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Oriented diameter of the complete tripartite graph (II)

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Abstract. For a graph G, let $\mathbb{D}(G)$ denote the set of all strong orientations of G, and the oriented diameter of G is $f(G) = \min\{\mathrm{diam}(D) \mid D \in \mathbb{D}(G)\}$, which is the minimum value of the diameters $\mathrm{diam}(D)$ where $D \in \mathbb{D}(G)$. In this paper, we determine the oriented diameter of complete tripartite graphs $K(3,3,q)$ and $K(3,4,q)$, these are special cases that arise in determining the oriented diameter of $K(3,p,q)$.

1. Introduction

Let G be a finite undirected simple connected graph with vertex set $V(G)$ and edge set $E(G)$. The orientation of a graph G is a directed graph obtained from G by assigning each of its edges in G a direction. An orientation D of G is strong if for any two vertices u, v in D, there exists a directed path from u to v. For any $u, v \in V(G)$, the distance $d_G(u,v)$ denotes the length of a shortest (u,v) - path in G, which is the number of edges in a shortest path connecting u and v in G. The diamater of G is defined as $\mathrm{diam}(G) = \max\{d_G(u,v) \mid u, v \in V(G)\}$. An edge $e \in E(G)$ is called a bridge of a graph G if the subgraph obtained by deleting the edge e of the graph G is disconnected. A graph is called bridreless if it has no bridge.

Robbins' one-way street theorem [10] proves that a connected graph has a strong orientation if and only if it is bridgeless. Boesch and Tindell [1] proposed the notion $f(G)$ in order to extend Robbins' theorem [10].

Let G be a connected and bridgeless graph, and $\mathbb{D}(G)$ be the set of all strong orientations of G. Define the oriented diameter of G to be

$$f(G) = \min\{\mathrm{diam}(D) \mid D \in \mathbb{D}(G)\}.$$

For an arbitrary connected graph G, the problem of evaluting oriented diameter $f(G)$ is very difficult. In reality, Chvátal and Thomassen [2] demonstrate that the problem of deciding whether a graph admits an orientation of diameter two is NP-hard. Next, we will present some results on the oriented diameter that have been obtained in the literature.

Given any positive integers $n, p_1, p_2, \ldots, p_n$, let K_n denote the complete graph of order n, and $K(p_1, p_2, \ldots, p_n)$ denote the complete n-partite graph having p_i vertices in the i-th partite set V_i for each $i = 1, 2, \ldots, n$. Thus

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K_n is isomorphic to $K(p_1, p_2, \ldots, p_n)$ where $p_1 = p_2 = \cdots = p_n = 1$. The oriented diameter of complete graph K_n was obtained by Boesch and Tindell [1]:

$$f(K_n) = \begin{cases} 2, & \text{if } n \geq 3 \text{ and } n \neq 4; \\ 3, & \text{if } n = 4. \end{cases}$$

The oriented diameter of complete bipartite graph $K(p, q)$ was given by Šoltés [11]:

$$f(K(p,q)) = \begin{cases} 3, \text{ if } 2 \leq p \leq q \leq \binom{p}{\lfloor \frac{p}{2} \rfloor}; \\ 4, \text{ if } q > \binom{p}{\lfloor \frac{p}{2} \rfloor}; \end{cases}$$

where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. Let $n \geq 3$, Plesník [9], Gutin [3], and Koh and Tay [6] independently obtained the oriented diameter of complete n-partite graph, the specific result is as follows:

$$2 \leq f(K(p_1, p_2, \cdots, p_n)) \leq 3.$$

Let $p \geq 2$ and $n \geq 3$, Koh and Tan [4] also obtained:

$$f(K(\overbrace{p, p, \cdots p}^{n})) = 2;$$

$$f(K(\overbrace{p, p, \cdots p}^{r}, q)) = 2, (r \geqslant 3, p \geqslant 3, 1 \leqslant q \leqslant 2p).$$

They also got some other results on complete multipartite graphs.

Actually, a problem was proposed by Koh and Tan [6] : "given a complete multipartite graph $G = K(p_1, p_2, \cdots, p_n)$, classify it according to $f(G) = 2$ or $f(G) = 3$." We endeavor to classify complete multipartite graphs according to the oriented diameter 2 or 3. Koh and Tan [5] obtained:

$$f(K(2, p, q)) = 2 \text{ for } 2 \leq p \leq q \leq \binom{p}{\lfloor \frac{p}{2} \rfloor}.$$

and the authors and Rao and Zhang proved [7] that

$$f(K(2, p, q)) = 3 \text{ for } q > \binom{p}{\lfloor \frac{p}{2} \rfloor}.$$

So far, the oriented diameter of complete tripartite graph $K(2, p, q)$ is completely determined.

In this paper, we devote ourselves to deal with the case $K(3, p, q)$ for $p \geq 3$, and we first discuss special cases $K(3, 3, q)$ and $K(3, 4, q)$, the results are as follows:

$$f(K(3, 3, q)) = \begin{cases} 2, & \text{if } q \leq 6; \\ 3, & \text{if } q > 6; \end{cases}$$

$$f(K(3, 4, q)) = \begin{cases} 2, & \text{if } q \leq 11; \\ 3, & \text{if } q > 11. \end{cases}$$

In a long manuscript [8], we also have determined the oriented diameter of $K(3, p, q)$ for $p \geqslant 5$. Hence the problem posed by Koh and Tay for complete tripartite graphs $K(3, p, q)$ is completely solved.

2. Preliminaries

For a digraph D, we denote its vertex set by $V(D)$. Take any $u, v \in V(D)$, the distance $\partial_D(u,v)$ denotes the number of directed arcs in a shortest directed path from u to v in D. The diameter of D is defined as $\mathrm{diam}(D) = \max\{\partial_D(u,v) \mid u, v \in V(D)\}$. For $u, v \in V(D), U, V \subseteq V(D)$ and $U \cap V = \emptyset$, if the direction is from u to v, we write '$u \to v$'; if $a \to b$ for each $a \in U$ and each $b \in V$, we write '$U \to V$'; if $U = \{u\}$, we write '$u \to V$' for '$U \to V$'; if $V = \{v\}$, we write '$U \to v$' for '$U \to V$'. In addition, the set $N_D^+(u) = \{x \in V(D) \mid u \to x\}$, which is a collection of all out-neighbors of u; the set $N_D^-(v) = \{y \in V(D) \mid y \to v\}$, which is a collection of all in-neighbors of v.

For the complete tripartite graph $K(3,p,q)$, $p \in \{3,4\}$, let

$$V_1 = \{x_1, x_2, x_3\},$$
$$V_2 = \left\{y_1, y_2, \cdots, y_p\right\},$$
$$V_3 = \left\{z_1, z_2, \cdots, z_q\right\}.$$

be the three parts of the vertex set of $K(3,p,q)$. Let D be a strong orientation of $K(3,p,q)$. We consider the sets

$$\begin{aligned}
N_D^{+++} &= N_D^+(x_1) \cap N_D^+(x_2) \cap N_D^+(x_3),\\
N_D^{++-} &= N_D^+(x_1) \cap N_D^+(x_2) \cap N_D^-(x_3),\\
N_D^{+-+} &= N_D^+(x_1) \cap N_D^-(x_2) \cap N_D^+(x_3),\\
N_D^{-++} &= N_D^-(x_1) \cap N_D^+(x_2) \cap N_D^+(x_3),\\
N_D^{+--} &= N_D^+(x_1) \cap N_D^-(x_2) \cap N_D^-(x_3),\\
N_D^{-+-} &= N_D^-(x_1) \cap N_D^+(x_2) \cap N_D^-(x_3),\\
N_D^{--+} &= N_D^-(x_1) \cap N_D^-(x_2) \cap N_D^+(x_3),\\
N_D^{---} &= N_D^-(x_1) \cap N_D^-(x_2) \cap N_D^-(x_3),
\end{aligned}$$

For $i \in \{2,3\}$, the following eight sets

$$\begin{aligned}
V_i^{+++} &= V_i \cap N_D^{+++},\\
V_i^{++-} &= V_i \cap N_D^{++-},\\
V_i^{+-+} &= V_i \cap N_D^{+-+},\\
V_i^{-++} &= V_i \cap N_D^{-++},\\
V_i^{+--} &= V_i \cap N_D^{+--},\\
V_i^{-+-} &= V_i \cap N_D^{-+-},\\
V_i^{--+} &= V_i \cap N_D^{--+},\\
V_i^{---} &= V_i \cap N_D^{---}.
\end{aligned}$$

form a partition of V_i. For convenience, we will denote V_i^{+++}as V_i^+*and* V_i^{---}as V_i^-.

Lemma 2.1. *D is a strong orientation of G with diameter diam(D) = 2, suppose $\{i,j\} = \{2,3\}$, then the following properties hold.*
1. If $V_i^{+++} \neq \emptyset$, then $V_i^{+++} \to V_j$ and $\left|V_i^{+++}\right| = 1$; if $V_i^{---} \neq \emptyset$, then $V_j \to V_i^{---}$and $\left|V_i^{---}\right| = 1$.
2. If $V_i^{+++} \neq \emptyset$, then $V_j^{+++} = \emptyset$; if $V_i^{---} \neq \emptyset$, then $V_j^{---} = \emptyset$.

Proof. Suppose $V_i^{+++} \neq \emptyset$. Take any $y \in V_i^{+++}$and any $z \in V_j$, we have $\partial_D(y,z) \leqslant 2$. If $z \to y$, then $N_D^+(y) \subseteq V_j \backslash \{z\}$. We know $\partial_D(z', z) \geqslant 2$ for any $z' \in V_j \backslash \{z\}$, so $\partial_D(y,z) \geqslant 3$, a contradiction. Hence $y \to z$. This means $V_i^{+++} \to V_j$.

Suppose X is a strongly connected digraph. Let $u, v \in V(X)$ be two vertices of X. If $N_X^+(u) \cap N_X^-(v) = \emptyset$, then $\partial_X(u,v) \neq 2$. We assume $\partial_X(u,v) = 2$, then there exists $w \in V(X)$ such that $u \to w \to v$. So

$w \in N_X^+(u) \cap N_X^-(v) \neq \emptyset$, a contradiction. For distinct vertices $y_h, y_k \in V_i^{+++}$, we have $N_D^+(y_h) \subseteq V_j$ and $N_D^-(y_k) \subseteq V_1$. $V_1 \cap V_j = \emptyset$ implies $N_D^+(y_h) \cap N_D^-(y_k) = \emptyset$, so we get $\partial_D(y_h, y_k) \geqslant 3$, a contradiction. Thus $\left|V_i^{+++}\right| = 1$. The proof for the case $V_i^{---} \neq \emptyset$ is analogous.

Suppose $V_i^{+++} \neq \emptyset$ and $V_j^{+++} \neq \emptyset$, then $V_i^{+++} \to V_j$ and $V_j^{+++} \to V_i$, i.e., for $y \in V_i^{+++}$and $z \in V_j^{+++}$, we have $y \to z$ and $z \to y$, a contradiction. The proof for the case $V_i^{---} \neq \emptyset$ is analogous. ■

3. The oriented diameter of $K(3,3,q)$

Lemma 3.1. *For $3 \leq q \leq 6$, $f(K(3,3,q)) = 2$.*

Proof. When $q = 3$, it follows from Theorem 3 in Koh and Tan [4] (Discrete Mathematics, 1996) that $f(K(3,3,q)) = 2$.

When $q = 6$, let $V_3 = V_3^{+++} \cup V_3^{+--} \cup V_3^{-+-} \cup V_3^{---}$where $\left|V_3^{+++}\right| = \left|V_3^{---}\right| = 1$ and $\left|V_3^{+--}\right| = \left|V_3^{-+-}\right| = 2$. Orient $K(3,3,6)$ as follows : $\{y_2, y_3\} \to \{x_1, x_2\} \to y_1 \to x_3 \to \{y_2, y_3\}, V_3^{+++} \to y_1 \to V_3 \backslash V_3^{+++}$, $V_3^{+++} \to \{y_2, y_3\} \to V_3^{---}$and it is enough for $\{y_2, y_3\}$ to form a directed four-length circle with two vertices in V_3^{+--}or V_3^{-+-}. Let D_6 be the resulting orientation, it is easy to verify that $\mathrm{diam}(D_6) = 2$. On the basis of the above orientation D_6, if $V_3^{---} = \varnothing$, then we can get $f(K(3,3,5)) = 2$.

When $q = 4$, let $V_3 = V_3^{+++} \cup V_3^{++-} \cup V_3^{+-+} \cup V_3^{+--}$where $\left|V_3^{+++}\right| = \left|V_3^{++-}\right| = \left|V_3^{+-+}\right| = \left|V_3^{+--}\right| = 1$. Orient $K(3,3,4)$ as follows: $V_2 \to x_1$, $\{y_2, y_3\} \to x_2 \to y_1, \{y_1, y_3\} \to x_3 \to y_2, V_3^{+++} \cup V_3^{++-} \to y_1 \to V_3^{+-+} \cup V_3^{+--}$, $V_3^{+++} \cup V_3^{+-+} \to y_2 \to V_3^{++-} \cup V_3^{+--}$and $V_3 \to y_3$. Let D_4 be the resulting orientation, it is easy to verify that $\mathrm{diam}(D_4) = 2$. ■

Theorem 3.2. *Suppose $q \geq 3$, then*

$$f(K(3,3,q)) = \begin{cases} 2, & \text{if } q \leq 6; \\ 3, & \text{if } q > 6. \end{cases}$$

Proof. When $3 \leq q \leq 6$, we have shown in Lemma 3.1 that there exists an orientation of diameter 2 for $K(3,3,q)$. When $q > 6$, we prove it by contradiction. Assuming $f(K(3,3,q)) = 2$ when $q > 6$, then $K(3,3,q)$ has a strong orientation D with diameter $\mathrm{diam}(D) = 2$. Let $V_1 = \{x_1, x_2, x_3\}, V_2 = \{y_1, y_2, y_3\}$ and $V_3 = \left\{z_1, z_2, \cdots, z_q\right\}$ be the three parts of the vertex set of $K(3,3,q)$, and $i = \left|N_D^+(x_1) \cap V_2\right|, j = \left|N_D^+(x_2) \cap V_2\right|, k = \left|N_D^+(x_3) \cap V_2\right|$. For all $i, j, k \in \{0,1,2,3\}$, we may suppose $i \leq j \leq k$ to avoid dealing with similar cases according to different order of the vertices in V_1, so there are a total of 20 cases of (i, j, k), namely $(0,0,0)$, $(0,0,1)$, $(0,0,2)$, $(0,0,3)$, $(0,1,1)$, $(0,1,2)$, $(0,1,3)$, $(0,2,2)$, $(0,2,3)$, $(0,3,3)$, $(1,1,1)$, $(1,1,2)$, $(1,1,3)$, $(1,2,2)$, $(1,2,3)$, $(1,3,3)$, $(2,2,2)$, $(2,2,3)$, $(2,3,3)$ and $(3,3,3)$.

The case $(3,3,3)$ is the same as case $(0,0,0)$ by reversing directions of all the arcs in D.

The case $(2,3,3)$ is the same as case $(0,0,1)$ by reversing directions of all the arcs in D and interchanging vertices x_1 and x_3.

The case $(2,2,3)$ is the same as case $(0,1,1)$ by reversing directions of all the arcs in D and interchanging vertices x_1 and x_3.

The case $(2,2,2)$ is the same as case $(1,1,1)$ by reversing directions of all the arcs in D.

The case $(1,3,3)$ is the same as case $(0,0,2)$ by reversing directions of all the arcs in D and interchanging vertices x_1 and x_3.

The case $(1,2,3)$ is the same as case $(0,1,2)$ by reversing directions of all the arcs in D and interchanging vertices x_1 and x_3.

The case $(1,2,2)$ is the same as case $(1,1,2)$ by reversing directions of all the arcs in D and interchanging vertices x_1 and x_3.

The case $(1,1,3)$ is the same as case $(0,2,2)$ by reversing directions of all the arcs in D and interchanging vertices x_1 and x_3.

The case $(0,3,3)$ is the same as case $(0,0,3)$ by reversing directions of all the arcs in D and interchanging vertices x_1 and x_3.

The case $(0,2,3)$ is the same as case $(0,1,3)$ by reversing directions of all the arcs in D and interchanging vertices x_1 and x_3.

Thus, we have the following ten exhaustive cases Case(i,j,k)s to obtain the required contradiction.

(1) Case $(0,0,0)$. $V_2 \to V_1$.

Take any $x \in V_1, z \in V_3$, since $\partial_D(x,z) \le 2$, then we have $x \to z$, this means $V_1 \to V_3$. So $\partial_D(x_1,x_2) \ge 3$, a contradiction.

(2) Case $(0,0,1)$. $V_2 \to \{x_1,x_2\}, \{y_2,y_3\} \to x_3 \to y_1$.

Take any $x \in V_1 \backslash \{x_3\}$ and $z \in V_3$, since $\partial_D(x,z) \le 2$, then we have $x \to z$, this means $V_1 \backslash \{x_3\} \to V_3$. So $\partial_D(x_1,x_2) \ge 3$, a contradiction.

(3) Case $(0,0,2)$. $V_2 \to \{x_1,x_2\}, y_3 \to x_3 \to \{y_1,y_2\}$.

Take any $x \in V_1 \backslash \{x_3\}$ and $z \in V_3$, since $\partial_D(x,z) \le 2$, then we have $x \to z$, this means $V_1 \backslash \{x_3\} \to V_3$. So $\partial_D(x_1,x_2) \ge 3$, a contradiction.

(4) Case $(0,0,3)$. $V_2 \to \{x_1,x_2\}, x_3 \to V_2$.

Take any $x \in V_1 \backslash \{x_3\}$ and $z \in V_3$, since $\partial_D(x,z) \le 2$, then we have $x \to z$, this means $V_1 \backslash \{x_3\} \to V_3$. So $\partial_D(x_1,x_2) \ge 3$, a contradiction.

(5) Case $(0,1,1)$.

Subcase 1: $V_2 \to x_1, \{y_2,y_3\} \to \{x_2,x_3\} \to y_1$.

Take any $y \in V_2 \backslash \{y_1\}$ and $z \in V_3$, since $\partial_D(z,y) \le 2$, then we have $z \to y$, this means $V_3 \to V_2 \backslash \{y_1\}$. So $\partial_D(y_2,y_3) \ge 3$, a contradiction.

Subcase2: $V_2 \to x_1, \{y_2,y_3\} \to x_2 \to y_1, \{y_1,y_3\} \to x_3 \to y_2$.

We know $y_1 \in V_2^{-+-}, y_2 \in V_2^{--+}$and $y_3 \in V_2^{---}$. By Lemma 2.1, we can get $V_3^{-+-} = V_3^{--+} = V_3^{---} = \varnothing$. Take any $z \in V_3$, since $\partial_D(x_1,z) \le 2$ and $\partial_D(z,y_3) \le 2$, then we have $x_1 \to z$ and $z \to y_3$, this means $x_1 \to V_3$ and $V_3 \to y_3$. Thus $V_3 = V_3^{+++} \cup V_3^{++-} \cup V_3^{+-+} \cup V_3^{+--}$. Since $\partial_D\left(V_3^{++-}, y_i\right) \le 2$ where $i = 1,2,3,$ then we have $V_3^{++-} \to y_1$. Since $\partial_D\left(V_3^{+-+}, y_i\right) \le 2$ where $i = 1,2,3$, then we have $V_3^{+-+} \to y_2$. Since $\partial_D\left(x_3, V_3^{++-}\right) \le 2, \partial_D\left(x_2, V_3^{+-+}\right) \le 2, \partial_D\left(x_2, V_3^{+--}\right) \le 2$ and $\partial_D\left(x_3, V_3^{+--}\right) \le 2$, then we have $y_2 \to V_3^{++-}, y_1 \to V_3^{+-+}$and $\{y_1,y_2\} \to V_3^{+--}$. Let $\left|V_3^{++-}\right| = q_1, \left|V_3^{+-+}\right| = q_2$ and $\left|V_3^{+--}\right| = q_3$. If $q_1 \ge 2$, then there exists $z_i, z_j \in V_3^{++-}$such that $\partial_D\left(z_i,z_j\right) \ge 3$, a contradiction. Hence $q_1 \le 1$. If $q_2 \ge 2$, then there exists $z_i,z_j \in V_3^{+-+}$such that $\partial_D\left(z_i,z_j\right) \ge 3$, a contradiction. Hence $q_2 \le 1$. If $q_3 \ge 2$, then there exists $z_i,z_j \in V_3^{+--}$such that $\partial_D\left(z_i,z_j\right) \ge 3$, a contradiction. Hence $q_3 \le 1$. By Lemma 2.1, we also have $\left|V_3^{+++}\right| \le 1$, thus $|V_3| = q \le 1+1+1+1 = 4 \le 6$.

(6) Case $(0,1,2)$

Subcase 1: $V_2 \to x_1, \{y_2,y_3\} \to x_2 \to y_1, y_3 \to x_3 \to \{y_1,y_2\}$.

We know $y_1 \in V_2^{-++}, y_2 \in V_2^{--+}$and $y_3 \in V_2^{---}$. By Lemma 2.1, we can get $V_3^{-++} = V_3^{--+} = V_3^{---} = \varnothing$. Take any $z \in V_3$, since $\partial_D(x_1,z) \le 2$ and $\partial_D(z,y_3) \le 2$, then we have $x_1 \to z$ and $z \to y_3$, this means $x_1 \to V_3$ and $V_3 \to y_3$. Thus $V_3 = V_3^{+++} \cup V_3^{++-} \cup V_3^{+-+} \cup V_3^{+--}$. Since $\partial_D\left(V_3^{+-+}, y_i\right) \le 2$ where $i = 1,2,3$, then we have $V_3^{+-+} \to y_2$. Since $\partial_D\left(x_2, V_3^{+-+}\right) \le 2$ and $\partial_D\left(x_2, V_3^{+--}\right) \le 2$, we have $y_1 \to V_3^{+-+}$ and $y_1 \to V_3^{+--}$. Let $\left|V_3^{++-}\right| = q_1, \left|V_3^{+-+}\right| = q_2, \left|V_3^{+--}\right| = q_3$ and $F = D\left[V_2 \backslash \{y_3\} \cup V_3^{++-}\right]$. Then F is an orientation of $K(2,q_1)$ where $q_1 \le q$. If $q_1 \ge 3$, then there exists $z_i, z_j \in V_3^{++-}$such that $\partial_F\left(z_i,z_j\right) = 4$, so $\partial_D\left(z_i,z_j\right) \ge 3$, a contradiction. Hence $q_1 \le 2$. If $q_2 \ge 2$, then there exists $z_i,z_j \in V_3^{+-+}$such that $\partial_D\left(z_i,z_j\right) \ge 3$, a contradiction. Hence $q_2 \le 1$. If $q_3 \ge 2$, then there exists $z_i,z_j \in V_3^{+--}$such that $\partial_D\left(z_i,z_j\right) \ge 3$, a contradiction. Hence $q_3 \le 1$. By Lemma 2.1, we also have $\left|V_3^{+++}\right| \le 1$, thus $|V_3| = q \le 1+2+1+1 = 5 \le 6$.

Subcase 2: $V_2 \to x_1, \{y_2,y_3\} \to x_2 \to y_1 \to x_3 \to \{y_2,y_3\}$.

We know $y_1 \in V_2^{-+-}$and $y_2,y_3 \in V_2^{--+}$. By Lemma 2.1, we can get $V_3^{--+} = V_3^{-+-} = \varnothing$. Take any $z \in V_3$, since $\partial_D(x_1,z) \le 2$, then we have $x_1 \to z$, this means $x_1 \to V_3$. Thus $V_3 = V_3^{+++} \cup V_3^{++-} \cup V_3^{+-+} \cup V_3^{+--}$. Since $\partial_D\left(V_3^{++-}, y_i\right) \le 2$ where $i = 1,2,3$, then we have $V_3^{++-} \to y_1$. Since $\partial_D\left(V_3^{+-+}, y_i\right) \le 2$ where $i = 1,2,3$, then we have $V_3^{+-+} \to \{y_2,y_3\}$. Since $\partial_D\left(x_2, V_3^{+-+}\right) \le 2$ and $\partial_D\left(x_2, V_3^{+--}\right) \le 2$, then we have $y_1 \to V_3^{+-+}$and $y_1 \to V_3^{+--}$. If $V_3^{+-+} \ne \varnothing$, then $\partial_D\left(V_3^{+-+}, x_3\right) \ge 3$, a contradiction. So $V_3^{+-+} = \varnothing$. Let $\left|V_3^{++-}\right| = q_1, \left|V_3^{+--}\right| = q_2$

and $F_1 = D\left[V_2^{--+} \cup V_3^{++-}\right], F_2 = D\left[V_2^{--+} \cup V_3^{+--}\right]$. Then F_1 and F_2 are respectively an orientation of $K(2, q_1)$ and $K(2, q_2)$ where $q_1, q_2 \leq q$. If $q_1 \geq 3$, then there exists $z_i, z_j \in V_3^{++-}$such that $\partial_{F_1}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_1 \leq 2$. If $q_2 \geq 3$, then there exists $z_i, z_j \in V_3^{+--}$such that $\partial_{F_2}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_2 \leq 2$. By Lemma 2.1, we also have $\left|V_3^{+++}\right| \leq 1$, thus $|V_3| = q \leq 1 + 2 + 2 = 5 \leq 6$.
(7) Case(0,1,3). $V_2 \to x_1, \{y_2, y_3\} \to x_2 \to y_1, x_3 \to V_2$.

We know $y_1 \in V_2^{-++}$and $y_2, y_3 \in V_2^{--+}$. By Lemma 2.1, we can get $V_3^{-++} = V_3^{--+} = \varnothing$. Take any $z \in V_3$, since $\partial_D(x_1, z) \leq 2$ and $\partial_D(z, x_3) \leq 2$, then we have $x_1 \to z$ and $z \to x_3$, this means $x_1 \to V_3$ and $V_3 \to x_3$. Thus $V_3 = V_3^{++-} \cup V_3^{+--}$. Since $\partial_D\left(x_2, V_3^{+--}\right) \leq 2$, then we have $y_1 \to V_3^{+--}$. Let $\left|V_3^{++-}\right| = q_1 \left|V_3^{+--}\right| = q_2$ and $F_1 = \left[V_2 \cup V_3^{++-}\right], F_2 = \left[V_2^{--+} \cup V_3^{+--}\right]$. Then F_1 and F_2 are respectively an orientation of $K(2, q_1)$ and $K(2, q_2)$ where $q_1, q_2 \leq q$. If $q_1 \geq 4$, then there exists $z_i, z_j \in V_3^{++-}$such that $\partial_{F_1}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_1 \leq 3$. If $q_2 \geq 3$, then there exists $z_i, z_j \in V_3^{+--}$such that $\partial_{F_2}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_2 \leq 2$. Thus $|V_3| = q \leq 3 + 2 = 5 \leq 6$.
(8) Case $(0, 2, 2)$
Subcase 1: $V_2 \to x_1, y_3 \to \{x_2, x_3\} \to \{y_1, y_2\}$.

We know $y_1, y_2 \in V_2^{-++}$and $y_3 \in V_2^{---}$. By Lemma 2.1, we can get $V_3^{-++} = V_3^{---} = \varnothing$. Take any $z \in V_3$, since $\partial_D(x_1, z) \leq 2$ and $\partial_D(z, y_3) \leq 2$, then we have $x_1 \to z$ and $z \to y_3$, this means $x_1 \to V_3$ and $V_3 \to y_3$. Thus $V_3 = V_3^{+++} \cup V_3^{++-} \cup V_3^{+-+} \cup V_3^{+--}$. Let $\left|V_3^{++-}\right| = q_1, \left|V_3^{+-+}\right| = q_2, \left|V_3^{+--}\right| = q_3$ and $F_1 = D\left[V_2^{-++} \cup V_3^{++-}\right], F_2 = D\left[V_2^{-++} \cup V_3^{+-+}\right], F_3 = D\left[V_2^{-++} \cup V_3^{+--}\right]$. Then F_1, F_2 and F_3 are respectively an orientation of $K(2, q_1), K(2, q_2)$ and $K(2, q_3)$ where $q_1, q_2, q_3 \leq q$. If $q_1 \geq 3$, then there exists $z_i, z_j \in V_3^{++-}$such that $\partial_{F_1}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_1 \leq 2$. If $q_2 \geq 3$, then there exists $z_i, z_j \in V_3^{+-+}$such that $\partial_{F_2}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_2 \leq 2$. If $q_3 \geq 3$, then there exists $z_i, z_j \in V_3^{+--}$such that $\partial_{F_3}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_3 \leq 2$. Since $\partial_D\left(V_3^{++-}, V_3^{+--}\right) \leq 2$, if $q_1 = q_3 = 2$, then there exists $z_i \in V_3^{++-}, z_j \in V_3^{+--}$such that $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence we have $q_1 \leq 1$ or $q_3 \leq 1$. The argument for these two cases are similar, so we may assume $q_1 \leq 1$. By Lemma 2.1, we also have $\left|V_3^{+++}\right| \leq 1$, thus $|V_3| = q \leq 1 + 1 + 2 + 2 = 6 \leq 6$.
Subcase2: $V_2 \to x_1, y_3 \to x_2 \to \{y_1, y_2\}, y_1 \to x_3 \to \{y_2, y_3\}$.

We know $y_1 \in V_2^{-+-}, y_2 \in V_2^{-++}$and $y_3 \in V_2^{--+}$. By Lemma 2.1, we can get $V_3^{-+-} = V_3^{-++} = V_3^{--+} = \varnothing$. Take any $z \in V_3$, since $\partial_D(x_1, z) \leq 2$, then we have $x_1 \to z$, this means $x_1 \to V_3$. Thus $V_3 = V_3^{+++} \cup V_3^{++-} \cup V_3^{+-+} \cup V_3^{+--}$. Since $\partial_D\left(V_3^{++-}, y_i\right) \leq 2$ where $i = 1, 2, 3$, then we have $V_3^{++-} \to y_1$. Since $\partial_D\left(V_3^{+-+}, y_i\right) \leq 2$ where $i = 1, 2, 3$, then we have $V_3^{+-+} \to y_3$. Since $\partial_D\left(V_3^{++-}, x_2\right) \leq 2, \partial_D\left(x_3, V_3^{++-}\right) \leq 2, \partial_D\left(V_3^{+-+}, x_3\right) \leq 2$ and $\partial_D\left(x_2, V_3^{+-+}\right) \leq 2$, then we have $y_2 \to V_3^{++-} \to y_3$ and $y_2 \to V_3^{+-+} \to y_1$. Let $\left|V_3^{++-}\right| = q_1, \left|V_3^{+-+}\right| = q_2, \left|V_3^{+---}\right| = q_3$ and $F = D\left[V_2 \cup V_3^{+--}\right]$. Then F is an orientation of $K(2, q_3)$ where $q_3 \leq q$. If $q_3 \geq 4$, then there exists $z_i, z_j \in V_3^{+--}$such that $\partial_F\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_3 \leq 3$. If $q_1 \geq 2$, then there exists $z_i, z_j \in V_3^{++-}$such that $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_1 \leq 1$. If $q_2 \geq 2$, then there exists $z_i, z_j \in V_3^{+-+}$such that $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_2 \leq 1$. By Lemma 2.1, we also have $\left|V_3^{+++}\right| \leq 1$, thus $|V_3| = q \leq 1 + 1 + 1 + 3 = 6 \leq 6$.
(9) Case $(1, 1, 1)$.
Subcase1: $V_1 \to y_1, \{y_2, y_3\} \to V_1$.

Take any $y \in V_2 \backslash \{y_1\}$ and $z \in V_3$, since $\partial_D(z, y) \leq 2$, then we have $z \to y$, this means $V_3 \to \{y_2, y_3\}$. So $\partial_D(y_2, y_3) \geq 3$, a contradiction.
Subcase2: $\{y_2, y_3\} \to \{x_1, x_2\} \to y_1, \{y_1, y_3\} \to x_3 \to y_2$.

We know $y_1 \in V_2^{++-}, y_2 \in V_2^{--+}$and $y_3 \in V_2^{---}$. By Lemma 2.1, we can get $V_3^{++-} = V_3^{--+} = V_3^{---} = \varnothing$. Thus $V_3 = V_3^{+++} \cup V_3^{+-+} \cup V_3^{-++} \cup V_3^{+--} \cup V_3^{-+-}$. Take any $z \in V_3$, since $\partial_D(z, y_3) \leq 2$, then we have $z \to y_3$, this means $V_3 \to y_3$. Since $\partial_D\left(V_3^{+-+}, y_i\right) \leq 2$ where $i = 1, 2, 3$, then we have $V_3^{+-+} \to y_2$. Since $\partial_D\left(V_3^{-++}, y_i\right) \leq 2$

where $i = 1, 2, 3$, then we have $V_3^{-++} \to y_2$. Since $\partial_D\left(y_i, V_3^{+--}\right) \leq 2$ where $i = 1, 2, 3$, then we have $y_1 \to V_3^{+--}$. Since $\partial_D\left(y_i, V_3^{-+-}\right) \leq 2$ where $i = 1, 2, 3$, then we have $y_1 \to V_3^{-+-}$. Since $\partial_D\left(x_2, V_3^{+-+}\right) \leq 2, \partial_D\left(x_1, V_3^{-++}\right) \leq 2, \partial_D\left(x_3, V_3^{+--}\right) \leq 2$ and $\partial_D\left(x_3, V_3^{-+-}\right) \leq 2$, then we have $y_1 \to V_3^{+-+}, y_1 \to V_3^{-++}, y_2 \to V_3^{+--}$and $y_2 \to V_3^{-+-}$. Let $\left|V_3^{+-+}\right| = q_1, \left|V_3^{-++}\right| = q_2, \left|V_3^{+--}\right| = q_3$ and $\left|V_3^{-+-}\right| = q_4$. If $q_1 \geq 2$, then there exists $z_i, z_j \in V_3^{++-}$such that $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_1 \leq 1$. If $q_2 \geq 2$, then there exists $z_i, z_j \in V_3^{-++}$such that $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_2 \leq 1$. If $q_3 \geq 2$, then there exists $z_i, z_j \in V_3^{+--}$such that $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_3 \leq 1$. If $q_4 \geq 2$, then there exists $z_i, z_j \in V_3^{-+-}$such that $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_1 \leq 1$. By Lemma 2.1, we also have $\left|V_3^{+++}\right| \leq 1$, thus $|V_3| = q \leq 1 + 1 + 1 + 1 + 1 = 5 \leq 6$.
Subcase3: $\{y_2, y_3\} \to x_1 \to y_1, \{y_1, y_3\} \to x_2 \to y_2, \{y_1, y_2\} \to x_3 \to y_3$.

We know $y_1 \in V_2^{+--}, y_2 \in V_2^{-+-}$and $y_3 \in V_2^{--+}$. By Lemma 2.1, we can get $V_3^{+--} = V_3^{-+-} = V_3^{--+} = \emptyset$, thus $V_3 = V_3^{+++} \cup V_3^{++-} \cup V_3^{+-+} \cup V_3^{-++} \cup V_3^{---}$. Since $\partial_D\left(V_3^{++-}, y_i\right) \leq 2$ where $i = 1, 2, 3$, then we have $V_3^{++-} \to \{y_1, y_2\}$. Since $\partial_D\left(V_3^{+-+}, y_i\right) \leq 2$ where $i = 1, 2, 3$, then we have $V_3^{+-+} \to \{y_1, y_3\}$. Since $\partial_D\left(V_3^{-++}, y_i\right) \leq 2$ where $i = 1, 2, 3$, then we have $V_3^{-++} \to \{y_2, y_3\}$. Since $\partial_D\left(x_3, V_3^{++-}\right) \leq 2, \partial_D\left(x_2, V_3^{+-+}\right) \leq 2$ and $\partial_D\left(x_1, V_3^{-++}\right) \leq 2$, then we have $y_3 \to V_3^{++-}, y_2 \to V_3^{+++}$and $y_1 \to V_3^{-++}$. Let $\left|V_3^{++-}\right| = q_1, \left|V_3^{+++}\right| = q_2, \left|V_3^{-++}\right| = q_3$. If $q_1 \geq 2$, then there exists $z_i, z_j \in V_3^{++-}$such that $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_1 \leq 1$. If $q_2 \geq 2$, then there exists $z_i, z_j \in V_3^{+-+}$such that $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_2 \leq 1$. If $q_3 \geq 2$, then there exists $z_i, z_j \in V_3^{-++}$such that $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_3 \leq 1$. By Lemma 2.1, we also have $\left|V_3^{+++}\right| \leq 1$ and $\left|V_3^{---}\right| \leq 1$, thus $|V_3| = q \leq 1 + 1 + 1 + 1 + 1 = 5 \leq 6$.
(10) Case $(1, 1, 2)$.
Subcase1: $\{y_2, y_3\} \to \{x_1, x_2\} \to y_1, y_3 \to x_3 \to \{y_1, y_2\}$.

We know $y_1 \in V_2^{+++}, y_2 \in V_2^{--+}$and $y_3 \in V_2^{---}$. By Lemma 2.1, we can get $V_3^{+++} = V_3^{--+} = V_3^{---} = \emptyset$, thus $V_3 = V_3^{++-} \cup V_3^{+-+} \cup V_3^{-++} \cup V_3^{+--} \cup V_3^{-+-}$. Take any $z \in V_3$, since $\partial_D(y_1, z) \leq 2$ and $\partial_D(z, y_3) \leq 2$, we have $y_1 \to z$ and $z \to y_3$, this means $y_1 \to V_3$ and $V_3 \to y_3$. Since $\partial_D\left(V_3^{+-+}, y_i\right) \leq 2$ where $i = 1, 2, 3$, then we have $V_3^{+-+} \to y_2$. Since $\partial_D\left(V_3^{-++}, y_i\right) \leq 2$ where $i = 1, 2, 3$, then we have $V_3^{-++} \to y_2$. Let $\left|V_3^{++-}\right| = q_1, \left|V_3^{+-+}\right| = q_2, \left|V_3^{-++}\right| = q_3, \left|V_3^{+--}\right| = q_4$ and $\left|V_3^{-+-}\right| = q_5$. If $q_1 \geq 2$, then there exists $z_i, z_j \in V_3^{++-}$such that $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_1 \leq 1$. If $q_2 \geq 2$, then there exists $z_i, z_j \in V_3^{+-+}$such that $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_2 \leq 1$. If $q_3 \geq 2$, then there exists $z_i, z_j \in V_3^{-++}$such that $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_3 \leq 1$. If $q_4 \geq 2$, then there exists $z_i, z_j \in V_3^{+--}$such that $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_4 \leq 1$. If $q_5 \geq 2$, then there exists $z_i, z_j \in V_3^{-+-}$such that $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_5 \leq 1$. Thus $|V_3| = q \leq 1 + 1 + 1 + 1 + 1 = 5 \leq 6$.
Subcase2: $\{y_2, y_3\} \to \{x_1, x_2\} \to y_1 \to x_3 \to \{y_2, y_3\}$.

We know $y_1 \in V_2^{++-}$and $y_2, y_3 \in V_2^{--+}$. By Lemma 2.1, we can get $V_3^{++-} = V_3^{--+} = \emptyset$, thus $V_3 = V_3^{+++} \cup V_3^{-++} \cup V_3^{+-+} \cup V_3^{+--} \cup V_3^{-+-} \cup V_3^{---}$. Since $\partial_D\left(V_3^{+-+}, y_i\right) \leq 2$ where $i = 1, 2, 3$, then we have $V_3^{+-+} \to \{y_2, y_3\}$. Since $\partial_D\left(V_3^{-++}, y_i\right) \leq 2$ where $i = 1, 2, 3$, then we have $V_3^{-++} \to \{y_2, y_3\}$. Since $\partial_D\left(y_i, V_3^{+--}\right) \leq 2$ where $i = 1, 2, 3$, then we have $y_1 \to V_3^{+--}$. Since $\partial_D\left(y_i, V_3^{-+-}\right) \leq 2$ where $i = 1, 2, 3$, then we have $y_1 \to V_3^{-+-}$. Since $\partial_D\left(x_2, V_3^{+-+}\right) \leq 2$ and $\partial_D\left(x_1, V_3^{-++}\right) \leq 2$, we have $y_1 \to V_3^{+-+}$and $y_1 \to V_3^{-++}$. If $V_3^{+-+} \neq \emptyset$, then $\partial_D\left(V_3^{+-+}, x_3\right) \geq 3$, a contradiction. Hence $V_3^{+-+} = \emptyset$. Similarly, we have $V_3^{-++} = \emptyset$. Let $\left|V_3^{+--}\right| = q_1, \left|V_3^{-+-}\right| = q_2$ and $F_1 = D\left[V_2^{--+} \cup V_3^{+--}\right], F_1 = D\left[V_2^{--+} \cup V_3^{-+-}\right]$. Then F_1 and F_2 are respectively an orientation of $K(2, q_1)$ and $K(2, q_2)$ where $q_1, q_2 \leq q$. If $q_1 \geq 3$, then there exists $z_i, z_j \in V_3^{+--}$such that $\partial_{F_1}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_1 \leq 2$. If $q_2 \geq 3$, then there exists $z_i, z_j \in V_3^{-+-}$such that $\partial_{F_2}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_2 \leq 2$. By Lemma 2.1, we also have $\left|V_3^{+++}\right| \leq 1$

and $\left|V_3^{---}\right| \leq 1$, thus $|V_3| = q \leq 1+0+0+2+2+1 = 6 \leq 6$.
Subcase3: $\{y_2, y_3\} \to x_1 \to y_1, \{y_1, y_3\} \to x_2 \to y_2, y_3 \to x_3 \to \{y_1, y_2\}$.

We know $y_1 \in V_2^{+-+}, y_2 \in V_2^{-++}$and $y_3 \in V_2^{---}$. By Lemma 2.1, we can get $V_3^{+-+} = V_3^{-++} = V_3^{---} = \emptyset$, thus $V_3 = V_3^{+++} \cup V_3^{++-} \cup V_3^{+--} \cup V_3^{-+-} \cup V_3^{--+}$. Take any $z \in V_3$, since $\partial_D(z, y_3) \leq 2$, then we have $z \to y_3$, this means $V_3 \to y_3$. Since $\partial_D\left(y_i, V_3^{+--}\right) \leq 2$, where $i = 1,2,3$, then we have $y_1 \to V_3^{+--}$. Since $\partial_D\left(y_i, V_3^{-+-}\right) \leq 2$, where $i = 1,2,3$, then we have $y_2 \to V_3^{-+-}$. Since $\partial_D\left(y_i, V_3^{--+}\right) \leq 2$, where $i = 1,2,3$, then we have $\{y_1, y_2\} \to V_3^{--+}$. Let $\left|V_3^{++-}\right| = q_1, \left|V_3^{+--}\right| = q_2, \left|V_3^{-+-}\right| = q_3, \left|V_3^{--+}\right| = q_4$ and $F = D\left[V_2 \backslash \{y_3\} \cup V_3^{++-}\right]$. Then F is an orientation of $K(2, q_1)$ where $q_1 \leq q$. If $q_1 \geq 3$, then there exists $z_i, z_j \in V_3^{++-}$such that $\partial_F\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_1 \leq 2$. If $q_2 \geq 2$, then there exists $z_i, z_j \in V_3^{+--}$such that $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_2 \leq 1$. If $q_3 \geq 2$, then there exists $z_i, z_j \in V_3^{-+-}$such that $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_3 \leq 1$. If $q_4 \geq 2$, then there exists $z_i, z_j \in V_3^{--+}$such that $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_4 \leq 1$. By Lemma 2.1, we also have $\left|V_3^{+++}\right| \leq 1$, thus $|V_3| = q \leq 1+2+1+1+1 = 6 \leq 6$.
Subcase 4: $\{y_2, y_3\} \to x_1 \to y_1, \{y_1, y_3\} \to x_2 \to y_2, y_1 \to x_3 \to \{y_2, y_3\}$.

We know $y_1 \in V_2^{+--}, y_2 \in V_2^{-++}$and $y_3 \in V_2^{--+}$. By Lemma 2.1, we can get $V_3^{+--} = V_3^{-++} = V_3^{--+} = \emptyset$, thus $V_3 = V_3^{+++} \cup V_3^{++-} \cup V_3^{+-+} \cup V_3^{-+-} \cup V_3^{---}$. Since $\partial_D\left(V_3^{++-}, y_i\right) \leq 2$ where $i = 1,2,3$, we have $V_3^{++-} \to y_1$. Since $\partial_D\left(V_3^{+-+}, y_i\right) \leq 2$ where $i = 1,2,3$, we have $V_3^{+-+} \to \{y_1, y_3\}$. Since $\partial_D\left(y_i, V_3^{-+-}\right) \leq 2$ where $i = 1,2,3$, we have $y_2 \to V_3^{-+-}$. Since $\partial_D\left(x_2, V_3^{+-+}\right) \leq 2, \partial_D\left(x_1, V_3^{-+-}\right) \leq 2$ and $\partial_D\left(V_3^{-+-}, x_2\right) \leq 2$, then we have $y_2 \to V_3^{+-+}$and $y_1 \to V_3^{-+-} \to y_3$. Let $\left|V_3^{++-}\right| = q_1, \left|V_3^{+-+}\right| = q_2, \left|V_3^{-+-}\right| = q_3$ and $F = D\left[V_2 \backslash \{y_1\} \cup V_3^{++-}\right]$. Then F is an orientation of $K(2, q_1)$ where $q_1 \leq q$. If $q_1 \geq 3$, then there exists $z_i, z_j \in V_3^{++-}$such that $\partial_F\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_1 \leq 2$. If $q_2 \geq 2$, then there exists $z_i, z_j \in V_3^{+-+}$such that $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_2 \leq 1$. If $q_3 \geq 2$, then there exists $z_i, z_j \in V_3^{-+-}$such that $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_3 \leq 1$. By Lemma 2.1, we also have $\left|V_3^{+++}\right| \leq 1$ and $\left|V_3^{---}\right| \leq 1$, thus $|V_3| = q \leq 1+2+1+1+1 = 6 \leq 6$.

In summary, it can be concluded that if $f(K(3,3,q)) = 2$, then $q \leq 6$. Since when $q \leq 6$, we have found an orientation of diameter 2 of $K(3,3,q)$ in Lemma 3.1. Therefore, $f(K(3,3,q)) = 2$ if and only if $q \leq 6$. ■

4. The oriented diameter of $K(3,4,q)$

Lemma 4.1. *For* $4 \leq q \leq 11$, $f(K(3,4,q)) = 2$.

Proof. When $q = 11$, let $V_3 = V_3^{+++} \cup V_3^{++-} \cup V_3^{+-+} \cup V_3^{+--}$where $\left|V_3^{+++}\right| = 1$, $\left|V_3^{++-}\right| = \left|V_3^{+-+}\right| = 2$ and $\left|V_3^{+--}\right| = 6$. Orient $K(3,4,11)$ as follows: $V_2 \to x_1, y_4 \to x_2 \to V_2 \backslash \{y_4\}, y_1 \to x_3 \to V_2 \backslash \{y_1\}, V_3^{+++} \to V_2, V_3^{++-} \to \{y_1, y_4\}, V_3^{+-+} \to \{y_1, y_4\}$. For the set $\{y_2, y_3\}$, let it form a directed cycle of length four with the vertices in $V_3^{++-}$and V_3^{+-+}respectively. The orientation between V_2 and $V_3^{+--}$is the same as the orientation of $K(4,6)$. Let D_{11} be the resulting orientation, it is easy to verify that $\mathrm{diam}(D_{11}) = 2$.

When $q = 10$, let $V_3 = V_3^{+++} \cup V_3^{+-+} \cup V_3^{-++} \cup V_3^{+--} \cup V_3^{-+-} \cup V_3^{---}$where $\left|V_3^{+++}\right| = \left|V_3^{---}\right| = 1, \left|V_3^{+-+}\right| = \left|V_3^{-++}\right| = \left|V_3^{+--}\right| = \left|V_3^{-+-}\right| = 2$ and $V_3^{+++} = \{z_+\}, V_3^{---} = \{z_-\}, V_3^{+-+} = \{z_1, z_2\}, V_3^{-++} = \{z_3, z_4\}, V_3^{+--} = \{z_5, z_6\}, V_3^{-+-} = \{z_7, z_8\}$. Orient $K(3,4,10)$ as follows : $\{y_3, y_4\} \to \{x_1, x_2\} \to \{y_1, y_2\} \to x_3 \to \{y_3, y_4\}$, $V_3^{+++} \to V_2 \to V_3^{---}, \{y_1, y_2\} \to V_3^{+--} \cup V_3^{-+-}, V_3^{+-+} \cup V_3^{-++} \to \{y_3, y_4\}, y_1 \to z_1 \to y_2 \to z_2 \to y_1, y_1 \to z_3 \to y_2 \to z_4 \to y_1, y_3 \to z_5 \to y_4 \to z_6 \to y_3, y_3 \to z_7 \to y_4 \to z_8 \to y_3$. Let D_{10} be the resulting orientation, it is easy to verify that $\mathrm{diam}(D_{10}) = 2$.

Let D_9 be the orientation obtained by deleting vertex $z_-$from the above orientation D_{10}, it is easy to verify $\mathrm{diam}(D_9) = 2$, so $f(K(3,4,9)) = 2$.

Let D_8 be the orientation obtained by deleting vertex $z_-$and z_+ from the above orientation D_{10}, it is easy to verify $\mathrm{diam}(D_8) = 2$, so $f(K(3,4,8)) = 2$.

Let D_7 be the orientation obtained by deleting vertex set $\{z_7, z_8\}$ and vertex z_- from the above orientation D_{10}, it is easy to verify diam $(D_7) = 2$, so $f(K(3,4,7)) = 2$.

Let D_6 be the orientation obtained by deleting vertex set $\{z_7, z_8\}$ and vertex z_+, z_- from the above orientation D_{10}, it is easy to verify diam $(D_6) = 2$, so $f(K(3,4,6)) = 2$.

Let D_5 be the orientation obtained by deleting vertex set $\{z_1, z_2\}, \{z_7, z_8\}$ and vertex z_- from the above orientation D_{10}, it is easy to verify diam $(D_5) = 2$, so $f(K(3,4,5)) = 2$.

Let D_4 be the orientation obtained by deleting vertex set $\{z_1, z_2\}, \{z_7, z_8\}$ and vertex z_+, z_- from the above orientation D_{10}, it is easy to verify diam $(D_4) = 2$, so $f(K(3,4,4)) = 2$. ■

Theorem 4.2. *Suppose $q \geq 4$, then*

$$f(K(3,4,q)) = \begin{cases} 2, & \text{if } q \leq 11; \\ 3, & \text{if } q > 11. \end{cases}$$

Proof. When $4 \leq q \leq 11$, we have shown in Lemma 4.1 that there exists an orientation of diameter 2 for $K(3,4,q)$. When $q > 11$, we prove it by contradiction. Assuming $f(K(3,4,q)) = 2$ when $q > 11$, then $K(3,4,q)$ has a strong orientation D with diameter diam$(D) = 2$. Let $V_1 = \{x_1, x_2, x_3\}, V_2 = \{y_1, y_2, y_3, y_4\}$ and $V_3 = \left\{z_1, z_2, \cdots z_q\right\}$ be the three parts of the vertex set of $K(3,4,q)$, and $i = \left|N_D^+(x_1) \cap V_2\right|, j = \left|N_D^+(x_2) \cap V_2\right|, k = \left|N_D^+(x_3) \cap V_2\right|$. For all $i, j, k \in \{0,1,2,3,4\}$, we may suppose $i \leq j \leq k$ to avoid dealing with similar cases according to different order of the vertices in V_1, so there are a total of 35 cases of (i, j, k), namely $(0,0,0)$, $(0,0,1)$, $(0,0,2)$, $(0,0,3)$, $(0,0,4)$, $(0,1,1)$, $(0,1,2)$, $(0,1,3)$, $(0,1,4)$ $(0,2,2)$, $(0,2,3)$, $(0,2,4)$, $(0,3,3)$, $(0,3,4)$, $(0,4,4)$, $(1,1,1)$, $(1,1,2)$, $(1,1,3)$, $(1,1,4)$, $(1,2,2)$, $(1,2,3)$, $(1,2,4)$, $(1,3,3)$, $(1,3,4)$, $(1,4,4)$, $(2,2,2)$, $(2,2,3)$, $(2,2,4)$, $(2,3,3)$, $(2,3,4)$, $(2,4,4)$, $(3,3,3)$, $(3,3,4)$, $(3,4,4)$ and $(4,4,4)$.

The case $(4,4,4)$ is the same as case $(0,0,0)$ by reversing directions of all the arcs in D.

The case $(3,4,4)$ is the same as case $(0,0,1)$ by reversing directions of all the arcs in D and interchanging vertices x_1 and x_3.

The case $(3,3,4)$ is the same as case $(0,1,1)$ by reversing directions of all the arcs in D and interchanging vertices x_1 and x_3.

The case $(3,3,3)$ is the same as case $(1,1,1)$ by reversing directions of all the arcs in D.

The case $(2,4,4)$ is the same as case $(0,0,2)$ by reversing directions of all the arcs in D and interchanging vertices x_1 and x_3.

The case $(2,3,4)$ is the same as case $(0,1,2)$ by reversing directions of all the arcs in D and interchanging vertices x_1 and x_3.

The case $(2,3,3)$ is the same as case $(1,1,2)$ by reversing directions of all the arcs in D and interchanging vertices x_1 and x_3.

The case $(2,2,4)$ is the same as case $(0,2,2)$ by reversing directions of all the arcs in D and interchanging vertices x_1 and x_3.

The case $(2,2,3)$ is the same as case $(1,2,2)$ by reversing directions of all the arcs in D and interchanging vertices x_1 and x_3.

The case $(1,4,4)$ is the same as case $(0,0,3)$ by reversing directions of all the arcs in D and interchanging vertices x_1 and x_3.

The case $(1,3,4)$ is the same as case $(0,1,3)$ by reversing directions of all the arcs in D and interchanging vertices x_1 and x_3.

The case $(1,3,3)$ is the same as case $(1,1,3)$ by reversing directions of all the arcs in D and interchanging vertices x_1 and x_3.

The case $(1,2,4)$ is the same as case $(0,2,3)$ by reversing directions of all the arcs in D and interchanging vertices x_1 and x_3.

The case $(1,1,4)$ is the same as case $(0,3,3)$ by reversing directions of all the arcs in D and interchanging vertices x_1 and x_3.

The case $(0,4,4)$ is the same as case $(0,0,4)$ by reversing directions of all the arcs in D and interchanging vertices x_1 and x_3.

The case $(0,3,4)$ is the same as case $(0,1,4)$ by reversing directions of all the arcs in D and interchanging vertices x_1 and x_3.

Thus, we have the following nineteen exhaustive cases Case(i,j,k)s to obtain the required contradiction.
(1) Case $(0,0,0)$. $V_2 \to V_1$.

Take any $x \in V_1, z \in V_3$, since $\partial_D(x,z) \le 2$, then we have $x \to z$, this means $V_1 \to V_3$. So $\partial_D(x_1,x_2) \ge 3$, a contradiction.
(2) Case $(0,0,1)$. $V_2 \to \{x_1,x_2\}, V_2\backslash\{y_1\} \to x_3 \to y_1$.

Take any $x \in V_1\backslash\{x_3\}$ and $z \in V_3$, since $\partial_D(x,z) \le 2$, then we have $x \to z$, this means $V_1\backslash\{x_3\} \to V_3$. So $\partial_D(x_1,x_2) \ge 3$, a contradiction.
(3) Case($0,0,2$). $V_2 \to \{x_1,x_2\}, V_2\backslash\{y_1,y_2\} \to x_3 \to \{y_1,y_2\}$.

Take any $x \in V_1\backslash\{x_3\}$ and $z \in V_3$, since $\partial_D(x,z) \le 2$, then we have $x \to z$, this means $V_1\backslash\{x_3\} \to V_3$. So $\partial_D(x_1,x_2) \ge 3$, a contradiction.
(4) Case($0,0,3$). $V_2 \to \{x_1,x_2\}, y_4 \to x_3 \to V_2\backslash\{y_4\}$.

Take any $x \in V_1\backslash\{x_3\}$ and $z \in V_3$, since $\partial_D(x,z) \le 2$, then we have $x \to z$, this means $V_1\backslash\{x_3\} \to V_3$. So $\partial_D(x_1,x_2) \ge 3$, a contradiction.
(5) Case($0,0,4$). $V_2 \to \{x_1,x_2\}, x_3 \to V_2$.

Take any $x \in V_1\backslash\{x_3\}$ and $z \in V_3$, since $\partial_D(x,z) \le 2$, then we have $x \to z$, this means $V_1\backslash\{x_3\} \to V_3$. So $\partial_D(x_1,x_2) \ge 3$, a contradiction.
(6) Case $(0,1,1)$.
Subcase 1: $V_2 \to x_1, V_2\backslash\{y_1\} \to \{x_2,x_3\} \to y_1$.

Take any $y \in V_2\backslash\{y_1\}$ and $z \in V_3$, since $\partial_D(z,y) \le 2$, then we have $z \to y$, this means $V_3 \to V_2\backslash\{y_1\}$. So $\partial_D(y_2,y_3) \ge 3$, a contradiction.
Subcase 2: $V_2 \to x_1, V_2\backslash\{y_1\} \to x_2 \to y_1, V_2\backslash\{y_2\} \to x_3 \to y_2$.

Take any $y \in V_2\backslash\{y_1,y_2\}$ and $z \in V_3$, since $\partial_D(z,y) \le 2$, then we have $z \to y$, this means $V_3 \to V_2\backslash\{y_1,y_2\}$. So $\partial_D(y_3,y_4) \ge 3$, a contradiction.
(7) Case $(0,1,2)$.
Subcasse 1: $V_2 \to x_1, V_2\backslash\{y_1\} \to x_2 \to y_1, V_2\backslash\{y_1,y_2\} \to x_3 \to \{y_1,y_2\}$.

Take any $y \in V_2\backslash\{y_1,y_2\}$ and $z \in V_3$, since $\partial_D(z,y) \le 2$, then we have $z \to y$, this means $V_3 \to V_2\backslash\{y_1,y_2\}$. So $\partial_D(y_3,y_4) \ge 3$, a contradiction.
Subcase 2: $V_2 \to x_1, V_2\backslash\{y_1\} \to x_2 \to y_1, V_2\backslash\{y_2,y_3\} \to x_3 \to \{y_2,y_3\}$.

We know $y_1 \in V_2^{-+-}, y_4 \in V_2^{---}$and $y_2,y_3 \in V_2^{--+}$. By Lemma 2.1, we can get $V_3^{-+-} = V_3^{--+} = V_3^{---} = \emptyset$. Take any $z \in V_3$, since $\partial_D(x_1,z) \le 2$ and $\partial_D(z,y_4) \le 2$, then we have $x_1 \to z$ and $z \to y_4$, this means $x_1 \to V_3$ and $V_3 \to y_4$. Thus $V_3 = V_3^{+++} \cup V_3^{++-} \cup V_3^{+-+} \cup V_3^{+--}$. Since $\partial_D\left(V_3^{++-},y_i\right) \le 2$ where $i = 1,2,3$, then we have $V_3^{++-} \to y_1$. Since $\partial_D\left(V_3^{+-+},y_i\right) \le 2$ where $i = 1,2,3$, then we have $V_3^{+-+} \to \{y_2,y_3\}$. Since $\partial_D\left(x_2,V_3^{+-+}\right) \le 2$, then we have $y_1 \to V_3^{+-+}$. Since $\partial_D\left(x_2,V_3^{+--}\right) \le 2$, then we have $y_1 \to V_3^{+--}$. Let $\left|V_3^{++-}\right| = q_1, \left|V_3^{+-+}\right| = q_2, \left|V_3^{+--}\right| = q_3$ and $F_1 = D\left[V_2^{--+} \cup V_3^{++-}\right], F_2 = D\left[V_2^{--+} \cup V_3^{+--}\right]$. Then F_1 and F_2 are respectively an orientation of $K(2,q_1)$ and $K(2,q_3)$ where $q_1,q_3 \le q$. If $q_1 \ge 3$, then there exists $z_i,z_j \in V_3^{++-}$such that $\partial_{F_1}\left(z_i,z_j\right) = 4$, so $\partial_D\left(z_i,z_j\right) \ge 3$, a contradiction. Hence $q_1 \le 2$. If $q_3 \ge 3$, then there exists $z_i,z_j \in V_3^{+--}$such that $\partial_{F_2}\left(z_i,z_j\right) = 4$, so $\partial_D\left(z_i,z_j\right) \ge 3$, a contradiction. Hence $q_3 \le 2$. If $q_2 \ge 2$, then there exists $z_i,z_j \in V_3^{+-+}$such that $\partial_D\left(z_i,z_j\right) \ge 3$, a contradiction. Hence $q_2 \le 1$. By Lemma 2.1, we also have $\left|V_3^{+++}\right| \le 1$, thus $|V_3| = q \le 1+2+1+2 = 6 \le 11$.
(8) Case $(0,1,3)$.
Subcase 1: $V_2 \to x_1, V_2\backslash\{y_1\} \to x_2 \to y_1, y_4 \to x_3 \to V_2\backslash\{y_4\}$.

We know $y_1 \in V_2^{-++}, y_4 \in V_2^{---}$and $y_2,y_3 \in V_2^{--+}$. By Lemma 2.1, we can get $V_3^{-++} = V_3^{--+} = V_3^{---} = \emptyset$. Take any $z \in V_3$, since $\partial_D(x_1,z) \le 2$ and $\partial_D(z,y_4) \le 2$, then we have $x_1 \to z$ and $z \to y_4$, this means $x_1 \to V_3$ and $V_3 \to y_4$. Thus $V_3 = V_3^{+++} \cup V_3^{++-} \cup V_3^{+-+} \cup V_3^{+--}$. Since $\partial_D\left(V_3^{+-+},y_i\right) \le 2$ where $i = 1,2,3$, then we have $V_3^{+-+} \to \{y_2,y_3\}$. Since $\partial_D\left(x_2,V_3^{+-+}\right) \le 2$, then we have $y_1 \to V_3^{+-+}$. Since $\partial_D\left(x_2,V_3^{+--}\right) \le 2$, then we have $y_1 \to V_3^{+--}$. Let $\left|V_3^{++-}\right| = q_1, \left|V_3^{+-+}\right| = q_2, \left|V_3^{+--}\right| = q_3$ and $F_1 = D\left[V_2\backslash\{y_4\} \cup V_3^{++-}\right], F_2 = D\left[V_2^{--+} \cup V_3^{+--}\right]$.

Then F_1 and F_2 are respectively an orientation of $K(3, q_1)$ and $K(2, q_3)$ where $q_1, q_3 \le q$. If $q_1 \ge 4$, then there exists $z_i, z_j \in V_3^{++-}$such that $\partial_{F_1}(z_i, z_j) = 4$, so $\partial_D(z_i, z_j) \ge 3$, a contradiction. Hence $q_1 \le 3$. If $q_2 \ge 2$, then there exists $z_i, z_j \in V_3^{+-+}$such that $\partial_D(z_i, z_j) \ge 3$, a contradiction. Hence $q_2 \le 1$. If $q_3 \ge 3$, then there exists $z_i, z_j \in V_3^{+--}$such that $\partial_{F_2}(z_i, z_j) = 4$, so $\partial_D(z_i, z_j) \ge 3$, a contradiction. Hence $q_3 \le 2$. By Lemma 2.1, we also have $|V_3^{+++}| \le 1$, thus $|V_3| = q \le 1 + 3 + 1 + 2 = 7 \le 11$.
Subcase 2: $V_2 \to x_1, V_2 \backslash \{y_1\} \to x_2 \to y_1 \to x_3 \to V_2 \backslash \{y_1\}$.

We know $y_1 \in V_2^{-+-}$and $y_2, y_3, y_4 \in V_2^{--+}$. By Lemma 2.1, we can get $V_3^{-+-} = V_3^{--+} = \emptyset$. Take any $z \in V_3$, since $\partial_D(x_1, z) \le 2$, then we have $x_1 \to z$, this means $x_1 \to V_3$. Thus $V_3 = V_3^{+++} \cup V_3^{++-} \cup V_3^{+-+} \cup V_3^{+--}$. Since $\partial_D(V_3^{++-}, y_i) \le 2$ where $i = 1, 2, 3$, then we have $V_3^{++-} \to y_1$. Since $\partial_D(V_3^{+-+}, y_i) \le 2$ where $i = 1, 2, 3$, then we have $V_3^{+-+} \to \{y_2, y_3, y_4\}$. Since $\partial_D(x_2, V_3^{+-+}) \le 2$, then we have $y_1 \to V_3^{+-+}$. Since $\partial_D(x_2, V_3^{+--}) \le 2$,then we have $y_1 \to V_3^{+--}$. If $V_3^{+-+} \ne \emptyset$, then $\partial_D(V_3^{+-+}, x_3) \ge 3$, a contradiction, so $V_3^{+-+} = \emptyset$. Let $|V_3^{++-}| = q_1, |V_3^{+--}| = q_2$ and $F_1 = D[V_2^{--+} \cup V_3^{++-}], F_2 = D[V_2^{--+} \cup V_3^{+--}]$. Then F_1 and F_2 are respectively an orientation of $K(3, q_1)$ and $K(3, q_2)$ where $q_1, q_2 \le q$. If $q_1 \ge 4$, then there exists $z_i, z_j \in V_3^{++-}$such that $\partial_{F_1}(z_i, z_j) = 4$, so $\partial_D(z_i, z_j) \ge 3$, a contradiction. Hence $q_1 \le 3$. If $q_2 \ge 4$, then there exists $z_i, z_j \in V_3^{+--}$such that $\partial_{F_2}(z_i, z_j) = 4$, so $\partial_D(z_i, z_j) \ge 3$, a contradiction. Hence $q_2 \le 3$. By Lemma 2.1, we also have $|V_3^{+++}| \le 1$, thus $|V_3| = q \le 1 + 3 + 0 + 3 = 7 \le 11$.

(9) Case(0,1,4). $V_2 \to x_1, V_2 \backslash \{y_1\} \to x_2 \to y_1, x_3 \to V_2$.

We know $y_1 \in V_2^{-++}$and $y_2, y_3, y_4 \in V_2^{--+}$. By Lemma 2.1, we can get $V_3^{-++} = V_3^{--+} = \emptyset$. Take any $z \in V_3$, since $\partial_D(x_1, z) \le 2$ and $\partial_D(z, x_3) \le 2$, then we have $x_1 \to z$ and $z \to x_3$, this means $x_1 \to V_3$ and $V_3 \to x_3$. Thus $V_3 = V_3^{++-} \cup V_3^{+--}$. Since $\partial_D(x_2, V_3^{+--}) \le 2$, then we have $y_1 \to V_3^{+--}$. Let $|V_3^{++-}| = q_1, |V_3^{+--}| = q_2$ and $F_1 = D[V_2 \cup V_3^{++-}], F_2 = D[V_2^{--+} \cup V_3^{+--}]$. Then F_1 and F_2 are respectively an orientation of $K(4, q_1)$ and $K(3, q_2)$ where $q_1, q_2 \le q$. If $q_1 > \binom{4}{4/2} = 6$, then there exists $z_i, z_j \in V_3^{++-}$such that $\partial_{F_1}(z_i, z_j) = 4$, so $\partial_D(z_i, z_j) \ge 3$, a contradiction. Hence $q_1 \le 6$. If $q_2 \ge 4$, then there exists $z_i, z_j \in V_3^{+--}$such that $\partial_{F_2}(z_i, z_j) = 4$, so $\partial_D(z_i, z_j) \ge 3$, a contradiction. Hence $q_2 \le 3$. Thus $|V_3| = q \le 6 + 3 = 9 \le 11$.

(10) Case $(0, 2, 2)$.
Subcase 1: $V_2 \to x_1, V_2 \backslash \{y_1, y_2\} \to \{x_2, x_3\} \to \{y_1, y_2\}$.

We know $y_1, y_2 \in V_2^{-++}$and $y_3, y_4 \in V_2^{---}$. By Lemma 2.1, we can get $V_3^{-++} = V_3^{---} = \emptyset$. Take any $z \in V_3$, since $\partial_D(x_1, z) \le 2, \partial_D(z, y_3) \le 2$ and $\partial_D(z, y_4) \le 2$, then we have $x_1 \to z, z \to y_3$ and $z \to y_4$, this means $x_1 \to V_3$ and $V_3 \to \{y_3, y_4\}$. Thus $V_3 = V_3^{+++} \cup V_3^{++-} \cup V_3^{+-+} \cup V_3^{+--}$. Let $|V_3^{++-}| = q_1, |V_3^{+-+}| = q_2, |V_3^{+--}| = q_3$ and $F_1 = D[V_2^{-++} \cup V_3^{++-}], F_2 = D[V_2^{-++} \cup V_3^{+-+}], F_3 = D[V_2^{-++} \cup V_3^{+--}]$. Then F_1, F_2 and are respectively an orientation of $K(2, q_1), K(2, q_2)$ and $K(2, q_3)$ where $q_1, q_2, q_3 \le q$. If $q_1 \ge 3$, then there exists $z_i, z_j \in V_3^{++-}$such that $\partial_{F_1}(z_i, z_j) = 4$, so $\partial_D(z_i, z_j) \ge 3$, a contradiction. Hence $q_1 \le 2$. If $q_2 \ge 3$, then there exists $z_i, z_j \in V_3^{+-+}$such that $\partial_{F_2}(z_i, z_j) = 4$, so $\partial_D(z_i, z_j) \ge 3$, a contradiction. Hence $q_2 \le 2$. If $q_3 \ge 3$, then there exists $z_i, z_j \in V_3^{+--}$such that $\partial_{F_3}(z_i, z_j) = 4$, so $\partial_D(z_i, z_j) \ge 3$, a contradiction. Hence $q_3 \le 2$. By Lemma 2.1, we also have $|V_3^{+++}| \le 1$, thus $|V_3| = q \le 1 + 2 + 2 + 2 = 7 \le 11$.
Subcase 2: $V_2 \to x_1, V_2 \backslash \{y_1, y_2\} \to x_2 \to \{y_1, y_2\}, V_2 \backslash \{y_2, y_3\} \to x_3 \to \{y_2, y_3\}$.

We know $y_1 \in V_2^{-+-}, y_2 \in V_2^{-++}, y_3 \in V_2^{--+}$and $y_4 \in V_2^{---}$. By Lemma 2.1, we can get $V_3^{-+-} = V_3^{-++} = V_3^{--+} = V_3^{---} = \emptyset$. Take any $z \in V_3$, since $\partial_D(x_1, z) \le 2$ and $\partial_D(z, y_4) \le 2$, then we have $x_1 \to z$ and $z \to y_4$, this means $x_1 \to V_3$ and $V_3 \to y_4$. Thus $V_3 = V_3^{+++} \cup V_3^{++-} \cup V_3^{+-+} \cup V_3^{+--}$. Since $\partial_D(V_3^{++-}, y_i) \le 2$ where $i = 1, 2, 3$, then we have $V_3^{++-} \to y_1$. Since $\partial_D(V_3^{+-+}, y_i) \le 2$ where $i = 1, 2, 3$, then we have $V_3^{+-+} \to y_3$. Let $|V_3^{++-}| = q_1, |V_3^{+-+}| = q_2, |V_3^{+--}| = q_3$ and $F_1 = D[V_2 \backslash \{y_1, y_4\} \cup V_3^{++-}], F_2 = D[V_2 \backslash \{y_3, y_4\} \cup V_3^{+-+}], F_3 = D[V_2 \backslash \{y_4\} \cup V_3^{+--}]$. Then F_1, F_2 and F_3 are respectively an orientation of $K(2, q_1), K(2, q_2)$ and $K(3, q_3)$ where $q_1, q_2, q_3 \le q$. If $q_1 \ge 3$, then there exists $z_i, z_j \in V_3^{++-}$such that $\partial_{F_1}(z_i, z_j) = 4$, so $\partial_D(z_i, z_j) \ge 3$, a contradiction.

Hence $q_1 \le 2$. If $q_2 \ge 3$, then there exists $z_i, z_j \in V_3^{+-+}$such that $\partial_{F_2}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \ge 3$, a contradiction. Hence $q_2 \le 2$. If $q_3 \ge 4$, then there exists $z_i, z_j \in V_3^{+--}$such that $\partial_{F_3}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \ge 3$, a contradiction. Hence $q_3 \le 3$. By Lemma 2.1, we also have $\left|V_3^{+++}\right| \le 1$, thus $|V_3| = q \le 1 + 2 + 2 + 3 = 8 \le 11$.
Subcase 3: $V_2 \to x_1, V_2 \backslash \{y_1, y_2\} \to x_2 \to \{y_1, y_2\}, V_2 \backslash \{y_3, y_4\} \to x_3 \to \{y_3, y_4\}$.

We know $y_1, y_2 \in V_2^{-+-}$and $y_3, y_4 \in V_2^{--+}$. By Lemma 2.1, we can get $V_3^{-+-} = V_3^{--+} = \emptyset$. Take any $z \in V_3$, since $\partial_D(x_1, z) \le 2$, then we have $x_1 \to z$, this means $x_1 \to V_3$. Thus $V_3 = V_3^{+++} \cup V_3^{++-} \cup V_3^{+-+} \cup V_3^{+--}$. Since $\partial_D\left(V_3^{++-}, y_i\right) \le 2$ where $i = 1, 2, 3$, then we have $V_3^{++-} \to \{y_1, y_2\}$. Since $\partial_D\left(V_3^{+-+}, y_i\right) \le 2$ where $i = 1, 2, 3$, then we have $V_3^{+-+} \to \{y_3, y_4\}$. Let $\left|V_3^{++-}\right| = q_1, \left|V_3^{+-+}\right| = q_2, \left|V_3^{+--}\right| = q_3$ and $F_1 = D\left[V_2^{--+} \cup V_3^{++-}\right], F_2 = D\left[V_2^{-+-} \cup V_3^{+-+}\right], F_3 = D\left[V_2 \cup V_3^{+--}\right]$. Then F_1, F_2 and F_3 are respectively an orientation of $K(2, q_1), K(2, q_2)$ and $K(4, q_3)$ where $q_1, q_2, q_3 \le q$. If $q_1 \ge 3$, then there exists $z_i, z_j \in V_3^{++-}$such that $\partial_{F_1}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \ge 3$, a contradiction. Hence $q_1 \le 2$. If $q_2 \ge 3$, then there exists $z_i, z_j \in V_3^{+-+}$such that $\partial_{F_2}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \ge 3$, a contradiction. Hence $q_2 \le 2$. If $q_3 > \binom{4}{4/2} = 6$, then there exists $z_i, z_j \in V_3^{+--}$such that $\partial_{F_3}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \ge 3$, a contradiction. Hence $q_3 \le 6$. By Lemma 2.1, we also have $\left|V_3^{+++}\right| \le 1$, thus $|V_3| = q \le 1 + 2 + 2 + 6 = 11 \le 11$.
(11) Case $(0, 2, 3)$.
Subcase 1: $V_2 \to x_1, V_2 \backslash \{y_1, y_2\} \to x_2 \to \{y_1, y_2\}, y_4 \to x_3 \to V_2 \backslash \{y_4\}$.

We know $y_1, y_2 \in V_2^{-++}, y_3 \in V_2^{--+}$and $y_4 \in V_2^{---}$. By Lemma 2.1, we can get $V_3^{-++} = V_3^{--+} = V_3^{---} = \emptyset$. Take any $z \in V_3$, since $\partial_D(x_1, z) \le 2$ and $\partial_D(z, y_4) \le 2$, then we have $x_1 \to z$ and $z \to y_4$, this means $x_1 \to V_3$ and $V_3 \to y_4$. Thus $V_3 = V_3^{+++} \cup V_3^{++-} \cup V_3^{+-+} \cup V_3^{+--}$. Since $\partial_D\left(V_3^{+-+}, y_i\right) \le 2$ where $i = 1, 2, 3$, then we have $V_3^{+-+} \to y_3$. Let $\left|V_3^{++-}\right| = q_1, \left|V_3^{+-+}\right| = q_2, \left|V_3^{+--}\right| = q_3$ and $F_1 = D\left[V_2 \backslash \{y_4\} \cup V_3^{++-}\right], F_2 = D\left[V_2^{-++} \cup V_3^{+-+}\right], F_3 = D\left[V_2 \backslash \{y_4\} \cup V_3^{+--}\right]$. Then F_1, F_2 and F_3 are respectively an orientation of $K(3, q_1), K(2, q_2)$ and $K(3, q_3)$ where $q_1, q_2, q_3 \le q$. If $q_1 \ge 4$, then there exists $z_i, z_j \in V_3^{++-}$such that $\partial_{F_1}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \ge 3$, a contradiction. Hence $q_1 \le 3$. If $q_2 \ge 3$, then there exists $z_i, z_j \in V_3^{+-+}$such that $\partial_{F_2}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \ge 3$, a contradiction. Hence $q_2 \le 2$. If $q_3 \ge 4$, then there exists $z_i, z_j \in V_3^{+--}$such that $\partial_{F_3}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \ge 3$, a contradiction. Hence $q_3 \le 3$. By Lemma 2.1, we also have $\left|V_3^{+++}\right| \le 1$, thus $|V_3| = q \le 1 + 3 + 2 + 3 = 9 \le 11$.
Subcase 2: $V_2 \to x_1, V_2 \backslash \{y_1, y_2\} \to x_2 \to \{y_1, y_2\}, y_1 \to x_3 \to V_2 \backslash \{y_1\}$.

We know $y_1 \in V_2^{-+-}, y_2 \in V_2^{-++}$and $y_3, y_4 \in V_2^{--+}$. By Lemma 2.1, we can get $V_3^{-+-} = V_3^{-++} = V_3^{--+} = \emptyset$. Take any $z \in V_3$, since $\partial_D(x_1, z) \le 2$, then we have $x_1 \to z$, this means $x_1 \to V_3$. Thus $V_3 = V_3^{+++} \cup V_3^{++-} \cup V_3^{+-+} \cup V_3^{+--}$. Since $\partial_D\left(V_3^{++-}, y_i\right) \le 2$ where $i = 1, 2, 3$, then we have $V_3^{++-} \to y_1$. Since $\partial_D\left(V_3^{+-+}, y_i\right) \le 2$ where $i = 1, 2, 3$, then we have $V_3^{+-+} \to \{y_3, y_4\}$. Since $\partial_D\left(V_3^{+-+}, x_3\right) \le 2$ and $\partial_D\left(x_2, V_3^{+-+}\right) \le 2$, then we have $y_2 \to V_3^{+-+} \to y_1$. Let $\left|V_3^{++-}\right| = q_1, \left|V_3^{+-+}\right| = q_2, \left|V_3^{+--}\right| = q_3$ and $F_1 = D\left[V_2 \backslash \{y_1\} \cup V_3^{++-}\right], F_2 = D\left[V_2 \cup V_3^{+--}\right]$. Then F_1 and F_2 are respectively an orientation of $K(3, q_1)$ and $K(4, q_3)$ where $q_1, q_3 \le q$. If $q_1 \ge 4$, then there exists $z_i, z_j \in V_3^{++-}$such that $\partial_{F_1}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \ge 3$, a contradiction. Hence $q_1 \le 3$. If $q_2 \ge 2$, then there exists $z_i, z_j \in V_3^{+-+}$such that $\partial_D\left(z_i, z_j\right) \ge 3$, a contradiction. Hence $q_2 \le 1$. If $q_3 > \binom{4}{4/2} = 6$, then there exists $z_i, z_j \in V_3^{+--}$such that $\partial_{F_2}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \ge 3$, a contradiction. Hence $q_3 \le 6$. By Lemma 2.1, we also have $\left|V_3^{+++}\right| \le 1$, thus $|V_3| = q \le 1 + 3 + 1 + 6 = 11 \le 11$.
(12) Case(0,2,4). $V_2 \to x_1, V_2 \backslash \{y_1, y_2\} \to x_2 \to \{y_1, y_2\}, x_3 \to V_2$.

We know $y_1, y_2 \in V_2^{-++}$and $y_3, y_4 \in V_2^{--+}$. By Lemma 2.1, we can get $V_3^{-++} = V_3^{--+} = \emptyset$. Take any $z \in V_3$, since $\partial_D(x_1, z) \le 2$ and $\partial_D(z, x_3) \le 2$, then we have $x_1 \to z$ and $z \to x_3$, this means $x_1 \to V_3$ and $V_3 \to x_3$. Thus $V_3 = V_3^{++-} \cup V_3^{+--}$. Let $\left|V_3^{++-}\right| = q_1, \left|V_3^{+--}\right| = q_2$ and $F_1 = D\left[V_2 \cup V_3^{++-}\right], F_2 = D\left[V_2 \cup V_3^{+--}\right]$. Then F_1 and F_2 are respectively an orientation of $K(4, q_1)$ and $K(4, q_2)$ where $q_1, q_2 \le q$. If $q_1 > \binom{4}{4/2} = 6$, then there exists $z_i, z_j \in V_3^{++-}$such that $\partial_{F_1}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \ge 3$, a contradiction. Hence $q_1 \le 6$.

If $q_2 > \binom{4}{4/2} = 6$, then there exists $z_i, z_j \in V_3^{+--}$such that $\partial_{F_2}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_2 \leq 6$. Since the orientation is unique when making $f(K(4,6)) = 2$, if $q_1 = q_2 = 6$, then there exists $z_i \in V_3^{++-}, z_j \in V_3^{+--}$such that $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence we can get $q_1 \leq 5$ or $q_2 \leq 5$. The argument for these two cases are similar, so we may assume $q_2 \leq 5$. Thus $|V_3| = q \leq 6 + 5 = 11 \leq 11$.

(13) Case $(0, 3, 3)$.

Subcase 1: $V_2 \to x_1, y_4 \to \{x_2, x_3\} \to V_2 \backslash \{y_4\}$.

We know $y_1, y_2, y_3 \in V_2^{-++}$and $y_4 \in V_2^{---}$. By Lemma 2.1, we can get $V_3^{-++} = V_3^{---} = \varnothing$. Take any $z \in V_3$, since $\partial_D(x_1, z) \leq 2$ and $\partial_D(z, y_4) \leq 2$, then we have $x_1 \to z$ and $z \to y_4$, this means $x_1 \to V_3$ and $V_3 \to y_4$. Thus $V_3 = V_3^{+++} \cup V_3^{++-} \cup V_3^{+-+} \cup V_3^{+--}$. Let $\left|V_3^{++-}\right| = q_1, \left|V_3^{+-+}\right| = q_2, \left|V_3^{+--}\right| = q_3$ and $F_1 = D\left[V_2^{-++} \cup V_3^{++-}\right], F_2 = D\left[V_2^{-++} \cup V_3^{+-+}\right], F_3 = D\left[V_2^{-++} \cup V_3^{+--}\right]$. Then F_1, F_2 and F_3 are respectively an orientation of $K(3, q_1), K(3, q_2)$ and $K(3, q_3)$ where $q_1, q_2, q_3 \leq q$. If $q_1 \geq 4$, then there exists $z_i, z_j \in V_3^{++-}$such that $\partial_{F_1}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_1 \leq 3$. If $q_2 \geq 4$, then there exists $z_i, z_j \in V_3^{+-+}$such that $\partial_{F_2}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_2 \leq 3$. If $q_3 \geq 4$, then there exists $z_i, z_j \in V_3^{+--}$such that $\partial_{F_3}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_3 \leq 3$. By Lemma 2.1, we also have $\left|V_3^{+++}\right| \leq 1$, thus $|V_3| = q \leq 1 + 3 + 3 + 3 = 10 \leq 11$.

Subcase 2: $V_2 \to x_1, y_4 \to x_2 \to V_2 \backslash \{y_4\}, y_1 \to x_3 \to V_2 \backslash \{y_1\}$.

We know $y_1 \in V_2^{-+-}, y_2, y_3 \in V_2^{-++}$and $y_4 \in V_2^{--+}$. By Lemma 2.1, we can get $V_3^{-+-} = V_3^{-++} = V_3^{--+} = \varnothing$. Take any $z \in V_3$, since $\partial_D(x_1, z) \leq 2$, then we have $x_1 \to z$, this means $x_1 \to V_3$. Thus $V_3 = V_3^{+++} \cup V_3^{++-} \cup V_3^{+-+} \cup V_3^{+--}$. Since $\partial_D\left(V_3^{++-}, y_i\right) \leq 2$ where $i = 1, 2, 3$, then we have $V_3^{++-} \to y_1$. Since $\partial_D\left(V_3^{+-+}, y_i\right) \leq 2$ where $i = 1, 2, 3$, then we have $V_3^{+-+} \to y_4$. Since $\partial_D\left(V_3^{++-}, x_2\right) \leq 2$, then we have $V_3^{++-} \to y_4$. Since $\partial_D\left(V_3^{+-+}, x_3\right) \leq 2$, then we have $V_3^{+-+} \to y_1$. Let $\left|V_3^{++-}\right| = q_1, \left|V_3^{+-+}\right| = q_2, \left|V_3^{+--}\right| = q_3$ and $F_1 = D\left[V_2^{-++} \cup V_3^{++-}\right], F_2 = D\left[V_2^{-++} \cup V_3^{+-+}\right], F_3 = D\left[V_2 \cup V_3^{+--}\right]$. Then F_1, F_2 and F_3 are respectively an orientation of $K(2, q_1), K(2, q_2)$ and $K(4, q_3)$ where $q_1, q_2, q_3 \leq q$. If $q_1 \geq 3$, then there exists $z_i, z_j \in V_3^{++-}$such that $\partial_{F_1}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_1 \leq 2$. If $q_2 \geq 3$, then there exists $z_i, z_j \in V_3^{+-+}$such that $\partial_{F_2}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_2 \leq 2$. If $q_3 > \binom{4}{4/2} = 6$, then there exists $z_i, z_j \in V_3^{+--}$such that $\partial_{F_3}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_3 \leq 6$. By Lemma 2.1, we also have $\left|V_3^{+++}\right| \leq 1$, thus $|V_3| = q \leq 1 + 2 + 2 + 6 = 11 \leq 11$.

(14) Case $(1, 1, 1)$.

Subcase 1: $V_2 \backslash \{y_1\} \to V_1 \to y_1$.

Take any $y \in V_2 \backslash \{y_1\}$ and $z \in V_3$, since $\partial_D(z, y) \leq 2$, then we have $z \to y$, this means $V_3 \to V_2 \backslash \{y_1\}$. So $\partial_D(y_2, y_3) \geq 3$, a contradiction.

Subcase 2: $V_2 \backslash \{y_1\} \to \{x_1, x_2\} \to y_1, V_2 \backslash \{y_2\} \to x_3 \to y_2$.

Take any $y \in V_2 \backslash \{y_1, y_2\}$ and $z \in V_3$, since $\partial_D(z, y) \leq 2$, then we have $z \to y$, this means $V_3 \to V_2 \backslash \{y_1, y_2\}$. So $\partial_D(y_3, y_4) \geq 3$, a contradiction.

Subcase 3: $V_2 \backslash \{y_1\} \to x_1 \to y_1, V_2 \backslash \{y_2\} \to x_2 \to y_2, V_2 \backslash \{y_3\} \to x_3 \to y_3$.

We know $y_1 \in V_2^{+--}, y_2 \in V_2^{-+-}, y_3 \in V_2^{--+}$and $y_4 \in V_2^{---}$. By Lemma 2.1, we can get $V_3^{+--} = V_3^{-+-} = V_3^{--+} = V_3^{---} = \varnothing$. Thus $V_3 = V_3^{+++} \cup V_3^{++-} \cup V_3^{+-+} \cup V_3^{-++}$. Take any $z \in V_3$, since $\partial_D(z, y_4) \leq 2$, then we have $z \to y_4$, this means $V_3 \to y_4$. Since $\partial_D\left(V_3^{++-}, y_i\right) \leq 2$ where $i = 1, 2, 3$, then we have $V_3^{++-} \to \{y_1, y_2\}$. Since $\partial_D\left(V_3^{+-+}, y_i\right) \leq 2$ where $i = 1, 2, 3$, then we have $V_3^{+-+} \to \{y_1, y_3\}$. Since $\partial_D\left(V_3^{-++}, y_i\right) \leq 2$ where $i = 1, 2, 3$, then we have $V_3^{-++} \to \{y_2, y_3\}$. Since $\partial_D\left(x_3, V_3^{++-}\right) \leq 2$, then we have $y_3 \to V_3^{++-}$. Since $\partial_D\left(x_2, V_3^{+-+}\right) \leq 2$, then we have $y_2 \to V_3^{+-+}$. Since $\partial_D\left(x_1, V_3^{-++}\right) \leq 2$, then we have $y_1 \to V_3^{-++}$. Let $\left|V_3^{++-}\right| = q_1, \left|V_3^{+-+}\right| = q_2$ and $\left|V_3^{-++}\right| = q_3$. If $q_1 \geq 2$, then there exists $z_i, z_j \in V_3^{++-}$such that $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_1 \leq 1$. If $q_2 \geq 2$, then there exists $z_i, z_j \in V_3^{+-+}$such that $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_2 \leq 1$. If $q_3 \geq 2$, then there exists $z_i, z_j \in V_3^{-++}$such that $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_3 \leq 1$. By Lemma 2.1,

we also have $\left|V_3^{+++}\right| \leq 1,$ thus $|V_3| = q \leq 1+1+1+1 = 4 \leq 11.$
(15) Case $(1,1,2)$.
Subcase 1: $V_2\backslash\{y_1\} \to \{x_1,x_2\} \to y_1, V_2\backslash\{y_1,y_2\} \to x_3 \to \{y_1,y_2\}$.

Take any $y \in V_2\backslash\{y_1,y_2\}$ and $z \in V_3$, since $\partial_D(z,y) \leq 2$, then we have $z \to y$, this means $V_3 \to V_2\backslash\{y_1,y_2\}$. So $\partial_D(y_3,y_4) \geq 3$, a contradiction.
Subcase2: $V_2\backslash\{y_1\} \to \{x_1,x_2\} \to y_1, V_2\backslash\{y_2,y_3\} \to x_3 \to \{y_2,y_3\}$.

We know $y_1 \in V_2^{++-}, y_2,y_3 \in V_2^{--+}$and $y_4 \in V_2^{---}$. By Lemma 2.1, we can get $V_3^{++-} = V_3^{--+} = V_3^{---} = \varnothing$. Thus $V_3 = V_3^{+++} \cup V_3^{+-+} \cup V_3^{-++} \cup V_3^{+--} \cup V_3^{-+-}$. Take any $z \in V_3$, since $\partial_D(z,y_4) \leq 2$, then we have $z \to y_4$, this means $V_3 \to y_4$. Since $\partial_D\left(V_3^{+-+},y_i\right) \leq 2$ where $i = 1,2,3$, then we have $V_3^{+-+} \to \{y_2,y_3\}$. Since $\partial_D\left(V_3^{-++},y_i\right) \leq 2$ where $i = 1,2,3$, then we have $V_3^{-++} \to \{y_2,y_3\}$. Since $\partial_D\left(x_2,V_3^{+--}\right) \leq 2$, then we have $y_1 \to V_3^{+--}$. Since $\partial_D\left(x_1,V_3^{-+-}\right) \leq 2$, then we have $y_1 \to V_3^{-+-}$. Let $\left|V_3^{+-+}\right| = q_1, \left|V_3^{-++}\right| = q_2, \left|V_3^{+--}\right| = q_3, \left|V_3^{-+-}\right| = q_4$ and $F_1 = D\left[V_2^{--+} \cup V_3^{+--}\right], F_2 = D\left[V_2^{--+} \cup V_3^{-+-}\right]$. Then F_1 and F_2 are respectively an orientation of $K(2,q_3)$ and $K(2,q_4)$ where $q_3,q_4 \leq q$. If $q_1 \geq 2$, then there exists $z_i,z_j \in V_3^{+-+}$such that $\partial_D\left(z_i,z_j\right) \geq 3$, a contradiction. Hence $q_1 \leq 1$. If $q_2 \geq 2$, then there exists $z_i,z_j \in V_3^{-++}$such that $\partial_D\left(z_i,z_j\right) \geq 3$, a contradiction. Hence $q_2 \leq 1$. If $q_3 \geq 3$, then there exists $z_i,z_j \in V_3^{+--}$such that $\partial_{F_1}\left(z_i,z_j\right) = 4$, so $\partial_D\left(z_i,z_j\right) \geq 3$, a contradiction. Hence $q_3 \leq 2$. If $q_4 \geq 3$, then there exists $z_i,z_j \in V_3^{-+-}$such that $\partial_{F_2}\left(z_i,z_j\right) = 4$, so $\partial_D\left(z_i,z_j\right) \geq 3$, a contradiction. Hence $q_4 \leq 2$. By Lemma 2.1, we also have $\left|V_3^{+++}\right| \leq 1,$ thus $|V_3| = q \leq 1+1+1+2+2 = 7 \leq 11.$
Subcase 3: $V_2\backslash\{y_1\} \to x_1 \to y_1, V_2\backslash\{y_2\} \to x_2 \to y_2, V_2\backslash\{y_1,y_2\} \to x_3 \to \{y_1,y_2\}$.

Take any $y \in V_2\backslash\{y_1,y_2\}$ and $z \in V_3$, since $\partial_D(z,y) \leq 2$, then we have $z \to y$, this means $V_3 \to V_2\backslash\{y_1,y_2\}$. So $\partial_D(y_3,y_4) \geq 3$, a contradiction.
Subcase 4: $V_2\backslash\{y_1\} \to x_1 \to y_1, V_2\backslash\{y_2\} \to x_2 \to y_2, V_2\backslash\{y_2,y_3\} \to x_3 \to \{y_2,y_3\}$.

We know $y_1 \in V_2^{+--}, y_2 \in V_2^{-++}, y_3 \in V_2^{--+}$and $y_4 \in V_2^{---}$. By Lemma 2.1, we can get $V_3^{+--} = V_3^{-++} = V_3^{--+} = V_3^{---} = \varnothing$. Thus $V_3 = V_3^{+++} \cup V_3^{++-} \cup V_3^{+-+} \cup V_3^{-+-}$. Take any $z \in V_3$, since $\partial_D(z,y_4) \leq 2$, then we have $z \to y_4$, this means $V_3 \to y_4$. Since $\partial_D\left(V_3^{++-},y_i\right) \leq 2$ where $i = 1,2,3$, then we have $V_3^{++-} \to y_1$. Since $\partial_D\left(V_3^{+-+},y_i\right) \leq 2$ where $i = 1,2,3$, then we have $V_3^{+-+} \to \{y_1,y_3\}$. Since $\partial_D\left(y_i,V_3^{-+-}\right) \leq 2$ where $i = 1,2,3$, then we have $y_2 \to V_3^{-+-}$. Since $\partial_D\left(x_2,V_3^{+-+}\right) \leq 2$, then we have $y_2 \to V_3^{+-+}$. Since $\partial_D\left(x_1,V_3^{-+-}\right) \leq 2$, then we have $y_1 \to V_3^{-+-}$. Let $\left|V_3^{++-}\right| = q_1, \left|V_3^{+-+}\right| = q_2, \left|V_3^{-+-}\right| = q_3$ and $F = D\left[V_2\backslash\{y_1,y_4\} \cup V_3^{++-}\right]$. Then F is an orientation of $K(2,q_1)$ where $q_1 \leq q$. If $q_1 \geq 3$, then there exists $z_i,z_j \in V_3^{++-}$such that $\partial_F\left(z_i,z_j\right) = 4$, so $\partial_D\left(z_i,z_j\right) \geq 3$, a contradiction. Hence $q_1 \leq 2$. If $q_2 \geq 2$, then there exists $z_i,z_j \in V_3^{+-+}$such that $\partial_D\left(z_i,z_j\right) \geq 3$, a contradiction. Hence $q_2 \leq 1$. If $q_3 \geq 2$, then there exists $z_i,z_j \in V_3^{-+-}$such that $\partial_D\left(z_i,z_j\right) \geq 3$, a contradiction. Hence $q_3 \leq 1$. By Lemma 2.1, we also have $\left|V_3^{+++}\right| \leq 1$, thus ——$V_3$ $|= q \leq 1+2+1+1 = 5 \leq 11.$
Subcase 5: $V_2\backslash\{y_1\} \to x_1 \to y_1, V_2\backslash\{y_2\} \to x_2 \to y_2, V_2\backslash\{y_3,y_4\} \to x_3 \to \{y_3,y_4\}$.

We know $y_1 \in V_2^{+--}, y_2 \in V_2^{-+-}$and $y_3,y_4 \in V_2^{--+}$. By Lemma 2.1, we can get $V_3^{+--} = V_3^{-+-} = V_3^{--+} = \varnothing$. Thus $V_3 = V_3^{+++} \cup V_3^{++-} \cup V_3^{+-+} \cup V_3^{-++} \cup V_3^{---}$. Since $\partial_D\left(V_3^{++-},y_i\right) \leq 2$ where $i = 1,2,3$, then we have $V_3^{++-} \to \{y_1,y_2\}$. Since $\partial_D\left(V_3^{+-+},y_i\right) \leq 2$ where $i = 1,2,3$, then we have $V_3^{+-+} \to \{y_1,y_3,y_4\}$. Since $\partial_D\left(V_3^{-++},y_i\right) \leq 2$ where $i = 1,2,3$, then we have $V_3^{-++} \to \{y_2,y_3,y_4\}$. Let $\left|V_3^{++-}\right| = q_1, \left|V_3^{+-+}\right| = q_2, \left|V_3^{-++}\right| = q_3$ and $F = D\left[V_2^{--+} \cup V_3^{++-}\right]$. Then F is an orientation of $K(2,q_1)$ where $q_1 \leq q$. If $q_1 \geq 3$, then there exists $z_i,z_j \in V_3^{++-}$such that $\partial_F\left(z_i,z_j\right) = 4$, so $\partial_D\left(z_i,z_j\right) \geq 3$, a contradiction. Hence $q_1 \leq 2$. If $q_2 \geq 2$, then there exists $z_i,z_j \in V_3^{+-+}$such that $\partial_D\left(z_i,z_j\right) \geq 3$, a contradiction. Hence $q_2 \leq 1$. If $q_3 \geq 2$, then there exists $z_i,z_j \in V_3^{-++}$such that $\partial_D\left(z_i,z_j\right) \geq 3$, a contradiction. Hence $q_3 \leq 1$. By Lemma 2.1, we also have $\left|V_3^{+++}\right| \leq 1$ and $\left|V_3^{---}\right| \leq 1,$ thus $|V_3| = q \leq 1+2+1+1+1 = 6 \leq 11.$
(16) Case $(1,1,3)$.
Subcase 1: $V_2\backslash\{y_1\} \to \{x_1,x_2\} \to y_1, y_4 \to x_3 \to V_2\backslash\{y_4\}$.

We know $y_1 \in V_2^{+++}, y_2, y_3 \in V_2^{--+}$and $y_4 \in V_2^{---}$. By Lemma 2.1, we can get $V_3^{+++} = V_3^{--+} = V_3^{---} = \emptyset$. Thus $V_3 = V_3^{++-} \cup V_3^{+-+} \cup V_3^{-++} \cup V_3^{+--} \cup V_3^{-+-}$. Take any $z \in V_3$, since $\partial_D(y_1, z) \leq 2$ and $\partial_D(z, y_4) \leq 2$, then we have $y_1 \to z$ and $z \to y_4$, this means $y_1 \to V_3$ and $V_3 \to y_4$. Since $\partial_D\left(V_3^{+-+}, y_i\right) \leq 2$ where $i = 1, 2, 3$, then we have $V_3^{+-+} \to \{y_2, y_3\}$. Since $\partial_D\left(V_3^{-++}, y_i\right) \leq 2$ where $i = 1, 2, 3$, then we have $V_3^{-++} \to \{y_2, y_3\}$. Let $\left|V_3^{++-}\right| = q_1, \left|V_3^{+-+}\right| = q_2, \left|V_3^{-++}\right| = q_3, \left|V_3^{+--}\right| = q_4, \left|V_3^{-+-}\right| = q_5$ and $F_1 = D\left[V_2^{--+} \cup V_3^{++-}\right], F_2 = D\left[V_2^{--+} \cup V_3^{+--}\right], F_3 = D\left[V_2^{--+} \cup V_3^{-+-}\right]$. Then F_1, F_2 and F_3 are respectively an orientation of $K(2, q_1), K(2, q_4)$ and $K(2, q_5)$ where $q_1, q_4, q_5 \leq q$. If $q_1 \geq 3$, then there exists $z_i, z_j \in V_3^{++-}$such that $\partial_{F_1}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_1 \leq 2$. If $q_2 \geq 2$, then there exists $z_i, z_j \in V_3^{+-+}$such that $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_2 \leq 1$. If $q_3 \geq 2$, then there exists $z_i, z_j \in V_3^{-++}$such that $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_3 \leq 1$. If $q_4 \geq 3$, then there exists $z_i, z_j \in V_3^{+--}$such that $\partial_{F_2}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_4 \leq 2$. If $q_5 \geq 3$, then there exists $z_i, z_j \in V_3^{-+-}$such that $\partial_{F_3}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_5 \leq 2$. Thus $|V_3| = q \leq 2 + 1 + 1 + 2 + 2 = 8 \leq 11$.

Subcase 2: $V_2 \backslash \{y_1\} \to \{x_1, x_2\} \to y_1 \to x_3 \to V_2 \backslash \{y_1\}$.

We know $y_1 \in V_2^{++-}$and $y_2, y_3, y_4 \in V_2^{--+}$. By Lemma 2.1, we can get $V_3^{++-} = V_3^{--+} = \emptyset$. Thus $V_3 = V_3^{+++} \cup V_3^{+-+} \cup V_3^{-++} \cup V_3^{+--} \cup V_3^{-+-} \cup V_3^{---}$. Since $\partial_D\left(V_3^{+-+}, y_i\right) \leq 2$ where $i = 1, 2, 3$, then we have $V_3^{+-+} \to \{y_2, y_3, y_4\}$. Since $\partial_D\left(V_3^{-++}, y_i\right) \leq 2$ where $i = 1, 2, 3$, then we have $V_3^{-++} \to \{y_2, y_3, y_4\}$. Since $\partial_D\left(y_i, V_3^{+--}\right) \leq 2$ where $i = 1, 2, 3$, then we have $y_1 \to V_3^{+--}$. Since $\partial_D\left(y_i, V_3^{-+-}\right) \leq 2$ where $i = 1, 2, 3$, then we have $y_1 \to V_3^{-+-}$. Let $\left|V_3^{+-+}\right| = q_1, \left|V_3^{-++}\right| = q_2, \left|V_3^{+--}\right| = q_3, \left|V_3^{-+-}\right| = q_4$ and $F_1 = D\left[V_2^{--+} \cup V_3^{+--}\right], F_2 = D\left[V_2^{--+} \cup V_3^{-+-}\right]$. Then F_1 and F_2 are respectively an orientation of $K(3, q_3)$ and $K(3, q_4)$ where $q_3, q_4 \leq q$. If $q_1 \geq 2$, then there exists $z_i, z_j \in V_3^{+-+}$such that $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_2 \leq 1$. If $q_2 \geq 2$, then there exists $z_i, z_j \in V_3^{-++}$such that $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_2 \leq 1$. If $q_3 \geq 4$, then there exists $z_i, z_j \in V_3^{+--}$such that $\partial_{F_1}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_3 \leq 3$. If $q_4 \geq 4$, then there exists $z_i, z_j \in V_3^{-+-}$such that $\partial_{F_2}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_4 \leq 3$. By Lemma 2.1, we also have $\left|V_3^{+++}\right| \leq 1$ and $\left|V_3^{---}\right| \leq 1$, thus $|V_3| = q \leq 1 + 1 + 1 + 3 + 3 + 1 = 10 \leq 11$.

Subcase 3: $V_2 \backslash \{y_1\} \to x_1 \to y_1, V_2 \backslash \{y_2\} \to x_2 \to y_2, y_4 \to x_3 \to V_2 \backslash \{y_4\}$.

We know $y_1 \in V_2^{+-+}, y_2 \in V_2^{-++}, y_3 \in V_2^{--+}$and $y_4 \in V_2^{---}$. By Lemma 2.1, we can get $V_3^{+-+} = V_3^{-++} = V_3^{--+} = V_3^{---} = \emptyset$. Thus $V_3 = V_3^{+++} \cup V_3^{++-} \cup V_3^{+--} \cup V_3^{-+-}$. Take any $z \in V_3$, since $\partial_D(z, y_4) \leq 2$, then we have $z \to y_4$, this means $V_3 \to y_4$. Since $\partial_D\left(y_i, V_3^{+--}\right) \leq 2$ where $i = 1, 2, 3$, then we have $y_1 \to V_3^{+--}$. Since $\partial_D\left(y_i, V_3^{-+-}\right) \leq 2$ where $i = 1, 2, 3$, then we have $y_2 \to V_3^{-+-}$. Let $\left|V_3^{++-}\right| = q_1, \left|V_3^{+--}\right| = q_2, \left|V_3^{-+-}\right| = q_3$ and $F_1 = D\left[V_2 \backslash \{y_4\} \cup V_3^{++-}\right], F_2 = D\left[V_2 \backslash \{y_1, y_4\} \cup V_3^{+--}\right], F_3 = D\left[V_2 \backslash \{y_2, y_4\} \cup V_3^{-+-}\right]$. Then F_1, F_2 and F_3 are respectively an orientation of $K(3, q_1), K(2, q_2)$ and $K(2, q_3)$ where $q_1, q_2, q_3 \leq q$. If $q_1 \geq 4$, then there exists $z_i, z_j \in V_3^{++-}$such that $\partial_{F_1}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_1 \leq 3$. If $q_2 \geq 3$, then there exists $z_i, z_j \in V_3^{+--}$such that $\partial_{F_2}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_2 \leq 2$. If $q_3 \geq 3$, then there exists $z_i, z_j \in V_3^{-+-}$such that $\partial_{F_3}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_3 \leq 2$. By Lemma 2.1, we also have $\left|V_3^{+++}\right| \leq 1$, thus $|V_3| = q \leq 1 + 3 + 2 + 2 = 8 \leq 11$.

Subcase 4: $V_2 \backslash \{y_1\} \to x_1 \to y_1, V_2 \backslash \{y_2\} \to x_2 \to y_2, y_1 \to x_3 \to V_2 \backslash \{y_1\}$.

We know $y_1 \in V_2^{+--}, y_2 \in V_2^{-++}$and $y_3, y_4 \in V_2^{--+}$. By Lemma 2.1, we can get $V_3^{+--} = V_3^{-++} = V_3^{--+} = \emptyset$. Thus $V_3 = V_3^{+++} \cup V_3^{++-} \cup V_3^{+-+} \cup V_3^{-+-} \cup V_3^{---}$. Since $\partial_D\left(V_3^{++-}, y_i\right) \leq 2$ where $i = 1, 2, 3$, then we have $V_3^{++-} \to y_1$. Since $\partial_D\left(V_3^{+-+}, y_i\right) \leq 2$ where $i = 1, 2, 3$, then we have $V_3^{+-+} \to \{y_1, y_3, y_4\}$. Since $\partial_D\left(y_i, V_3^{-+-}\right) \leq 2$ where $i = 1, 2, 3$, then we have $y_2 \to V_3^{-+-}$. Let $\left|V_3^{++-}\right| = q_1, \left|V_3^{+-+}\right| = q_2, \left|V_3^{-+-}\right| = q_3$ and $F_1 = D\left[V_2 \backslash \{y_1\} \cup V_3^{++-}\right], F_2 = D\left[V_2 \backslash \{y_2\} \cup V_3^{-+-}\right]$. Then F_1 and F_2 are respectively an orientation of

$K(3,q_1)$ and $K(3,q_3)$ where $q_1,q_3 \le q$. If $q_1 \ge 4$, then there exists $z_i,z_j \in V_3^{++-}$such that $\partial_{F_1}\left(z_i,z_j\right)=4$, so $\partial_D\left(z_i,z_j\right)\ge 3$, a contradiction. Hence $q_1 \le 3$. If $q_2 \ge 2$, then there exists $z_i,z_j \in V_3^{+-+}$such that $\partial_D\left(z_i,z_j\right)\ge 3$, a contradiction. Hence $q_2 \le 1$. If $q_3 \ge 4$, then there exists $z_i,z_j \in V_3^{-+-}$such that $\partial_{F_2}\left(z_i,z_j\right)=4$, so $\partial_D\left(z_i,z_j\right)\ge 3$, a contradiction. Hence $q_3 \le 3$. By Lemma 2.1, we also have $\left|V_3^{+++}\right|\le 1$ and $\left|V_3^{---}\right|\le 1$, thus $|V_3|=q\le 1+3+1+3+1=9\le 11$.

(17) Case(1,2,2).

Subcase 1: $V_2\backslash\{y_1\}\to x_1\to y_1, V_2\backslash\{y_1,y_2\}\to\{x_2,x_3\}\to\{y_1,y_2\}$.

Take any $z\in V_3$, since $\partial_D(z,y_3)\le 2$ and $\partial_D(z,y_4)\le 2$, then we have $z\to y_3$ and $z\to y_4$, this means $V_3\to\{y_3,y_4\}$. So $\partial_D(y_3,y_4)\ge 3$, a contradiction.

Subcase 2: $V_2\backslash\{y_1\}\to x_1\to y_1,\{y_3,y_4\}\to x_2\to\{y_1,y_2\},\{y_2,y_4\}\to x_3\to\{y_1,y_3\}$.

We know $y_1\in V_2^{+++}, y_2\in V_2^{-+-}, y_3\in V_2^{--+}$and $y_4\in V_2^{---}$. By Lemma 2.1, we can get $V_3^{+++}=V_3^{-+-}=V_3^{--+}=V_3^{---}=\varnothing$. Thus $V_3=V_3^{++-}\cup V_3^{+-+}\cup V_3^{-++}\cup V_3^{+--}$. Take any $z\in V_3$, since $\partial_D(y_1,z)\le 2$ and $\partial_D(z,y_4)\le 2$, then we have $y_1\to z$ and $z\to y_4$, this means $y_1\to V_3$ and $V_3\to y_4$. Since $\partial_D\left(V_3^{++-},y_i\right)\le 2$ where $i=1,2,3$, then we have $V_3^{++-}\to y_2$. Since $\partial_D\left(V_3^{+-+},y_i\right)\le 2$ where $i=1,2,3$, then we have $V_3^{+-+}\to y_3$. Since $\partial_D\left(V_3^{-++},y_i\right)\le 2$ where $i=1,2,3$, then we have $V_3^{-++}\to\{y_2,y_3\}$. Let $\left|V_3^{++-}\right|=q_1,\left|V_3^{+-+}\right|=q_2,\left|V_3^{-++}\right|=q_3,\left|V_3^{+--}\right|=q_4$ and $F=D\left[V_2\backslash\{y_1,y_4\}\cup V_3^{+--}\right]$. Then F is an orientation of $K(2,q_4)$ where $q_4\le q$. If $q_1\ge 2$, then there exists $z_i,z_j\in V_3^{++-}$such that $\partial_D\left(z_i,z_j\right)\ge 3$, a contradiction. Hence $q_1\le 1$. If $q_2\ge 2$, then there exists $z_i,z_j\in V_3^{+-+}$such that $\partial_D\left(z_i,z_j\right)\ge 3$, a contradiction. Hence $q_2\le 1$. If $q_3\ge 2$, then there exists $z_i,z_j\in V_3^{-++}$such that $\partial_D\left(z_i,z_j\right)\ge 3$, a contradiction. Hence $q_3\le 1$. If $q_4\ge 3$, then there exists $z_i,z_j\in V_3^{+--}$such that $\partial_F\left(z_i,z_j\right)=4$, so $\partial_D\left(z_i,z_j\right)\ge 3$, a contradiction. Hence $q_4\le 2$. Thus $|V_3|=q\le 1+1+1+2=5\le 11$.

Subcase 3: $V_2\backslash\{y_1\}\to x_1\to y_1,\{y_3,y_4\}\to x_2\to\{y_1,y_2\},\{y_1,y_4\}\to x_3\to\{y_2,y_3\}$.

We know $y_1\in V_2^{++-}, y_2\in V_2^{-++}, y_3\in V_2^{--+}$and $y_4\in V_2^{---}$. By Lemma 2.1, we can get $V_3^{++-}=V_3^{-++}=V_3^{--+}=V_3^{---}=\varnothing$. Thus $V_3=V_3^{+++}\cup V_3^{+-+}\cup V_3^{+--}\cup V_3^{-+-}$. Take any $z\in V_3$, since $\partial_D(z,y_4)\le 2$, then we have $z\to y_4$, this means $V_3\to y_4$. Since $\partial_D\left(V_3^{+-+},y_i\right)\le 2$ where $i=1,2,3$, then we have $V_3^{+-+}\to y_3$. Since $\partial_D\left(y_i,V_3^{+--}\right)\le 2$ where $i=1,2,3$, then we have $y_1\to V_3^{+--}$. Since $\partial_D\left(y_i,V_3^{-+-}\right)\le 2$ where $i=1,2,3$, then we have $\{y_1,y_2\}\to V_3^{-+-}$. Let $\left|V_3^{+-+}\right|=q_1,\left|V_3^{+--}\right|=q_2,\left|V_3^{-+-}\right|=q_3$ and $|\ F_1=D\left[V_2\backslash\{y_3,y_4\}\cup V_3^{+-+}\right], F_2=D\left[V_2\backslash\{y_1,y_4\}\cup V_3^{+--}\right]$. Then F_1 and F_2 are respectively an orientation of $K(2,q_1)$ and $K(2,q_2)$ where $q_1,q_2\le q$. If $q_1\ge 3$, then there exists $z_i,z_j\in V_3^{+-+}$such that $\partial_{F_1}\left(z_i,z_j\right)=4$, so $\partial_D\left(z_i,z_j\right)\ge 3$, a contradiction. Hence $q_1\le 2$. If $q_2\ge 3$, then there exists $z_i,z_j\in V_3^{+--}$such that $\partial_{F_2}\left(z_i,z_j\right)=4$, so $\partial_D\left(z_i,z_j\right)\ge 3$, a contradiction. Hence $q_2\le 2$. If $q_3\ge 2$, then there exists $z_i,z_j\in V_3^{-+-}$such that $\partial_D\left(z_i,z_j\right)\ge 3$, a contradiction. Hence $q_3\le 1$. By Lemma 2.1, we also have $\left|V_3^{+++}\right|\le 1$, thus $|V_3|=q\le 1+2+2+1=6\le 11$.

Subcase 4: $V_2\backslash\{y_1\}\to x_1\to y_1,\{y_3,y_4\}\to x_2\to\{y_1,y_2\}\to x_3\to\{y_3,y_4\}$.

We know $y_1\in V_2^{++-}, y_2\in V_2^{-+-}$and $y_3,y_4\in V_2^{--+}$. By Lemma 2.1, we can get $V_3^{++-}=V_3^{-+-}=V_3^{--+}=\varnothing$. Thus $V_3=V_3^{+++}\cup V_3^{+-+}\cup V_3^{-++}\cup V_3^{+--}\cup V_3^{---}$. Since $\partial_D\left(V_3^{+-+},y_i\right)\le 2$ where $i=1,2,3$, then we have $V_3^{+-+}\to\{y_3,y_4\}$. Since $\partial_D\left(V_3^{-++},y_i\right)\le 2$ where $i=1,2,3$, then we have $V_3^{-++}\to\{y_2,y_3,y_4\}$. Since $\partial_D\left(y_i,V_3^{+--}\right)\le 2$ where $i=1,2,3$, then we have $y_1\to V_3^{+--}$. Let $\left|V_3^{+-+}\right|=q_1,\left|V_3^{-++}\right|=q_2,\left|V_3^{+--}\right|=q_3$ and $F_1=D\left[V_2\backslash\{y_3,y_4\}\cup V_3^{+-+}\right], F_2=D\left[V_2\backslash\{y_1\}\cup V_3^{+--}\right]$. Then F_1 and F_2 are respectively an orientation of $K(2,q_1)$ and $K(3,q_3)$ where $q_1,q_3\le q$. If $q_1\ge 3$, then there exists $z_i,z_j\in V_3^{+-+}$such that $\partial_{F_1}\left(z_i,z_j\right)=4$, so $\partial_D\left(z_i,z_j\right)\ge 3$, a contradiction. Hence $q_1\le 2$. If $q_2\ge 2$, then there exists $z_i,z_j\in V_3^{-++}$such that $\partial_D\left(z_i,z_j\right)\ge 3$, a contradiction. Hence $q_2\le 1$. If $q_3\ge 4$, then there exists $z_i,z_j\in V_3^{+--}$such that $\partial_{F_2}\left(z_i,z_j\right)=4$, so $\partial_D\left(z_i,z_j\right)\ge 3$, a contradiction. Hence $q_3\le 3$. By Lemma 2.1, we also have $\left|V_3^{+++}\right|\le 1$ and $\left|V_3^{---}\right|\le 1$, thus

$|V_3| = q \le 1+2+1+3+1 = 8 \le 11$.

Subcase 5: $V_2\backslash\{y_1\} \to x_1 \to y_1, V_2\backslash\{y_2,y_3\} \to \{x_2,x_3\} \to \{y_2,y_3\}$.

We know $y_1 \in V_2^{+--}, y_2, y_3 \in V_2^{-++}$and $y_4 \in V_2^{---}$. By Lemma 2.1, we can get $V_3^{+--} = V_3^{-++} = V_3^{---} = \varnothing$. Thus $V_3 = V_3^{+++} \cup V_3^{++-} \cup V_3^{+-+} \cup V_3^{-+-} \cup V_3^{--+}$. Take any $z \in V_3$, since $\partial_D(z,y_4) \le 2$, then we have $z \to y_4$, this means $V_3 \to y_4$. Since $\partial_D\left(V_3^{++-}, y_i\right) \le 2$ where $i=1,2,3$, then we have $V_3^{++-} \to y_1$. Since $\partial_D\left(V_3^{+-+}, y_i\right) \le 2$ where $i=1,2,3$, then we have $V_3^{+-+} \to y_1$. Since $\partial_D\left(y_i, V_3^{-+-}\right) \le 2$ where $i=1,2,3$, then we have $\{y_2,y_3\} \to V_3^{-+-}$. Since $\partial_D\left(y_i, V_3^{--+}\right) \le 2$ where $i=1,2,3$, then we have $\{y_2,y_3\} \to V_3^{--+}$. Let $\left|V_3^{++-}\right| = q_1, \left|V_3^{+-+}\right| = q_2, \left|V_3^{-+-}\right| = q_3, \left|V_3^{--+}\right| = q_4$ and $F_1 = D\left[V_2^{-++} \cup V_3^{++-}\right], F_2 = D\left[V_2^{-++} \cup V_3^{+-+}\right]$. Then F_1 and F_2 are respectively an orientation of $K(2,q_1)$ and $K(2,q_2)$ where $q_1,q_2 \le q$. If $q_1 \ge 3$, then there exists $z_i,z_j \in V_3^{++-}$such that $\partial_{F_1}\left(z_i,z_j\right) = 4$, so $\partial_D\left(z_i,z_j\right) \ge 3$, a contradiction. Hence $q_1 \le 2$. If $q_2 \ge 3$, then there exists $z_i,z_j \in V_3^{+-+}$such that $\partial_{F_2}\left(z_i,z_j\right) = 4$, so $\partial_D\left(z_i,z_j\right) \ge 3$, a contradiction. Hence $q_2 \le 2$. If $q_3 \ge 2$, then there exists $z_i,z_j \in V_3^{-+-}$such that $\partial_D\left(z_i,z_j\right) \ge 3$, a contradiction. Hence $q_3 \le 1$. If $q_4 \ge 2$, then there exists $z_i,z_j \in V_3^{--+}$such that $\partial_D\left(z_i,z_j\right) \ge 3$, a contradiction. Hence $q_4 \le 1$. By Lemma 2.1, we also have $\left|V_3^{+++}\right| \le 1$, thus $|V_3| = q \le 1+2+2+1+1 = 7 \le 11$.

Subcase 6: $V_2\backslash\{y_1\} \to x_1 \to y_1, \{y_1,y_4\} \to x_2 \to \{y_2,y_3\}, \{y_1,y_2\} \to x_3 \to \{y_3,y_4\}$.

We know $y_1 \in V_2^{+--}, y_2 \in V_2^{-+-}, y_3 \in V_2^{-++}$and $y_4 \in V_2^{--+}$. By Lemma 2.1, we can get $V_3^{+--} = V_3^{-+-} = V_3^{-++} = V_3^{--+} = \varnothing$. Thus $V_3 = V_3^{+++} \cup V_3^{++-} \cup V_3^{+-+} \cup V_3^{---}$. Since $\partial_D\left(V_3^{++-}, y_i\right) \le 2$ where $i=1,2,3$, then we have $V_3^{++-} \to \{y_1,y_2\}$. Since $\partial_D\left(V_3^{+-+}, y_i\right) \le 2$ where $i=1,2,3$, then we have $V_3^{+-+} \to \{y_1,y_4\}$. Let $\left|V_3^{++-}\right| = q_1, \left|V_3^{+-+}\right| = q_2$ and $F_1 = D\left[V_2\backslash\{y_1,y_2\} \cup V_3^{++-}\right], F_2 = D\left[V_2\backslash\{y_1,y_4\} \cup V_3^{+-+}\right]$. Then F_1 and F_2 are respectively an orientation of $K(2,q_1)$ and $K(2,q_2)$ where $q_1,q_2 \le q$. If $q_1 \ge 3$, then there exists $z_i,z_j \in V_3^{++-}$such that $\partial_{F_1}\left(z_i,z_j\right) = 4$, so $\partial_D\left(z_i,z_j\right) \ge 3$, a contradiction. Hence $q_1 \le 2$. If $q_2 \ge 3$, then there exists $z_i,z_j \in V_3^{+-+}$such that $\partial_{F_2}\left(z_i,z_j\right) = 4$, so $\partial_D\left(z_i,z_j\right) \ge 3$, a contradiction. Hence $q_2 \le 2$. By Lemma 2.1, we also have $\left|V_3^{+++}\right| \le 1$ and $\left|V_3^{---}\right| \le 1$, thus $|V_3| = q \le 1+2+2+1 = 6 \le 11$.

(18) Case $(1,2,3)$.

Subcase 1: $V_2\backslash\{y_1\} \to x_1 \to y_1, V_2\backslash\{y_1,y_2\} \to x_2 \to \{y_1,y_2\}, y_4 \to x_3 \to V_2\backslash\{y_4\}$.

We know $y_1 \in V_2^{+++}, y_2 \in V_2^{-++}, y_3 \in V_2^{--+}$and $y_4 \in V_2^{---}$. By Lemma 2.1, we can get $V_3^{+++} = V_3^{-++} = V_3^{--+} = V_3^{---} = \varnothing$. Thus $V_3 = V_3^{++-} \cup V_3^{+-+} \cup V_3^{+--} \cup V_3^{-+-}$. Take any $z \in V_3$, since $\partial_D(y_1,z) \le 2$ and $\partial_D(z,y_4) \le 2$, then we have $y_1 \to z$ and $z \to y_4$, this means $y_1 \to V_3$ and $V_3 \to y_4$. Since $\partial_D\left(V_3^{+-+}, y_i\right) \le 2$ where $i=1,2,3$, then we have $V_3^{+-+} \to y_3$. Since $\partial_D\left(y_i, V_3^{-+-}\right) \le 2$ where $i=1,2,3$, then we have $y_2 \to V_3^{-+-}$. Let $\left|V_3^{++-}\right| = q_1, \left|V_3^{+-+}\right| = q_2, \left|V_3^{+--}\right| = q_3, \left|V_3^{-+-}\right| = q_4$ and $F_1 = D\left[V_2\backslash\{y_1,y_4\} \cup V_3^{++-}\right], F_2 = D\left[V_2\backslash\{y_1,y_4\} \cup V_3^{+--}\right]$. Then F_1 and F_2 are respectively an orientation of $K(2,q_1)$ and $K(2,q_3)$ where $q_1,q_3 \le q$. If $q_1 \ge 3$, then there exists $z_i,z_j \in V_3^{++-}$such that $\partial_{F_1}\left(z_i,z_j\right) = 4$, so $\partial_D\left(z_i,z_j\right) \ge 3$, a contradiction. Hence $q_1 \le 2$. If $q_2 \ge 2$, then there exists $z_i,z_j \in V_3^{+-+}$such that $\partial_D\left(z_i,z_j\right) \ge 3$, a contradiction. Hence $q_2 \le 1$. If $q_3 \ge 3$, then there exists $z_i,z_j \in V_3^{+--}$such that $\partial_{F_2}\left(z_i,z_j\right) = 4$, so $\partial_D\left(z_i,z_j\right) \ge 3$, a contradiction. Hence $q_3 \le 2$. If $q_4 \ge 2$, then there exists $z_i,z_j \in V_3^{-+-}$such that $\partial_D\left(z_i,z_j\right) \ge 3$, a contradiction. Hence $q_4 \le 1$. Thus $|V_3| = q \le 2+1+2+1 = 6 \le 11$.

Subcase 2: $V_2\backslash\{y_1\} \to x_1 \to y_1, V_2\backslash\{y_1,y_2\} \to x_2 \to \{y_1,y_2\}, y_2 \to x_3 \to V_2\backslash\{y_2\}$.

We know $y_1 \in V_2^{+++}, y_2 \in V_2^{-+-}$and $y_3,y_4 \in V_2^{--+}$. By Lemma 2.1, we can get $V_3^{+++} = V_3^{-+-} = V_3^{--+} = \varnothing$. Thus $V_3 = V_3^{++-} \cup V_3^{+-+} \cup V_3^{-++} \cup V_3^{+--} \cup V_3^{---}$. Take any $z \in V_3$, since $\partial_D(y_1,z) \le 2$, then we have $y_1 \to z$, this means $y_1 \to V_3$. Since $\partial_D\left(V_3^{++-}, y_i\right) \le 2$ where $i=1,2,3$, then we have $V_3^{++-} \to y_2$. Since $\partial_D\left(V_3^{+-+}, y_i\right) \le 2$ where $i=1,2,3$, then we have $V_3^{+-+} \to \{y_3,y_4\}$. Since $\partial_D\left(V_3^{-++}, y_i\right) \le 2$ where $i=1,2,3$, then we have $V_3^{-++} \to \{y_2,y_3,y_4\}$. Let $\left|V_3^{++-}\right| = q_1, \left|V_3^{+-+}\right| = q_2, \left|V_3^{-++}\right| = q_3, \left|V_3^{+--}\right| = q_4$ and $F_1 = D\left[V_2\backslash\{y_1,y_2\} \cup V_3^{++-}\right], F_2 = D\left[V_2\backslash\{y_1\} \cup V_3^{+--}\right]$. Then F_1 and F_2 are respectively an orientation of

$K(2,q_1)$ and $K(3,q_4)$ where $q_1,q_4 \le q$. If $q_1 \ge 3$, then there exists $z_i,z_j \in V_3^{++-}$such that $\partial_{F_1}\left(z_i,z_j\right) = 4$, so $\partial_D\left(z_i,z_j\right) \ge 3$, a contradiction. Hence $q_1 \le 2$. If $q_2 \ge 2$, then there exists $z_i,z_j \in V_3^{-+}$such that $\partial_D\left(z_i,z_j\right) \ge 3$, a contradiction. Hence $q_2 \le 1$. If $q_3 \ge 2$, then there exists $z_i,z_j \in V_3^{-++}$such that $\partial_D\left(z_i,z_j\right) \ge 3$, a contradiction. Hence $q_3 \le 1$. If $q_4 \ge 4$, then there exists $z_i,z_j \in V_3^{+--}$such that $\partial_{F_2}\left(z_i,z_j\right) = 4$, so $\partial_D\left(z_i,z_j\right) \ge 3$, a contradiction. Hence $q_4 \le 3$. By Lemma 2.1, we also have $\left|V_3^{---}\right| \le 1$, thus $|V_3| = q \le 2+1+1+3+1 = 8 \le 11$.

Subcase 3: $V_2\backslash\{y_1\} \to x_1 \to y_1, V_2\backslash\{y_1,y_2\} \to x_2 \to \{y_1,y_2\}, y_1 \to x_3 \to V_2\backslash\{y_1\}$.

We know $y_1 \in V_2^{++-}, y_2 \in V_2^{-++}$and $y_3,y_4 \in V_2^{--+}$. By Lemma 2.1, we can get $V_3^{++-} = V_3^{-++} = V_3^{--+} = \emptyset$. Thus $V_3 = V_3^{+++} \cup V_3^{+-+} \cup V_3^{+--} \cup V_3^{-+-} \cup V_3^{---}$. Since $\partial_D\left(V_3^{+-+},y_i\right) \le 2$ where $i = 1,2,3$, then we have $V_3^{+-+} \to \{y_3,y_4\}$. Since $\partial_D\left(y_i,V_3^{+--}\right) \le 2$ where $i = 1,2,3$, then we have $y_1 \to V_3^{+--}$. Since $\partial_D\left(y_i,V_3^{-+-}\right) \le 2$ where $i = 1,2,3$, then we have $\{y_1,y_2\} \to V_3^{-+-}$. Let $\left|V_3^{+-+}\right| = q_1, \left|V_3^{+--}\right| = q_2, \left|V_3^{-+-}\right| = q_3$ and $F_1 = D\left[V_2\backslash\{y_3,y_4\} \cup V_3^{+-+}\right], F_2 = D\left[V_2\backslash\{y_1\} \cup V_3^{+--}\right], F_3 = D\left[V_2\backslash\{y_1,y_2\} \cup V_3^{-+-}\right]$. Then F_1,F_2 and F_3 are respectively an orientation of $K(2,q_1), K(3,q_2)$ and $K(2,q_3)$ where $q_1,q_2,q_3 \le q$. If $q_1 \ge 3$, then there exists $z_i,z_j \in V_3^{+-+}$such that $\partial_{F_1}\left(z_i,z_j\right) = 4$, so $\partial_D\left(z_i,z_j\right) \ge 3$, a contradiction. Hence $q_1 \le 2$. If $q_2 \ge 4$, then there exists $z_i,z_j \in V_3^{+--}$such that $\partial_{F_2}\left(z_i,z_j\right) = 4$, so $\partial_D\left(z_i,z_j\right) \ge 3$, a contradiction. Hence $q_2 \le 3$. If $q_3 \ge 3$, then there exists $z_i,z_j \in V_3^{-+-}$such that $\partial_{F_3}\left(z_i,z_j\right) = 4$, so $\partial_D\left(z_i,z_j\right) \ge 3$, a contradiction. Hence $q_3 \le 2$. By Lemma 2.1, we also have $\left|V_3^{+++}\right| \le 1$ and $\left|V_3^{---}\right| \le 1$, thus $|V_3| = q \le 1+2+3+2+1 = 9 \le 11$.

Subcase 4: $V_2\backslash\{y_1\} \to x_1 \to y_1, V_2\backslash\{y_2,y_3\} \to x_2 \to \{y_2,y_3\}, y_4 \to x_3 \to V_2\backslash\{y_4\}$.

We know $y_1 \in V_2^{+-+}, y_2,y_3 \in V_2^{-++}$and $y_4 \in V_2^{---}$. By Lemma 2.1, we can get $V_3^{+-+} = V_3^{-++} = V_3^{---} = \emptyset$. Thus $V_3 = V_3^{+++} \cup V_3^{++-} \cup V_3^{+--} \cup V_3^{-+-} \cup V_3^{--+}$. Take any $z \in V_3$, since $\partial_D(z,y_4) \le 2$, then we have $z \to y_4$, this means $V_3 \to y_4$. Since $\partial_D\left(y_i,V_3^{+--}\right) \le 2$ where $i = 1,2,3$, then we have $y_1 \to V_3^{+--}$. Since $\partial_D\left(y_i,V_3^{-+-}\right) \le 2$ where $i = 1,2,3$, then we have $\{y_2,y_3\} \to V_3^{-+-}$. Since $\partial_D\left(y_i,V_3^{--+}\right) \le 2$ where $i = 1,2,3$, then we have $\{y_1,y_2,y_3\} \to V_3^{--+}$. Let $\left|V_3^{++-}\right| = q_1, \left|V_3^{+--}\right| = q_2, \left|V_3^{-+-}\right| = q_3, \left|V_3^{--+}\right| = q_4$ and $F_1 = D\left[V_2\backslash\{y_4\} \cup V_3^{++-}\right], F_2 = D\left[V_2\backslash\{y_1,y_4\} \cup V_3^{+--}\right]$. Then F_1 and F_2 are respectively an orientation of $K(3,q_1)$ and $K(2,q_2)$ where $q_1,q_2 \le q$. If $q_1 \ge 4$, then there exists $z_i,z_j \in V_3^{++-}$such that $\partial_{F_1}\left(z_i,z_j\right) = 4$, so $\partial_D\left(z_i,z_j\right) \ge 3$, a contradiction. Hence $q_1 \le 3$. If $q_2 \ge 3$, then there exists $z_i,z_j \in V_3^{+--}$such that $\partial_{F_2}\left(z_i,z_j\right) = 4$, so $\partial_D\left(z_i,z_j\right) \ge 3$, a contradiction. Hence $q_2 \le 2$. If $q_3 \ge 2$, then there exists $z_i,z_j \in V_3^{-+-}$such that $\partial_D\left(z_i,z_j\right) \ge 3$, a contradiction. Hence $q_3 \le 1$. If $q_4 \ge 2$, then there exists $z_i,z_j \in V_3^{--+}$such that $\partial_D\left(z_i,z_j\right) \ge 3$, a contradiction. Hence $q_4 \le 1$. By Lemma 2.1, we also have $\left|V_3^{+++}\right| \le 1$, thus $|V_3| = q \le 1+3+2+1+1 = 8 \le 11$.

Subcase 5: $V_2\backslash\{y_1\} \to x_1 \to y_1, V_2\backslash\{y_2,y_3\} \to x_2 \to \{y_2,y_3\}, y_2 \to x_3 \to V_2\backslash\{y_2\}$.

We know $y_1 \in V_2^{+-+}, y_2 \in V_2^{-+-}, y_3 \in V_2^{-++}$and $y_4 \in V_2^{--+}$. By Lemma 2.1, we can get $V_3^{+-+} = V_3^{-+-} = V_3^{-++} = V_3^{--+} = \emptyset$. Thus $V_3 = V_3^{+++} \cup V_3^{++-} \cup V_3^{+--} \cup V_3^{---}$. Since $\partial_D\left(V_3^{++-},y_i\right) \le 2$ where $i = 1,2,3$, then we have $V_3^{++-} \to y_2$. Since $\partial_D\left(y_i,V_3^{+--}\right) \le 2$ where $i = 1,2,3$, then we have $y_1 \to V_3^{+--}$. Let $\left|V_3^{++-}\right| = q_1, \left|V_3^{+--}\right| = q_2$ and $F_1 = D\left[V_2\backslash\{y_2\} \cup V_3^{++-}\right], F_2 = D\left[V_2\backslash\{y_1\} \cup V_3^{+--}\right]$. Then F_1 and F_2 are respectively an orientation of $K(3,q_1)$ and $K(3,q_2)$ where $q_1,q_2 \le q$. If $q_1 \ge 4$, then there exists $z_i,z_j \in V_3^{++-}$such that $\partial_{F_1}\left(z_i,z_j\right) = 4$, so $\partial_D\left(z_i,z_j\right) \ge 3$, a contradiction. Hence $q_1 \le 3$. If $q_2 \ge 4$, then there exists $z_i,z_j \in V_3^{+--}$such that $\partial_{F_2}\left(z_i,z_j\right) = 4$, so $\partial_D\left(z_i,z_j\right) \ge 3$, a contradiction. Hence $q_2 \le 3$. By Lemma 2.1, we also have $\left|V_3^{+++}\right| \le 1$ and $\left|V_3^{---}\right| \le 1$, thus $|V_3| = q \le 1+3+3+1 = 8 \le 11$.

Subcase 6: $V_2\backslash\{y_1\} \to x_1 \to y_1, V_2\backslash\{y_2,y_3\} \to x_2 \to \{y_2,y_3\}, y_1 \to x_3 \to V_2\backslash\{y_1\}$.

We know $y_1 \in V_2^{+--}, y_2,y_3 \in V_2^{-++}$and $y_4 \in V_2^{--+}$. By Lemma 2.1, we can get $V_3^{+--} = V_3^{-++} = V_3^{--+} = \emptyset$. Thus $V_3 = V_3^{+++} \cup V_3^{++-} \cup V_3^{+-+} \cup V_3^{-+-} \cup V_3^{---}$. Since $\partial_D\left(V_3^{++-},y_i\right) \le 2$ where $i = 1,2,3$, then we have $V_3^{++-} \to y_1$. Since $\partial_D\left(V_3^{+-+},y_i\right) \le 2$ where $i = 1,2,3$, then we have $V_3^{+-+} \to \{y_1,y_4\}$. Since $\partial_D\left(y_i,V_3^{-+-}\right) \le 2$ where $i = 1,2,3$, then we have $\{y_2,y_3\} \to V_3^{-+-}$. Let $\left|V_3^{++-}\right| = q_1, \left|V_3^{+-+}\right| = q_2, \left|V_3^{-+-}\right| = q_3$

and $F_1 = D\left[V_2 \backslash \{y_1\} \cup V_3^{++-}\right], F_2 = D\left[V_2 \backslash \{y_1, y_4\} \cup V_3^{+-+}\right], F_3 = D\left[V_2 \backslash \{y_2, y_3\} \cup V_3^{-+-}\right]$. Then F_1, F_2 and F_3 are respectively an orientation of $K(3, q_1), K(2, q_2)$ and $K(2, q_3)$ where $q_1, q_2, q_3 \le q$. If $q_1 \ge 4$, then there exists $z_i, z_j \in V_3^{++-}$such that $\partial_{F_1}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \ge 3$, a contradiction. Hence $q_1 \le 3$. If $q_2 \ge 3$, then there exists $z_i, z_j \in V_3^{+-+}$such that $\partial_{F_2}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \ge 3$, a contradiction. Hence $q_2 \le 2$. If $q_3 \ge 3$, then there exists $z_i, z_j \in V_3^{-+-}$such that $\partial_{F_3}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \ge 3$, a contradiction. Hence $q_3 \le 2$. By Lemma 2.1, we also have $\left|V_3^{+++}\right| \le 1$ and $\left|V_3^{---}\right| \le 1$, thus $|V_3| = q \le 1 + 3 + 2 + 2 + 1 = 9 \le 11$.

(19) Case $(2, 2, 2)$.

Subcase 1: $V_2 \backslash \{y_1, y_2\} \to V_1 \to \{y_1, y_2\}$.

Take any $y \in \{y_1, y_2\}$ and $z \in V_3$, since $\partial_D(y, z) \le 2$, then we have $y \to z$, this means $\{y_1, y_2\} \to V_3$. So $\partial_D(y_1, y_2) \ge 3$, a contradiction.

Subcase 2: $\{y_3, y_4\} \to \{x_1, x_2\} \to \{y_1, y_2\}, \{y_2, y_4\} \to x_3 \to \{y_1, y_3\}$.

We know $y_1 \in V_2^{+++}, y_2 \in V_2^{++-}, y_3 \in V_2^{--+}$and $y_4 \in V_2^{---}$. By Lemma 2.1, we can get $V_3^{+++} = V_3^{++-} = V_3^{--+} = V_3^{---} = \varnothing$. Thus $V_3 = V_3^{+-+} \cup V_3^{-++} \cup V_3^{+--} \cup V_3^{-+-}$. Take any $z \in V_3$, since $\partial_D(y_1, z) \le 2$ and $\partial_D(z, y_4) \le 2$, then we have $y_1 \to z$ and $z \to y_4$, this means $y_1 \to V_3$ and $V_3 \to y_4$. Since $\partial_D\left(V_3^{+-+}, y_i\right) \le 2$ where $i = 1, 2, 3$, then we have $V_3^{+-+} \to y_3$. Since $\partial_D\left(V_3^{-++}, y_i\right) \le 2$ where $i = 1, 2, 3$, then we have $V_3^{-++} \to y_3$. Since $\partial_D\left(y_i, V_3^{+--}\right) \le 2$ where $i = 1, 2, 3$, then we have $y_2 \to V_3^{+--}$. Since $\partial_D\left(y_i, V_3^{-+-}\right) \le 2$ where $i = 1, 2, 3$, then we have $y_2 \to V_3^{-+-}$. Let $\left|V_3^{+-+}\right| = q_1, \left|V_3^{-++}\right| = q_2, \left|V_3^{+--}\right| = q_3$ and $\left|V_3^{-+-}\right| = q_4$. If $q_1 \ge 2$, then there exists $z_i, z_j \in V_3^{+-+}$such that $\partial_D\left(z_i, z_j\right) \ge 3$, a contradiction. Hence $q_1 \le 1$. If $q_2 \ge 2$, then there exists $z_i, z_j \in V_3^{-++}$such that $\partial_D\left(z_i, z_j\right) \ge 3$, a contradiction. Hence $q_2 \le 1$. If $q_3 \ge 2$, then there exists $z_i, z_j \in V_3^{+--}$such that $\partial_D\left(z_i, z_j\right) \ge 3$, a contradiction. Hence $q_3 \le 1$. If $q_4 \ge 2$, then there exists $z_i, z_j \in V_3^{-+-}$such that $\partial_D\left(z_i, z_j\right) \ge 3$, a contradiction. Hence $q_4 \le 1$. Thus $|V_3| = q \le 1 + 1 + 1 + 1 = 4 \le 11$.

Subcase 3: $\{y_3, y_4\} \to \{x_1, x_2\} \to \{y_1, y_2\} \to x_3 \to \{y_3, y_4\}$.

We know $y_1, y_2 \in V_2^{++-}$and $y_3, y_4 \in V_2^{--+}$. By Lemma 2.1, we can get $V_3^{++-} = V_3^{--+} = \varnothing$. Thus $V_3 = V_3^{+++} \cup V_3^{+-+} \cup V_3^{-++} \cup V_3^{+--} \cup V_3^{-+-} \cup V_3^{---}$. Since $\partial_D\left(V_3^{+-+}, y_i\right) \le 2$ where $i = 1, 2, 3$, then we have $V_3^{+-+} \to \{y_3, y_4\}$. Since $\partial_D\left(V_3^{-++}, y_i\right) \le 2$ where $i = 1, 2, 3$, then we have $V_3^{-++} \to \{y_3, y_4\}$. Since $\partial_D\left(y_i, V_3^{+--}\right) \le 2$ where $i = 1, 2, 3$, then we have $\{y_1, y_2\} \to V_3^{+--}$. Since $\partial_D\left(y_i, V_3^{-+-}\right) \le 2$ where $i = 1, 2, 3$, then we have $\{y_1, y_2\} \to V_3^{-+-}$. Let $\left|V_3^{+-+}\right| = q_1, \left|V_3^{-++}\right| = q_2, \left|V_3^{+--}\right| = q_3, \left|V_3^{-+-}\right| = q_4$ and $F_1 = D\left[V_2^{++-} \cup V_3^{+-+}\right], F_2 = D\left[V_2^{++-} \cup V_3^{-++}\right], F_3 = D\left[V_2^{--+} \cup V_3^{+--}\right], F_4 = D\left[V_2^{--+} \cup V_3^{-+-}\right]$. Then F_1, F_2, F_3 and F_4 are respectively an orientation of $K(2, q_1), K(2, q_2), K(2, q_3)$ and $K(2, q_4)$ where $q_1, q_2, q_3, q_4 \le q$. If $q_1 \ge 3$, then there exists $z_i, z_j \in V_3^{+-+}$such that $\partial_{F_1}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \ge 3$, a contradiction. Hence $q_1 \le 2$. If $q_2 \ge 3$, then there exists $z_i, z_j \in V_3^{-++}$such that $\partial_{F_2}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \ge 3$, a contradiction. Hence $q_2 \le 2$. If $q_3 \ge 3$, then there exists $z_i, z_j \in V_3^{+--}$such that $\partial_{F_3}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \ge 3$, a contradiction. Hence $q_3 \le 2$. If $q_4 \ge 3$, then there exists $z_i, z_j \in V_3^{-+-}$such that $\partial_{F_4}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \ge 3$, a contradiction. Hence $q_4 \le 2$. By Lemma 2.1, we also have $\left|V_3^{+++}\right| \le 1$ and $\left|V_3^{---}\right| \le 1$, thus $|V_3| = q \le 1 + 2 + 2 + 2 + 2 + 1 = 10 \le 11$.

Subcase 4: $\{y_3, y_4\} \to x_1 \to \{y_1, y_2\}, \{y_2, y_4\} \to x_2 \to \{y_1, y_3\}, \{y_2, y_3\} \to x_3 \to \{y_1, y_4\}$.

We know $y_1 \in V_2^{+++}, y_2 \in V_2^{+--}, y_3 \in V_2^{-+-}$and $y_4 \in V_2^{--+}$. By Lemma 2.1, we can get $V_3^{+++} = V_3^{+--} = V_3^{-+-} = V_3^{--+} = \varnothing$. Thus $V_3 = V_3^{++-} \cup V_3^{+-+} \cup V_3^{-++} \cup V_3^{---}$. Take any $z \in V_3$, since $\partial_D(y_1, z) \le 2$, then we have $y_1 \to z$, this means $y_1 \to V_3$. Since $\partial_D\left(V_3^{++-}, y_i\right) \le 2$ where $i = 1, 2, 3$, then we have $V_3^{++-} \to \{y_2, y_3\}$. Since $\partial_D\left(V_3^{+-+}, y_i\right) \le 2$ where $i = 1, 2, 3$, then we have $V_3^{+-+} \to \{y_2, y_4\}$. Since $\partial_D\left(V_3^{-++}, y_i\right) \le 2$ where $i = 1, 2, 3$, then we have $V_3^{-++} \to \{y_3, y_4\}$. Let $\left|V_3^{++-}\right| = q_1, \left|V_3^{+-+}\right| = q_2$ and $\left|V_3^{-++}\right| = q_3$. If $q_1 \ge 2$, then there exists $z_i, z_j \in V_3^{++-}$ such that $\partial_D\left(z_i, z_j\right) \ge 3$, a contradiction. Hence $q_1 \le 1$. If $q_2 \ge 2$, then there exists $z_i, z_j \in V_3^{+-+}$such that $\partial_D\left(z_i, z_j\right) \ge 3$, a contradiction. Hence $q_2 \le 1$. If $q_3 \ge 2$, then there exists

$z_i, z_j \in V_3^{-++}$such that $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_3 \leq 1$. By Lemma 2.1, we also have $\left|V_3^{---}\right| \leq 1$, thus $|V_3| = q \leq 1+1+1+1 = 4 \leq 11$.

Subcase 5: $\{y_3, y_4\} \to x_1 \to \{y_1, y_2\}, \{y_2, y_4\} \to x_2 \to \{y_1, y_3\}, \{y_1, y_4\} \to x_3 \to \{y_2, y_3\}$.

We know $y_1 \in V_2^{++-}, y_2 \in V_2^{+-+}, y_3 \in V_2^{-++}$and $y_4 \in V_2^{---}$. By Lemma 2.1, we can get $V_3^{++-} = V_3^{+-+} = V_3^{-++} = V_3^{---} = \emptyset$. Thus $V_3 = V_3^{+++} \cup V_3^{+--} \cup V_3^{-+-} \cup V_3^{--+}$. Take any $z \in V_3$, since $\partial_D(z, y_4) \leq 2$, then we have $z \to y_4$, this means $V_3 \to y_4$. Since $\partial_D\left(y_i, V_3^{+--}\right) \leq 2$ where $i = 1,2,3$, then we have $\{y_1, y_2\} \to V_3^{+--}$. Since $\partial_D\left(y_i, V_3^{-+-}\right) \leq 2$ where $i = 1,2,3$, then we have $\{y_1, y_3\} \to V_3^{-+-}$. Since $\partial_D\left(y_i, V_3^{--+}\right) \leq 2$ where $i = 1,2,3$, then we have $\{y_2, y_3\} \to V_3^{--+}$. Let $\left|V_3^{+--}\right| = q_1, \left|V_3^{-+-}\right| = q_2$ and $\left|V_3^{--+}\right| = q_3$. If $q_1 \geq 2$, then there exists $z_i, z_j \in V_3^{+--}$ such that $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_1 \leq 1$. If $q_2 \geq 2$, then there exists $z_i, z_j \in V_3^{-+-}$such that $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_2 \leq 1$. If $q_3 \geq 2$, then there exists $z_i, z_j \in V_3^{--+}$such that $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_3 \leq 1$. By Lemma 2.1, we also have $\left|V_3^{+++}\right| \leq 1$, thus $|V_3| = q \leq 1+1+1+1 = 4 \leq 11$.

Subcase 6 : $\{y_3, y_4\} \to x_1 \to \{y_1, y_2\}, \{y_2, y_4\} \to x_2 \to \{y_1, y_3\}, \{y_1, y_2\} \to x_3 \to \{y_3, y_4\}$.

We know $y_1 \in V_2^{++-}, y_2 \in V_2^{+--}, y_3 \in V_2^{-++}$and $y_4 \in V_2^{--+}$. By Lemma 2.1, we can get $V_3^{++-} = V_3^{+--} = V_3^{-++} = V_3^{--+} = \emptyset$. Thus $V_3 = V_3^{+++} \cup V_3^{+-+} \cup V_3^{-+-} \cup V_3^{---}$. Since $\partial_D\left(V_3^{+-+}, y_i\right) \leq 2$ where $i = 1,2,3$, then we have $V_3^{+-+} \to \{y_2, y_4\}$. Since $\partial_D\left(y_i, V_3^{-+-}\right) \leq 2$ where $i = 1,2,3$, then we have $\{y_1, y_3\} \to V_3^{-+-}$. Let $\left|V_3^{+-+}\right| = q_1, \left|V_3^{-+-}\right| = q_2$ and $F_1 = D\left[V_2 \backslash \{y_2, y_4\} \cup V_3^{+-+}\right], F_2 = D\left[V_2 \backslash \{y_1, y_3\} \cup V_3^{-+-}\right]$. Then F_1 and F_2 are respectively an orientation of $K(2, q_1)$ and $K(2, q_2)$ where $q_1, q_2 \leq q$. If $q_1 \geq 3$, then there exists $z_i, z_j \in V_3^{+-+}$such that $\partial_{F_1}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_1 \leq 2$. If $q_2 \geq 3$, then there exists $z_i, z_j \in V_3^{-+-}$such that $\partial_{F_2}\left(z_i, z_j\right) = 4$, so $\partial_D\left(z_i, z_j\right) \geq 3$, a contradiction. Hence $q_2 \leq 2$. By Lemma 2.1, we also have $\left|V_3^{+++}\right| \leq 1$ and $\left|V_3^{---}\right| \leq 1$, thus $|V_3| = q \leq 1+2+2+1 = 6 \leq 11$.

In summary, it can be concluded that if $f(K(3,4,q)) = 2$, then $q \leq 11$. Since when $q \leq 11$, we have found an orientation of diameter 2 of $K(3,4,q)$ in Lemma 4.1. Therefore, $f(K(3,4,q)) = 2$ if and only if $q \leq 11$. ■

References

[1] F. Boesch and R. Tindell. Robbins's theorem for mixed multigraphs. *The American Mathematical Monthly*, 87(9):716–719, 1980.
[2] V. Chvátal and C. Thomassen. Distances in orientations of graphs. *Journal of Combinatorial Theory, Series B*, 24(1):61–75, 1978.
[3] G. Gutin. Minimizing and maximizing the diameter in orientations of graphs. *Graphs and Combinatorics*, 10:225–230, 1994.
[4] K. M. Koh and B. Tan. The diameter of an orientation of a complete multipartite graph. *Discrete Mathematics*, 149(1-3):131–139, 1996.
[5] K. M. Koh and B. Tan. The minimum diameter of orientations of complete multipartite graphs. *Graphs and Combinatorics*, 12:333–339, 1996.
[6] K. M. Koh and E. G. Tay. Optimal orientations of graphs and digraphs: a survey. *Graphs and Combinatorics*, 18(4):745–756, 2002.
[7] J. Liu, G. Rao, J. Zhang, and H. Zhou. Oriented diameter of the complete tripartite graph. *Filomat*, 39(2):675–681, 2025.
[8] J. Liu, G. Rao, and H. Zhou. Oriented diameter of the complete tripartite graph (iii). *arXiv:2502.14903*, 2025.
[9] J. Plesnik. Remarks on diameters of orientations of graphs. *Acta Math. Univ. Comenian*, 46(47):225–236, 1985.
[10] H. E. Robbins. A theorem on graphs, with an application to a problem of traffic control. *The American Mathematical Monthly*, 46(5):281–283, 1939.
[11] L. Šoltés. Orientations of graphs minimizing the radius or the diameter. *Mathematica Slovaca*, 36(3):289–296, 1986.