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Classification of homothetic production models whose isoquants are minimal

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Abstract. In this article, we give the complete classification of homothetic production functions with minimal isoquants, solving an open problem raised in [C.-D. Neacşu et al., *Math. Methods Appl. Sci.* 47 (9) (2024), pp. 7532-7545] and generalizing some recent results obtained in [Y. Fu, Y. Luo, *J. Math. Anal. Appl.* 541 (2025) 128670].

1. Introduction

Production functions are fundamental tools used in economic analysis to mathematically model production processes. According to [29], these functions were introduced by the German scientist J. V. von Thünen, one of the founders of mathematical economics and of econometrics, in the seminal work [33]. A production function is nothing but a map f of n variables $(x_1, x_2, \dots, x_n)$, defined on an open domain $\mathcal{D}$ of $\mathbb{R}^n_+$, i.e. $f : \mathcal{D} \subset \mathbb{R}^n_+ \to \mathbb{R}_+$, $f = f(x_1, x_2, \dots, x_n)$, where n is the number of inputs involved in the process of production, $x_1, x_2, \dots, x_n$ are the input factors, while f represents the output. Over time, in addition to classic production functions, such as Cobb-Douglas and CES (for more details and other examples see [2, 15, 24, 26, 29]), numerous general classes of production functions have been investigated, the most important of which are the homogeneous, weighted-homogeneous, quasi-sum, quasi-product, and homothetic models. Investigating the geometric properties of these production models through the associated production hypersurfaces is a topic of great interest because curvature conditions are essential in economic analysis, these appearing even in the celebrated theory of welfare economics developed by Debreu (see [16]). Note that various results on the differential geometry of production models were established in [1, 3, 5–14, 17, 19, 22, 28, 34–36].

A basic concept used in production function theory is that of the isoquant. Note that the isoquants play a crucial role in production function theory because they represent the combinations of inputs that produce the same level of output. Specifically, an isoquant shows all the possible combinations of production factors (such as labor and capital) that result in a constant amount of output. These curves are analogous to indifference curves in consumer theory but are applied to production. The importance of isoquants in production function theory mainly consists in resource optimization, they helping in understanding

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how a firm can combine available resources (such as labor, capital, land, technology) to produce a specific quantity of goods or services. By studying the geometry of isoquants, a firm can analyze the efficiency of input combinations. In particular, the minimality of isoquants is related to the phenomenon of substitution between production factors [18].

Recently, Neacşu et al. [30] investigated the isoquants of quasi-product production functions, providing a full classification of such models possessing minimal isoquants. At the end of the mentioned paper, four challenging open problems were proposed for further research, namely the classification of homogeneous, weighted-homogeneous, quasi-sum, and homothetic production functions with minimal isoquants. Two of the four proposed problems were solved by Fu and Luo in [21], namely the classification of homogeneous production functions with n inputs ($n \geq 2$) and quasi-sum production functions with two inputs having minimal isoquants. The aim of the present paper is to solve a third problem on the list, namely the classification of homothetic production models with minimal isoquants. Recall that homothetic production functions, originally introduced by Shephard in [31] and expanded upon in [32], hold significant importance. It was demonstrated in [32] that this type of production structure is both a necessary and sufficient condition for the associated cost function to decompose into a product of output and a factor price index. Later, in [20], these functions, which increase strictly along rays in the input space, were defined through a functional equation. Note that important results on the geometry of homothetic production functions were obtained in [4, 10, 11]. In this work, we are going to obtain some new result concerning the geometry of these production models. We will prove that there exist a single family of homothetic production functions with minimal isoquants, generalizing in particular [21, Theorem 3.1]. A direct consequence of this result is presented in the last part of the paper, along with some examples to illustrate the main result.

2. Preliminaries

A production function f associated with a production process with n inputs $x_1, x_2, \ldots, x_n$ ($n \geq 2$) is said to be *homothetic* if it has the form [11]

$$f(x_1, x_2, \ldots, x_n) = F\left(h(x_1, x_2, \ldots, x_n)\right), \tag{1}$$

where h is a homogeneous function of some degree $p \neq 0$ and F is a positive and monotonically increasing function. It is clear that if F is the identity map, then the homothetic function f reduces to a classical homogeneous function. In the following, we suppose that F and h are twice differentiable.

Let us denote by $\mathcal{H}$ the production hypersurface associated with f, i.e. the hypersurface of the Euclidean space of dimension $(n+1)$ given by [36]

$$\mathcal{H} = \{(x_1, x_2 ..., x_n, f(x_1, x_2, ..., x_n)) \mid (x_1, \ldots, x_n) \in D\}. \tag{2}$$

This hypersurface offers the most appropriate setting to derive the geometric properties of production models, these being essential in economic analysis. An isoquant $\mathcal{H}_{f^*}$ of $\mathcal{H}$ consists in all combinations of the production factors $(x_1, x_2, \ldots, x_n)$ producing the same output f^*. Hence, we have

$$\mathcal{H}_{f^*} = \{(x_1, x_2, \ldots, x_n) \in \mathcal{D} \mid f(x_1, \ldots, x_n) = f^*\} \tag{3}$$

and geometrically, $\mathcal{H}_{f^*}$ is a hypersurfaces of $\mathcal{H}$. When $n = 2$, then $\mathcal{H}_{f^*}$ is a curve on the production surface $\mathcal{H}$. Isoquants play a crucial role in economics as they help firms optimize their input combinations during production process to maximize profit.

Next, we will investigate the isoquants $\mathcal{H}_{f^*}$ of the production hypersurfaces $\mathcal{H}$ associated with the homothetic production models f defined by (1). First we remark that denoting $u = h(x_1, x_2, \ldots, x_n)$, we obtain

$$f_i = F' h_i, \tag{4}$$

where $f_i = \frac{\partial f}{\partial x_i}$, $F' = \frac{\mathrm{d}F}{\mathrm{d}u}$ and $h_i = \frac{\partial h}{\partial x_i}$, $i = 1, \ldots, n$. From the definition of the homothetic function f, we derive that $\mathrm{grad} f \neq 0$. Hence it follows that the unit vector field $\mathcal{V} = \frac{\mathrm{grad} f}{\|\mathrm{grad} f\|}$ is normal on the isoquant $\mathcal{H}_{f^*}$. Therefore, the mean curvature H of $\mathcal{H}_{f^*}$ is [25]

$$H = \frac{1}{n-1} \sum_{i=1}^{n-1} < A_{\mathcal{V}} F_i, F_i >, \tag{5}$$

where $A_{\mathcal{V}}$ denotes the Weingarten operator associated with $\mathcal{V}$, $< \cdot, \cdot >$ is the induced metric on $\mathcal{H}_{f^*}$, while $\{F_1, \ldots, F_{n-1}\}$ is a locally orthonormal frame on $\mathcal{H}_{f^*}$. The use of the Weingarten formula in (5), combined with the properties of the Levi-Civita connection, leads to next simple formula for H:

$$H = \frac{1}{(n-1)\| \mathrm{grad} f \|} \sum_{i=1}^{n-1} f_{ii}, \tag{6}$$

where f_{ii}, $i = 1, \ldots, n-1$, stand for the second partial derivatives of f (for details and alternative ways to deduce this formula see [21, 23, 30]). From (6), we conclude that $\mathcal{H}_{f^*}$ has vanishing mean curvature if and only if

$$\sum_{i=1}^{n-1} f_{ii} = 0. \tag{7}$$

A production model f is called with *minimal isoquants* if all isoquants $\mathcal{H}_{f^*}$ of the production hypersurface $\mathcal{H}$ associated with f have vanishing mean curvature. In [30], it was proved that there are 9 quasi-product production models with minimal isoquants, while in [21] it was demonstrated that there are 6 quasi-sum production models with minimal isoquants and there exist a unique family of homogeneous production models with minimal isoquants, this being related to the class of harmonic functions of $(n-1)$ real variables. Recall that a function H of $(n-1)$ real variables is called *harmonic* if it satisfies the Laplace equation $\Delta H = 0$, where Δ is the Laplace operator defined by $\Delta H = \sum_{i=1}^{n-1} H_{ii}$.

In the next section, we will show that the last mentioned classification can be extended to the more general class of homothetic production functions, proving the next result.

Theorem 2.1. *An homothetic production model f is with minimal isoquants if and only if, modulo a suitable translation of the inputs, f reduces to*

$$f(x_1, x_2, \ldots, x_n) = A x_n^p H(v_1, v_2, \ldots, v_{n-1}) + B, \tag{8}$$

where $(v_1, v_2, \ldots, v_{n-1}) = (\frac{x_1}{x_n}, \frac{x_2}{x_n}, \ldots, \frac{x_{n-1}}{x_n})$, $H(v_1, v_2, \ldots, v_{n-1})$ is any harmonic function, while A and B are real constants with $A > 0$ such that $f > 0$.

3. Proof of Theorem 2.1

We start by proving the forward implication. Let us suppose that f is an homothetic production model given by (1). Letting $u = h(x_1, x_2, \ldots, x_n)$, we derive from (4) that

$$f_{ii} = F'' h_i^2 + F' h_{ii}, \tag{9}$$

where $f_{ii} = \frac{\partial^2 f}{\partial x_i^2}$, $F'' = \frac{\mathrm{d}^2 F}{\mathrm{d}u^2}$ and $h_i = \frac{\partial h}{\partial x_i}$, for $i = 1, \ldots, n$.

Inserting (9) in (7), we get

$$F'' \sum_{i=1}^{n-1} h_i^2 + F' \sum_{i=1}^{n-1} h_{ii} = 0. \tag{10}$$

But, as the function h is homogeneous of some degree $p \neq 0$, from Euler equation

$$\sum_{i=1}^{n} x_i f_i = p \cdot f,$$

we derive using the method of characteristics that there exists at least one index $j \in \{1, 2, \ldots, n\}$ such that h can be expressed by

$$h(x_1, x_2, ..., x_n) = x_j^p \cdot H\left(\frac{x_1}{x_j}, \frac{x_2}{x_j}, \ldots, \frac{x_n}{x_j}\right),$$

where H is some function of $(n-1)$ variables $(\frac{x_1}{x_j}, \frac{x_2}{x_j}, \ldots, \frac{x_n}{x_j})$. Possibly using a renumbering of variables, we can assume now that $j = n$ and therefore, we have

$$h(x_1, x_2, ..., x_n) = x_n^p \cdot H(v_1, v_2, \ldots, v_{n-1}), \tag{11}$$

where $(v_1, v_2, \ldots, v_{n-1}) = (\frac{x_1}{x_n}, \frac{x_2}{x_n}, \ldots, \frac{x_{n-1}}{x_n})$.

Now, it is easy to check that

$$h_i = x_n^{p-1} H_i, \ h_{ii} = x_n^{p-2} H_{ii}, \tag{12}$$

where $H_i = \frac{\partial H}{\partial v_i}$ and $H_{ii} = \frac{\partial^2 H}{\partial v_i^2}$, for $i = 1, \ldots, n-1$.

Replacing (12) in (10), we obtain

$$F'' x_n^p \sum_{i=1}^{n-1} H_i^2 + F' \sum_{i=1}^{n-1} H_{ii} = 0. \tag{13}$$

We split the proof into two cases as follows.

Case I: If $F'' = 0$, then it follows immediately that

$$F(u) = Au + B, \tag{14}$$

for some real constant A and B, with $A > 0$. In this case, (13) reduces to

$$A \sum_{i=1}^{n-1} H_{ii} = 0$$

and therefore we derive that function H is harmonic. Hence, in view of (1), (11), (14) we derive that f has the form (8).

Case II: If $F'' \neq 0$, then we introduce the next variables:

$$V = x_n^p \cdot H, \ G = \frac{1}{(F')^2}, \ X^i = H_i^2, \ i = 1, \ldots, n-1. \tag{15}$$

Now, we easily derive from (15) that

$$V_i = x_n^{p-1} H_i, \ G' = -\frac{2F''}{(F')^3}, \ X_i^i = \frac{2}{x_n^p} H_{ii}, \tag{16}$$

where $V_i = \frac{\partial V}{\partial x_i}$, $G' = \frac{\mathrm{d}G}{\mathrm{d}u}$ and $X_i^i = \frac{\partial X^i}{\partial V}$.

Dividing (13) by $(F')^3$, we get

$$\frac{F''}{(F')^3} \cdot x_n^p \sum_{i=1}^{n-1} H_i^2 + \frac{1}{(F')^2} \sum_{i=1}^{n-1} H_{ii} = 0 \tag{17}$$

and using (15) and (16) in (17), we obtain

$$G'\sum_{i=1}^{n-1} X^i - G\sum_{i=1}^{n-1} X_i^i = 0. \tag{18}$$

Now, it is necessary to split Case II in two subcases, as follows.

Subcase II.1. If $G' = 0$, then we have $G(u) = k$, for some positive constant k. Then we derive from (15) that $F' = \frac{1}{\sqrt{k}}$ and we find immediately

$$F(u) = \frac{1}{\sqrt{k}}u + K, \tag{19}$$

where K is a real constant. In this case, due to the fact that $G \neq 0$, (18) reduces to

$$\sum_{i=1}^{n-1} X_i^i = 0$$

and in view of (16) we derive

$$\frac{2}{x_n^p}\sum_{i=1}^{n-1} H_{ii} = 0.$$

Thus, we conclude that H is an harmonic function and in view of (1), (11) and (19) we deduce that f has the form (8), where $A = \frac{1}{\sqrt{k}}$ and $B = K$.

Subcase II.2. If $G' \neq 0$, then it follows immediately from (18) that

$$\frac{G'}{G} = \frac{\sum_{i=1}^{n-1} X_i^i}{\sum_{i=1}^{n-1} X^i}, \tag{20}$$

because according to (15), we have $\sum_{i=1}^{n-1} X^i = \sum_{i=1}^{n-1} H_i^2 \neq 0$. Denoting $\alpha = \frac{G'}{G}$ and $\beta = \frac{\sum_{i=1}^{n-1} X_i^i}{\sum_{i=1}^{n-1} X^i}$, we have from (20) that

$$\alpha(x_n^p \cdot H(v_1, v_2, \dots, v_{n-1})) = \beta(v_1, v_2, \dots, v_{n-1}), \tag{21}$$

for all $(v_1, v_2, \dots, v_{n-1})$, where $v_i = \frac{x_i}{x_n}$, $i = 1, \dots, n-1$. But, due to fact that $p \neq 0$, we deduce that (21) is valid if and only if α and β are equal constant functions. Hence

$$\frac{\sum_{i=1}^{n-1} X_i^i}{\sum_{i=1}^{n-1} X^i} = C \tag{22}$$

and

$$\frac{G'}{G} = C, \tag{23}$$

where C is a non-zero constant (as we are in the subcase $G' \neq 0$).

Using now (15) and (16) in (22), we obtain

$$\frac{2}{x_n^p}\sum_{i=1}^{n-1} H_{ii} = C\sum_{i=1}^{n-1} H_i^2. \tag{24}$$

Denoting $\gamma = \sum_{i=1}^{n-1} H_{ii}$ and $\delta = C\sum_{i=1}^{n-1} H_i^2$, we have from (24) that

$$\frac{2}{x_n^p}\gamma(v_1, v_2, \ldots, v_{n-1}) = \delta(v_1, v_2, \ldots, v_{n-1}), \tag{25}$$

for all $(v_1, v_2, \ldots, v_{n-1})$, where $v_i = \frac{x_i}{x_n}$, $i = 1, \ldots, n-1$. But (25) cannot be valid because $p \neq 0$. Hence Subcase II.2 is not possible.

Consequently, the forward implication is proved.

Conversely, if we suppose that the production model f is given by (8), then it follows immediately that

$$f_{ii} = x_n^{p-2} H_{ii}.$$

Consequently, due to the fact that H is an harmonic function, we have

$$\sum_{i=1}^{n-1} f_{ii} = x_n^{p-2} \sum_{i=1}^{n-1} H_{ii} = 0.$$

Hence any isoquant $\mathcal{H}_{f^*}$ has vanishing mean curvature and, obviously, the production model f is with minimal isoquants.

4. Final remarks and examples

Rewriting Theorem 2.1 in the case of the main two inputs used in economics, namely the labor and the capital, denoted by L and K, respectively, and taking into account that an harmonic function of a single real variable has the form $H(u) = Cu + D$, where C, D are real constants, we derive the following result.

Theorem 4.1. *An homothetic production model f with two inputs is with minimal isoquants if and only if f reduces to*

$$f(L, K) = AK^{p-1}(CL + DK) + B, \tag{26}$$

or

$$f(L, K) = AL^{p-1}(CK + DL) + B, \tag{27}$$

where A, B, C, D are real constants with $A > 0$ such that $f > 0$.

Remark 4.2. *Due to the fact that the family of homothetic production models contains in particular the family of homogeneous production models, it follows that Theorem 2.1 generalizes [21, Theorem 3.1], while Theorem 4.1 generalizes [21, Theorem 3.2].*

Next, as a direct illustration of Theorem 2.1, we will give some examples of homothetic production models with minimal isoquants.

Example 4.3. *It is clear that the function H defined by*

$$H(v_1, v_2) = v_1 + v_2$$

is harmonic and we deduce that the homothetic production model f with three inputs (x_1, x_2, x_3), defined by

$$f(x_1, x_2, x_3) = x_3^p H\left(\frac{x_1}{x_3}, \frac{x_2}{x_3}\right),$$

i.e.

$$f(x_1, x_2, x_3) = x_3^{p-1}(x_1 + x_2)$$

possesses minimal isoquants, for any real number $p \neq 0$.

Example 4.4. *It is easy to check that the function H defined by*

$$H(v_1, v_2, v_3) = (v_1^2 + v_2^2 + v_3^2)^{-\frac{1}{2}}$$

is harmonic and we deduce that the homothetic production model f with four inputs (x_1, x_2, x_3, x_4)*, defined by*

$$f(x_1, x_2, x_3, x_4) = x_4^p H\left(\frac{x_1}{x_4}, \frac{x_2}{x_4}, \frac{x_3}{x_4}\right),$$

i.e.

$$f(x_1, x_2, x_3, x_4) = x_4^{p+1}(x_1^2 + x_2^2 + x_3^2)^{-\frac{1}{2}}$$

has minimal isoquants, for any real number $p \neq 0$.

Example 4.5. *One can easily verify that the function H defined by*

$$H(v_1, v_2, v_3, v_4) = v_1^3 - 3v_1v_2^2 + v_1v_2(v_3 + v_4)$$

is harmonic and we deduce that the homothetic production model f with five inputs $(x_1, x_2, x_3, x_4, x_5)$*, defined by*

$$f(x_1, x_2, x_3, x_4, x_5) = x_5^p H\left(\frac{x_1}{x_5}, \frac{x_2}{x_5}, \frac{x_3}{x_5}, \frac{x_4}{x_5}\right),$$

i.e.

$$f(x_1, x_2, x_3, x_4, x_5) = x_5^{p-3}[x_1^3 - 3x_1x_2^2 + x_1x_2(x_3 + x_4)]$$

is with minimal isoquants, for any real number $p \neq 0$.

Example 4.6. *We consider the function H defined by*

$$H(v_1, v_2, \ldots, v_{n-1}) = \sum_{i=1}^{n-1} \alpha_i v_i^2,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ *are real numbers such that* $\sum_{i=1}^{n-1} \alpha_i = 0$. *Then one can easily check that H is an harmonic map and we deduce that the homothetic production model f with n inputs* $(x_1, x_2, \ldots, x_n)$*, defined by*

$$f(x_1, x_2, \ldots, x_n) = x_n^p H\left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \ldots, \frac{x_{n-1}}{x_n}\right),$$

i.e.

$$f(x_1, x_2, \ldots, x_n) = x_n^{p-2} \sum_{i=1}^{n-1} \alpha_i x_i^2$$

has minimal isoquants, for any real number $p \neq 0$.

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