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Existence and uniqueness of entropy solution for nonlinear periodic parabolic problem with Orlicz growth and L^1 data

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Abstract. In this paper we prove the existence and uniqueness of entropy solution for a nonlinear periodic parabolic problem in the setting of Orlicz spaces represented by the following equation :

$$\frac{\partial u}{\partial t} + A(u) = f(x,t)$$

where A is a Leray-Lions operator defined on a subset of $W_0^{1,x}L_M(Q_T)$ and f belongs to $L^1(Q_T)$.

1. Introduction

In this paper, our aim is to investigate the existence and uniqueness of a periodic solution to a nonlinear parabolic problem with an Orlicz growth condition and L^1 data, represented as follows:

$$\begin{cases} \frac{\partial u}{\partial t} - div(a(x,t,u,\nabla u) = f(x,t) & \text{in} \quad Q_T, \\ u(x,t) = 0, & \text{on} \quad (0,T)\times\partial\Omega, \\ u(x,T) = u(x,0) & \text{in} \quad \Omega. \end{cases} \tag{1}$$

Here, Ω is a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ with a smooth boundary $\partial\Omega$, the measurable function f belongs to $L^1(Q_T)$, and $a : \Omega \times (0,T) \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function satisfying the classical Leray Lions assumptions of Orlicz type. The motivation for studying partial differential equations in Orlicz spaces come from various physical phenomena, particularly the problems related to Non-Newtonian fluids exhibiting strongly inhomogeneous behaviors with a high ability to increase viscosity under different stimuli such as shear rate, magnetic or electric fields (see, for example [10], [16], [23],[24] and [32]).
In the classical Sobolev space setting, Lions in [26] established the existence, regularity, and uniqueness of weak periodic solutions to the problem (1) when f belongs to $L^p(0,T,W^{-1,p'}(\Omega))$. This was achieved using

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the theory of maximal monotone operators. In [11], Duel and Hess extended the results of [26] to a class of quasilinear periodic equations with critical growth and nonlinearity with respect to the gradient, using the sub and super-solution method. Further exploration of related topics can be found in ([6], [9], [15], [14], [13] and [12]).

When replacing $L^p(0, T, W^{1,p}(\Omega))$ with an inhomogeneous Sobolev space $W^{1,x}L_M(Q_T)$, where M is the N-function related to the actual growth of a. A large literature exists on this setting and numerous researchers have expressed interest in this area. Cite Mahi and Meskine in [19] demonstrated the existence of solutions for the following parabolic initial-boundary value problem:

$$\begin{cases} \frac{\partial u}{\partial t} - div(a(x,t,u,\nabla u) + g(x,t,u,\nabla u) = f(x,t) & \text{in} \quad Q_T, \\ u(x,t) = 0, & \text{on} \quad (0,T) \times \partial\Omega, \\ u(x,0) = u_0 & \text{in} \quad \Omega. \end{cases}$$

Where f belongs to $L^1(Q_T)$ and without assuming the Δ_2-condition. Their approach utilized the concept of entropy solutions, originally introduced by P. Bénilan and al. in [7] for the analysis of nonlinear elliptic problems. For additional instances, refer to [4] where $g = 0$, a depends only on u, and $u \in \mathcal{K}$, with $\mathcal{K}$ representing a closed convex space. Further references on this topic can be found in [2],[29] and [30] .

However, when we shift our focus to the periodic condition instead of the initial one, the existence of a solution for problem (1) has been established by [21], where f belongs to $W^{-1,x}E_M(Q_T)$, a depends only on u, and there are no restrictions on M, the proof relies on classical approximating methods and Leray Schauder's fixed point theorem to guarantee the existence of the periodic approximating problem. Furthermore, we refer to [20] for the case where f belongs to L^1 and with an additional term $g = g(x,t,u,\nabla u)$ satisfies a growth condition with respect to the gradient.

Our innovation in this paper lies in providing an existence result for entropy solutions to the problem (1) within the framework of inhomogeneous Orlicz Sobolev spaces. One of the challenges arises from the dependence on u of the operator a, resulting in the loss of uniqueness of the approximating problem, This uniqueness is crucial for applying Leray Schauder's theorem to prove the existence of these approximating problems. Another challenge involves showing the uniqueness of the entropy solution, which is not guaranteed in general. Additionally, further difficulties appears due to the lack of reflexivity of these spaces .

This paper is organized as follows: In the second section, we are going to recall some important definitions and results of Orlicz-Sobolev spaces. In the third section we specifies the essential assumptions, we introduce the Definition of an entropy solution and the main result. In the fourth section, we establish the uniqueness of the entropy solution. We conclude this work by proving the existence of the approximating problem of (1) in Appendix 1.

2. Preliminaries

2.1. *Orlicz-Sobolev Spaces-Notations and Properties*

1. let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function, i.e continuous, convex, with $M(t) > 0$ for $t > 0$, $M(t)/t \to 0$ as $t \to 0$ and $M(t)/t \to \infty$ as $t \to \infty$.

   Equivalently, M admits the representation: $M(t) = \int_0^t m(\tau)d\tau$ where $m : \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing, right continuous, with $m(0) = 0$, $m(t) > 0$ for $t > 0$ and $m(t) \to \infty$ as $t \to \infty$.

   The N-function $\overline{M}$ conjugate to M is defined by $\overline{M}(t) = \int_0^t \overline{m}(\tau)d\tau$ where $\overline{m} : \mathbb{R}^+ \to \mathbb{R}^+$ is given by $m(t) = \sup\{s : m(s) \le t\}$.

   The N-function M is said to satisfy a Δ_2 condition if, for some $k > 0$:

   $$M(2t) \le kM(t) \quad \forall t \ge 0$$

   When this inequality holds only for $t \ge t_0 > 0$, M is said to satisfy the Δ_2-condition near infinity .

2. Let Ω be an open subset of $\mathbb{R}^N$. The Orlicz class $\mathcal{L}_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real-valued measurable functions u on Ω such that $\int_\Omega M(u(x))dx < +\infty$ (resp. $\int_\Omega M(u(x)/\lambda)dx < +\infty$ for some $\lambda > 0$).
$L_M(\Omega)$ is a Banach space under the norm:
$$\|u\|_{M,\Omega} = \inf\left\{\lambda > 0 : \int_\Omega M\left(\frac{u(x)}{\lambda}\right)dx \leq 1\right\}$$
and $\mathcal{L}_M(\Omega)$ is a convex subset of $L_M(\Omega)$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$.
The equality $E_M(\Omega) = L_M(\Omega)$ holds if and only if M satisfies the Δ_2 condition, for all t or for t large according to whether Ω has infinite measure or not.
The dual of $E_M(\Omega)$ can be identified with $L_M(\Omega)$ by means of the pairing $\int_\Omega u(x)v(x)dx$, and the dual norm on $L_{\overline{M}}(\Omega)$ is equivalent to $\|\cdot\|_{\overline{M},\Omega}$.
The space $L_M(\Omega)$ is reflexive if and only if M and $\overline{M}$ satisfy the Δ_2 condition (near infinity only if Ω has finite measure).

3. We now turn to the Orlicz-Sobolev spaces. $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp. $E_M(\Omega)$). It is a Banach space under the norm:
$$\|u\|_{1,M,\Omega} = \sum_{|\alpha|\leq 1} \|D^\alpha u\|_{M,\Omega}.$$
Thus $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspace of the product of $(N+1)$ copies of $L_M(\Omega)$. Denoting this product by ΠL_M, we will use the weak topologies $\sigma\left(\Pi L_M, \Pi E_{\overline{M}}\right)$ and $\sigma\left(\Pi L_M, \Pi L_{\overline{M}}\right)$.
The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma\left(\Pi L_M, \Pi E_{\overline{M}}\right)$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$.
4. We say that u_n converges to u for the modular convergence in $W^1L_M(\Omega)$ if for some $\lambda > 0$
$$\int_\Omega M\left(\left(D^\alpha u_n - D^\alpha u\right)/\lambda\right)dx \to 0 \text{ for all } |\alpha| \leq 1$$
This implies convergence for $\sigma\left(\Pi L_M, \Pi L_{\overline{M}}\right)$. Note that, if $u_n \to u$ in $L_M(\Omega)$ for the modular convergence and $v_n \to v$ in $L_M(\Omega)$ for the modular convergence, we have
$$\int_\Omega u_n v_n dx \to \int_\Omega uv dx \quad \text{as } n \to \infty$$

2.2. Inhomogeneous Orlicz-Sobolev spaces

Let Ω be a bounded open subset of $\mathbb{R}^N$, $T > 0$ and set $Q_T = \Omega\times]0,T[$. Let M be an N-function. For each $\alpha \in \mathbb{N}^N$, denote by D_x^α the distributional derivative on Q of order α with respect to the variable $x \in \mathbb{R}^N$. The homogeneous Orlicz-Sobolev spaces of order 1 are defined as follows
$$W^{1,x}L_M(Q_T) = \{u \in L_M(Q_T) : D_x^\alpha u \in L_M(Q_T), \forall|\alpha| \leq 1\}$$
and
$$W^{1,x}E_M(Q_T) = \{u \in E_M(Q_T) : D_x^\alpha u \in E_M(Q_T), \forall|\alpha| \leq 1\}$$
The latter space is a subspace of the former. Both are Banach spaces under the norm
$$\|u\| = \sum_{|\alpha|\leq 1} \left\|D_x^\alpha u\right\|_{M,Q}.$$

The space $W_0^{1,x}L_M(Q_T)$ is defined as the (norm) closure in $W^{1,x}L_M(Q_T)$ of $\mathcal{D}(Q_T)$ and we have .

$$W_0^{1,x}L_M(Q_T) = \overline{\mathcal{D}(Q_T)}^{\sigma(\Pi L_M, \Pi L_{\overline{M}})}.$$

Furthermore, $W_0^{1,x}E_M(Q_T) = W_0^{1,x}L_M(Q_T) \cap \Pi E_M$.

Poincare's inequality also holds in $W_0^{1,x}L_M(Q_T)$ and then there is a constant $C > 0$ such that for all $u \in W_0^{1,x}L_M(Q_T)$ one has

$$\sum_{|\alpha|\leq 1} \left\|D_x^\alpha u\right\|_{M,Q} \leq C \sum_{|\alpha|=1} \left\|D_x^\alpha u\right\|_{M,Q}$$

thus both sides of the last inequality are equivalent norms on $W_0^{1,x}L_M(Q_T)$. We have then the following complementary system

$$\begin{pmatrix} W_0^{1,x}L_M(Q_T) & F \\ W_0^{1,x}E_M(Q_T) & F_0 \end{pmatrix}$$

F being the dual space of $W_0^{1,x}L_M(Q_T)$. It is also, up to an isomorphism, the quotient of $\Pi L_{\overline{M}}$ by the polar set $W_0^{1,x}E_M(Q_T)^{\perp}$, and will be denoted by $F = W^{-1,x}L_{\overline{M}}(Q_T)$ and it is shown that

$$W^{-1,x}L_{\overline{M}}(Q_T) = \left\{ f = \sum_{|\alpha|\leq 1} D_x^\alpha f_\alpha \ : \ f_\alpha \in L_{\overline{M}}(Q_T) \right\}.$$

This space will be equipped with the usual quotient norm:

$$\|f\| = \inf \sum_{|\alpha|\leq 1} \left\|f_\alpha\right\|_{\overline{M},Q}$$

where the inf is taken over all possible decomposition $f = \sum_{|\alpha|\leq 1} D_x^\alpha f_\alpha, f_\alpha \in L_{\overline{M}}(Q_T)$. The space F_0 is then given by

$$F_0 = \left\{ f = \sum_{|\alpha|\leq 1} D_x^\alpha f_\alpha : f_\alpha \in E_{\overline{M}}(Q_T) \right\},$$

and is denoted by $F_0 = W^{-1,x}E_{\overline{M}}(Q_T)$.

2.3. Compactness results

Theorem 2.1. *Let B be a Banach space and let $T > 0$ be a fixed real number. If $F \subset L^1(0,T;B)$ is such that*

$$\left\{ \int_{t_1}^{t_2} f(t)dt \right\}_f \text{ is relatively compact in } B, \quad \text{for all } 0 < t_1 < t_2 < T. \tag{2}$$

$$\left\|\tau_h f - f\right\|_{L^1(0,T;B)} \to 0 \text{ uniformly in } f \in F, \text{ when } h \to 0. \tag{3}$$

Then F is relatively compact in $L^1(0,T;B)$.

Lemma 2.2. *(see [17]) Let Y be a Banach space such that $L^1(\Omega) \subset Y$ with continuous embedding. If F is bounded in $W_0^{1,x}L_M(Q_T)$ and is relatively compact in $L^1(0,T;Y)$ then F is relatively compact in $L^1(Q_T)$.*

Theorem 2.3. *(see [27]) Let M be an N-function. If F is bounded in $W_0^{1,x}L_M(Q_T)$ and $\left\{\frac{\partial f}{\partial t} : f \in F\right\}$ is bounded in $W^{-1,x}L_{\overline{M}}(Q_T)$ then F is relatively compact in $L^1(Q_T)$.*

Theorem 2.4. *(see [27]) If $u \in W^{1,x}L_M(Q_T) \cap L^1(Q_T)$ (resp. $W_0^{1,x}L_M(Q_T)$ $\cap L^1(Q_T)$) and $\partial u/\partial t \in W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T)$ then there exists a sequence (v_j) in $\mathcal{D}(\overline{Q})$ such that*

$$v_j \to u \text{ in } W^{1,x}L_M(Q_T) \quad \text{and} \quad \frac{\partial v_j}{\partial t} \to \frac{\partial u}{\partial t} \text{ in } W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T)$$

for the modular convergence.

Corollary 2.5. *(see [27]) Let M be an N-function and u_n be a sequence of $W^{1,x}L_M(Q_T)$ such that*

$$u_n \rightharpoonup u \quad \text{weakly in} \quad W^{1,x}L_M(Q_T) \quad \text{for} \quad \sigma(\Pi L_M, \Pi E_{\overline{M}})$$

and

$$\frac{\partial u_n}{\partial t} = h_n + k_n \quad \text{in} \quad \mathcal{D}'(Q_T)$$

with h_n bounded in $W^{-1,x}L_{\overline{M}}(Q_T)$ and k_n bounded in the space $\mathcal{M}(Q_T)$ of measures on Q_T then :

1. *$u_n \to u$ strongly in $L^1_{Loc}(Q_T)$*
2. *If further $u_n \in W_0^{1,x}L_M(Q_T)$ then $u_n \to u$ strongly in $L^1(Q_T)$*

Lemma 2.6. *(see [22]) For all $v \in W_0^{1,x}L_M(Q_T)$ there exist two positive constants δ and λ such that*

$$\int_{Q_T} M(v)dxdt \leq \delta \int_{Q_T} M(\lambda|\nabla v|)dxdt. \tag{4}$$

Lemma 2.7. *[27] Let Ω be a bounded open subset of $\mathbb{R}^N$. Then,*

$$\left\{u \in W_0^{1,x}L_M(Q_T) : \frac{\partial u}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T)\right\} \subset C\left([0,T], L^1(\Omega)\right).$$

3. Assumptions and main result

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ with the segment property, and Q_T be the cylinder $\Omega \times (0,T)$ with some given $T > 0$. Let M and P be two N-function such that $P << M$.
Consider the second order operator $A : D(A) \subset W_0^{1,x}L_M(Q_T) \to$ $W^{-1,x}L_{\overline{M}}(Q_T)$ of the form :

$$A(u) = -div(a(x,t,u,\nabla u))$$

where $a : \Omega \times (0,T) \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carateodory function satisfying for almost every $(x,t) \in \Omega \times (0,T)$, for all $s \in \mathbb{R}$ and all $\xi \neq \xi^* \in \mathbb{R}^N$ we have the following assumptions :

$$|a(x,t,s,\xi)| \leq \beta(h_1(x,t) + \overline{M}^{-1}P(\delta|s|) + \overline{M}^{-1}M(\delta|\xi|)), \tag{5}$$

$$[a(x,t,s,\xi) - a(x,t,s,\xi^*)][\xi - \xi^*] > 0, \tag{6}$$

$$a(x,t,s,\xi)\xi \geq \alpha M\Big(\frac{|\xi|}{\lambda}\Big). \tag{7}$$

Where $h_1(x,t) \in E_{\overline{M}}(Q_T)$, $c_1 \geq 0$ and , $\beta, \alpha, \lambda > 0$.

$$|a(x,t,s,\xi) - a(x,t,\overline{s},\xi^*)| \leq (d(x,t) + B(|\xi|))|s - \overline{s}|, \tag{8}$$

where $d \in L^\infty(Q_T)$ and $B \in L^\infty(\mathbb{R}^N)$.

$$f \in L^1(Q_T). \tag{9}$$

Furthemore, let $s^{'} > 0$ such that $s \geq s^{'}$, and for a positive constant δ_0 we have

$$|s|^2 \leq \delta_0 M(|s|). \tag{10}$$

Throughout this paper $\langle \cdot, \cdot \rangle$ means for either the pairing between $W_0^{1,x}L_M(Q_T)$ and $W^{-1,x}L_{\overline{M}}(Q_T)$, or between $W_0^{1,x}L_M(Q_T) \cap L^\infty(Q_T)$ and $W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T)$.
Now, let as defined the set $\mathcal{W}(0,T)$ by :

$$\mathcal{W}(0,T) = \{v \in W_0^{1,x}L_M(Q_T) / \frac{\partial v}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q_T) \text{ and } v(x,0) = v(x,T)\},$$

equipped with the norm

$$\|u\|_{\mathcal{W}(0,T)} = \|u\|_{W_0^{1,x}L_M(Q_T)} + \left\|\frac{\partial v}{\partial t}\right\|_{W^{-1,x}L_{\overline{M}}(Q_T)}$$

Consider then the following periodic parabolic problem :

$$\begin{cases} \frac{\partial u}{\partial t} - div(a(x,t,u,\nabla u)) = f(x,t) & \text{in} \quad Q, \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ u(x,0) = u(x,T) & \text{in} \quad \Omega \, . \end{cases} \tag{11}$$

Let us now precise in which sense the problem (11) will be solved.

Definition 3.1. *A measurable function $u : \Omega \times (0,T) \to \mathbb{R}$ is called entropy solution of (11) if u belongs to $L^\infty(0,T,L^1(\Omega))$, $T_k(u)$ belongs to $D(A) \cap W_0^{1,x}L_M(\Omega)$ for every $k > 0$*

$$\langle \frac{\partial v}{\partial t}, T_k(u-v)\rangle_{Q_T} + \int_{Q_T} a(x,t,u,\nabla u)\nabla T_k(u-v)dxdt \leq \int_{Q_T} fT_k(u-v)dxdt, \tag{12}$$

for all $v \in \mathcal{W}(0,T) \cap L^\infty(Q_T)$. T_k is defined by $T_k(s) = min(k; max(s,-k))$.

Remark 3.2. *Equation (12) is obtained pointwise multiplication by $T_k(u-v)$ in (11) and using the periodicity condition .*
Note that, all terms of equation (12) make sense, since $T_k(u-v)$ belongs to $W_0^{1,x}L_M(Q_T) \cap L^\infty(Q_T)$. Moreover lemma 2.7 implies that $v \in C(0,T,L^1(\Omega))$ and then the periodicity condition is well defined in $L^1(\Omega)$.

Theorem 3.3. *Assume that (5)-(10) hold true, then there exist a unique entropy solution $u \in C(0,T;L^1(\Omega))$ for problem (11) satisfying $u(x,0) = u(x,T)$ a.e in Ω.*

3.1. Proof of the main result

We divide the proof into five steps:

Proof. Step1: Approximate problem
Let f_n be a sequence of smooth functions, such that $\|f_n\|_{L^1(Q_T)} \leq \|f\|_{L^1(Q_T)}$ and $f_n \to f$ in $L^1(Q_T)$, let $a_n(x,t,s,\xi) = a(x,t,T_n(s),\xi)$. Consider the following approximate problem :

$$\begin{cases} u_n \in \mathcal{W}(0,T) \cap C(0,T,L^2(\Omega)), \\ \frac{\partial u_n}{\partial t} - div(a_n(x,t,u_n,\nabla u_n)) = f_n & \text{in} \quad Q_T, \\ u_n(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T). \end{cases} \tag{13}$$

The existence of weak soltuion to (13) be based on the research of fixed points for the following mapping

$$\begin{array}{rcl} \Psi : L^2(\Omega) & \to & L^2(\Omega) \\ u_{0n} & \mapsto & u_n(T) \end{array}$$

Where u_{0n} is the initial condition of the following problem:

$$\begin{cases} u_n \in D(A) \cap W_0^{1,x}L_M(Q_T) \cap C(0,T,L^2(\Omega)), \\ \frac{\partial u_n}{\partial t} - div(a_n(x,t,u_n,\nabla u_n)) = f_n & \text{in} \quad Q_T, \\ u_n(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ u_n(x,0) = u_{0n} & \text{in} \quad \Omega. \end{cases} \tag{14}$$

The existence of solution of (14) is proved in [17]. See the prove of uniqueness in (4.1), then the mapping Ψ is well defined.
To prove the existence of a fixed point we will use Schauder's fixed point theorem (i.e we will prove that Ψ is continous and compact)

Proof. See the proof in (4.1). □

Steps 2: A priori estimates
By taking $T_k(u_n)$ as a test function in (13) we get

$$\int_{Q_T} \frac{\partial u_n}{\partial t} T_k(u_n)dxdt + \int_{Q_T} a_n(x,t,u_n,\nabla u_n)\nabla T_k(u_n)dxdt = \int_{Q_T} f_n T_k(u_n)dxdt.$$

Using periodicity condition and the fact that $\|f_n\|_{L^1} \leq \|f\|_{L^1}$ we deduce easily

$$\int_{Q_T} a_n(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dxdt \leq Ck, \tag{15}$$

where C is a positive constant independent of n.
Using (7) we have by lemma (2.6) we get,

$$\int_{Q_T} M\Big(\frac{T_k(u_n)}{\lambda}\Big)dxdt \leq C_1 k. \tag{16}$$

which imply that

$$meas\{(x,t) \in Q_T, |u_n| > k\} \leq \frac{C_2 k}{M(\frac{k}{\lambda})},$$

and so ,

$$\lim_{k\to+\infty} meas\{(x,t) \in Q_T, |u_n| > k\}) = 0 \quad \text{uniformaly with respect to } n. \tag{17}$$

Moerever, consider a $C^2(\mathbb{R})$, nondeacreasing function $\overline{T}_k$ such that $\overline{T}_k(s) = s$ for $|s| \leq \frac{k}{2}$ and $\overline{T}_k(s) = s\ sign(s)$ for $|s| \geq k$. Multiplying the approximating equation by $\overline{T}_k^{'}(u_n)$, we get

$$\tfrac{\partial}{\partial t}(\overline{T}_k(u_n)) - div(a_n(x,t,u_n,\nabla u_n)\overline{T}_k^{'}(u_n)) - \overline{T}_k^{''}(u_n)a_n(x,t,u_n,\nabla u_n)\nabla u_n = f_n\overline{T}_k^{'}(u_n),$$

in the sense of distributions. This implies, thanks to (15) and the fact that $\overline{T}_k^{'}(u_n)$ has a compact support that, $\overline{T}_k(u_n)$ is bounded in $W_0^{1,x}L_M(Q_T)$, while it's time derivative $\frac{\partial}{\partial t}(\overline{T}_k(u_n))$ is bounded in $W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T)$, hence Corollary 2.5 gives $\overline{T}_k(u_n)$ is compact in $L^1(Q_T)$. We can deduce by using the same argument as [28] that for a subsequences still denote u_n that

$$T_k(u_n) \rightharpoonup T_k(u) \text{ in } W_0^{1,x}L_M(Q_T) \text{ for } \sigma(\Pi L_M, \Pi E_{\overline{M}}),$$

strongly in $L^1(Q_T)$ and a.e in Q_T.

Now we turn to prove that $a(x,t,T_k(u_n),\nabla T_k(u_n))$ is a bounded in $(L_{\overline{M}}(Q_T))^N$.
Let $\varphi \in (E_{\overline{M}}(Q_T))^N$ with $\|\varphi\|_{M,Q_T} = 1$, by (6) we have

$$\int_{Q_T} [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\varphi)][\nabla T_k(u_n) - \varphi]dxdt \geq 0,$$

which gives

$$\int_{Q_T} a(x,t,T_k(u_n),\varphi)\varphi dxdt \leq \int_{Q_T} a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dxdt - \int_{Q} a(x,t,T_k(u_n),\varphi)[\nabla T_k(u_n) - \varphi]dxdt,$$

On the one hand, using (5), we see that

$$\overline{M}\Big(\frac{a(x,t,T_k(u_n),\varphi)}{\beta}\Big) \leq C' + M^{-1}(P(\delta k)) + M^{-1}(M(\delta|\varphi|)),$$

hence, $a(x,t,T_k(u_n),\varphi)$ is bounded in $(L_{\overline{M}}(Q_T))^N$, implying that, since $T_k(u_n)$ is bounded in $W_0^{1,x}L_M(Q_T)$

$$\Big|\int_{Q} a(x,t,T_k(u_n),\varphi)[\nabla T_k(u_n) - \varphi]\Big| \leq C''. \tag{18}$$

Finally, using (15) and (18) we can deduce that $a(x,t,T_k(u_n),\nabla T_k(u_n))$ is bounded in $(L_{\overline{M}}(Q_T))^N$. Then for some $h_k \in (L_{\overline{M}}(Q_T))^N$ we get

$$a(x,t,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup h_k \text{ in } (L_{\overline{M}}(Q_T))^N \text{ for } \sigma(\Pi L_{\overline{M}}, \Pi E_M). \tag{19}$$

Steps 3: Almost everywhere convergence of the gradients.
The main tool in this step proves

$$\int_{Q_T} (a(x,t,T_k(u_n),\nabla T_k(u_n) - a(x,t,T_k(u),\nabla T_k(u))(\nabla T_k(u_n) - \nabla T_k(u))dxdt = 0$$

which gives by the same argument as in [8] and adapted to the parabolic case that $\nabla u_n \to \nabla u$ a.e in Q_T.
Using the following regularization principle $\omega_j^\nu = T_k(v_j)_\nu + \exp(-\nu t)z_\nu$, where $v_j \in \mathcal{D}(Q_T)$ such that $v_j \to T_k(u)$ with the modular convergence in $W_0^{1,x}L_M(Q_T)$, $T_k(v_j)_\nu$ is the mollification with respect to time of $T_k(v_j)$ and z_ν is a sequence of functions such that

$$\begin{cases} z_\nu \in W_0^{1,x}L_M(\Omega) \cap L^\infty(\Omega); \quad \|z_\nu\|_\infty \leq k; \forall \nu > 0, \\ z_\nu \to T_k(\nu)(T) \quad \text{a.e in } \Omega \text{ when } n \to +\infty, \\ \lim_{\nu\to+\infty} \frac{1}{\nu}\|z_\nu\|_{W_0^{1,x}L_M(\Omega)} = 0. \end{cases}$$

Note that ω_j^ν satisfy the following properties

$$\begin{cases} \frac{\partial}{\partial t}(\omega_j^\nu) = \nu(T_k(v_j) - \omega_j^\nu), \quad \omega_j^\nu(0) = z_\nu, \quad |\omega_j^\nu| \leq k, \\ \omega_j^\nu \to T_k(u)_\nu + \exp(-\nu t)z_\nu \quad \text{in } W_0^{1,x}L_M(Q_T) \text{ for the modular convergence when } j \to +\infty, \\ T_k(u)_\nu + \exp(-\nu t) \to T_k(u) \quad \text{in } W_0^{1,x}L_M(Q_T) \text{ for the modular convergence when } \nu \to +\infty. \end{cases}$$

We denote by $\varepsilon(n,j,\nu,s)$ all quantities (possibly different) such that

$$\lim_{s\to\infty}\lim_{\nu\to\infty}\lim_{j\to\infty}\lim_{n\to\infty} \varepsilon(n,j,\nu,s) = 0,$$

and this will be the order in which the parameters we use will tend to infinity, that is, first n, then j, and finally ν. Similarly, we will write only $\varepsilon(n)$, or $\varepsilon(n,j),\ldots$ to mean that the limits are only on the specified

parameters.
Consider next the function φ_m defined on $\mathbb{R}$ by

$$\varphi_m(s) = \begin{cases} 1 & \text{if } |s| \leq m, \\ m+1-|s| & \text{if } m \leq |s| \leq m+1, \\ 0 & \text{if } |s| \geq m+1. \end{cases}$$

Where $m > k$. We take $\eta = (T_k(u_n) - \omega_\nu^j)\varphi_m(u_n)$ as a test function in (13) we have

$$\langle \frac{\partial u_n}{\partial t}, \eta \rangle + \int_{Q_T} a_n(x,t,u_n,\nabla u_n)(\nabla T_k(u_n) - \nabla \omega_\nu^j)\varphi_m(u_n)dxdt +$$

$$+ \int_{Q_T} a_n(x,t,u_n,\nabla u_n)\nabla u_n(T_k(u_n) - \omega_\nu^j)\varphi_m'(u_n)dxdt = \int_{Q_T} f_n \eta dxdt, \tag{20}$$

since $u_n \in W_0^{1,x}L_M(Q_T)$, there exist a smooth function $u_{n\sigma}$ such that

$$u_{n\sigma} \to u_n \text{ for the modular convergence in } W_0^{1,x}L_M(Q_T) \textit{ and}$$
$$\frac{\partial u_{n\sigma}}{\partial t} \to \frac{\partial u_n}{\partial t} \text{ for the modular convergence in } W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T).$$

For the first term in the left hand side we have

$$\begin{aligned} \langle \frac{\partial u_n}{\partial t}, \eta \rangle &= \lim_{\sigma \to 0^+} \int_{Q_T} (u_{n\sigma})'(T_k(u_{n\sigma}) - \omega_\nu^j)\varphi_m(u_{n\sigma})dxdt \\ &= \lim_{\sigma \to 0^+} \Big\{ \int_{Q_T} [R_m(u_{n\sigma}) - T_k(u_{n\sigma})]'(T_k(u_{n\sigma}) - \omega_\nu^j)dxdt + \int_{Q_T} T_k(u_{n\sigma})'(T_k(u_{n\sigma}) - \omega_\nu^j))dxdt \Big\}. \end{aligned}$$

Where $R_m(s) = \int_0^s \varphi_m(r)dr$. Integrating by part and using the periodicity condition it is easy to see that,

$$\begin{aligned} \langle \frac{\partial u_n}{\partial t}, \eta \rangle &= -\lim_{\sigma \to 0^+} \int_{Q_T} [R_m(u_{n\sigma}) - T_k(u_{n\sigma})](T_k(u_{n\sigma}) - \omega_\nu^j)' dxdt + \lim_{\sigma \to 0^+} \int_{Q_T} T_k(u_{n\sigma})'(T_k(u_{n\sigma}) - \omega_\nu^j))dxdt, \\ &= \lim_{\sigma \to 0^+} I_1(\sigma) + I_2(\sigma). \end{aligned}$$

<u>**Claim 1:**</u>$\lim_{\sigma \to 0^+} I_1(\sigma) \geq \varepsilon(n, j, \nu)$:
by the definition of $T_k(u_n)$ we remark that $T_k(u_{n\sigma})' = 0$ if $\{u_{n\sigma} > k\}$, and if $\{u_{n\sigma} \geq k\}$ we have that $R_m(u_{n\sigma}) = \int_0^{u_{n\sigma}} \varphi_m(r)dr > \int_0^k \varphi_m(r)dr$ so the definition of φ_m imply that $R_m(u_{n\sigma}) > k \geq |\omega_\nu^j|$ (since $|\omega_\nu^j| \leq k$) , then $0 \leq (R_m(u_{n\sigma}) - k)(k - \omega_\nu^j)' \chi_{\{u_{n\sigma} > k\}} = (R_m(u_{n\sigma}) - T_k(u_{n\sigma}))(T_k(u_{n\sigma}) - \omega_\nu^j)' \chi_{\{u_{n\sigma} > k\}}$, so we obtain

$$I_1(\sigma) = -\int_{\{u_{n\sigma} \leq k\}} [R_m(u_{n\sigma}) - T_k(u_{n\sigma})](T_k(u_{n\sigma}) - \omega_\nu^j)' dxdt + \int_{\{u_{n\sigma} > k\}} [R_m(u_{n\sigma}) - T_k(u_{n\sigma})](\omega_\nu^j)' dxdt$$

Since $m > k$, we see that $R_m(u_{n\sigma}) = T_k(u_{n\sigma})$ on $\{u_{n\sigma} \leq k\}$ then

$$\begin{aligned} I_1(\sigma) &= \nu \int_{\{u_{n\sigma} > k\}} [R_m(u_{n\sigma}) - T_k(u_{n\sigma})](T_k(v_j) - \omega_\nu^j)dxdt, \\ &\geq \nu \int_{\{u_{n\sigma} > k\}} [R_m(u_{n\sigma}) - T_k(u_{n\sigma})](T_k(v_j) - T_k(u_{n\sigma}))dxdt \end{aligned}$$

Finally, by letting $\sigma \to 0^+$, we obtain when n then j tends to $+\infty$

$$\lim_{\sigma \to 0^+} I_1(\sigma) \geq \varepsilon(n, j). \tag{21}$$

Claim 2 : For $I_2(\sigma)$ we have

$$I_3(\sigma) = \int_{Q_T} (T_k(u_{n\sigma}) - \omega_\nu^j)'(T_k(u_{n\sigma}) - \omega_\nu^j)dxdt + \int_{Q_T} (\omega_\nu^j)'(T_k(u_{n\sigma}) - \omega_\nu^j)dxdt$$

$$= \int_\Omega \Big[\frac{(T_k(u_{n\sigma}) - \omega_\nu^j)^2}{2}\Big]_0^T dx + \nu \int_{Q_T} (T_k(v_j) - \omega_\nu^j)(T_k(u_{n\sigma}) - \omega_\nu^j)dxdt.$$

Using the periodicity condition, then letting n, j, ν to $+\infty$ we get

$$\lim_{\sigma\to 0^+} I_2(\sigma) = \varepsilon(n, j, \nu). \tag{22}$$

Combining (21) and (22) we conclude

$$\langle \frac{\partial u_n}{\partial t}, \eta\rangle \geq \varepsilon(n, j, \nu). \tag{23}$$

We will now treat the second and the third term in the right hand side. First we have

$$\begin{aligned}
&\int_{Q_T} a_n(x, t, u_n, \nabla u_n)(\nabla T_k(u_n) - \nabla\omega_\nu^j)\varphi_m(u_n)dxdt = \\
&= \int_{\{|u_n|\leq k\}} a_n(x, t, u_n, \nabla u_n)(\nabla T_k(u_n) - \nabla\omega_\nu^j)\varphi_m(u_n)dxdt \\
&+ \int_{\{|u_n|>k\}} a_n(x, t, u_n, \nabla u_n)(\nabla T_k(u_n) - \nabla\omega_\nu^j)\varphi_m(u_n)dxdt,
\end{aligned}$$

The fact that, $m > k$ we have $\varphi_m(u_n) = 1$ on $\{|u_n| \leq k\}$ imply that

$$\begin{aligned}
&\int_{Q_T} a_n(x, t, u_n, \nabla u_n)(\nabla T_k(u_n) - \nabla\omega_\nu^j)\varphi_m(u_n)dxdt = \\
&= \int_{Q_T} a(x, t, T_k(u_n), \nabla T_k(u_n))(\nabla T_k(u_n) - \nabla\omega_\nu^j)dxdt \\
&+ \int_{\{|u_n|>k\}} a_n(x, t, u_n, \nabla u_n)(\nabla T_k(u_n) - \nabla\omega_\nu^j)\varphi_m(u_n)dxdt.
\end{aligned}$$

fix a reel number $s > 0$, we denote by χ_j^s and χ^s the characteristic functions of $Q_j^s = \Big\{(x, t) \in Q : \Big|\nabla T_k(v_j)\Big| \leqslant s\Big\}$ and $Q^s = \{(x, t) \in Q : |\nabla T_k(u)| \leqslant s\}$, respectively. Then we can write

$$\int_{Q_T} a_n(x, t, u_n, \nabla u_n)(\nabla T_k(u_n) - \nabla\omega_\nu^j)\varphi_m(u_n)dxdt =$$

$$\begin{aligned}
&\int_{Q_T} (a(x, t, T_k(u_n), \nabla T_k(u_n) - a(x, t, T_k(u_n), \nabla T_k(v_j)\chi_j^s)(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s)dxdt \\
&+ \int_{Q_T} a(x, t, T_k(u_n), \nabla T_k(v_j)\chi_j^s)(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s)dxdt \\
&+ \int_{Q_T} a(x, t, T_k(u_n), \nabla T_k(u_n))\nabla T_k(v_j)\chi_j^s)dxdt \\
&- \int_{Q_T} a_n(x, t, u_n, \nabla u_n)\nabla\omega_\nu^j\varphi_m(u_n)dxdt = I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

We shall goes to the limit as n, j, ν and s goes to $+\infty$. Starting with I_2, by letting n to $+\infty$

$$I_2 = \int_{Q_T} a(x, t, T_k(u), \nabla T_k(v_j)\chi_j^s)(\nabla T_k(u) - \nabla T_k(v_j)\chi_j^s)dxdt + \varepsilon(n),$$

Since $a(x, t, T_k(u_n), \nabla T_k(v_j)\chi_j^s) \to a(x, t, T_k(u), \nabla T_k(v_j)\chi_j^s)$ strongly in $(E_{\overline{M}}(Q_T))^N$ by (5) and Lebesgue theorem while $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ in $(L_M(Q_T))^N$ by (15). Letting now j then s to $+\infty$ and using the fact that

$a(x,t,T_k(u),\nabla T_k(v_j)\chi_j^s) \to a(x,t,T_k(u),\nabla T_k(u)\chi^s)$ strongly in $(E_{\overline{M}}(Q_T))^N$ and $\nabla T_k(v_j)\chi_j^s \to \nabla T_k(u)\chi^s$ strongly in $(L_M(Q_T))^N$ we get

$$I_2 = \varepsilon(n,j,s). \tag{24}$$

For I_3, by letting $n \to +\infty$ we have

$$I_3 = \int_{Q_T} h_k \nabla T_k(v_j)\chi_j^s dxdt + \varepsilon(n),$$

when $j \to +\infty$ we get

$$I_3 = \int_{Q_T} h_k \nabla T_k(u)\chi s dxdt + \varepsilon(n,j), \tag{25}$$

About I_4, recall that $\varphi_m(s) = 0$ if $|s| \geq m+1$ then ,

$$\begin{aligned} I_4 &= -\int_{\{|u_n|\leq m+1\}} a_n(x,t,u_n,\nabla u_n)\nabla\omega_\nu^j \varphi_m(u_n)dxdt \\ &= -\int_{\{|u_n|\leq k\}} a_n(x,t,u_n,\nabla u_n)\nabla\omega_\nu^j \varphi_m(u_n)dxdt - \int_{k\leq\{|u_n|\leq m+1\}} a_n(x,t,u_n,\nabla u_n)\nabla\omega_\nu^j \varphi_m(u_n)dxdt \\ &= -\int_{\{|u_n|\leq k\}} a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla\omega_\nu^j \varphi_m(u_n)dxdt \\ &\quad -\int_{k\leq\{|u_n|\leq m+1\}} a_n(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\nabla\omega_\nu^j \varphi_m(u_n)dxdt \\ &= -\int_{Q_T} h_k\nabla\omega_\nu^j dxdt - \int_{\{k\leq|u|\leq m+1\}} h_{m+1}\nabla\omega_\nu^j \varphi_m(u)dxdt + \varepsilon(n) \\ &= -\int_{Q_T} h_k\nabla T_k(u) dxdt - \int_{\{k\leq|u|\leq m+1\}} h_{m+1}\nabla T_k(u) \varphi_m(u)dxdt + \varepsilon(n,j,\nu), \end{aligned}$$

which imply

$$I_4 = -\int_{Q_T} h_k \nabla T_k(u)dxdt + \varepsilon(n,j,\nu). \tag{26}$$

Combining (25),(26) and (24) we conclude

$$\int_{Q_T} a_n(x,t,u_n,\nabla u_n)(\nabla T_k(u_n) - \nabla\omega_\nu^j)\varphi_m(u_n)dxdt = I_1 + \varepsilon(n,j,\nu). \tag{27}$$

For what we concerne the third term in the right hand side we have
$= \int_{Q_T} a(x,t,u_n,\nabla u_n)\nabla u_n (T_k(u_n) - \omega_\nu^j)\varphi_m'(u_n)dxdt =$

$$\begin{aligned} &\int_{\{m\leq|u_n|\leq m+1\}} a(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\nabla T_{m+1}(u_n)(T_k(u_n) - \omega_\nu^j)dxdt, \\ &= \int_{Q_T} a(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\nabla T_{m+1}(u_n)(k - \omega_\nu^j)dxdt \geq 0. \end{aligned} \tag{28}$$

Now we turn to the term in the right handside, since $\|f_n\| \leq \|f\|$, using lebesgue theorem with respect to n, j, ν we can easily see that

$$\int_{Q_T} f_n(T_k(u_n) - \omega_\nu^j)dxdt = \varepsilon(n,j,\nu). \tag{29}$$

finally, by (20), (23), (27), (28) and (29) we deduce

$$\int_{Q_T} (a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(v_j)\chi_j^s)(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s)dxdt \leq \varepsilon(n,j,\nu,s) \tag{30}$$

On the other hand we have

$$\begin{aligned}
&\int_{Q_T}(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u)\chi^s))(\nabla T_k(u_n)-\nabla T_k(u)\chi^s)dxdt\\
&-\int_{Q_T}(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(v_j)\chi_j^s))(\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s)dxdt\\
&=\int_{Q_T}a(x,t,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(v_j)\chi_j^s-\nabla T_k(u)\chi^s)dxdt\\
&-\int_{Q_T}a(x,t,T_k(u_n),\nabla T_k(u)\chi^s)(\nabla T_k(u_n)-\nabla T_k(u)\chi^s)dxdt\\
&+\int_{Q_T}a(x,t,T_k(u_n),\nabla T_k(v_j)\chi_j^s)(\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s)dxdt.
\end{aligned}$$

As it can be easily seen that each term in the right hand side is of the form $\varepsilon(n,j,s)$ we get

$$\int_{Q_T}(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u)\chi^s))(\nabla T_k(u_n)-\nabla T_k(u)\chi^s)dxdt$$

$$=\int_{Q_T}(a(x,t,T_k(u_n),\nabla T_k(u_n)-a(x,t,T_k(u_n),\nabla T_k(v_j)\chi_j^s))(\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s)dxdt+\varepsilon(n,j,s). \tag{31}$$

For $r<s$ we have

$$\begin{aligned}
0 &\le \int_{Q_T^r}[a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n)-\nabla T_k(u)]dxdt\\
&\le \int_{Q_T^s}[a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n)-\nabla T_k(u)]dxdt\\
&\le \int_{Q_T}[a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u)\chi^s)][\nabla T_k(u_n)-\nabla T_k(u)\chi^s]dxdt\\
&= \int_{Q_T}(a(x,t,T_k(u_n),\nabla T_k(u_n)-a(x,t,T_k(u_n),\nabla T_k(v_j)\chi_j^s))(\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s)dxdt+\varepsilon(n,j,s).
\end{aligned}$$

hence by passing to the limit sup over n, we get

$$\limsup_{n\to+\infty}\int_{Q_T^r}[a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n)-\nabla T_k(u)]dxdt\le\limsup_{n\to+\infty}\varepsilon(n,j,\nu,s).$$

Finally, we obtain by letting n, j, ν then s to infinity

$$\limsup_{n\to+\infty}\int_{Q_T^r}[a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u))]$$

and thus, as in the elliptic case [8], there exists a subsequence also denoted by u_n such that

$$\nabla u_n\to\nabla u \text{ a.e in } Q_T. \tag{32}$$

We deduce then that

$$a(x,t,T_k(u_n),\nabla T_k(u_n))\to a(x,t,T_k(u),\nabla T_k(u)) \text{ in } (L_{\overline{M}}(Q_T))^N, \text{ for } \sigma(\Pi L_{\overline{M}},\Pi E_M).$$

Step 4: Modular convergence of the gradient
Indeed, thanks to (30) and (31) we have

$$\int_{Q_T} a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dxdt \leq \int_{Q_T} a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)\chi^s dxdt$$

$$+\int_{Q_T} a(x,t,T_k(u_n),\nabla T_k(u)\chi^s)(\nabla T_k(u_n)-\nabla T_k(u)\chi^s)dxdt + \varepsilon(n,j,\nu,s),$$

then

$$\limsup_{n\to+\infty} a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dxdt$$
$$\leq \int_{Q_T} a(x,t,T_k(u),\nabla T_k(u))\nabla T_k(u)\chi^s dxdt + \limsup_{n\to+\infty}\varepsilon(n,j,\nu,s),$$

in which we can pass to the limit as j,ν,s to infinity to obtain

$$\limsup_{n\to+\infty} a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dxdt \leq \int_{Q_T} a(x,t,T_k(u),\nabla T_k(u))\nabla T_k(u)dxdt,$$

by applying Fatou's lemma we get

$$\int_{Q_T} a(x,t,T_k(u),\nabla T_k(u))\nabla T_k(u)dxdt \leq \liminf_{n\to+\infty} a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dxdt.$$

and thus ,

$$a(x,t,T_k(u_n),\nabla T_k(u_n)) \to a(x,t,T_k(u),\nabla T_k(u)) \text{ in } L^1(Q_T). \tag{33}$$

Using (7) and Vitali's theorem we conclude that

$$\nabla T_k(u_n) \to \nabla T_k(u) \text{ for the modular convergence in } (L_M(Q_T))^N. \tag{34}$$

Step 5: Passage to the limit
The passage to the limit is an easy task by taking $v \in \mathcal{W}(0,T)\cap L^\infty(Q_T)$ and $T_k(u_n - v)$ as test function in (13). Furthemore, we have $u_n(x,0) = u_n(x,T)$ using lemma 2.7 and we pass to the limit we get

$$u(x,0) = u(x,T) \text{ a.e in } \Omega.$$

□

4. Uniqueness of entropy solution

The uniqueness of entropy solution for problem (11) is not satisfied under the assumptions (5)-)(10), due to the dependence of u for the operator a, To address this issue, an additional condition must be introduced.

Proposition 4.1. *Assume that a satisfying in addition to (5)-(10) the following condition*

$$|a(x,t,s,\xi)| \geq \delta_1 M(\frac{|u|}{\lambda}), \tag{35}$$

then problem (11) admit one and only one entropy solution .

Proof. Let u_1 and u_2 be two entropy solution for problem (11) we shal prove that $u_1 = u_2$. By using $T_h(u_2) \in \mathcal{W}(0,T)L^\infty(Q_T)$ in (12) with solution u_1, we have

$$\int_0^T \langle \frac{\partial T_h(u_2)}{\partial t}, T_k(u_1 - T_h(u_2))\rangle_\Omega dt + \int_{Q_T} a(x,t,u_1,\nabla u_1)\nabla T_k(u_1 - T_h(u_2))dxdt$$

$$\leq \int_{Q_T} f T_k(u_1 - T_h(u_2) dxdt,$$

and using $T_h(u_1) \in \mathcal{W}(0,T) \cap L^\infty(Q_T)$ in (12) with solution u_2, we have

$$\int_0^T \langle \frac{\partial T_h(u_1)}{\partial t}, T_k(u_2 - T_h(u_1)) \rangle_\Omega dt + \int_{Q_T} a(x,t,u_2,\nabla u_2)\nabla T_k(u_2 - T_h(u_1))dxdt$$

$$\leq \int_{Q_T} f T_k(u_2 - T_h(u_1) dxdt.$$

By adding these two inequalities, we get

$$\int_0^T \Big\langle \frac{\partial T_h(u_2)}{t}, T_k(u_1 - T_h(u_2)) \Big\rangle_\Omega + \int_0^T \Big\langle \frac{\partial T_h(u_1)}{\partial t}, T_k(u_2 - T_h(u_1)) \Big\rangle_\Omega dt$$

$$+ \int_{Q_T} a(x,t,u_1,\nabla u_1)\nabla T_k(u_1 - T_h(u_2))dxdt + \int_{Q_T} a(x,t,u_2,\nabla u_2)\nabla T_k(u_2 - T_h(u_1))dxdt$$

$$\leq \int_{Q_T} f T_k(u_1 - T_h(u_2) dxdt + \int_{Q_T} f T_k(u_2 - T_h(u_1) dxdt.$$

For the first and term in the left hand side we can easly see that

$$\int_0^T \Big\langle \frac{\partial T_h(u_2)}{\partial t}, T_k(u_1 - T_h(u_2)) \Big\rangle_\Omega + \int_0^T \Big\langle \frac{\partial T_h(u_1)}{\partial t}, T_k(u_2 - T_h(u_1)) \Big\rangle_\Omega dt$$

$$= \int_0^T \int_{\{|u_2| \leq h\} \cap \{|u_1| \leq k\}} \frac{\partial (u_2 - u_1)}{\partial t} T_k(u_1 - u_2) dxdt = 0$$

On the other hand, we decompose the third integral in the left hand side as follow

$$\int_{Q_T} a(x,t,u_1,\nabla u_1)\nabla T_k(u_1 - T_h(u_2))dxdt$$

$$= \int_0^T \int_{\{|u_1 - T_h(u_2)| \leq k\}} a(x,t,u_1,\nabla u_1)(\nabla u_1 - \nabla T_h(u_2))dxdt,$$

$$= \int_0^T \int_{\{\{|u_1 - u_2| \leq k\} \cap \{|u_2| \leq h\}\}} a(x,t,u_1,\nabla u_1)(\nabla u_1 - \nabla u_2)dxdt$$

$$+ \int_0^T \int_{\{\{|u_1 - u_2| \leq k\} \cap \{|u_2| > h\}\}} a(x,t,u_1,\nabla u_1)\nabla u_1 dxdt$$

$$\geq \int_0^T \int_{\{\{|u_1 - u_2| \leq k\} \cap \{|u_2| \leq h\} \cap \{|u_1| \leq h\}\}} a(x,t,u_1,\nabla u_1)(\nabla u_1 - \nabla u_2)dxdt$$

$$+ \int_0^T \int_{\{\{|u_1 - u_2| \leq k\} \cap \{|u_2| \leq h\} \cap \{|u_1| > h\}\}} a(x,t,u_1,\nabla u_1)(\nabla u_1 - \nabla u_2)dxdt. \tag{36}$$

Similarly, we get

$$\int_{Q_T} a(x,t,u_2,\nabla u_2)\nabla T_k(u_2 - T_h(u_1))dxdt \geq$$

$$\geq \int_0^T \int_{\{\{|u_2 - u_1| \leq k\} \cap \{|u_1| \leq h\} \cap \{|u_2| \leq h\}\}} a(x,t,u_2,\nabla u_2)(\nabla u_2 - \nabla u_1)dxdt$$

$$+\int_0^T\int_{\{\{|u_2-u_1|\le k\}\cap\{|u_1|\le h\}\cap\{|u_2|>h\}\}} a(x,t,u_2,\nabla u_2)(\nabla u_2-\nabla u_1)dxdt. \quad (37)$$

Combining (36) and (37), we obtain

$$\int_{Q_T} a(x,t,u_1,\nabla u_1)\nabla T_k(u_1-T_h(u_2))dxdt+\int_{Q_T} a(x,t,u_2,\nabla u_2)\nabla T_k(u_2-T_h(u_1))dxdt$$

$$\ge\int_0^T\int_{\{\{|u_2-u_1|\le k\}\cap\{|u_1|\le h\}\cap\{|u_2|\le h\}\}}(a(x,t,u_1,\nabla u_1)-a(x,t,u_2,\nabla u_2))(\nabla u_1-\nabla u_2)dxdt$$

$$+\int_0^T\int_{\{\{|u_2-u_1|\le k\}\cap\{|u_2|\le h\}\cap\{|u_1|>h\}\}} a(x,t,u_1,\nabla u_1)(\nabla u_1-\nabla u_2)dxdt$$

$$+\int_0^T\int_{\{\{|u_2-u_1|\le k\}\cap\{|u_1|\le h\}\cap\{|u_2|>h\}\}} a(x,t,u_2,\nabla u_2)(\nabla u_1-\nabla u_2)dxdt.$$

Using the result above, we conclude that

$$\int_0^T\int_{\{\{|u_2-u_1|\le k\}\cap\{|u_1|\le h\}\cap\{|u_2|\le h\}\}}(a(x,t,u_1,\nabla u_1)-a(x,t,u_2,\nabla u_2))(\nabla u_1-\nabla u_2)dxdt$$

$$\le\int_{Q_T} f(T_k(u_1-T_h(u_2)dxdt+T_k(u_2-T_h(u_1))dxdt$$

$$-\int_0^T\int_{\{\{|u_2-u_1|\le k\}\cap\{|u_1|\le h\}\cap\{|u_2|>h\}\}} a(x,t,u_2,\nabla u_2)(\nabla u_1-\nabla u_2)dxdt$$

$$-\int_0^T\int_{\{\{|u_2-u_1|\le k\}\cap\{|u_1|\le h\}\cap\{|u_2|>h\}\}} a(x,t,u_2,\nabla u_2)(\nabla u_1-\nabla u_2)dxdt.$$

The first term on the right hand side, reads as

$$\Big|\int_{Q_T} f(T_k(u_1-T_h(u_2)dxdt+T_k(u_2-T_h(u_1))dxdt\Big|$$

$$\le\int_{\{|u_1|\le h\};\{|u_2[\le h\}}|f||T_k(u_1-u_2)+T_k(u_2-u_1)|dxdt$$

$$+\int_{\{|u_1|>h\}}|f||T_k(u_1-T_h(u_2))+T_k(u_2-T_h(u_1))|dxdt$$

$$+\int_{\{|u_2|>h\}}|f||T_k(u_1-T_h(u_2))+T_k(u_2-T_h(u_1))|dxdt$$

$$\le 2k\int_{\{|u_1|>h\}}|f|dxdt+2k\int_{\{|u_2|>h\}}|f|dxdt,$$

Since $f\in L^1(Q_T)$ and $meas\{u_i\ge k\}\to 0$ as $h\to+\infty$ for $i=1,2$, it follow that

$$\int_{Q_T} f(T_k(u_1-T_h(u_2)dxdt+T_k(u_2-T_h(u_1))dxdt\to 0 \quad \text{as } \to+\infty \quad (38)$$

For what we concerne the second term in the right hand side, by taking $T_h(u_1)$ as test function in (12), we have

$$\int_{Q_T} T_h(u_1)T_k(u_1-T_h(u_1))dxdt+\int_{Q_T} a(x,t,u_1,\nabla u_1)\nabla T_k(u_1-T_h(u_1))dxdt$$

$$\leq \int_{Q_T} fT_k(u_1 - T_h(u_1))dxdt,$$

which imply, in view of (7) that

$$\alpha \int_{\{h\leq|u_1|\leq h+k\}} M(\frac{|\nabla u|}{\lambda}dxdt \leq \int_{\{h\leq|u_1|\leq h+k\}} a(x,t,u_1,\nabla u_1)\nabla u_1 dxdt$$

$$\leq k \int_{\{|u_1|\geq h\}} |f|dxdt.$$

Since $k\int_{\{|u_1|\geq h\}} |f|dxdt \to 0$ as $h \to +\infty$, we can obtain that

$$\alpha \int_{\{h\leq|u_1|\leq h+k\}} M(\frac{|\nabla u_1|}{\lambda})dxdt \to 0,$$

using the fact that $\int_{Q_T} M(v)dxdt \leq \delta \int_{Q_T} M(\lambda|\nabla v|)dxdt$ (see lemma 2.6) we have .

$$\int_{\{h\leq|u_1|\leq h+k\}} M(|u_1|)dxdt \to 0.$$

Similarly, using the above method, we get

$$\alpha \int_{\{h\leq|u_2|\leq h+k\}} M(\frac{|\nabla u_2|}{\lambda})dxdt \to 0,$$

Furthemore, we remark that
$\{|u_1 - u_2| \leq k\} \cap \{|u_2| \leq h\} \cap \{|u_1| > h\} \subseteq \{h \leq |u_1| \leq h+k\} \cap \{h-k \leq |u_2| \leq h\}$,
then (5) and Young inequality lead us to write,

$$\int_{\{|u_1-u_2|\leq k\}\cap\{|u_2|\leq h\}\cap\{|u_1|>h\}} a(x,t,u_1,\nabla u_1)(\nabla u_1 - \nabla u_2)dxdt$$

$$\leq \beta \int_{\{|u_1-u_2|\leq k\}\cap\{|u_2|\leq h\}\cap\{|u_1|>h\}} (h_1(x,t) + \overline{M}^{-1}P(\delta|u_1|) + \overline{M}^{-1}(M(\delta|\nabla u_1))(|\nabla u_1| + |\nabla u_2|)dxdt$$

$$\leq 2\beta \int_{\{|u_1|>h\}} (\overline{M}(h_1(x,t))dxdt + 2\beta \int_{\{h<|u_1|\leq h+k\}} P(\delta|u_1|)dxdt + \int_{\{h<|u_1|\leq h+k\}} (M(\delta|\nabla u_1))dxdt$$

$$+3\beta \int_{\{h<|u_1|\leq h+k\}} M(|\nabla u_1|)dxdt + 3\int_{\{h-k<|u_1|\leq h\}} M(|\nabla u_2|)dxdt \longrightarrow 0$$

$$\text{as } h \to +\infty.$$

Similarly, we can prove that

$$\int_{\{|u_1-u_2|\leq k\}\cap\{|u_2|\leq h\}\cap\{|u_1|>h\}} a(x,t,u_2,\nabla u_2)(\nabla u_2 - \nabla u_1)dxdt \to 0 \text{ as } h \to +\infty,$$

finally, we obtain

$$\int_0^T \int_{\{|u_2-u_1|\leq k\}} (a(x,t,u_1,\nabla u_1) - a(x,t,u_2,\nabla u_2))(\nabla u_1 - \nabla u_2)dxdt = 0$$

On the other hand we can write

$$\int_0^T \int_{\{|u_2-u_1|\leq k\}} (a(x,t,u_1,\nabla u_1) - a(x,t,u_2,\nabla u_1))(\nabla u_1 - \nabla u_2)dxdt$$

$$+\int_0^T \int_{\{|u_2-u_1|\le k\}} (a(x,t,u_2,\nabla u_1) - a(x,t,u_2,\nabla u_2))(\nabla u_1 - \nabla u_2)dxdt = 0$$

by using (7), (10) and (35), there exists c_1 and c_2 two positive constants such that

$$c_1 \int_0^T \int_{\{|u_2-u_1|\le k\}} |\nabla u_1 - \nabla u_2|^2 dxdt + c_2 \int_0^T \int_{\{|u_2-u_1|\le k\}} |u_1 - u_2|^2 |\nabla u_1 - \nabla u_2| dxdt \le 0. \tag{39}$$

Using the fact that the both terms in (39) is positive we conclude that $\nabla u_1 = \nabla u_2$ a.e in Q_T, since $u_1 = u_2 = 0$ on $\partial\Omega \times (0,T)$, thus $u_1 = u_2$ a.e in Q_T, which achieve the prove. □

4.1. Uniqueness and existence of approximate problem

Proposition 4.2. *Assume that (5)- (10) holds, then problem (14) admits a unique weak solution* $u \in D(A) \cap W_0^{1,x} L_M(Q_T) \cap C(0,T,L^2(\Omega))$.

Proof. Assume that u and v are two weak solutions of (14),then u and v satisfying the equation of problem (14)

$$\frac{\partial u}{\partial t} - div(a(x,t,u,\nabla u)) = f, \tag{40}$$

and

$$\frac{\partial v}{\partial t} - div(a(x,t,v,\nabla v)) = f, \tag{41}$$

subtracting equality (40) from (41) and using $(u - v)$ as a test function we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u(t) - v(t))^2 dx + \int_\Omega (a(x,t,u,\nabla u) - a(x,t,v,\nabla v))(\nabla u - \nabla v)dx = 0,$$

it is easy to see that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u(t) - v(t))^2 dx + \int_\Omega (a(x,t,v,\nabla u) - a(x,t,v,\nabla v))(\nabla u - \nabla v)dx =$$

$$-\int_\Omega a(x,t,u,\nabla u) - a(x,t,v,\nabla u))(\nabla u - \nabla v)dx,$$

then we can write

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u(t) - v(t))^2 dx + I_1 = -I_2. \tag{42}$$

for I_1 by using (7) and 10 we have

$$I_1 \ge \alpha \int_\Omega M(\frac{\nabla u - \nabla v}{\lambda})dxdt \ge \alpha c_1 \int_\Omega |\nabla u - \nabla v|^2 dx. \tag{43}$$

For I_2 by using (3) we get

$$\begin{aligned} |I_2| &\le \int_\Omega |u - v|(d(x,t) + B(\nabla u))|\nabla u - \nabla v| dx \\ &\le c_2 \int_\Omega |u - v|^2 dxdt + c_3 \int_\Omega |\nabla u - \nabla v|^2 dx. \end{aligned} \tag{44}$$

Combining (42),(4.1), (44), and we choose $\alpha c_1 = c_3$ we deduce that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u(t) - v(t))^2 dx \le c_2 \int_\Omega |u - v|^2 dx,$$

we apply Gronwall lemma to obtain

$$\int_{\Omega}(u(t)-v(t))^2dx \leq c_4 \int_{\Omega}|u(x,0)-v(x,0)|^2dx.$$

Finally, initial condition allows us to have $u = v$. □

Now, we pass to prove the existence of problem (13):

Proof. ***Continuity of*** Ψ Let u_n and u_m satisfying (13), we can write by using $u_n - u_m$ as test function

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}(u_n(t)-u_m(t))^2dx + \int_{\Omega}a(x,t,u_n,\nabla u_n) - a(x,t,u_m,\nabla u_m)(\nabla u_n - \nabla u_m)dx$$

$$= \int_{\Omega}(f_n - f_m)(u_n - u_m)dx$$

using the same argument as above and we get

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}(u_n(t)-u_m(t))^2dx + \alpha c_1\int_{\Omega}|\nabla u_n - \nabla u_m|^2dx \leq \frac{1}{\epsilon}\|f_n - f_m\|_{\overline{M},\Omega}$$

$$+\varepsilon\|u_n - u_m\|_{M,\Omega} + c_2\int_{\Omega}|u_n - u_m|^2dxdt + c_3\int_{\Omega}|\nabla u_n - \nabla u_m|^2dx.$$

using (10) and taking $c_3 = \alpha c_2$ we have

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}(u_n(t)-u_m(t))^2dx + \varepsilon c_4\|u_n - u_m\|^2 \leq \frac{1}{\epsilon}\|f_n - f_m\|_{\overline{M},\Omega} + c_2\|u_n - u_m\|^2,$$

taking $\varepsilon = \frac{c_2}{2c_4}$ we get

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}(u_n(t)-u_m(t))^2dx \leq c_5 + \frac{c_2}{2}\|u_n - u_m\|^2,$$

integrating between 0 and T, and using Gronwall lemma we deduce

$$\|u_n(x,T) - u_m(x,T)\|^2_{L^2(\Omega)} \leq c_6\|u_{0n} - u_{0m}\|^2_{L^2(\Omega)}.$$

Finally, we obtain

$$\|\Psi(u_{0n}) - \Psi(u_{0m})\| \leq c_6\|u_{0n} - u_{0m}\|.$$

The continuity of Ψ is achieved.

Compactness of Ψ***.***

we prove that $\Psi(\mathcal{B}(0,R)) \subset \mathcal{B}(0,R)$ such that $B(0,R)$ is the ball of $L^2(\Omega)$ with radius R. (i.e we will find R such that if $|u_{0n}| \leq R$ we obtain $u_n(x,T) \leq R$) , using u_n as a test function in (13) we have

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}(u_n(t))^2dx + \int_{\Omega}a(x,t,u_n,\nabla u_n)\nabla u_n dxdt = \int_{\Omega}f_n u_n dx,$$

by (7) and Young inequality we get

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}(u_n(t))^2dx + \alpha\int_{\Omega}M(\frac{|\nabla u_n|}{\lambda})dx \leq \frac{1}{\varepsilon}\|f_n\|_{\overline{M},\Omega} + \varepsilon\|u_n\|_{M,\Omega},$$

using (10) we have

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}(u_n(t))^2dx + \frac{\alpha}{\lambda^2\delta_0}\|\nabla u_n\|^2_{L^2(\Omega)} + \frac{\varepsilon}{\delta_0}\|u_n\|^2_{L^2(\Omega)} \leq \frac{1}{\varepsilon}\|f_n\|_{\overline{M},\Omega}$$

and so, by Poincarré inequality there exist $c > 0$ such that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u_n(t))^2 dx + \frac{\alpha\lambda^2+\varepsilon}{\lambda\delta_0}\|u_n\|^2_{L^2(\Omega)} \leq \frac{1}{\varepsilon}\|f_n\|_{\overline{M},\Omega}$$

we set $c_3 = \frac{\alpha\lambda^2+\varepsilon}{\lambda\delta_0}$ and multiplying by $\exp(c_3 t)$ we can write

$$\exp(c_3 t)\frac{1}{2}\frac{d}{dt}\int_\Omega (u_n(t))^2 dx + c_3\exp(c_3 t)\|u_n\|^2_{L^2(\Omega)} \leq \exp(c_3 t)\frac{1}{\varepsilon}\|f_n\|_{\overline{M},\Omega}$$

integrating between 0 and T we obtain

$$\exp(c_3 T)\|u_n(T)\|^2_{L^2(\Omega)} \leq c_4 + R^2.$$

Finally, we choose R such that $R^2 \geq \frac{c_4\exp(c_3 T)}{1-\exp(c_3 T)}$ to deduce the result. □

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