_j)_{j\in J}$. The product of two nPS-systems is defined as follows: Define set of indices of the form $T = I \times J$ and:

$$\mathcal{H}_{i,j} = \{F \times G \colon F \in \mathcal{F}_i \wedge G \in \mathcal{G}_j\}.$$

We call the system $(\mathcal{H}_{i,j})_{(i,j)\in I\times J}$ the product nPS-system.

Example 3.15. We immediately obtain: $\mathcal{F}(\text{nPS}) \times \mathcal{F}(\text{nPS}) = \mathcal{F}(\text{nPS}_{n\times m})$.

3.4. Other examples of the standard ideals

The notion of nPS-representable is similar, though not identical, to the notion of Marczewski-Burstin countable representability. Recall the definition from [11].

Definition 3.16. We call an ideal $\mathcal{I} \subseteq P(X)$ Marczewski-Burstin countably representable (briefly: MBC) if there exists a countable family $\mathcal{F}$ of infinite subsets of X, such that $\mathcal{I} = \mathcal{S}_0(\mathcal{F})$.

Note that this definition requires that all sets from the family $\mathcal{F}$ should be infinite, but in our case of nPS-representable ideals we require that all sets from $\mathcal{F}_j$ should be finite sets.

Let us recall the definitions of classical ideals defined on ω:

Definition 3.17. Family $\mathcal{I}_d$ is the ideal of sets of asymptotic density zero, i.e.

$$\mathcal{I}_d = \left\{ A \subset \omega : \limsup_{n\to\infty} \frac{|A \cap \{0, 1, \ldots, n-1\}|}{n} = 0 \right\}.$$

Note that this definition requires that all sets from the family $\mathcal{F}$ should be infinite, but in our case of nPS-representable ideals we require that all sets from $\mathcal{F}_j$ should be finite sets.

Definition 3.18. Let E_n denote the family of subsets of naturals which do not contain any arithmetic sequence of length n. Put

$$\mathrm{W} = \bigcup_n E_n.$$

W is an ideal called the van der Waerden ideal.

Definition 3.19. Let $(\alpha_n)_{n\in\omega}$ be a sequence of positive reals such that $\sum_{n\in\omega} \alpha_n = \infty$. Then

$$\mathcal{I}_{(\alpha_n)} = \left\{ A \subset \omega \colon \sum_{n\in A} \alpha_n < \infty \right\}$$

is its corresponding summable ideal.

Definition 3.20. For an infinite set $D \subseteq \omega$ denote

$$FS(D) = \left\{ \sum_{n \in D^*} n : D^* \subset D,\ 0 < |D^*| < \aleph_0 \right\}.$$

We define the Hindman ideal $\mathcal{H}$ in the following way. $A \in \mathcal{H}$ if there is no infinite set D such that $FS(D) \subseteq A$.

The ideals W and $\mathcal{I}_{\frac{1}{n}}$ are F_σ ideals (treated as subsets of 2^ω via characteristic functions); $\mathcal{I}_d$ is an $F_{\sigma\delta}$ ideal, whereas $\mathcal{H}$ is not even Borel (see [13]).

At first note that in [11] it was proven that (see Proposition 3.2) the ideals $\mathcal{I}_d$, W, $\mathcal{I}_{\frac{1}{n}}$ and $\mathcal{H}$ are not Marczewski-Burstin countably representable. In contrast of this let us prove:

Theorem 3.21. *For each F_σ ideal $\mathcal{I}$ there exists a nPS-system $(\mathcal{F}_i)_{i\in I}$ indexed by $I = \omega$ such that $\mathcal{F}_i \subseteq \mathrm{Fin}$ and $\nabla((\mathcal{F}_i)_{i\in I}) = \mathcal{I}$.*

Proof. Since $\mathcal{I}$ is an F_σ ideal we know that there exists a function $\mu\colon \mathrm{Fin} \to \omega$ such that $a \subseteq b \implies \mu(a) \leq \mu(b)$, $\mu(a \cup b) \leq \mu(a) + \mu(b)$, $\lim_{n\to\infty} \mu(\{0, \dots n-1\}) = \infty$, and $\mathcal{I} = \{A \subseteq \omega\colon \lim_{n\to\infty} \mu(A \cap \{0, \dots n-1\}) < \infty\}$ (see [12]).

Define $\mathcal{F}_j = \{a \in \mathrm{Fin}\colon \mu(a) \geq j\}$. Let us check that $\mathcal{I} = \nabla((\mathcal{F}_i)_{i\in I})$. Suppose $A \in \mathcal{I}$. Then $\lim_{n\to\infty} \mu(A \cap \{0, \dots, n-1\}) = K$ for some $K \in \omega$. Let $j_0 \in \omega_+$. Define $j_1 := j_0 + K$ and choose $b \in \mathcal{F}_{j_1}$. Then $\mu(b) \geq j_1$. Let us fix n_0 such that $\mu(A \cap \{0, \dots, n_0 - 1\}) = K$ and $b \subseteq \{0, \dots, n_0 - 1\}$. We have

$$\mu(\{0, \dots, n_0 - 1\} \cap A \cap b) + \mu(\{0, \dots, n_0 - 1\} \cap b \setminus A) \geq \mu(\{0, \dots, n_0 - 1\} \cap b) = \mu(b).$$

Hence $\mu(A \cap b) + \mu(b \setminus A) \geq \mu(b)$. But $\mu(A \cap b) \leq \mu(A \cap \{0, \dots, n_0 - 1\}) = K$ thus $K + \mu(b \setminus A) \geq \mu(b)$, so $\mu(b \setminus A) \geq \mu(b) - K \geq j_1 - K = j_0$. It proves that $b \setminus A \in \mathcal{F}_{j_0}$, but since $(b \setminus A) \cap A = \emptyset$ the condition $\nabla((\mathcal{F}_i)_{i\in I})$ holds.

On the other hand, assume that $A \in \nabla((\mathcal{F}_i)_{i\in I})$ but, towards a contradiction, suppose that $A \notin \mathcal{I}$. Then $\lim_{n\to\infty} \mu(A \cap \{0, \dots n-1\}) = \infty$ and this means that $\forall_{j\in\omega} \exists_{a\in\mathrm{Fin}} a \subseteq A \wedge \mu(a) \geq j$, so $\forall_{j\in\omega} \exists_{a\in\mathcal{F}_j} a \subseteq A$ hence A cannot satisfy the condition $\nabla((\mathcal{F}_i)_{i\in I})$ □

Note that if $|I| \leq \omega$ and $\mathcal{F}_j \subseteq \mathrm{Fin}$, then $\nabla((\mathcal{F}_i)_{i\in I})$ is Borel. This immediately implies that the Hindman ideal cannot be represented as $\nabla((\mathcal{F}_i)_{i\in I})$ for such sets $\mathcal{F}_j$.

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