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Rectangle rule to compute hypersingular integral on a circle

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Abstract. Consider the hypersingular integral on circle

$$I(c,s,f) = \int_c^{c+2\pi} \frac{f(x)}{\sin^2 \frac{x-s}{2}} dx, \quad f(x) \in C^\infty[c, c+2\pi], c \in R, s \in (c, c+2\pi)$$

defined as the Hadamard finite-part integral. Classical rectangle rule

$$I_n(c,s,f) = h \sum_{i=0}^{n-1} \frac{f_C(\hat{x}_i)}{\sin^2 \frac{\hat{x}_i - s}{2}}$$

with $f_C(\hat{x}_i) = f(\hat{x}_i), h = 2\pi/n, \hat{x}_i = x_i + h/2$ is the middle of subinterval which can not be used to compute hypersingular integral as there are the divergence part. In order to give the simple rectangle rule, we present the modify rectangle rule as

$$\tilde{I}_n(c,s,f) = h \sum_{i=0}^{n-1} \frac{f_C(\hat{x}_i)}{\sin^2 \frac{\hat{x}_i - s}{2}} - \frac{4f(s)\pi^2}{h \sin^2 \frac{\xi\pi}{2}}, \xi \in [-1, 1]$$

and

$$\hat{I}_n(c,s,f) = h \sum_{i=0}^{n-1} \frac{f_C(\hat{x}_i)}{\sin^2 \frac{\hat{x}_i - s}{2}} - \frac{4f(s)\pi^2}{h \sin^2 \frac{\xi\pi}{2}} - 4f'(s)\pi \tan \frac{(\xi+1)\pi}{2}$$

We get the numerical quadrature formulas $\tilde{I}_n(c,s,f)$ have the spectral accuracy as

$$\hat{I}_n(c,s,f) - I(c,s,f) = O(h^\mu), \mu \geq 0$$

with the special function $\frac{\pi^2}{\sin^2 \frac{\xi\pi}{2}}$ and $\tan \frac{(\xi+1)\pi}{2}$ equals to zero. Numerical examples are provided to valid our theorem.

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1. Introduction

With the development of boundary element methods(BEM) [2], especially the natura boundary element methods (NBEM) [30], how to compute hypersingular integral become important. The definition of hypersingular integral have been investigated by Hadamard [12] in 1923. Hypersingular integral defined on the interval takes

$$\int_a^b \frac{f(x)}{(x-s)^{p+1}}dx = g(s) \quad s \in (a,b), p = 1,2 \tag{1}$$

There are definition of derivative Cauchy principal integral, finite part definition and singular part separation which can be proved the equability with each other. In the following, we adopt

$$\int_a^b \frac{f(x)}{(x-s)^{p+1}}dx = \lim_{\varepsilon\to 0}\left\{\left(\int_a^{s-\varepsilon} + \int_{s+\varepsilon}^b\right)\frac{f(x)}{(x-s)^{p+1}}dx - \frac{2f^{(p-1)}(s)}{\varepsilon}\right\}, \tag{2}$$

where $\int_a^b$ denotes a hypersingualr integral and s the singular point.

Because of the hypersingularity, classical numerical methods can not be directly applied to calculated hypersingular integral. In the years of 1973, Gaussian method [23] was presented by Kutt with new Gauss points and Gauss weight was recalculated. Then equation (1) be approximated by Gaussian method [7, 9, 10, 25] with derivative Cauchy principal integral or finite part definition. There are also transformation method [3, 5] and some other methods [11, 24] developed to investigate (1).

Composite trapezoidal and Simpson's rule were firstly suggested by Linz [19] with the density function numerical approximated and hypersingular kernel analysis calculated which belonged to semi-discrete methods for $p = 1$. In [30], for the case of singular point coincide with the mesh-joint, the trapezoidal rule is studied. Du [4] studied the composite Simpson's rule and showed the optimal global convergence rate is $O(h)$ for $p = 2$. In recent researches, superconvergence phenomenon of trapezoidal rule [27], Simpson's rule [29] and Newton-Cotes rules [28] for hypersingular integrals on interval were found and $O(h^{k+1})$ convergence rate was obtained. For hypersingular integrals defined on circle, Newton-Cotes rules and the corresponding superconvergence were discussed in [32, 33].

In this paper, we consider the hypersingular integral defined in a circle which have been paid less attention to it.

$$\begin{aligned} I(c,s,f) : &= \int_c^{c+2\pi} \frac{f(x)}{\sin^2\frac{x-s}{2}}dx \\ &= \lim_{\epsilon\to 0}\left\{\int_c^{s-\epsilon} \frac{f(x)}{\sin^2\frac{x-s}{2}}\,dx + \int_{s+\epsilon}^{c+2\pi} \frac{f(x)}{\sin^2\frac{x-s}{2}}\,dx - \frac{8f(s)}{\epsilon}\right\}. \end{aligned} \tag{3}$$

Hypersingular integral defined on the circle [18] have not been paid much attention until NBEM was proposed, then professor Yu[31] firstly calculated (3) by expansion the singular kernel into infinity series to solve unbounded problems. In reference [31], trapezoidal and adaptive methods have been studied to approximated density function with the singular kernel calculated analysis. In recent years, superconvergence phenomenon of (3) by trapezoidal rule [32] and Newton-Cote rule [33] have been considered. Then in the paper of Li [13] et.al, extrapolation scheme based on trapezoidal rule was constructed hypersingular integral on circle. Different from the semi-discrete methods with density function approximated, fully discrete methods with both the density function and singular kernel was approximated at the same time. Rectangle rule [16] and trapezoidal rule [15] for computing Cauchy principal value integral on interval and circle were considered. Asidi [26], have presented the generalized Euler-Maclaurin formula with the singular point coincide with the mesh-joint. In reference [8, 20–22] asymptotic error with the singular point located at the mesh-joint for hypersingular integrals on interval was studied. By using middle rectangle rule, a generalized Euler-Maclaurin [17] for Hadamard finite part values integrals is shown.

In this paper, we take fully discrete methods to compute hypersingular integrals on circle by middle rectangle rule. Fully discrete methods by rectangle rule can not be used to approximate hypersingular integrals directly because of the first part of error expansion is divergence. By performing the Taylor expansion at the singular point s, the asymptotic error is obtained. With the help of special function, first term and second term of asymptotic error have been expressed as $\frac{\pi^2}{\sin^2\frac{\tau\pi}{2}}$ and $\tan\frac{(\tau+1)\pi}{2}$. Two modify rectangle rule have been proposed to get higher convergence rate. At last, numerical examples are given to illustrate our numerical analysis.

2. Main result

Let $c = x_0 < x_1 < \cdots < x_{n-1} < x_n = c + 2\pi$ be a uniform partition of the interval $[c, c + 2\pi]$ with mesh size $h = 2\pi/n$ and $f_C(\hat{x}_j)$ the piecewise interpolant for $f(x)$:

$$f_C(\hat{x}_j) = f(\hat{x}_j),\ \ \hat{x}_j = x_j + \frac{h}{2},\quad x \in [x_j, x_{j+1}],\ \ 0 \le j \le n-1, \tag{4}$$

and a transformation

$$x = \hat{x}_j(\tau) := (\xi + 1)(x_{j+1} - x_j)/2 + x_j,\ \ \xi \in [-1, 1], \tag{5}$$

changes subinterval $[x_j, x_1]$ into $[-1, 1]$.

The composite rectangle rule gives by $f_C(\hat{x}_j)$ replace $f(x)$:

$$I_n(c, s, f) := h\sum_{j=0}^{n-1} \frac{f_C(\hat{x}_j)}{\sin^2\frac{\hat{x}_j - s}{2}} = \sum_{j=0}^{n-1} \omega_j(s) f(\hat{x}_j) = I(c, s, f) - E_n(c, s, f), \tag{6}$$

where

$$\omega_j(s) = \frac{h}{\sin^2\frac{\hat{x}_j - s}{2}}$$

is the Cotes coefficients, see reference [14] and $E_n(c, s, f)$ denotes the error functional.

Before presenting the main results, we firstly define $K_s(x)$ as follows

$$K_s(x) = \begin{cases} \dfrac{(x-s)^2}{\sin^2\frac{x-s}{2}} & x \neq s, \\ 4, & x = s. \end{cases} \tag{7}$$

We present our main results below.

Theorem 1. *Assume $f(x) \in C^\infty[c, c + 2\pi]$. For the rectangle rule $I_n(f; s)$ defined in (6), there exists a positive constant c_k, independent of h and s, such that*

$$E_n(c, s, f) = \frac{4f(s)\pi^2}{h\sin^2\frac{\xi\pi}{2}} - 4f'(s)\pi\tan\frac{(\xi+1)\pi}{2} + \sum_{k=1}^{\infty} c_k h^{2k}, \tag{8}$$

where $s = x_{m-1} + (1 + \xi)h/2$.

Remmark 1. *For the first part of (8), if*

$$\frac{4\pi^2}{h\sin^2\frac{\xi\pi}{2}} = 0 \tag{9}$$

there are no zero roots. For the second part of (8), if

$$\pi\tan\frac{(\xi+1)\pi}{2} = 0 \tag{10}$$

with $\xi \in [-1, 1]$. We get $\xi = \pm 1$ is the roots.

Setting

$$I^1_{n,j}(s)=\begin{cases}\displaystyle ⨍_{x_m}^{x_{m+1}}\left[\frac{1}{\sin^2\frac{x-s}{2}}-\frac{1}{\sin^2\frac{\hat{x}_m-s}{2}}\right]dx, & j=m,\\ \displaystyle\int_{x_j}^{x_{j+1}}\left[\frac{1}{\sin^2\frac{x-s}{2}}-\frac{1}{\sin^2\frac{x_j-s}{2}}\right]dx, & j\neq m.\end{cases}\tag{11}$$

and

$$I^0_{n,j}(s)=\begin{cases}\displaystyle ⨍_{x_m}^{x_{m+1}}\left[\frac{x-s}{\sin^2\frac{x-s}{2}}-\frac{\hat{x}_m-s}{\sin^2\frac{\hat{x}_m-s}{2}}\right]dx, & j=m,\\ \displaystyle\int_{x_j}^{x_{j+1}}\left[\frac{x-s}{\sin^2\frac{x-s}{2}}-\frac{\hat{x}_j-s}{\sin^2\frac{x_j-s}{2}}\right]dx, & j\neq m.\end{cases}\tag{12}$$

Lemma 1. *Assume $s=x_m+(\xi+1)h/2$. Let $I_{n,j}(s)$ be defined by (11), then there holds that*

$$I^1_{n,j}(s) = 2\sum_{k=1}^{\infty}\Big(\sin k(x_{j+1}-s)-\sin k(x_j-s)\Big)-h\sum_{k=1}^{\infty}k\cos k(\hat{x}_j-s).\tag{13}$$

Proof For $i=m$,by equation (3), we have

$$\begin{aligned}I^1_{n,m}(s) &= \lim_{\varepsilon\to0}\left(\int_{x_m}^{s-\varepsilon}+\int_{s+\varepsilon}^{x_{m+1}}\right)\left[\frac{1}{\sin^2\frac{x-s}{2}}-\frac{1}{\sin^2\frac{\hat{x}_m-s}{2}}\right]dx\\ &= 2\cot\frac{x_m-s}{2}-2\cot\frac{x_{m+1}-s}{2}-\frac{h}{\sin^2\frac{\hat{x}_m-s}{2}}\end{aligned}\tag{14}$$

For $i\neq m$,by taking integration by parts, we have

$$I^1_{n,j}(s) = 2\cot\frac{x_j-s}{2}-2\cot\frac{x_{j+1}-s}{2}-\frac{h}{\sin^2\frac{\hat{x}_j-s}{2}}\tag{15}$$

Now, by using the well-known identity

$$\frac{1}{2}\cot\frac{x}{2}=\sum_{n=1}^{\infty}\sin nx\tag{16}$$

and

$$\frac{1}{\sin^2\frac{x}{2}}=\sum_{n=1}^{\infty}n\cos nx\tag{17}$$

The proof is completed.

Lemma 2. *Assume $s=x_m+(\xi+1)h/2$. Let $I_{n,j}(s)$ be defined by (11), then there holds that*

$$\sum_{j=0}^{n-1}I^1_{n,j}(s) = \frac{4\pi^2}{h\sin^2(\xi\pi/2)}\tag{18}$$

Proof: By (13), we have

$$\begin{aligned}\sum_{j=0}^{n-1} I_{n,j}^1(s) &= \sum_{j=0}^{n-1} 2\left[\sum_{k=1}^{\infty}\Big(\sin k(x_{j+1}-s)-\sin k(x_j-s)\Big)-h\sum_{k=1}^{\infty} k\cos k(\hat{x}_j-s)\right]\\ &= \sum_{k=1}^{\infty}\left[2\sum_{j=0}^{n-1}\Big(\sin k(x_{j+1}-s)-\sin k(x_j-s)\Big)-h\sum_{j=0}^{n-1} k\cos k(\hat{x}_j-s)\right]\\ &= -h\sum_{k=1}^{\infty}\sum_{j=0}^{n-1} k\cos k(\hat{x}_j-s)\\ &= -h\sum_{j=1}^{\infty} n^2 j\cos nj(\hat{x}_0-s)\\ &= -h\sum_{j=1}^{\infty} n^2 j\cos[j\xi\pi]\\ &= -\frac{4\pi^2}{h}\sum_{j=1}^{\infty} j\cos[j\xi\pi]\\ &= -\frac{4\pi^2}{h\sin^2(\xi\pi/2)} \end{aligned} \tag{19}$$

where we have used

$$\sum_{j=0}^{n-1} k\cos k(\hat{x}_j-s) = \begin{cases} n^2 j\cos[k(\hat{x}_0-s)], & k=nj,\\ 0, & otherwise\end{cases} \tag{20}$$

Lemma 3. *Assume $s = x_m + (\xi+1)h/2$ with $\xi \in (-1,1)$. Let $I_{n,j}(s)$ be defined by (11), then there holds that*

$$\sum_{j=0}^{n-1} I_{n,j}^0(s) = -4\pi\tan\frac{(\xi+1)\pi}{2} \tag{21}$$

Proof: By the identity in [1]

$$\frac{\pi^2}{\sin^2\pi t} = \sum_{l=-\infty}^{l=\infty}\frac{1}{(t+l)^2}, \tag{22}$$

then we get

$$\frac{1}{\sin^2\frac{x-s}{2}} = \frac{4}{(x-s)^2} + 4\sum_{l=1}^{\infty}\frac{1}{(x-s-2l\pi)^2} + 4\sum_{l=1}^{\infty}\frac{1}{(x-s+2l\pi)^2} \tag{23}$$

$$\frac{1}{\sin^2\frac{\hat{x}_j-s}{2}} = \frac{4}{(\hat{x}_j-s)^2} + 4\sum_{l=1}^{\infty}\frac{1}{(\hat{x}_j-s-2l\pi)^2} + 4\sum_{l=1}^{\infty}\frac{1}{(\hat{x}_j-s+2l\pi)^2} \tag{24}$$

and

$$\begin{aligned}&\frac{x-s}{\sin^2\frac{x-s}{2}} - \frac{\hat{x}_j-s}{\sin^2\frac{\hat{x}_j-s}{2}}\\ =\ & \frac{4}{x-s} - \frac{4}{\hat{x}_j-s} + 4\sum_{l=1}^{\infty}\left[\frac{1}{x-s-2l\pi+\frac{4l^2\pi^2}{x-s}} - \frac{1}{\hat{x}_j-s-2l\pi+\frac{4l^2\pi^2}{\hat{x}-s}}\right]\end{aligned}$$

$$+4\sum_{l=1}^{\infty}\left[\frac{1}{x-s+2l\pi+\frac{4l^2\pi^2}{x-s}}-\frac{1}{\hat{x}_j-s+2l\pi+\frac{4l^2\pi^2}{\hat{x}-s}}\right]$$
$$=\frac{4}{\xi-c_j}-\frac{4}{c_j}+4\sum_{l=1}^{\infty}\left[\frac{1}{\xi-c_j-2la_1}-\frac{1}{c_j-2la_1}\right]+4\sum_{l=1}^{\infty}\left[\frac{1}{\xi-c_j+2la_1}-\frac{1}{c_j+2la_1}\right]$$

which means

$$\begin{aligned}T_{ik}(\xi) &= \sum_{j=0}^{n-1} I_{n,j}^{0}(s)\\ &= 4\sum_{i=1}^{n}\phi_0(2(m-i)+\xi)+8\sum_{i=1}^{n}\sum_{l=1}^{\infty}\phi_0(2(m-i-nl)+\xi)+4\sum_{i=1}^{n}\sum_{l=1}^{\infty}\phi_0(2(m-i+nl)+\xi)\\ &= 4\sum_{l=-\infty}^{\infty}\sum_{i=1}^{n}\phi_0(2(m-i+nl)+\xi)\\ &= 4[\phi_0(\xi)+\sum_{l=1}^{\infty}\phi_0(2l+\xi)+\sum_{l=1}^{\infty}\phi_0(-2l+\xi)]\\ &= -4\pi\tan\frac{(\xi+1)\pi}{2} \end{aligned} \tag{25}$$

where

$$\phi_0(t)=\begin{cases}\fint_{-1}^{1}\frac{\xi}{(\xi-t)t}\,d\xi, & |t|<1,\\ \int_{-1}^{1}\frac{\xi}{(\xi-t)t}\,d\xi, & |t|>1,\end{cases} \tag{26}$$

Lemma 4. *Under the same assumptions of theorem 1, it holds that*

$$\frac{f(x)}{\sin^2\frac{x-s}{2}}-\frac{f_C(\hat{x}_j)}{\sin^2\frac{\hat{x}_j-s}{2}}=\left[\frac{1}{\sin^2\frac{x-s}{2}}-\frac{1}{\sin^2\frac{\hat{x}_j-s}{2}}\right]f(s)+\sum_{i=1}^{\infty}\frac{f^{(i)}(s)}{i!}\left[\frac{(x-s)^i}{\sin^2\frac{x-s}{2}}-\frac{(\hat{x}_j-s)^i}{\sin^2\frac{\hat{x}_j-s}{2}}\right]$$

Proof: Performing Taylor expansion of $f_C(\hat{x}_j)$ at the point s, we have

$$f_C(\hat{x}_j)=f(s)+\sum_{i=1}^{\infty}\frac{f^{(i)}(s)}{i!}(\hat{x}_j-s)^i \tag{27}$$

and

$$f(x)=f(s)+\sum_{i=1}^{\infty}\frac{f^{(i)}(s)}{i!}(x-s)^i \tag{28}$$

Combining (27) and (28) together we get the results. *Proof of Theorem 1:* we have

$$\begin{aligned}
&\left(\int_c^{x_m}+\int_{x_{m+1}}^{c+2\pi}\right)\frac{f(x)}{\sin^2\frac{x-s}{2}}dx-\sum_{j=0,j\neq m}^{n-1}h\frac{f_C(\hat{x}_j)}{\sin^2\frac{\hat{x}_j-s}{2}}\\
&=\sum_{j=0,j\neq m}^{n-1}\int_{x_j}^{x_{j+1}}\left[\frac{f(x)}{\sin^2\frac{x-s}{2}}-\frac{f_C(\hat{x}_j)}{\sin^2\frac{\hat{x}_j-s}{2}}\right]dx\\
&=f(s)\sum_{j=0,j\neq m}^{n-1}\int_{x_j}^{x_{j+1}}\left[\frac{1}{\sin^2\frac{x-s}{2}}-\frac{1}{\sin^2\frac{\hat{x}_j-s}{2}}\right]dx\\
&+f'(s)\sum_{j=0,j\neq m}^{n-1}\int_{x_m}^{x_{m+1}}\left[\frac{x-s}{\sin^2\frac{x-s}{2}}-\frac{\hat{x}_j-s}{\sin^2\frac{\hat{x}_j-s}{2}}\right]dx\\
&+\sum_{i=2}^{\infty}\frac{f^{(i)}(s)}{i!}\sum_{j=0,j\neq m}^{n-1}\int_{x_j}^{x_{j+1}}\left[\frac{(x-s)^i}{\sin^2\frac{x-s}{2}}-\frac{(\hat{x}_j-s)^i}{\sin^2\frac{\hat{x}_j-s}{2}}\right]dx
\end{aligned}\tag{29}$$

For $i=m$, we have

$$\begin{aligned}
&⨎_{x_m}^{x_{m+1}}\frac{f(x)}{\sin^2\frac{x-s}{2}}dx-h\frac{f_C(\hat{x}_m)}{\sin^2\frac{\hat{x}_m-s}{2}}\\
&=⨎_{x_m}^{x_{m+1}}\frac{f(x)}{\sin^2\frac{x-s}{2}}-\frac{f_C(\hat{x}_m)}{\sin^2\frac{\hat{x}_m-s}{2}}dx\\
&=f(s)⨎_{x_m}^{x_{m+1}}\left[\frac{1}{\sin^2\frac{x-s}{2}}-\frac{1}{\sin^2\frac{\hat{x}_m-s}{2}}\right]dx+f'(s)⨎_{x_m}^{x_{m+1}}\left[\frac{x-s}{\sin^2\frac{x-s}{2}}-\frac{\hat{x}_m-s}{\sin^2\frac{\hat{x}_m-s}{2}}\right]dx\\
&+\sum_{i=2}^{\infty}\frac{f^{(i)}(s)}{i!}\int_{x_m}^{x_{m+1}}\left[\frac{(x-s)^i}{\sin^2\frac{x-s}{2}}-\frac{(\hat{x}_m-s)^i}{\sin^2\frac{\hat{x}_m-s}{2}}\right]dx
\end{aligned}\tag{30}$$

Putting (29) and(30) together yields

$$\begin{aligned}
&⨎_c^{c+2\pi}\frac{f(x)}{\sin^2\frac{x-s}{2}}dx-\sum_{j=0}^{n-1}h\frac{f_C(\hat{x}_j)}{\sin^2\frac{\hat{x}_j-s}{2}}\\
&=\sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}}\left[\frac{f(x)}{\sin^2\frac{x-s}{2}}-\frac{f_C(\hat{x}_j)}{\sin^2\frac{\hat{x}_j-s}{2}}\right]dx\\
&=f(s)\sum_{j=0}^{n-1}⨎_{x_j}^{x_{j+1}}\left[\frac{1}{\sin^2\frac{x-s}{2}}-\frac{1}{\sin^2\frac{\hat{x}_j-s}{2}}\right]dx+f'(s)\sum_{j=0}^{n-1}⨎_{x_m}^{x_{m+1}}\left[\frac{x-s}{\sin^2\frac{x-s}{2}}-\frac{\hat{x}_j-s}{\sin^2\frac{\hat{x}_j-s}{2}}\right]dx\\
&+\sum_{i=2}^{2l-1}\frac{f^{(i)}(s)}{i!}\sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}}\left[\frac{(x-s)^i}{\sin^2\frac{x-s}{2}}-\frac{(\hat{x}_j-s)^i}{\sin^2\frac{\hat{x}_j-s}{2}}\right]dx\\
&+\sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}}\left[\frac{f^{(2l)}(s_1)}{(2l)!}\frac{(x-s)^{2l}}{\sin^2\frac{x-s}{2}}-\frac{f^{(2l)}(s_2)}{(2l)!}\frac{(\hat{x}_j-s)^{2l}}{\sin^2\frac{\hat{x}_j-s}{2}}\right]dx
\end{aligned}\tag{31}$$

$$= f(s)S_0'(\xi) + f'(s)S_0(\xi) + \sum_{i=2}^{\infty} \frac{f^{(i)}(s)}{i!} \sum_{j=0}^{n-1} \int_{x_j}^{x_{j+1}} \left[\frac{(x-s)^i}{\sin^2 \frac{x-s}{2}} - \frac{(\hat{x}_j - s)^i}{\sin^2 \frac{\hat{x}_j - s}{2}} \right] dx$$

Here

$$S_0'(\xi) = \sum_{j=0}^{n-1} \fint_{x_m}^{x_{m+1}} \left[\frac{1}{\sin^2 \frac{x-s}{2}} - \frac{1}{\sin^2 \frac{\hat{x}_j - s}{2}} \right] dx,$$

$$S_0(\xi) = \sum_{j=0}^{n-1} \fint_{x_m}^{x_{m+1}} \left[\frac{x-s}{\sin^2 \frac{x-s}{2}} - \frac{\hat{x}_j - s}{\sin^2 \frac{\hat{x}_j - s}{2}} \right] dx$$

by transformation (2), for the last part of

$$\sum_{j=0}^{n-1} \int_{x_j}^{x_{j+1}} \left[\frac{(x-s)^i}{\sin^2 \frac{x-s}{2}} - \frac{(\hat{x}_j - s)^i}{\sin^2 \frac{\hat{x}_j - s}{2}} \right] dx$$

by the identity of [1], we get

$$\begin{aligned}
\frac{(t-s)^i}{\sin^2 \frac{t-s}{2}} &= 4(\xi - c_j)^{i-2} + 4\sum_{l=1}^{\infty} \frac{(\xi - c_j)^i}{(\xi - c_j - 4l\pi/h)^2} + 4\sum_{l=1}^{\infty} \frac{(\xi - c_j)^i}{(\xi - c_j + 4l\pi/h)^2} \\
&= 4(\xi - c_j)^{i-2} + 4\sum_{l=1}^{\infty} \frac{(\xi - c_j)^{i-2}}{(1 - 2ln/(\xi - c_j))^2} + 4\sum_{l=1}^{\infty} \frac{(\xi - c_j)^{i-2}}{(1 + 2ln/(\xi - c_j))^2} \\
&= 4(\xi - c_j)^{i-2} + 4\sum_{l=1}^{\infty} \frac{(\xi - c_j)^{i-2}}{(1 - lc)^2} + 4\sum_{l=1}^{\infty} \frac{(\xi - c_j)^{i-2}}{(1 + lc)^2}
\end{aligned}$$

$$\begin{aligned}
\frac{(t_j-s)^i}{\sin^2 \frac{t_j-s}{2}} &= 4(c_j)^{i-2} + 4\sum_{l=1}^{\infty} \frac{(c_j)^i}{(c_j - 4l\pi/h)^2} + 4\sum_{l=1}^{\infty} \frac{(c_j)^i}{(c_j + 4l\pi/h)^2} \\
&= 4(c_j)^{i-2} + 4\sum_{l=1}^{\infty} \frac{(c_j)^{i-2}}{(1 - 2ln/c_j)^2} + 4\sum_{l=1}^{\infty} \frac{(c_j)^{i-2}}{(1 + 2ln/c_j)^2} \\
&= 4(c_j)^{i-2} + 4\sum_{l=1}^{\infty} \frac{(c_j)^{i-2}}{(1 - lc)^2} + 4\sum_{l=1}^{\infty} \frac{(c_j)^{i-2}}{(1 + lc)^2}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{j=0}^{n-1} \int_{x_j}^{x_{j+1}} \left[\frac{(x-s)^i}{\sin^2 \frac{x-s}{2}} - \frac{(\hat{x}_j - s)^i}{\sin^2 \frac{\hat{x}_j - s}{2}} \right] dx \\
= \;& 4\sum_{j=0}^{n-1} h^{i-1} \int_{-1}^{1} ((\xi - c_j)^{i-2} - (c_j)^{i-2}) \left[1 + \sum_{l=1}^{\infty} \left(\frac{1}{(1-lc)^2} + \frac{1}{(1+lc)^2} \right) \right] d\xi \\
= \;& Ch^{i-2}
\end{aligned}$$

where $i \geq 2$, which can be considered as the error estimate of middle rectangle rule for the integral $\int_c^{c+2\pi} \frac{(x-s)^i}{\sin^2 \frac{x-s}{2}} dx$, $i \geq 2$. Obviously, by the Theorem, it can be expanded by the Euler-Maclaurin expansions we complete the proof.

Remmark 2. *Based on the Theorem 1, assume $f(x) \in C^4[c, c+2\pi]$, we present the modify rectangle rule I*

$$\tilde{I}_n(c,s,f) = I_n(c,s,f) - \frac{4f(s)\pi^2}{h\sin^2\frac{\xi\pi}{2}}, \tag{32}$$

and modify rectangle rule II

$$\hat{I}_n(c,s,f) = \tilde{I}_n(c,s,f) + 4f'(s)\pi\tan\frac{(\xi+1)\pi}{2}, \tag{33}$$

and

$$\tilde{E}_n(c,s,f) = \int_c^{c+2\pi} \frac{f(x)}{\sin^2\frac{x-s}{2}}dx - \tilde{I}_n^2(c,s,f) \tag{34}$$

$$\hat{E}_n(c,s,f) = \int_c^{c+2\pi} \frac{f(x)}{\sin^2\frac{x-s}{2}}dx - \hat{I}_n^2(c,s,f) \tag{35}$$

then we have

Corollary 1. *Under the same assumption of Theorem 1,we have*

$$\tilde{E}_n(c,s,f) = -4f'(s)\pi\tan\frac{(\xi+1)\pi}{2} + \sum_{k=1}^{\infty} c_k h^{2k}, \tag{36}$$

and

$$\hat{E}_n(c,s,f) = \sum_{k=1}^{\infty} c_k h^{2k}, \tag{37}$$

Remmark 3. *For the Cauchy principle integral*

$$\int_c^{c+2\pi} \cot\frac{x-s}{2} f(x)dx = g(s), s \in (0, 2\pi) \tag{38}$$

with middle rectangle rule

$$\int_c^{c+2\pi} \cot\frac{x-s}{2} f(x)dx := \sum_{i=1}^{n} \omega_j(s) f(\hat{x}_j), \tag{39}$$

where $E_n(c,s,f)$ denotes the error functional and $\omega_i(s)$ is the Cote coefficients given by

$$\omega_j(s) = h\cot\frac{\hat{x}_j - s}{2}. \tag{40}$$

there holds that

$$\int_c^{c+2\pi} \cot\frac{x-s}{2} f(x)dx - \sum_{i=1}^{n} \omega_j(s) f(\hat{x}_j) = -f(s)\pi\tan\frac{\pi(\xi+1)}{2} + \sum_{k=1}^{\infty} c_k h^{2k}. \tag{41}$$

3. Numerical example

In this section, numerical examples are given to test our theoretical analysis.

Example 1. *Consider integral*

$$\int_{c}^{c+2\pi} \frac{1+2\cos(x)}{\sin^2 \frac{x-s}{2}} dx = -8\pi \cos(s), s \in (c, c+2\pi) \tag{42}$$

Table 1: Errors of the $\hat{I}_n(c,s,f)$ with $s = x_{[n/4]} + (1+\xi)h/2$

n	$\xi = 1$	$\xi = \frac{1}{2}$	$\xi = \frac{2}{3}$	$\xi = -\frac{1}{3}$
16	0	2.8422e-14	2.8422e-14	1.1369e-13
32	4.2633e-14	2.8422e-14	5.6843e-14	6.2528e-13
64	2.8422e-14	3.9790e-13	1.4211e-13	9.6634e-13
128	4.5475e-13	2.5011e-12	6.8212e-13	9.8908e-12
256	9.6634e-13	4.5475e-12	2.2737e-13	1.8872e-11
512	1.3642e-12	3.9563e-11	7.0486e-12	1.2824e-10

Table 2: Errors of $\tilde{I}_n(c,s,f)$ with $s = x_{[n/4]} + (1+\xi)h/2$

n	$\xi = 1$	$\xi = \frac{1}{2}$	$\xi = \frac{2}{3}$	$\xi = -\frac{1}{3}$
16	0	9.6179e+00	3.7556e+00	3.7699e+01
32	4.2633e-14	2.0897e+01	1.1512e+01	4.2048e+01
64	2.8422e-14	2.4051e+01	1.3740e+01	4.3159e+01
128	4.5475e-13	2.4861e+01	1.4317e+01	4.3438e+01
256	9.6634e-13	2.5065e+01	1.4462e+01	4.3508e+01
512	1.3642e-12	2.5116e+01	1.4498e+01	4.3525e+01

Table 3: Errors of $I_n(c,s,f)$ with $s = x_{[n/4]} + (1+\xi)h/2$

n	$\xi = 1$	$\xi = \frac{1}{2}$	$\xi = \frac{2}{3}$	$\xi = -\frac{1}{3}$
16	7.5398e+01	1.3353e+02	9.4492e+01	2.3876e+02
32	1.2135e+02	1.9134e+02	1.3711e+02	3.4719e+02
64	1.7747e+02	2.9374e+02	2.0647e+02	5.5026e+02
128	2.7951e+02	4.9527e+02	3.4110e+02	9.5289e+02
256	4.8095e+02	8.9751e+02	6.0934e+02	1.7573e+03
512	8.8317e+02	1.7018e+03	1.1455e+03	3.3658e+03

Table 4: Errors of $\hat{I}_n(c,s,f)$ with $s = x_{[n/6]} + (1+\xi)h/2$

n	$\xi = 1$	$\xi = \frac{1}{2}$	$\xi = \frac{2}{3}$	$\xi = -\frac{1}{3}$
24	4.2633e-14	1.1369e-13	2.8422e-14	2.4158e-13
48	1.2790e-13	1.5632e-13	4.2633e-14	1.8474e-13
96	1.4211e-13	1.1369e-13	2.8422e-14	9.8055e-13
192	2.7001e-13	8.1002e-13	4.2633e-14	1.8474e-13
384	6.8212e-13	2.6148e-12	2.8422e-14	4.1780e-12
512	6.5370e-13	2.5722e-12	9.8055e-13	4.5617e-12

Table 5: Errors of $\tilde{I}_n(c,s,f)$ with $s = x_{[n/6]} + (1+\xi)h/2$

n	$\xi = 1$	$\xi = \frac{1}{2}$	$\xi = \frac{2}{3}$	$\xi = -\frac{1}{3}$
24	4.2633e-14	2.4276e+01	1.3635e+01	4.2870e+01
48	1.2790e-13	2.4918e+01	1.4455e+01	4.0906e+01
96	1.4211e-13	2.3799e+01	1.3839e+01	3.9453e+01
192	2.7001e-13	2.2893e+01	1.3282e+01	3.8613e+01
384	6.8212e-13	2.2356e+01	1.2943e+01	3.8165e+01
512	6.5370e-13	2.2067e+01	1.2760e+01	3.7934e+01

Table 6: Errors of $I_n(c,s,f)$ with $s = x_{[n/6]} + (1+\xi)h/2$

n	$\xi = 1$	$\xi = \frac{1}{2}$	$\xi = \frac{2}{3}$	$\xi = -\frac{1}{3}$
16	7.5398e+01	9.0151e+01	7.1014e+01	1.4130e+02
32	7.5398e+01	8.6513e+01	6.8552e+01	1.3620e+02
64	7.2738e+01	8.3906e+01	6.6302e+01	1.3280e+02
128	7.0764e+01	8.2415e+01	6.4934e+01	1.3090e+02
256	6.9621e+01	8.1624e+01	6.4191e+01	1.2990e+02
512	6.9012e+01	8.1217e+01	6.3805e+01	1.2939e+02

For the case of $s = x_{[n/4]}+(1+\xi)h/2$, Table 1, Table 2 and Table 3 show the error of $\hat{I}_n(c,s,f), \tilde{I}_n(c,s,f), I_n(c,s,f)$ respectively. From table 1, When $\xi = 1$, the error is 0, and for other ξ values, the error decreases rapidly as n increases (approaching machine precision), verifying that the modified rule $\hat{I}_n$ in Theorem 1 has high-order convergence of $O(h^{\mu})$ ($\mu \geq 0$). When $\xi = 1$, $\tan\frac{(\xi+1)\pi}{2} = 0$, eliminating the second term error and resulting in an error of 0 (superconvergence).

From table 2, when $\xi = 1$, the error is 0 (the first term is corrected), but for other ξ values, the error increases with n (the second term error is not corrected). For $\xi \neq \pm 1$, the error shows $O(1)$ divergence, consistent with the remainder term $-4f'(s)\pi\tan\frac{(\xi+1)\pi}{2}$ in Corollary 1.

From table 3, the error significantly increases with n, showing $O(h^{-1})$ divergence (consistent with the error expansion of the original rule in Theorem 1). The original rule cannot handle the divergent part of the hypersingular integral, leading to uncontrolled error.

For the case of $s = x_{[n/6]}+(1+\xi)h/2$, table 4, table 5 and table 6 shows the error of $\hat{I}_n(c,s,f), \tilde{I}_n(c,s,f), I_n(c,s,f)$. The results for the singular point s located at different positions (such as $x_{[n/6]}$ and x_0) are consistent with the above trend.

When $s = x_{[n/6]} + (1+\xi)h/2$, the modified rule $\hat{I}_n$ still maintains high accuracy, while the original rule I_n shows divergent errors.

As there are no influence of boundary condition, for $s = x_0 + (1+\xi)h/2$, from table 7, table 8 and table 9 shows that errors of $\hat{I}_n(c,s,f), \tilde{I}_n(c,s,f), I_n(c,s,f)$ is similar as case of $s = x_{[n/4]} + (1+\xi)h/2$.

Table 7: Errors of $\hat{I}_n(c,s,f)$ with $s = x_0 + (1+\xi)h/2$

n	$\xi = 1$	$\xi = -1$	$\xi = -\frac{2}{3}$	$\xi = \frac{2}{3}$
24	5.1964e-15	7.1054e-14	3.5527e-14	2.1316e-14
48	2.8422e-14	2.8422e-14	2.2737e-13	1.4211e-13
96	1.4211e-13	8.2423e-13	1.7053e-13	1.4211e-13
192	2.8422e-13	2.1600e-12	1.3074e-12	3.4106e-13
384	9.0949e-13	1.0004e-11	6.1391e-12	1.7053e-12
512	1.2278e-11	5.3205e-11	3.7517e-11	3.4106e-11

Table 8: Errors of $\tilde{I}_n(c,s,f)$ with $s = x_0 + (1+\xi)h/2$

n	$\xi = 1$	$\xi = -1$	$\xi = -\frac{2}{3}$	$\xi = \frac{2}{3}$
24	2.5309e-15	7.1054e-14	2.5197e+00	1.1116e+01
48	2.8422e-14	2.8422e-14	1.2647e+00	6.1324e+00
96	1.4211e-13	8.2423e-13	6.3293e-01	3.1406e+00
192	2.8422e-13	2.1600e-12	3.1654e-01	1.5797e+00
384	9.0949e-13	1.0004e-11	1.5828e-01	7.9103e-01
512	1.2278e-11	5.3205e-11	7.9142e-02	3.9566e-01

Table 9: Errors of $I_n(c,s,f)$ with $s = x_0 + (1+\xi)h/2$

n	$\xi = 1$	$\xi = -1$	$\xi = -\frac{2}{3}$	$\xi = \frac{2}{3}$
24	1.4211e-14	3.7699e+01	4.6218e+01	2.5470e+01
48	5.5195e+01	7.5398e+01	9.8501e+01	8.7825e+01
96	1.4052e+02	1.5080e+02	2.0005e+02	1.9467e+02
192	2.9643e+02	3.0159e+02	4.0162e+02	3.9892e+02
384	6.0060e+02	6.0319e+02	8.0399e+02	8.0265e+02
512	1.2051e+03	1.2064e+03	1.6084e+03	1.6077e+03

The numerical examples support the theoretical analysis in the following ways: $\hat{I}_n(c,s,f)$ achieves spectral accuracy (errors close to machine precision) by subtracting the first two error terms. When $\xi = \pm 1$, $\tan\frac{(\xi+1)\pi}{2} = 0$, eliminating the second error term and resulting in zero error. The unmodified $I_n(c,s,f)$ diverges due to the singular term, in line with theoretical predictions. These results demonstrate that the modified trapezoidal rule can effectively handle hypersingular integrals on a circle and achieve high-precision calculations through error expansion.

Conclusion

In this paper, hypersingular integral on circle have been calculated by classical rectangle rule, the error expansion shows the two special function and $c_k h^{2k}, k = 1, 2, \cdots$ were performed. Based on the error expansion, we can construct extrapolation algorithm to accelerate the convergence rate. In fact, numerical examples shows that the convergence rate can reach spectral accuracy which is an interesting phenomenon. We will give deep explanation in future works.

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Conflicts of Interest

The authors declare that they have no conflicts of interest.

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