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Embankment surfaces according to the Bishop frame in Euclidean 3-space

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Abstract. This article is devoted to embankment, embankment-like and tubembankment-like surfaces determined by regular space curve according to the Bishop frame in Euclidean 3−space. First, parametric equations, quaternionic and matrix representations of them are given. Moreover, geometric properties of them are examined and some theorems and conclusions are obtained. At the end of the paper, the theory of the paper is supported by the examples and their illustrated figures are drawn by the programme Mathematica.

1. Introduction

Surface theory has been studied for a long time in Euclidean 3−space. Surfaces of revolution have a very important place in geometry. Ruled surfaces, canal surfaces, and embankment surfaces are of this type surfaces. Ruled surfaces from this family are generated by moving a line along a curve. The canal and embankment surfaces are constructed by the movement of spheres and cones along a curve, respectively.

Ruled surfaces are important from a geometrical point of view. Many mathematicians have given parametrizations of these surfaces according to the Frenet frames of curves and examined their geometric properties. These surfaces were first introduced and studied by Monge [27]. Catalan studied on ruled minimal surface [9]. The theory of ruled surfaces was developed using the E-Study transformation [32]. The geometric properties of the ruled surfaces were investigated by Izumiya and Takeuchi [21]. Similarly, the characterization and properties of ruled surfaces were investigated by Yoon [35].

Another important family of surfaces is the canal surfaces first defined by Monge [28]. A canal surface is generated by envelope of a moving sphere of variable radius. It's special case is a tubular surface whose radius is constant. Canal surfaces and tubular surfaces have been studied extensively in geometry. Gray, mentioned canal surfaces with their equations and draw their graphs via the programme Mathematica [15]. Gross has studied canal surfaces from an analytical perspective [16]. Uçum and İlarslan have also examined the canal surfaces from a different view [33]. Xu has also investigated the analytical and algebraic properties of canal surfaces [34]. Karacan, investigated geodesics of tubular surfaces, and also examined Weingarten type tubular surfaces with their geometric properties in some spaces [22, 23]. Maekawa and others investigated the analysis and some applications of tube surfaces, which are special cases of canal

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surfaces [25]. Blaga examined tubular surfaces in computer graphics [6]. Canal and tubular surfaces are very useful for engineering and computer visualization. Moreover, these surfaces have many applications in daily life.

Especially for curves, geometric properties have been examined using the Frenet frame. However, this frame may not be defined for all points along each curve. In 1975, Bishop [5] introduced that orthonormal frames, which he called relatively parallel adapted frames, exist the Frenet frame and compared their properties with the Frenet frame. The Bishop frame can provide these desired properties for space curves. Therefore, it is important for areas such as mathematical analysis and computer graphs. In recent years, it has been widely used, especially in curve and surface theory. Some of the good studies, close to the topic given in this article, on this frame related to curves and surfaces in geometry are [4, 7, 12]. Babadağ and Atasoy took a new approach to curve couples with Bishop frame [4]. Bükçü and Karacan defined slant helices in the Bishop frame [7]. Doğan, characterized the tube and canal surfaces according to this frame [12].

Parametric expressions of surfaces created by movements along the curve can be expressed in a short way with the help of homothetic motions and matrices. In addition, quaternions defined by Hamilton have a very useful structure in terms of movements [19]. Thus, surface equations can be expressed more simply with the help of quaternions corresponding to matrices. Quaternions, which have an important place in the geometry of motion, are widely used, especially in terms of expressing surfaces more easily and providing ease of operation. Detailed information, applications and examples on this subject are given in [18] and [29]. Babaarslan and Yaylı examined constant slope surfaces with quaternion [3]. Aslan and Yaylı expressed the canal surfaces created with the help of a regular curve with real quaternions and investigated and exemplified their properties [1]. Then, Gök examined canal and tubular surfaces constructed by spherical indicatrices of regular curve with alternative moving frame in Euclidean 3-Space. Moreover, he gives matrix and real quaternionic representation of them [14]. Çanakçı and others give the construction of circular surfaces with real quaternions [8]. In PhD thesis of Doğan the canal surfaces are characterized in different spaces with their properties [11]. He also considered tubular surfaces from a geometrical point of view by characterizing them according to the Bishop frame [12]. Ateş et al. examined tubular surfaces constructed by spherical indicatrices of regular curve with the alternative moving frame in the Euclidean 3-Space [2]. They gave some properties of these tube surfaces with illustrated examples.

Embankment surface is generated by the envelope of a moving cone of variable radius. Since its parametrization depends on its own component, it is difficult to express. For this reason, embankment-like surfaces created depending on arbitrary function and tubembankment-like surfaces created with the help of constant function are defined in [20].

These surfaces have found significant use and application areas, especially in geological engineering. They are used for dam structures such as roads and dams to prevent landslides. It is also important to derive geometric expressions and equations for easier construction of such structures. Detailed information on engineering applications and usage areas and the importance of these surfaces is given in the study titled "Guidelines for Embankment Construction"[13]. However, while there are many articles dealing with the geometric aspects of canal and tube surfaces, embankment surfaces have been studied very little. One of the most important studies of these studies is the study conducted by Kazan and Karadağ [24]. They investigated embankment, embankment-like and tubembankment-like surfaces constructed by regular curves in Euclidean 3-Space. Moreover, they gave quaternionic representations and some properties of them. Finally, each surface was exemplified and their graphs were drawn with Mathematica. Another important article dealing with these surfaces from a geometrical perspective is the work done by Mahmoud and Abd ElHafez [26]. They characterized isotropic Weingarten embankment surfaces with their geometrical properties and drawn graphs by giving examples for each surface.

In this study, first of all we give some basic definitions and properties of frames, real quaternions, and matrices corresponding to homothetic motions. After some information on the background, we define embankment, embankment-like and tubembankment-like surfaces determined by regular space curve according to Bishop frame in Euclidean 3−space. We begin by creating parametric equations of these surfaces, and then use matrices and quaternions to give simpler and more useful forms of these equations. Moreover, we examine geometric properties of them and have given some theorems and conclusions. At the end of the paper, we give some related examples of these surfaces along with their figures drawn by the programme Mathematica.

2. Background material

Let $\mathbb{E}^3$ be 3−dimensional Euclidean space endowed with the standart metric given by

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + x_3y_3,$$

where $\overrightarrow{x} = (x_1, x_2, x_3)$, $\overrightarrow{y} = (y_1, y_2, y_3)$ are vectors in $\mathbb{R}^3$ [17].

The norm of the vector $\overrightarrow{x} \in \mathbb{R}^3$ is defined by

$$\left\|\overrightarrow{x}\right\| = \sqrt{\left\langle \overrightarrow{x}, \overrightarrow{x}\right\rangle}$$

and the vector product is given by

$$\overrightarrow{x} \wedge \overrightarrow{y} = (x_2y_3 - y_2x_3, x_3y_1 - y_3x_1, x_1y_2 - y_1x_2)$$

[17].

We recall elementary properties of the surface in $\mathbb{R}^3$.

The Gauss map of the surface $\Upsilon(s, \theta)$ is the following as

$$\overrightarrow{U} = \frac{\Upsilon_s \wedge \Upsilon_\theta}{\|\Upsilon_s \wedge \Upsilon_\theta\|},$$

where $\{\Upsilon_s, \Upsilon_\theta\}$ is the natural bases [17].

Then the first and the second fundamental forms of the surface $\Upsilon(s, \theta)$ are defined by

$$\mathbf{I} = g_{11}ds^2 + 2g_{12}dsd\theta + g_{22}d\theta^2,$$
$$\mathbf{II} = L_{11}ds^2 + 2L_{12}dsd\theta + L_{22}d\theta^2,$$

respectively [10]. Note that

$$\begin{aligned} g_{11} &= \langle \Upsilon_s, \Upsilon_s\rangle, g_{12} = \langle \Upsilon_s, \Upsilon_\theta\rangle, g_{22} = \langle \Upsilon_\theta, \Upsilon_\theta\rangle, \\ L_{11} &= \left\langle \Upsilon_{ss}, \overrightarrow{U}\right\rangle, L_{12} = \left\langle \Upsilon_{s\theta}, \overrightarrow{U}\right\rangle, L_{22} = \left\langle \Upsilon_{\theta\theta}, \overrightarrow{U}\right\rangle. \end{aligned}$$

We can compute Gauss and mean curvature functions as, respectively

$$K = \frac{L_{11}L_{22} - L_{12}^2}{g_{11}g_{22} - g_{12}^2}, H = \frac{g_{22}L_{11} - 2g_{12}L_{12} + g_{11}L_{22}}{2(g_{11}g_{22} - g_{12}^2)}.$$

We know that, a surface is called (H, K)-Weingarten surface if it satisfies $\Phi(H, K) = H_sK_\theta - H_\theta K_s = 0$ [30].

Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a regular unit speed curve with Frenet apparatus $\{T(s), N(s), B(s), \kappa(s), \tau(s)\}$ in Euclidean space $\mathbb{E}^3$. Here, $\kappa(s)$ and $\tau(s)$ are called the first and the second curvature functions of the curve α, respectively. Let $T(s)$ be tangent to $\alpha(s)$ and let $N_1(s)$ be arbitrary orthogonal unit vector to $T(s)$. If $N_2(s) = T(s) \wedge N_1(s)$, namely $\{T(s), N_1(s), N_2(s)\}$ construct an orthogonal frame is called Bishop frame (relatively parallel adapted frame). If we rotate the Bishop frame by the angle around the tangent vector T, we obtain the Frenet frame as below

$$\begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Theta & \sin\Theta \\ 0 & -\sin\Theta & \cos\Theta \end{bmatrix} \begin{bmatrix} T(s) \\ N_1(s) \\ N_2(s) \end{bmatrix}. \tag{1}$$

The derivatives of the Bishop frame is given by the following formulae:

$$\begin{bmatrix} T'(s) \\ N_1'(s) \\ N_2'(s) \end{bmatrix} = \begin{bmatrix} 0 & k_1(s) & k_2(s) \\ -k_1(s) & 0 & 0 \\ -k_2(s) & 0 & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N_1(s) \\ N_2(s) \end{bmatrix}, \tag{2}$$

where $k_1 = \kappa \cos\Theta, k_2 = \kappa \sin\Theta$, $\Theta = arctan\left(\frac{k_2}{k_1}\right)$ and $\Theta = \int \tau(s)ds.$

The real quaternion algebra $\mathbb{H}$ were consicely defined by Hamilton [19] in the following form

$$\mathbb{H} = \{q = a_0 + a_1 i + a_2 j + a_3 k \mid a_0, a_1, a_2, a_3 \in \mathbb{R}\},$$

where i, j, k are imaginary units satisfying the following multiplication rules:

$$i^2 = j^2 = k^2 = i \wedge j \wedge k = -1, i \wedge j = -j \wedge i = k.$$

This real quaternion algebra is associative and not commutative. A basis of real quaternions is $\{1, i, j, k\}$ and identity element of $\mathbb{H}$ is 1. Also, i, j and k are standard orthonormal basis in $\mathbb{R}^3$. A real quaternion can be written as $q = a_0 + a_1 i + a_2 j + a_3 k$ or $q = (a_0, \overrightarrow{v})$, where the vector and the scalar component of q are $V(q) = \overrightarrow{v} \in \mathbb{R}^3$ and $S(q) = a_0 \in \mathbb{R}$, respectively. So the real quaternion can be written as $q = S(q) + V(q)$. If $S(q) = 0$, the real quaternion is called pure real quaternion. The conjugate of a real quaternion, summation of two real quaternions and multiplication of a real quaternion with a scalar λ can be given, respectively as

$$\begin{aligned} \overline{q} &= S(q) - V(q), \\ q + p &= (S(q) + S(p)) + (V(q) + V(p)), \\ \lambda q &= \lambda S(q) + \lambda V(q). \end{aligned}$$

By using dot and cross-product of Euclidean 3-space, we can give the real quaternion product of two real quaternions q and p as

$$q \times p = S(q)S(p) - < V(q), V(p) > + S(q)V(p) + S(p)V(q) + V(q) \wedge V(p), \tag{3}$$

where $\times$ is the real quaternion product. The norm of a real quaternion is $\left|q\right|^2 = q \times \overline{q} = \overline{q} \times q$ and $\left|q\right| = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}$. If norm of a real quaternion is 1, it is called a unit real quaternion. The unit real quaternion can be written as $q = cos\theta + \sin\theta \overrightarrow{v}$, where $\overrightarrow{v} \in \mathbb{R}^3$ and $\left\|\overrightarrow{v}\right\| = 1$ The inverse of q can be given as

$$q^{-1} = \frac{\overline{q}}{\left|q\right|^2}, \ \left|q\right| \neq 0.$$

One parameter homothetic motion in the 3−dimensional Euclidean space can be given by following translation as

$$y = hMx + C,$$

where y and x are the position vectors of a same point of the fixed space R' and the moving space R, respectively. h, M and C are a homothetic scalar, an orthogonal matrix and a translation vector, respectively. Also, they are continuously differentiable functions dependent on parameter s.

Let $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear mapping and $\phi(v) = q \times \overrightarrow{v} \times q^{-1}$, where q is a unit real quaternion and $\overrightarrow{v}$ is a pure real quaternion (a vector in $\mathbb{R}^3$). So, for every unit real quaternion $q = a_0 + a_1 i + a_2 j + a_3 k$, we can give matrix representation M of ϕ by using pure real quaternion basis elements of $\mathbb{H}$ as

$$M = \begin{bmatrix} a_0^2 + a_1^2 - a_2^2 - a_3^2 & -2a_0a_3 + 2a_1a_2 & 2a_0a_2 + 2a_1a_3 \\ 2a_0a_3 + 2a_1a_2 & a_0^2 + a_2^2 - a_1^2 - a_3^2 & -2a_0a_1 + 2a_2a_3 \\ -2a_0a_2 + 2a_1a_3 & 2a_0a_1 + 2a_2a_3 & a_0^2 + a_3^2 - a_2^2 - a_1^2 \end{bmatrix} \tag{4}$$

,where M is orthogonal matrix (namely $MM^T = I$ and $detM = 1$) [29]. Thus we can say that the linear mapping ϕ is a rotation in 3−dimensional space. Additionally, ϕ is given as $\phi(v) = q \times \overrightarrow{v} \times q^{-1} = M\overrightarrow{v}$.

3. Embankment surfaces according to the Bishop frame in $\mathbf{E}^3$

In this section, initially, the definitions required for the construction of embankment surfaces will be mentioned, then the equations of these surfaces according to the Bishop frame will be obtained. Subsequently, the expressions of these surface equations with matrices and real quaternions will be given.

Given $\Lambda : (r(t), z(t)), t \in [a,b]$, is a parametric planar curve with $r > 0$, then $\Phi := (r(t)\cos\varphi, r(t)\sin\varphi, z(t))$, $t \in [a,b], \varphi \in [0, 2\pi]$, is called parametric surface of revolution. For any implicit planar curve $\Lambda : f(r,z) = 0$ with $r > 0$, the surface $\Phi := \left(\sqrt{x^2+y^2}, z\right)$ is called implicit surface of revolution [20].

Given a one parameter family of regular implicit surfaces $\Phi_c : f(X,c) = 0$, $c \in [c_1, c_2]$. The intersection curve of two neighbored surfaces Φ_c and $\Phi_{c+\Delta c}$ fulfills the two equations $f(X,c) = 0$ and $f(X, c+\Delta c) = 0$. We consider the limit for $\Delta c \to 0$ and get $f_c(X,c) = \lim\limits_{\Delta c \to 0} \dfrac{f(X,c) - f(X,c+\Delta c)}{\Delta c} = 0$ [20]. The last equation motivates the following definitions.

Definition 3.1. Let $\Phi_c : f(X,c) = 0$, $c \in [c_1, c_2]$ be a one parameter family of regular implicit C^2 surfaces. The surface which is defined the two equations $f(X,c) = 0$ and $f_c(X,c) = 0$ is called envelope of the given family of surfaces [20].

Definition 3.2. Let $\Lambda : X = \alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s))$ be a regular space curve and $m \in \mathbb{R}$, $m > 0$ with $\left|m\alpha_3'\right| < \sqrt{\left(\alpha_1'\right)^2 + \left(\alpha_2'\right)^2}$. Then, the envelope of the one parameter family of cones

$$f(X; s) := (x - \alpha_1(s))^2 + (y - \alpha_2(s))^2 - m^2 (z - \alpha_3(s))^2 = 0 \tag{5}$$

is called an embankment surface and Λ its directrix [20].

The stages of creating embankment surfaces that constructed by a regular space curve can be seen as in Figure 1[20].

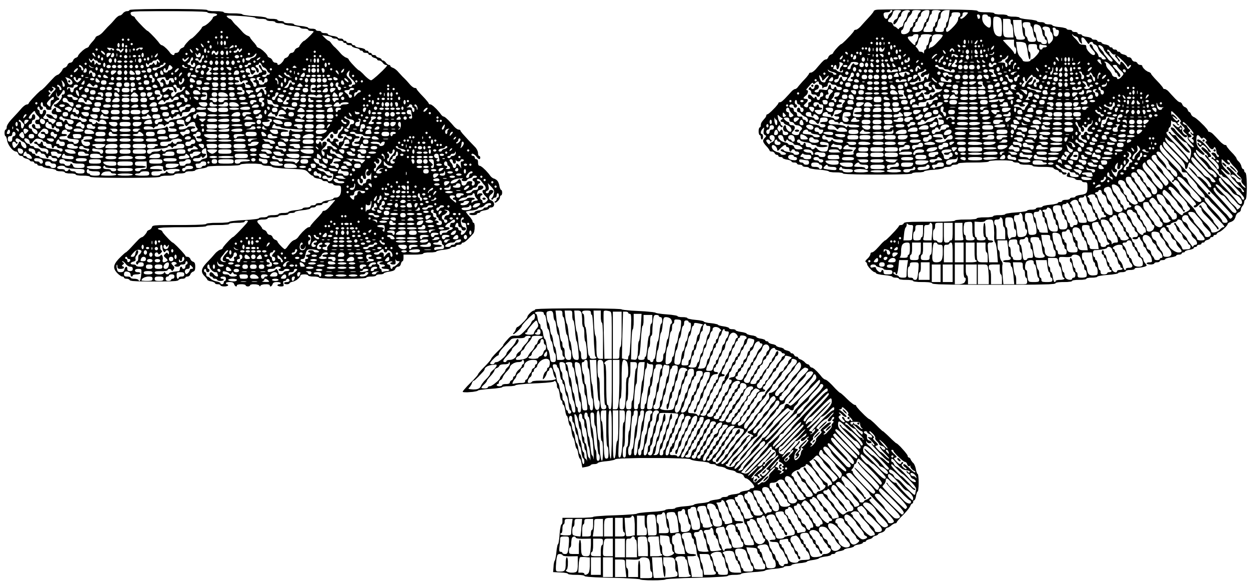

Figure 1: Embankment surface

Let Υ be a parametrization of the envelope of cones defining the embankment surface given by $\Upsilon^{E}(s,\theta)=\left(\Upsilon^{E_1}(s,\theta),\Upsilon^{E_2}(s,\theta),\Upsilon^{E_3}(s,\theta)\right)$. Moreover, let $\alpha(s)=(\alpha_1(s),\alpha_2(s),\alpha_3(s))$ be a regular space curve which is diretrix of $\Upsilon^{E}(s,\theta)$ and $m\in\mathbb{R}$, $m>0$ with $\left|m\alpha_3'\right|<\sqrt{\left(\alpha_1'\right)^2+\left(\alpha_2'\right)^2}$. Then, the embankment surface can be given as

$$\begin{aligned}&\left(\Upsilon^{E_1}(s,\theta)-\alpha_1(s)\right)^2+\left(\Upsilon^{E_2}(s,\theta)-\alpha_2(s)\right)^2-m^2\left(\Upsilon^{E_3}(s,\theta)-\alpha_3(s)\right)^2\\&\qquad+\left(\Upsilon^{E_3}(s,\theta)-\alpha_3(s)\right)^2-\left(\Upsilon^{E_3}(s,\theta)-\alpha_3(s)\right)^2=0.\end{aligned}\tag{6}$$

Thereafter, we have the next one;

$$\left\langle\Upsilon^{E}(s,\theta)-\alpha(s),\Upsilon^{E}(s,\theta)-\alpha(s)\right\rangle=\left(m^2+1\right)\left(\Upsilon^{E_3}(s,\theta)-\alpha_3(s)\right)^2.\tag{7}$$

However, the parametric equation of $\Upsilon^{E}(s,\theta)$ with Bishop frame is given by

$$\Upsilon^{E}(s,\theta)-\alpha(s)=\lambda_1(s,\theta)T(s)+\lambda_2(s,\theta)N_1(s)+\lambda_3(s,\theta)N_2(s),\tag{8}$$

where λ_1,λ_2 and λ_3 are differentiable functions of s and θ on the interval I.

Moreover, we have the following equality;

$$\left\langle\Upsilon^{E}(s,\theta)-\alpha(s),\Upsilon^{E}(s,\theta)-\alpha(s)\right\rangle=\lambda_1^2(s,\theta)+\lambda_2^2(s,\theta)+\lambda_3^2(s,\theta).\tag{9}$$

Hence, via (7) and (9) we have

$$\lambda_1^2(s,\theta)+\lambda_2^2(s,\theta)+\lambda_3^2(s,\theta)=\left(m^2+1\right)\left(\Upsilon^{E_3}(s,\theta)-\alpha_3(s)\right)^2.\tag{10}$$

If we differentiate (10) with respect to s and θ, then we get

$$\begin{aligned}&\lambda_1(s,\theta)\lambda_1(s,\theta)_s+\lambda_2(s,\theta)\lambda_2(s,\theta)_s+\lambda_3(s,\theta)\lambda_3(s,\theta)_s\\=\;&\left(m^2+1\right)\left(\Upsilon^{E_3}(s,\theta)-\alpha_3(s)\right)\left(\Upsilon^{E_3}(s,\theta)-\alpha_3(s)\right)_s\end{aligned}\tag{11}$$

and

$$\begin{aligned}&\lambda_1(s,\theta)\lambda_1(s,\theta)_\theta+\lambda_2(s,\theta)\lambda_2(s,\theta)_\theta+\lambda_3(s,\theta)\lambda_3(s,\theta)_\theta\\=\;&\left(m^2+1\right)\left(\Upsilon^{E_3}(s,\theta)-\alpha_3(s)\right)\left(\Upsilon^{E_3}(s,\theta)\right)_\theta,\end{aligned}\tag{12}$$

respectively. Also, if we differentiate the equation (8) with respect to s and θ, then we have

$$\begin{aligned}\Upsilon^{E}(s,\theta)_s=&\left(\|\alpha'(s)\|+\lambda_1(s,\theta)_s-k_1\lambda_2(s,\theta)_s-k_2\lambda_3(s,\theta)_s\right)T(s)\\&+\left(k_1\lambda_1(s,\theta)+\lambda_2(s,\theta)_s\right)N_1(s)+\left(k_2\lambda_1(s,\theta)+\lambda_3(s,\theta)_s\right)N_2(s)\end{aligned}\tag{13}$$

and

$$\Upsilon^{E}(s,\theta)_\theta=\lambda_1(s,\theta)_\theta T(s)+\lambda_2(s,\theta)_\theta N_1(s)+\lambda_3(s,\theta)_\theta N_2(s),\tag{14}$$

respectively.

Now, let us consider that

$$\left\langle\Upsilon^{E}(s,\theta)-\alpha(s),\Upsilon^{E}(s,\theta)_s\right\rangle=0$$

is satisfied on the surface $\Upsilon^{E}(s,\theta)$. Next, by the help of equations (8) and (13) we get

$$\lambda_1(s,\theta)\|\alpha'(s)\|+\lambda_1(s,\theta)\lambda_1(s,\theta)_s+\lambda_2(s,\theta)\lambda_2(s,\theta)_s+\lambda_3(s,\theta)\lambda_3(s,\theta)_s=0.\tag{15}$$

Hence, by using (11) and (15) we obtain

$$\lambda_1(s,\theta) = -\frac{\left(m^2+1\right)}{\|\alpha'(s)\|}\left(\Upsilon^{E_3}(s,\theta)-\alpha_3(s)\right)\left(\Upsilon^{E_3}(s,\theta)-\alpha_3(s)\right)_s. \tag{16}$$

Thus, thanks to properties of trigonometric functions and (10) we can choose

$$\begin{aligned}\lambda_2(s,\theta) &= \pm\sqrt{m^2+1}\left(\Upsilon^{E_3}(s,\theta)-\alpha_3(s)\right)\sqrt{1-\frac{\left(m^2+1\right)}{\|\alpha'(s)\|^2}\left(\Upsilon^{E_3}(s,\theta)-\alpha_3(s)\right)_s^2}\cos\theta\\ \lambda_3(s,\theta) &= \pm\sqrt{m^2+1}\left(\Upsilon^{E_3}(s,\theta)-\alpha_3(s)\right)\sqrt{1-\frac{\left(m^2+1\right)}{\|\alpha'(s)\|^2}\left(\Upsilon^{E_3}(s,\theta)-\alpha_3(s)\right)_s^2}\sin\theta.\end{aligned} \tag{17}$$

So, we can construct our main theorem given as following:

Theorem 3.3. *In Euclidean 3−space E^3, an embankment surface is the envelope of a one-parameter family of cones centered at the spine curve $\alpha(s)$. The embankment surface is can be parametrized as*

$$\begin{aligned}\Upsilon^E(s,\theta) &= \alpha(s) - \frac{m^2+1}{\|\alpha'(s)\|}\psi_3(s,\theta)\,\psi_3(s,\theta)_s\overrightarrow{T}(s)\\ &\pm\sqrt{m^2+1}\psi_3(s,\theta)\sqrt{1-\frac{\left(m^2+1\right)}{\|\alpha'(s)\|^2}\psi_3(s,\theta)_s^2}(\cos\theta\overrightarrow{N_1}(s)+\sin\theta\overrightarrow{N_2}(s)),\end{aligned} \tag{18}$$

where $\left\{\overrightarrow{T},\overrightarrow{N_1},\overrightarrow{N_2}\right\}$ is the Bishop frame of $\alpha(s)$, $\psi_3(s,\theta) = \Upsilon^{E_3}(s,\theta)-\alpha_3(s)$ and $m\in\mathbb{R}$, $m>0$ with $\left|m\alpha_3'\right| < \sqrt{\left(\alpha_1'\right)^2+\left(\alpha_2'\right)^2}$.

Theorem 3.4. *Let $\alpha : I\subset\mathbb{R}\to\mathbb{R}^3$ be a unit speed diretrix curve of embankment surface $\Upsilon^E(s,\theta)$ and $q(s,\theta) = \cos\theta+\sin\theta\overrightarrow{T}(s)$ be a unit real quaternion in $S^3\subset\mathbb{R}^4$. Then, the parametric equation of embankment surface $\Upsilon^E(s,\theta)$ generated by the curve α can be given*

(*i*) *via the real quaternion product $q(s,\theta)\times\overrightarrow{N_1}(s)$*

$$\Upsilon^E(s,\theta) = \alpha(s)-\left(m^2+1\right)\psi_3(s,\theta)\,\psi_3(s,\theta)_s\overrightarrow{T}(s)\pm\sqrt{m^2+1}\psi_3(s,\theta)\sqrt{1-(m^2+1)\,\psi_3(s,\theta)_s^2}q(s,\theta)\times\overrightarrow{N_1}(s) \tag{19}$$

(*ii*) *via the matrix representation of the map $\phi:\mathbb{R}^3\to\mathbb{R}^3$ for the unit real quaternion $q(s,\theta)$ and $\Upsilon^E(s,\theta)$ can be obtained by the homothetic motion as*

$$\Upsilon^E(s,\theta) = \alpha(s)-\left(m^2+1\right)\psi_3(s,\theta)\,\psi_3(s,\theta)_s\overrightarrow{T}(s)\pm\sqrt{m^2+1}\psi_3(s,\theta)\sqrt{1-(m^2+1)\,\psi_3(s,\theta)_s^2}M_qN_1(s) \tag{20}$$

where $\left\{\overrightarrow{T},\overrightarrow{N_1},\overrightarrow{N_2}\right\}$ is the Bishop frame of $\alpha(s)$, M_q is the matrix representation of the map ϕ for $q(s,\theta)$, $\psi_3(s,\theta) = \Upsilon^{E_3}(s,\theta)-\alpha_3(s)$ and $m\in\mathbb{R}$, $m>0$ with $\left|m\alpha_3'\right| < \sqrt{\left(\alpha_1'\right)^2+\left(\alpha_2'\right)^2}$.

Proof. Assume that $\Upsilon^E(s,\theta)$ is a embankment surface generated by the unit speed curve $\alpha(s)$. By using (18) we can easily obtain

$$\begin{aligned}\Upsilon^E(s,\theta) &= \alpha(s)-\left(m^2+1\right)\psi_3(s,\theta)\,\psi_3(s,\theta)_s\overrightarrow{T}(s)\\ &\pm\sqrt{m^2+1}\psi_3(s,\theta)\sqrt{1-(m^2+1)\,\psi_3(s,\theta)_s^2}(\cos\theta\overrightarrow{N_1}(s)+\sin\theta\overrightarrow{N_2}(s)).\end{aligned}$$

The definition of the real quaternion product for the unit real quaternion $q(s,\theta) = \cos\theta + \sin\theta \overrightarrow{T}(s)$ and the pure real quaternion $N_1(s)$, we get $q(s,\theta) \times \overrightarrow{N_1}(s) = \cos\theta \overrightarrow{N_1}(s) + \sin\theta \overrightarrow{N_2}(s)$.

On the other hand, if $\alpha(s) - \left(m^2+1\right)\psi_3(s,\theta)\,\psi_3(s,\theta)_s \overrightarrow{T}(s)$, $\pm\sqrt{m^2+1}\psi_3(s,\theta)\sqrt{1-(m^2+1)\,\psi_3(s,\theta)_s^2}$ and M_q are defined as translation vector, the homothetic scalar and the orthogonal matrix of the homothetic motion, respectively. Following this, (20) is the homothetic motion. This completes the proof. □

4. Embankment-like surfaces according to the Bishop frame in E^3

In this section, first, embankment-like surfaces, which are special cases of embankment surfaces, will be defined. Then, the parametric equations of these surfaces according to the Bishop frame will be given by means of quaternion product and matrix representation as homothetic motion.

Definition 4.1. Let $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$ be a regular unit speed space curve. Then, the surface $\Upsilon^{EL}(s,\theta) = \left(\Upsilon^{EL_1}(s,\theta), \Upsilon^{EL_2}(s,\theta), \Upsilon^{EL_3}(s,\theta)\right)$ which can be given by

$$\begin{aligned}\Upsilon^{EL}(s,\theta) &= \alpha(s) - \left(m^2+1\right)\psi(s,\theta)\,\psi(s,\theta)_s \overrightarrow{T}(s)\\ &\pm\sqrt{m^2+1}\psi(s,\theta)\sqrt{1-(m^2+1)\,\psi(s,\theta)_s^2}(\cos\theta\overrightarrow{N_1}(s) + \sin\theta\overrightarrow{N_2}(s))\end{aligned} \tag{21}$$

is called an embankment-like surface, where $\left\{\overrightarrow{T}, \overrightarrow{N_1}, \overrightarrow{N_2}\right\}$ is the Bishop frame apparatus of $\alpha(s)$, $\psi(s,\theta) = \Omega(s,\theta) - \alpha_3(s)$ and $m \in \mathbb{R}$, $m > 0$ with $\left|m\alpha_3'\right| < \sqrt{\left(\alpha_1'\right)^2 + \left(\alpha_2'\right)^2}$ and $\Omega(s,\theta)$ is an arbitrary function of s and θ.

Corollary 4.2. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$ be a unit speed diretrix curve of embankment-like surface $\Upsilon^{EL}(s,\theta)$ and $q(s,\theta) = \cos\theta + \sin\theta\overrightarrow{T}(s)$ be a unit real quaternion in $S^3 \subset \mathbb{R}^4$. Then, the parametric equation of embankment-like surface $\Upsilon^{EL}(s,\theta)$ generated by the curve α can be given*

(*i*) *via the real quaternion product $q(s,\theta) \times \overrightarrow{N_1}(s)$*

$$\Upsilon^{EL}(s,\theta) = \alpha(s) - \left(m^2+1\right)\psi(s,\theta)\,\psi(s,\theta)_s \overrightarrow{T}(s) \pm \sqrt{m^2+1}\psi(s,\theta)\sqrt{1-(m^2+1)\,\psi(s,\theta)_s^2}q(s,\theta)\times\overrightarrow{N_1}(s) \tag{22}$$

(*ii*) *via the matrix representation of the map $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ for the unit real quaternion $q(s,\theta)$ and $\Upsilon^{EL}(s,\theta)$ can be obtained by the homothetic motion as*

$$\Upsilon^{EL}(s,\theta) = \alpha(s) - \left(m^2+1\right)\psi(s,\theta)\,\psi(s,\theta)_s \overrightarrow{T}(s) \pm \sqrt{m^2+1}\psi(s,\theta)\sqrt{1-(m^2+1)\,\psi_3(s,\theta)_s^2}M_q N_1(s), \tag{23}$$

where $\left\{\overrightarrow{T}, \overrightarrow{N_1}, \overrightarrow{N_2}\right\}$ is the Bishop frame of $\alpha(s)$, $\psi(s,\theta) = \Omega(s,\theta) - \alpha_3(s)$, M_q is the matrix representation of the map ϕ for $q(s,\theta)$, $\Omega(s,\theta)$ is an arbitrary function of s and θ, $m \in \mathbb{R}$, $m > 0$ with $\left|m\alpha_3'\right| < \sqrt{\left(\alpha_1'\right)^2 + \left(\alpha_2'\right)^2}$.

Proof. The proof is easily obtained by choosing $\Upsilon^{E_3}(s,\theta) = \Omega(s,\theta)$ in the Theorem 3.4 in Section 3, where $\Omega(s,\theta)$ is an arbitrary function of s and θ. □

5. Tubembankment-like surfaces according to the Bishop frame in E^3

In this section, tubembankment-like surface, which is the most special case of the embankment surfaces obtained with the help of the constant ψ=c function, will be introduced. Then, the parametric equations of these surfaces according to the Bishop frame will be given by means of quaternion product and matrix representation as homothetic motion. Subsequently, the geometric properties and equations of these surfaces will be given.

Definition 5.1. In the embankment-like surface given by (21), if ψ=c is a constant function, then this surface called a tubembankment-like surface.

Corollary 5.2. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$ be a unit speed diretrix curve of the tubembankment-like surface $\Upsilon^{TEL}(s,\theta)$. Then, the parametrization of the tubembankment-like surface generated by $\alpha(s)$ given by*

$$\Upsilon^{TEL}(s,\theta) = \alpha(s) \pm c\sqrt{m^2+1}(\cos\theta\overrightarrow{N_1}(s) + \sin\theta\overrightarrow{N_2}(s)), \tag{24}$$

where $\left\{\overrightarrow{T}, \overrightarrow{N_1}, \overrightarrow{N_2}\right\}$ is the Bishop frame of $\alpha(s)$.

Corollary 5.3. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$ be a unit speed diretrix curve of the tubembankment-like surface $\Upsilon^{TEL}(s,\theta)$ given by and $q(s,\theta) = \cos\theta + \sin\theta\overrightarrow{T}(s)$ be a unit real quaternion in $S^3 \subset \mathbb{R}^4$. Then, the parametric equation of the tubembankment-like surface $\Upsilon^{TEL}(s,\theta)$ generated by the curve α can be given*

(i) via the real quaternion product $q(s,\theta) \times \overrightarrow{N_1}(s)$

$$\Upsilon^{TEL}(s,\theta) = \alpha(s) \pm c\sqrt{m^2+1}q(s,\theta) \times \overrightarrow{N_1}(s) \tag{25}$$

(ii) via the matrix representation of the map $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ for the unit real quaternion $q(s,\theta)$ and $\Upsilon^{TEL}(s,\theta)$ can be obtained by the homothetic motion as

$$\Upsilon^{TEL}(s,\theta) = \alpha(s) \pm c\sqrt{m^2+1}M_qN_1(s), \tag{26}$$

where $\left\{\overrightarrow{T}, \overrightarrow{N_1}, \overrightarrow{N_2}\right\}$ is the Bishop frame of $\alpha(s)$, M_q is the matrix representation of the map ϕ for $q(s,\theta)$, $\psi(s,\theta) = c$ constant function, $m \in \mathbb{R}$, $m > 0$ with $\left|m\alpha_3'\right| < \sqrt{\left(\alpha_1'\right)^2 + \left(\alpha_2'\right)^2}$.

The unit normal vector $\overrightarrow{U}$ of the tubembankment-like surface is given as

$$\overrightarrow{U} = -\cos\theta\overrightarrow{N_1} - \sin\theta\overrightarrow{N_2} \tag{27}$$

The first and second order derivatives of the tubembankment-like surface Υ^{TEL} with respect to the parameters s and θ can be easily calculated as follows:

$$\begin{aligned}
\Upsilon_s^{TEL} &= AT,\\
\Upsilon_\theta^{TEL} &= -c\sqrt{m^2+1}\sin\theta N_1 + c\sqrt{m^2+1}\cos\theta N_2,\\
\Upsilon_{ss}^{TEL} &= \left(-c\sqrt{m^2+1}k_1'\cos\theta - c\sqrt{m^2+1}k_2'\sin\theta\right)T + Ak_1N_1 + Ak_2N_2,\\
\Upsilon_{s\theta}^{TEL} &= \left(c\sqrt{m^2+1}k_1\sin\theta - c\sqrt{m^2+1}k_2\cos\theta\right)T,\\
\Upsilon_{\theta\theta}^{TEL} &= -c\sqrt{m^2+1}\cos\theta N_1 - c\sqrt{m^2+1}\sin\theta N_2.
\end{aligned} \tag{28}$$

In order to the surface properties of the tubembankment-like surface, it is useful to have expressions for the coefficients of the first and second fundamental forms are given by

$$\begin{aligned}
g_{11} = A^2, g_{12} = 0,\ g_{22} = c^2\left(m^2+1\right),\\
L_{11} = A\left(-k_1\cos\theta - k_2\sin\theta\right), L_{12} = 0,\ L_{22} = c\sqrt{m^2+1}.
\end{aligned} \tag{29}$$

The Gauss and mean curvature of the tubembankment-like surface is given as follows;

$$\begin{aligned}
K &= \frac{(-k_1\cos\theta - k_2\sin\theta)}{Ac\sqrt{m^2+1}},\\
H &= \frac{c\sqrt{m^2+1}\,(-k_1\cos\theta - k_2\sin\theta) + A}{2Ac\sqrt{m^2+1}},\\
H &= \frac{1}{2}\left(\frac{1}{c\sqrt{m^2+1}} + c\sqrt{m^2+1}K\right),
\end{aligned} \tag{30}$$

where $A = \left(1 - c\sqrt{m^2+1}k_1\cos\theta - c\sqrt{m^2+1}k_2\sin\theta\right)$.

Moreover, we can easily see $H_sK_\theta - H_\theta K_s = 0$. Because of this the tubembankment-like surface $\Upsilon^{TEL}(s,\theta)$ is a (H,K)-Weingarten surface.

Proposition 5.4. *The tubembankment-like surface $\Upsilon^{TEL}(s,\theta)$ given by (24) is a regular surface if and only if the inequality $k_1\cos\theta + k_2\sin\theta \neq \dfrac{1}{c\sqrt{m^2+1}}$ is provided.*

Proof. The surface to be regular should provide inequality $g_{11}g_{22} - g_{12}^2 \neq 0$. By using (29), we have

$$g_{11}g_{22} - g_{12}^2 = c^2\left(m^2+1\right)A^2.$$

Since $c > 0$ and from the last equation we can calculate

$$A^2c^2\left(m^2+1\right) \neq 0,$$

where $A = \left(1 - c\sqrt{m^2+1}k_1\cos\theta - c\sqrt{m^2+1}k_2\sin\theta\right)$.

Therefore, the proof is easily completed. □

Theorem 5.5. *The Gaussian curvature K of the regular tubembankment-like $\Upsilon^{TEL}(s,\theta)$ is zero if and only if the spine curve $\alpha(s)$ is a planar curve and slant helix according to Bishop Frame for the s parameter curves.*

Proof. Let K be zero. Then,

$$K = \frac{(-k_1\cos\theta - k_2\sin\theta)}{Ac\sqrt{m^2+1}} = 0,$$

where $A = \left(1 - c\sqrt{m^2+1}k_1\cos\theta - c\sqrt{m^2+1}k_2\sin\theta\right)$.

Hence, we conclude that $k_1\cos\theta + k_2\sin\theta = 0$. Therefore, $\dfrac{k_1}{k_2}$ is constant for s parameter curves. Because of the fact that it is a constant for the s parameter curves, the normal development (k_1, k_2) lies on a line through the origin. Moreover, according to [5] for the s parameter curves $\dfrac{k_1}{k_2}$ is constant and we know that $\Theta = arctan\left(\dfrac{k_2}{k_1}\right)$ *and* $\Theta = \int \tau(s)ds$. Therefore Θ is constant, namely $\tau = 0$. So, $\alpha(s)$ is a planar curve, also since $\dfrac{k_1}{k_2}$ is a constant, for the s parameter curves the spine curve $\alpha(s)$ is a slant helix according to the Bishop frame [7]. The sufficiency part of the proof is obvious. See for detail [5] and [7]. □

Theorem 5.6. *The parameter curves of tubembankment-like surface $\Upsilon^{TEL}(s,\theta)$ are lines of curvature.*

Proof. For the tubembankment-like surface $\Upsilon^{TEL}(s,\theta)$, by using (29) we have $g_{12} = L_{12} = 0$. Therefore, from theorem of line of curvature the parameter curves of $\Upsilon^{TEL}(s,\theta)$ are lines of curvature. □

Theorem 5.7. *Let $\Upsilon^{TEL}(s,\theta)$ be a regular tubembankment-like surface in E^3 given by (24). Section curves of the tubembankment-like surface $\Upsilon^{TEL}(s,\theta)$ has the following properties.*

(i) The $s-$parameter curve of the surface $\Upsilon^{TEL}(s,\theta)$ is an asymptotic curve if and only if the spine curve $\alpha(s)$ is a planar curve and slant helix according to Bishop frame.

(ii) The $\theta-$parameter curve of the surface $\Upsilon^{TEL}(s,\theta)$ can not be an asymptotic curve.

Proof. (i) A curve γ on a surface in E^3 is an asymptotic curve if and only if the vector field γ'' is tangent to the surface for all points of γ, that is $\left\langle \overrightarrow{U}, \gamma'' \right\rangle = 0$. For the $s-$parameter curve, via (27) and (28) we have

$$\left\langle \overrightarrow{U}, \Upsilon_{ss}^{TEL} \right\rangle = A\left(-k_1\cos\theta - k_2\sin\theta\right) = 0,$$

where $A = \left(1 - c\sqrt{m^2+1}k_1\cos\theta - c\sqrt{m^2+1}k_2\sin\theta\right)$.

Since $\Upsilon^{TEL}(s,\theta)$ is a regular tubembankment-like surface, $k_1\cos\theta + k_2\sin\theta \neq \dfrac{1}{c\sqrt{m^2+1}}$. Then, by using (27) and (28), $\left\langle \overrightarrow{U}, \Upsilon^{TEL}_{ss}\right\rangle = 0 \Longleftrightarrow k_1\cos\theta + k_2\sin\theta = 0$. Therefore, $\dfrac{k_1}{k_2}$ is constant for s parameter curves. From this, it follows that the spine curve $\alpha(s)$ is a slant helix according to Bishop Frame [7]. Moreover, with the same idea with proof of Theorem 5.5, Θ is constant namely $\tau = 0$. So, $\alpha(s)$ is a planar curve.
(ii) Because of the following, an inequality is provided by (27) and (28)

$$\left\langle \overrightarrow{U}, \Upsilon^{TEL}_{\theta\theta}\right\rangle = c\sqrt{m^2+1} \neq 0$$

for $\theta-$parameter curve, this curve can not be an asymptotic curve. □

Theorem 5.8. *Let $\Upsilon^{TEL}(s,\theta)$ be a regular tubembankment-like surface in E^3 given by* (24) *. Section curves of the tubembankment-like surface $\Upsilon^{TEL}(s,\theta)$ have the following properties.*
(i) The $\theta-$parameter curve of the surface $\Upsilon^{TEL}(s,\theta)$ is a geodesic curve.
(ii) The $s-$parameter curve of the surface $\Upsilon^{TEL}(s,\theta)$ can not be a geodesic curve.

Proof. (i) A curve γ lying on a surface is a geodesic curve if and only if the acceleration vector field γ'' and the surface normal $\overrightarrow{U}$ are linearly dependent namely $\overrightarrow{U} \times \gamma'' = 0$. In this case, for the $\theta-$parameter curve by using (27) and (28) we can easily see that

$$\overrightarrow{U} \times \Upsilon^{TEL}_{\theta\theta} = 0.$$

Hence, the $\theta-$parameter curve on the surface is always a geodesic curve.
(ii) For the $s-$ parameter curve, by using (27) and (28) we compute

$$\begin{aligned}\overrightarrow{U} \times \Upsilon^{TEL}_{ss} &= \left[A\left(k_1\sin\theta - k_2\cos\theta\right)\right]T \\ &\quad + \left[c\sqrt{m^2+1}\sin\theta\left(k_1'\cos\theta + k_2'\sin\theta\right)\right]N_1 \\ &\quad + \left[-c\sqrt{m^2+1}\cos\theta\left(k_1'\cos\theta + k_2'\sin\theta\right)\right]N_2,\end{aligned}$$

where $A = \left(1 - c\sqrt{m^2+1}k_1\cos\theta - c\sqrt{m^2+1}k_2\sin\theta\right)$.

Since, $\{T, N_1, N_2\}$ is a triply orthonormal system and $\overrightarrow{U} \times \Upsilon^{TEL}_{ss} = 0$, the following equalities are written

$$\begin{cases} A\left(k_1\sin\theta - k_2\cos\theta\right) = 0, \\ c\sqrt{m^2+1}\sin\theta\left(k_1'\cos\theta + k_2'\sin\theta\right) = 0, \\ -c\sqrt{m^2+1}\cos\theta\left(k_1'\cos\theta + k_2'\sin\theta\right) = 0, \end{cases}$$

where $A = \left(1 - c\sqrt{m^2+1}k_1\cos\theta - c\sqrt{m^2+1}k_2\sin\theta\right)$.

Therefore, $\Upsilon^{TEL}(s,\theta)$ is a regular tubembankment-like surface and we know that $sin\theta$ and $cos\theta$ cannot be zero at the same time. Then, we have

$$\begin{aligned} k_1\sin\theta - k_2\cos\theta &= 0 \\ k_1'\cos\theta + k_2'\sin\theta &= 0 \end{aligned}$$

We have $\dfrac{k_1}{k_2} = \cot\theta$ and $\dfrac{k_1'}{k_2'} = -tan\theta$. Here, by taking the derivative of the first equation with respect to s and performing the necessary simplifications, the equation $\dfrac{k_1}{k_2} = \dfrac{k_1'}{k_2'}$ is obtained. Then, using this equality and the previous two equations, we obtain $\cot\theta + \tan\theta = 0$. From this, it follows that $\dfrac{1}{\sin\theta\cos\theta} \neq 0$. This is a contradiction, i.e., the system of equations does not have a solution. Then, the s parameter curve of $\Upsilon^{TEL}(s,\theta)$ cannot be a geodesic curve. □

A minimal surface which has zero mean curvature is the surface of the smallest area spanned by a given space curve. Minimal surface theory has been developed rapidly in recent time. Also, there is a curve on the surface called a minimal curve containing the minimal points. The notion of minimal curves was given in the following proposition by Semin [31].

Proposition 5.9. *[31] On the surface, a hyperbolic point is a minimal point if and only if the coefficients of the first and second fundamental forms at this point is satisfying the following equation*

$$g_{22}L_{11} - 2g_{12}L_{12} + g_{11}L_{22} = 0. \tag{31}$$

Theorem 5.10. *The minimal curves of regular tubembankment-like surface given by the equation*

$$\Upsilon^{TEL}(s,\theta) = \alpha(s) + c\sqrt{m^2+1}(\cos\theta\overrightarrow{N_1}(s) + \sin\theta\overrightarrow{N_2}(s)) \tag{32}$$

are as follows:

$$\beta(s) = \alpha(s) + c\sqrt{m^2+1}(\cos\theta\overrightarrow{N_1}(s) + \sin\theta\overrightarrow{N_2}(s)), \tag{33}$$

where $\theta = \int \tau(s)\,ds - \arccos\left(\dfrac{1}{2\kappa(s)\,c\sqrt{m^2+1}}\right)$, $\kappa \neq 0$, *and here* κ *and* τ *are the curvature functions of the curve* $\alpha(s)$.

Proof. From the equations (29) and (31) we have

$$2c\sqrt{m^2+1}\,(-k_1\cos\theta - k_2\sin\theta) + 1 = 0$$

Therefore, we obtain

$$cos(\Theta - \theta) = \frac{1}{2\kappa(s)\,c\sqrt{m^2+1}}$$

and so

$$\theta = \int \tau(s)\,ds - \arccos\left(\frac{1}{2\kappa(s)\,c\sqrt{m^2+1}}\right),$$

where $k_1 = \kappa\cos\Theta$, $k_2 = \kappa\sin\Theta$ and $\Theta = \int\tau(s)ds$. This completes the proof. □

6. Visualizations for the embankment-like and tubembankment-like surfaces in E^3

In this section, some examples for embankment-like and tubembankment-like surfaces generated by the unit speed curve $\alpha(s)$ with the Bishop frame are given. We assume positive sign for examples, however similarly we can construct these surface with negative sign. Then, the parametric equations of these surfaces are given. Moreover, their graphs by using the Mathematica Programme are drawn.

Let $\alpha(s)$ be a unit speed curve. The curve is given as follows

$$\alpha(s) = \left(\frac{\sqrt{3}}{4}\sin 2s, -\frac{\sqrt{3}}{4}\cos 2s, -\frac{s}{2}\right)$$

vector fields and curvatures of Bishop frame $\{T, N_1, N_2, k_1, k_2\}$ of the curve α can be calculated as follows

$$\begin{aligned}
T(s) &= (t_1, t_2, t_3) = \left(\frac{\sqrt{3}}{2}\cos 2s, \frac{\sqrt{3}}{2}\sin 2s, -\frac{1}{2}\right) \\
N_1(s) &= \left(-\cos s \sin 2s + \frac{1}{2}\sin s \cos 2s, \cos s \cos 2s + \frac{1}{2}\sin s \sin 2s, \frac{\sqrt{3}}{2}\sin s\right) \\
N_2(s) &= \left(\sin s \sin 2s + \frac{1}{2}\cos s \cos 2s, -\sin s \cos 2s + \frac{1}{2}\cos s \sin 2s, \frac{\sqrt{3}}{2}\cos s\right) \\
k_1 &= \sqrt{3}\cos\Theta, k_2 = \sqrt{3}\sin\Theta, \Theta = \int \tau(s)\, ds.
\end{aligned}$$

Example 6.1. Let us consider that $\Omega(s,\theta) = 1-\theta-\frac{s}{2}, m = \frac{1}{2}, \theta \neq 1$. Then we can construct the embankment-like surface as following,

(i) via the real quaternion product;

$$\Upsilon^{EL}(s,\theta) = \alpha(s) + \frac{\sqrt{5}}{2}(1-\theta)\, q(s,\theta) \times N_1(s),$$

where $q(s,\theta) = \cos\theta + \sin\theta \overrightarrow{T}(s)$ is a unit real quaternion.

(ii) as homothetic motion via matrix representation of $q(s,\theta)$;

$$\Upsilon^{EL}(s,\theta) = \alpha(s) + \frac{\sqrt{5}}{2}(1-\theta)\, M_q N_1,$$

where M_q is a matrix equivalent of the unit real quaternion $q(s,\theta)$ as given following;

$$M_q = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

$$M_q = \begin{bmatrix} \cos^2\theta + \sin^2\theta\left(t_1^2 - t_2^2 - t_3^2\right) & -t_3\sin 2\theta + 2t_1t_2\sin^2\theta & t_2\sin 2\theta + 2t_1t_3\sin^2\theta \\ t_3\sin 2\theta + 2t_1t_2\sin^2\theta & \cos^2\theta + \sin^2\theta\left(t_2^2 - t_1^2 - t_3^2\right) & -t_1\sin 2\theta + 2t_2t_3\sin^2\theta \\ -t_2\sin 2\theta + 2t_1t_3\sin^2\theta & t_1\sin 2\theta + 2t_2t_3\sin^2\theta & \cos^2\theta + \sin^2\theta\left(t_3^2 - t_1^2 - t_2^2\right) \end{bmatrix}.$$

First column of M_q as given

$$\begin{bmatrix} m_{11} \\ m_{21} \\ m_{31} \end{bmatrix} = \begin{bmatrix} \cos^2\theta - \frac{1}{4}\sin^2\theta + \frac{3}{4}\sin^2\theta\cos 4s \\ -\frac{1}{2}\sin 2\theta + \frac{3}{4}\sin^2\theta\sin 4s \\ -\frac{\sqrt{3}}{2}\left(\sin 2\theta\sin 2s + \sin^2\theta\cos 2s\right) \end{bmatrix},$$

second column of M_q as given

$$\begin{bmatrix} m_{12} \\ m_{22} \\ m_{32} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sin 2\theta + \frac{3}{4}\sin^2\theta\sin 4s \\ \cos^2\theta - \frac{1}{4}\sin^2\theta - \frac{3}{4}\sin^2\theta\cos 4s \\ \frac{\sqrt{3}}{2}\left(\sin 2\theta\cos 2s - \sin^2\theta\sin 2s\right) \end{bmatrix},$$

and third column of M_q as given

$$\begin{bmatrix} m_{13} \\ m_{23} \\ m_{33} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2}\left(\sin 2\theta \sin 2s - \sin^2\theta \cos 2s\right) \\ -\frac{\sqrt{3}}{2}\left(\sin 2\theta \cos 2s + \sin^2\theta \sin 2s\right) \\ \cos^2\theta - \frac{1}{2}\sin^2\theta \end{bmatrix}.$$

(*iii*) in the form of parametric equation;

$$\Upsilon^{EL}(s,\theta) = \begin{pmatrix} \frac{\sqrt{3}}{4}\sin 2s + \frac{\sqrt{5}}{2}(1-\theta)\sin 2s \cos(\theta+s) + \frac{\sqrt{5}}{4}(1-\theta)\cos 2s \sin(\theta+s), \\ -\frac{\sqrt{3}}{4}\cos 2s + \frac{\sqrt{5}}{2}(1-\theta)\cos 2s \cos(\theta+s) + \frac{\sqrt{5}}{4}(1-\theta)\sin 2s \sin(\theta+s), \\ -\frac{s}{2} + \frac{\sqrt{15}}{4}(1-\theta)\sin(\theta+s) \end{pmatrix}.$$

The graph of this surface can be found in Figure 2.

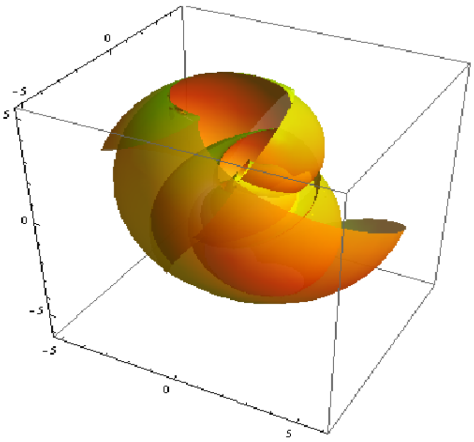

Figure 2: Embankment-like surface $\Upsilon^{EL}(s,\theta)$

Example 6.2. Let us consider that $\Omega(s,\theta) = \theta, m = \frac{1}{2}$. Then we can construct the embankment-like surface as following,

(*i*) via the real quaternion product;

$$\Upsilon^{EL}(s,\theta) = \alpha(s) - \frac{5}{8}\left(\theta + \frac{s}{2}\right)T(s) + \frac{\sqrt{55}}{8}\left(\theta + \frac{s}{2}\right)q(s,\theta) \times N_1(s),$$

where $q(s,\theta) = \cos\theta + \sin\theta \overrightarrow{T}(s)$ is a unit real quaternion.

(*ii*) as homothetic motion via matrix representation of $q(s,\theta)$;

$$\Upsilon^{EL}(s,\theta) = \alpha(s) - \frac{5}{8}\left(\theta + \frac{s}{2}\right)T(s) + \frac{\sqrt{55}}{8}\left(\theta + \frac{s}{2}\right)M_q N_1,$$

where M_q is a matrix equivalent of the unit real quaternion $q(s,\theta)$.

(*iii*) in the form of parametric equation;

$$\Upsilon^{EL}(s,\theta)=\begin{pmatrix} \frac{\sqrt{3}}{4}\sin 2s-\frac{\sqrt{15}}{16}\left(\theta+\frac{s}{2}\right)\cos 2s+\frac{\sqrt{55}}{8}\left(\theta+\frac{s}{2}\right)\sin 2s\cos(\theta+s) \\ +\frac{\sqrt{55}}{16}\left(\theta+\frac{s}{2}\right)\cos 2s\sin(\theta+s), \\ -\frac{\sqrt{3}}{4}\cos 2s-\frac{\sqrt{15}}{16}\left(\theta+\frac{s}{2}\right)\sin 2s+\frac{\sqrt{55}}{8}\left(\theta+\frac{s}{2}\right)\cos 2s\cos(\theta+s) \\ +\frac{\sqrt{55}}{16}\left(\theta+\frac{s}{2}\right)\sin 2s\sin(\theta+s), \\ -\frac{s}{2}+\frac{\sqrt{5}}{16}\left(\theta+\frac{s}{2}\right)+\frac{\sqrt{165}}{16}\left(\theta+\frac{s}{2}\right)\sin(\theta+s) \end{pmatrix}$$

The graph of this surface can be found in Figure 3.

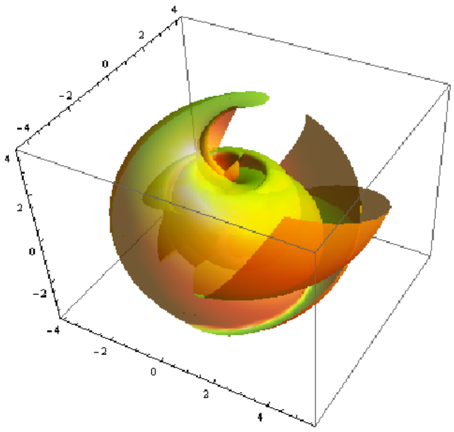

Figure 3: Embankment-like surface $\Upsilon^{EL}(s,\theta)$

Example 6.3. Let us consider that $\Psi(s,\theta)=1, m=\frac{3}{4}$. Then we can construct the tubembankment-like surface as following,

(*i*) via the real quaternion product;

$$\Upsilon^{TEL}(s,\theta)=\alpha(s)+\frac{5}{4}q(s,\theta)\times N_1(s),$$

where $q(s,\theta)=\cos\theta+\sin\theta\overrightarrow{T}(s)$ is a unit real quaternion.

(*ii*) as homothetic motion via matrix representation of $q(s,\theta)$;

$$\Upsilon^{TEL}(s,\theta)=\alpha(s)+\frac{5}{4}M_qN_1,$$

where M_q is a matrix equivalent of the unit real quaternion $q(s,\theta)$.

(*iii*) in the form of parametric equation;

$$\Upsilon^{TEL}(s,\theta)=\begin{pmatrix}\frac{\sqrt{3}}{4}\sin 2s-\frac{5}{4}\sin 2s\cos(\theta+s)+\frac{5}{8}\cos 2s\sin(\theta+s),\\ -\frac{\sqrt{3}}{4}\cos 2s+\frac{5}{4}\cos 2s\cos(\theta+s)+\frac{5}{8}\sin 2s\sin(\theta+s),\\ -\frac{s}{2}+\frac{5\sqrt{3}}{8}\sin(\theta+s)\end{pmatrix}.$$

The Gauss and mean curvature of surface $\Upsilon^{TEL}(s,\theta)$ is given as;

$$K=\frac{16\sqrt{3}\cos(\theta+s)}{25\sqrt{3}\cos(\theta+s)-20},\ H=\frac{8-20\sqrt{3}\cos(\theta+s)}{20-25\sqrt{3}\cos(\theta+s)}.$$

In this surface, if $\theta=-s-\arccos\left(\frac{2}{5\sqrt{3}}\right)$ or $\theta=-s-\arccos\left(\frac{4}{5\sqrt{3}}\right)$ is taken, then we obtain minimal curves of this surface.

The graph of this surface can be found in Figure 4.

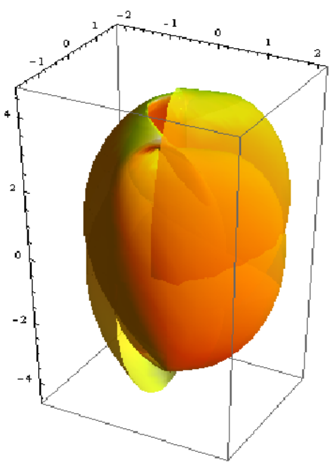

Figure 4: Tubembankment-like surface $\Upsilon^{TEL}(s,\theta)$

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