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On polynomial roots of class A operators

Sungeun Jung[a]

[a]*Department of Mathematics, Hankuk University of Foreign Studies, Gyeonggi-do, 17035, South Korea*

Abstract. In this paper, we show that if $T \in \mathcal{L}(\mathcal{H})$ is a polynomial root of a class A operator, i.e. $p(T)$ belongs to class A for some nonconstant polynomial p, then T is subscalar. As a corollary, we obtain that such an operator with thick spectrum has a nontrivial invariant subspace. We also prove that an algebraic extension of a polynomial root of a class A operator is subscalar and provide its several properties.

1. Introduction

Let $\mathcal{H}$ be a complex separable Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H})$, we write $\sigma_p(T)$, $\sigma(T)$, $\sigma_{le}(T)$, $\sigma_{re}(T)$, and $\sigma_e(T)$ for the point spectrum, spectrum, left essential spectrum, right essential spectrum, and essential spectrum of T, respectively.

If $T = U|T|$ is the polar decomposition of an operator $T \in \mathcal{L}(\mathcal{H})$, we define the *Aluthge transform* of T, denoted throughout this paper by $\widetilde{T}$, as $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. For an arbitrary operator $T \in \mathcal{L}(\mathcal{H})$, the sequence $\{\widetilde{T}^{(n)}\}$ of Aluthge iterates of T is given by $\widetilde{T}^{(0)} = T$ and $\widetilde{T}^{(n+1)} = \widetilde{(\widetilde{T}^{(n)})}$ for every positive integer n.

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *p-hyponormal* if $(TT^*)^p \leq (T^*T)^p$, where $0 < p < \infty$. In particular, if $p = 1$ or $p = \frac{1}{2}$, then T is called *hyponormal* or *semi-hyponormal*, respectively. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *w-hyponormal* if $|\widetilde{T}| \geq |T| \geq |\widetilde{T}^*|$ (see [2]). We say that an operator $T \in \mathcal{L}(\mathcal{H})$ is a *class A operator* (or *belongs to class A*) if the absolute condition $|T|^2 \leq |T^2|$ holds. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *normaloid* if $\|T\| = r(T)$. It is well-known from [12] that

$$\begin{aligned}\{\text{hyponormal operators}\} &\subset \{p\text{-hyponormal operators}\}\ (0 < p \leq 1)\\ &\subset \{w\text{-hyponormal operators}\}\\ &\subset \{\text{class } A \text{ operators}\}\\ &\subset \{\text{normaloid operators}\}.\end{aligned}$$

There are a lot of meaningful results concerning class A operators ([11], [12], [14], [15], [16], [25], etc). Especially, T. Furuta gave several examples of class A operators in [11]. M. Ito and T. Yamazaki showed in [14] that if $T \in \mathcal{L}(\mathcal{H})$ belongs to class A, then so does T^n for each positive integer n. In particular, the square of a class A operator is always w-hyponormal. It was proved in [16] that every class A operator is subscalar.

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This research was supported by Hankuk University of Foreign Studies Research Fund.
Email address: sungeun@hufs.ac.kr (Sungeun Jung)
ORCID iD: https://orcid.org/0000-0003-3942-1550 (Sungeun Jung)

An operator $T \in \mathcal{L}(\mathcal{H})$ is called a *polynomial root of a class A operator* if there is a nonconstant polynomial p such that $p(T)$ belongs to class A. It is evident that every class A operator is a polynomial root of a class A operator, but the converse fails to hold. For example, let $T := N \oplus S$ where $N \in \mathcal{L}(\mathcal{H})$ is a nilpotent operator of order 2 and $S \in \mathcal{L}(\mathcal{H})$ is a class A operator. Since N is not a class A operator, neither is T. On the other hand, $T^2 = 0 \oplus S^2$ belongs to class A by [14, Corollary 5], and hence T is a polynomial root of a class A operator. We refer to [5], [15], [17], and [19] for more details about polynomial roots of class A operators.

An operator $S \in \mathcal{L}(\mathcal{H})$ is called *scalar* of order m if it possesses a spectral distribution of order m, i.e. if there is a continuous unital morphism of topological algebras

$$\Phi : C_0^m(\mathbb{C}) \to \mathcal{L}(\mathcal{H})$$

such that $\Phi(z) = S$, where as usual z stands for the identical function on $\mathbb{C}$ and $C_0^m(\mathbb{C})$ for the space of all compactly supported functions continuously differentiable of order m, $0 \leq m \leq \infty$. We say that an operator is *subscalar* of order m if it is similar to the restriction of a scalar operator of order m to an invariant subspace.

In 1984, M. Putinar proved that every hyponormal operator is subscalar of order 2 (see [27]). S. Brown used this result to show that any hyponormal operator with thick spectrum has a nontrivial invariant subspace (see [4]). This nice functional model has been motivating the study on subscalar operators. For instance, E. Ko verified in [19] that if T^k is p-hyponormal for some positive integer k, then T is subscalar of order $4k$. Furthermore, it turned out in [17] that if $p(T)$ is hyponormal for some nonconstant polynomial p and $\sigma(T)$ contains no zeros of p', then T is subscalar of order 2.

In this paper, we show that if $T \in \mathcal{L}(\mathcal{H})$ is a polynomial root of a class A operator, then it is subscalar. As a corollary, we obtain that such an operator with thick spectrum has a nontrivial invariant subspace. We also prove that an algebraic extension of a polynomial root of a class A operator is subscalar and provide its several properties.

2. Preliminaries

An operator $T \in \mathcal{L}(\mathcal{H})$ is called *left semi-Fredholm* if T has closed range and $\dim(\ker(T)) < \infty$, and T is called *right semi-Fredholm* if T has closed range and $\dim(\mathcal{H}/\mathrm{ran}(T)) < \infty$. When T is either left semi-Fredholm or right semi-Fredholm, T is called *semi-Fredholm*. In this case, the *Fredholm index* of T is defined by $\mathrm{ind}(T) := \dim(\ker(T)) - \dim(\mathcal{H}/\mathrm{ran}(T))$. We say that T is *Fredholm* if it is both left and right semi-Fredholm. In particular, a Fredholm operator of index zero is said to be *Weyl*. The *Weyl spectrum* is given by $\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}$ and we write $\pi_{00}(T) := \{\lambda \in \mathrm{iso}(\sigma(T)) : 0 < \dim(\ker(T - \lambda)) < \infty\}$ where $\mathrm{iso}(\sigma(T))$ denotes the set of all isolated points of $\sigma(T)$. We say that *Weyl's theorem holds for T* if $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$. A *hole* in $\sigma_e(T)$ is a nonempty bounded component of $\mathbb{C} \setminus \sigma_e(T)$, and a *pseudohole* in $\sigma_e(T)$ is a nonempty component of $\sigma_e(T) \setminus \sigma_{le}(T)$ or $\sigma_e(T) \setminus \sigma_{re}(T)$. The *spectral picture* of T is the structure consisting of $\sigma_e(T)$, the collection of holes and pseudoholes in $\sigma_e(T)$, and it is denoted by $SP(T)$ (see [26] for more details).

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have the *single-valued extension property*, abbreviated SVEP, if for every open subset G of $\mathbb{C}$, the only analytic solution $f : G \to \mathcal{H}$ of the equation $(T - z)f(z) \equiv 0$ on G is the zero function on G. For example, every operator $T \in \mathcal{L}(\mathcal{H})$ whose point spectrum $\sigma_p(T)$ has empty interior satisfies SVEP. For $T \in \mathcal{L}(\mathcal{H})$ and $x \in \mathcal{H}$, *the local resolvent set* $\rho_T(x)$ of T at x is defined to be the union of every open set G in $\mathbb{C}$ on which $(T - z)f(z) \equiv x$ for some analytic function $f : G \to \mathcal{H}$. We denote the complement of $\rho_T(x)$ by $\sigma_T(x)$, called the *local spectrum* of T at x. It is trivial that every local resolvent set $\rho_T(x)$ contains $\rho(T)$, since $(T - z)[(T - z)^{-1}x] = x$ for all $z \in \rho(T)$. Hence $\sigma_T(x)$ is a closed subset of $\sigma(T)$. If T has SVEP, then there exists a unique analytic extension $f : \rho_T(x) \to \mathcal{H}$ of the function $(T - z)^{-1}x : \rho(T) \to \mathcal{H}$ such that $(T - z)f(z) = x$ for all $z \in \rho_T(x)$.

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *Bishop's property* (β) if for every open subset G of $\mathbb{C}$ and every sequence $f_n : G \to \mathcal{H}$ of $\mathcal{H}$-valued analytic functions such that $\{(T - z)f_n(z)\}$ converges uniformly to 0 in norm on compact subsets of G, the sequence $\{f_n(z)\}$ converges uniformly to 0 in norm on compact subsets of G. The *local spectral subspace* of T is given by $H_T(F) = \{x \in \mathcal{H} : \sigma_T(x) \subset F\}$ for each subset F of $\mathbb{C}$. We say that $T \in \mathcal{L}(\mathcal{H})$ has *Dunford's property* (C) if $H_T(F)$ is closed for each closed subset F of $\mathbb{C}$. We know that

$$\text{Bishop's property } (\beta) \Rightarrow \text{Dunford's property } (C) \Rightarrow \text{SVEP}$$

and each of the converse implications fails to hold, in general.

We say that an operator $T \in \mathcal{L}(\mathcal{H})$ is *decomposable* provided that for every open cover $\{G_1, G_2\}$ of $\mathbb{C}$, there are T-invariant subspaces $\mathcal{M}_1$ and $\mathcal{M}_2$ such that $\mathcal{H} = \mathcal{M}_1 + \mathcal{M}_2$ and $\sigma(T|_{\mathcal{M}_j}) \subset G_j$ for $j = 1, 2$. For an operator $T \in \mathcal{L}(\mathcal{H})$ and a closed subset F of $\mathbb{C}$, the *glocal spectral subspace* $\mathcal{H}_T(F)$ of T is defined to consist of all $x \in \mathcal{H}$ such that there is an analytic function $f : \mathbb{C} \setminus F \to \mathcal{H}$ for which $(T - z)f(z) \equiv x$ on $\mathbb{C} \setminus F$. Clearly, if T has SVEP, then $H_T(F) = \mathcal{H}_T(F)$ for any closed subset F of $\mathbb{C}$. We say that an operator $T \in \mathcal{L}(\mathcal{H})$ has *the decomposition property* (δ) if the decomposition $\mathcal{H} = \mathcal{H}_T(\overline{G_1}) + \mathcal{H}_T(\overline{G_2})$ holds for every open cover $\{G_1, G_2\}$ of $\mathbb{C}$. If T has property (β), then its adjoint T^* has property (δ). We also point out that if T is decomposable, then T has properties (β) and (δ), and vice versa. We refer to [6] or [20] for further details on local spectral theory.

Let z be the coordinate in $\mathbb{C}$, and let $d\mu(z)$, or simply $d\mu$, be the planar Lebesgue measure. Let U be a bounded open subset of $\mathbb{C}$. We denote by $L^2(U, \mathcal{H})$ the Hilbert space of measurable functions $f : U \to \mathcal{H}$ such that

$$\|f\|_{2,U} = \left(\int_U \|f(z)\|^2 d\mu(z) \right)^{\frac{1}{2}} < \infty.$$

We denote the space $L^2(U, \mathcal{H}) \cap O(U, \mathcal{H})$ by $A^2(U, \mathcal{H})$, where $O(U, \mathcal{H})$ is the Fréchet space of $\mathcal{H}$-valued analytic functions on U. Then $A^2(U, \mathcal{H})$ is a closed subspace, and the orthogonal projection of $L^2(U, \mathcal{H})$ onto this space will be denoted by P.

For a bounded open subset U of $\mathbb{C}$ and a fixed nonnegative integer m, let $W^m(U, \mathcal{H})$ denote the vector-valued Sobolev space of functions $f \in L^2(U, \mathcal{H})$ whose derivatives $\bar{\partial} f, \bar{\partial}^2 f, \cdots \bar{\partial}^m f$ in the sense of distributions still belong to $L^2(U, \mathcal{H})$. Endowed with the norm

$$\|f\|^2_{W^m} = \sum_{i=0}^{m} \|\bar{\partial}^i f\|^2_{2,U},$$

$W^m(U, \mathcal{H})$ becomes a Hilbert space contained continuously in $L^2(U, \mathcal{H})$. The linear operator M of multiplication by z on $W^m(U, \mathcal{H})$ is continuous and it has a spectral distribution u of order m defined by the following relation; for $\varphi \in C_0^m(\mathbb{C})$ and $f \in W^m(U, \mathcal{H})$, $u(\varphi)f = \varphi f$. Hence M is a scalar operator of order m.

We shall use the Cauchy-Pompeiu formula, given as follows, for the case when D is a bounded open disk in $\mathbb{C}$: if $f \in C^2(\bar{D}, \mathcal{H})$ and $z \in D$, then

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} \, d\zeta + \bar{\partial} f * (-\frac{1}{\pi z})$$

where $*$ stands for the convolution product. Remark that the first integral in the right-hand side is in $A^2(D, \mathcal{H})$ and that $\int_{|z|<r} \frac{1}{|z|} \, d\mu = 2\pi r$ for $r > 0$.

3. Subscalarity

In this section, we show that every polynomial root of a class A operator has a scalar extension. We begin with the following proposition, which is a generalization of [27, Proposition 2.1] and [19, Theorem 3.1].

Proposition 3.1. *Let p be any polynomial of degree k, and let D be any bounded disk in $\mathbb{C}$. Then there exists a constant C_D, depending only on D, such that for all $S \in \mathcal{L}(\mathcal{H})$ and $f \in W^{2k}(D,\mathcal{H})$ we have*

$$\|(I-P)f\|_{2,D} \le C_D \sum_{i=k}^{2k} \|(S-p(z))^* \bar{\partial}^i f\|_{2,D}$$

where P denotes the orthogonal projection of $L^2(D,\mathcal{H})$ onto $A^2(D,\mathcal{H})$.

Proof. Let $f \in W^{2k}(D,\mathcal{H})$ be given, and let $s_j \in C_0^\infty(\bar{D},\mathcal{H})$ be such that $s_j \equiv 1$ on $D-D$ for $j=1,2,\cdots,k$. Since $C^\infty(\bar{D},\mathcal{H})$ is dense in $W^{2k}(D,\mathcal{H})$, there exists a sequence $\{f_n\}_{n=1}^\infty$ in $C^\infty(\bar{D},\mathcal{H})$ such that

$$\lim_{n\to\infty} \|f-f_n\|_{W^{2k}} = 0.$$

For any fixed $n \in \mathbb{N}$, observe that

$$\begin{aligned}\bar{\partial}^k[f_n + \frac{1}{k!}(S-p(z))^*\bar{\partial}^k f_n] &= \bar{\partial}^k f_n + \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}\bar{\partial}^j(S-p(z))^*\bar{\partial}^{2k-j}f_n \\ &= \frac{1}{k!}\sum_{j=0}^{k-1}\binom{k}{j}\bar{\partial}^j(S-p(z))^*\bar{\partial}^{2k-j}f_n. \qquad (1)\end{aligned}$$

Using Cauchy-Pompeiu formula and (1), we obtain that

$$\begin{aligned}&\bar{\partial}^{k-1}[f_n + \frac{1}{k!}(S-p(z))^*\bar{\partial}^k f_n] \\ =\ & g_{1,n} + \Big[\frac{1}{k!}\sum_{j=0}^{k-1}\binom{k}{j}\bar{\partial}^j(S-p(z))^*\bar{\partial}^{2k-j}f_n\Big] * (-\frac{s_1}{\pi z}). \qquad (2)\end{aligned}$$

where

$$g_{1,n}(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{\bar{\partial}^{k-1}[f_n(\zeta) + \frac{1}{k!}(S-p(\zeta))^*\bar{\partial}^k f_n(\zeta)]}{\zeta - z}d\zeta \in A^2(D,\mathcal{H}).$$

Claim I. For $t = 1,2,\cdots,k$, we have

$$\begin{aligned}&\bar{\partial}^{k-t}[f_n + \frac{1}{k!}(S-p(z))^*\bar{\partial}^k f_n] \\ =\ & g_{t,n} + g_{t-1,n} * (-\frac{s_t}{\pi z}) + g_{t-2,n} * (-\frac{s_{t-1}}{\pi z}) * (-\frac{s_t}{\pi z}) + \cdots + g_{1,n} * (-\frac{s_2}{\pi z}) * \cdots * (-\frac{s_t}{\pi z}) \\ +\ & \Big[\frac{1}{k!}\sum_{j=0}^{k-1}\binom{k}{j}\bar{\partial}^j(S-p(z))^*\bar{\partial}^{2k-j}f_n\Big] * (-\frac{s_1}{\pi z}) * \cdots * (-\frac{s_t}{\pi z})\end{aligned}$$

where

$$g_{r,n} = \frac{1}{2\pi i}\int_{\partial D}\frac{\bar{\partial}^{k-r}[f_n(\zeta) + \frac{1}{k!}(S-p(\zeta))^*\bar{\partial}^k f_n(\zeta)]}{\zeta - z}d\zeta \in A^2(D,\mathcal{H}) \text{ if } r > 0$$

and $g_{r,n} = 0$ if $r \le 0$.

We will use induction on t. The case when $t=1$ was already proved in (2). Now assume that the claim holds for some $t=r$ with $0 \le r < k$. Then Cauchy-Pompeiu formula ensures that

$$\begin{aligned}&\bar{\partial}^{k-r-1}[f_n + \frac{1}{k!}(S-p(z))^*\bar{\partial}^k f_n] \\ =\ & g_{r+1,n} + \bar{\partial}^{k-r}[f_n + \frac{1}{k!}(S-p(z))^*\bar{\partial}^k f_n] * (-\frac{s_{r+1}}{\pi z}).\end{aligned}$$

where

$$g_{r+1,n}(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{\bar{\partial}^{k-r-1}[f_n(\zeta)+\frac{1}{k!}(S-p(\zeta))^*\bar{\partial}^k f_n(\zeta)]}{\zeta - z}d\zeta \in A^2(D,\mathcal{H}).$$

By the induction hypothesis,

$$\begin{aligned} & \bar{\partial}^{k-r-1}[f_n + \frac{1}{k!}(S-p(z))^*\bar{\partial}^k f_n] \\ = \;& g_{r+1,n} + g_{r,n} * (-\frac{s_{r+1}}{\pi z}) + g_{r-1,n} * (-\frac{s_r}{\pi z}) * (-\frac{s_{r+1}}{\pi z}) + \cdots + g_{1,n} * (-\frac{s_2}{\pi z}) * \cdots * (-\frac{s_r}{\pi z}) * (-\frac{s_{r+1}}{\pi z}) \\ + \;& \Big[\frac{1}{k!}\sum_{j=0}^{k-1}\binom{k}{j}\bar{\partial}^j(S-p(z))^*\bar{\partial}^{2k-j}f_n\Big] * (-\frac{s_1}{\pi z}) * \cdots * (-\frac{s_r}{\pi z}) * (-\frac{s_{r+1}}{\pi z}), \end{aligned}$$

completing the proof of Claim I.

From Claim I with $t = k$, we obtain that

$$\begin{aligned} & f_n + \frac{1}{k!}(S-p(z))^*\bar{\partial}^k f_n \\ = \;& g_{k,n} + g_{k-1,n} * (-\frac{s_k}{\pi z}) + g_{k-2,n} * (-\frac{s_{k-1}}{\pi z}) * (-\frac{s_k}{\pi z}) + \cdots + g_{1,n} * (-\frac{s_2}{\pi z}) * \cdots * (-\frac{s_k}{\pi z}) \\ + \;& \Big[\frac{1}{k!}\sum_{j=0}^{k-1}\binom{k}{j}\bar{\partial}^j(S-p(z))^*\bar{\partial}^{2k-j}f_n\Big] * (-\frac{s_1}{\pi z}) * \cdots * (-\frac{s_k}{\pi z}). \end{aligned}$$

Put

$$g_n = g_{k,n} + g_{k-1,n} * (-\frac{s_k}{\pi z}) + g_{k-2,n} * (-\frac{s_{k-1}}{\pi z}) * (-\frac{s_k}{\pi z}) + \cdots + g_{1,n} * (-\frac{s_2}{\pi z}) * \cdots * (-\frac{s_k}{\pi z}).$$

Then $g_n \in A^2(D,\mathcal{H})$ and

$$\begin{aligned} & f_n + \frac{1}{k!}(S-p(z))^*\bar{\partial}^k f_n \\ = \;& g_n + \Big[\frac{1}{k!}\sum_{j=0}^{k-1}\binom{k}{j}\bar{\partial}^j(S-p(z))^*\bar{\partial}^{2k-j}f_n\Big] * (-\frac{s_1}{\pi z}) * \cdots * (-\frac{s_k}{\pi z}). \end{aligned} \tag{3}$$

Claim II. It holds for all $j = 0, 1, 2, \cdots, k-1$ that

$$\begin{aligned} & [\bar{\partial}^j(S-p(z))^*\bar{\partial}^{2k-j}f_n] * (-\frac{s_1}{\pi z}) * \cdots * (-\frac{s_k}{\pi z}) \\ = \;& \sum_{t=0}^{j}(-1)^t\binom{j}{t}\Big[(S-p(z))^*\bar{\partial}^{2k-(j-t)}f_n\Big] * (-\frac{s_{j-t+1}}{\pi z}) * \cdots * (-\frac{s_k}{\pi z}). \end{aligned}$$

This claim is clearly true for $j = 0$. Suppose that Claim II holds for $j = r$. We know that for any $F \in C^\infty(\bar{D}, \mathcal{L}(\mathcal{H}))$ and $\varphi \in C^\infty(\bar{D}, \mathcal{H})$,

$$((\bar{\partial}F)\varphi) * (-\frac{1}{\pi z}) = F\varphi - (F(\bar{\partial}\varphi)) * (-\frac{1}{\pi z}). \tag{4}$$

According to (4) and the induction hypothesis, we obtain that

$$\begin{aligned}
& [\bar{\partial}^{r+1}(S-p(z))^*\bar{\partial}^{2k-r-1}f_n]*(-\frac{s_1}{\pi z})*\cdots*(-\frac{s_k}{\pi z}) \\
= & \left\{\bar{\partial}^{r}(S-p(z))^*\bar{\partial}^{2k-r-1}f_n - \left[\bar{\partial}^{r}(S-p(z))^*\bar{\partial}^{2k-r}f_n\right]*(-\frac{s_1}{\pi z})\right\}*(-\frac{s_2}{\pi z})*\cdots*(-\frac{s_k}{\pi z}) \\
= & \left[\bar{\partial}^{r}(S-p(z))^*\bar{\partial}^{2k-r-1}f_n\right]*(-\frac{s_2}{\pi z})*\cdots*(-\frac{s_k}{\pi z}) \\
- & \sum_{t=0}^{r}(-1)^t\binom{r}{t}\left[(S-p(z))^*\bar{\partial}^{2k-(r-t)}f_n\right]*(-\frac{s_{r-t+1}}{\pi z})*\cdots*(-\frac{s_k}{\pi z}).
\end{aligned} \tag{5}$$

In addition, from (4) we can show that for all $\ell = 0, 1, 2, \cdots, r$,

$$\begin{aligned}
& \left[\bar{\partial}^{r}(S-p(z))^*\bar{\partial}^{2k-r-1}f_n\right]*(-\frac{s_2}{\pi z})*\cdots*(-\frac{s_k}{\pi z}) \\
= & \sum_{t=0}^{\ell}(-1)^t\binom{\ell}{t}\left[\bar{\partial}^{r-\ell}(S-p(z))^*\bar{\partial}^{2k-(r+1-t)}f_n\right]*(-\frac{s_{\ell-t+2}}{\pi z})*\cdots*(-\frac{s_k}{\pi z}).
\end{aligned}$$

In particular, taking $\ell = r$, we obtain that

$$\begin{aligned}
& \left[\bar{\partial}^{r}(S-p(z))^*\bar{\partial}^{2k-r-1}f_n\right]*(-\frac{s_2}{\pi z})*\cdots*(-\frac{s_k}{\pi z}) \\
= & \sum_{t=0}^{r}(-1)^t\binom{r}{t}\left[(S-p(z))^*\bar{\partial}^{2k-(r+1-t)}f_n\right]*(-\frac{s_{r-t+2}}{\pi z})*\cdots*(-\frac{s_k}{\pi z}).
\end{aligned} \tag{6}$$

Applying (5) and (6), we have

$$\begin{aligned}
& [\bar{\partial}^{r+1}(S-p(z))^*\bar{\partial}^{2k-r-1}f_n]*(-\frac{s_1}{\pi z})*\cdots*(-\frac{s_k}{\pi z}) \\
= & \sum_{t=0}^{r}(-1)^t\binom{r}{t}\left[(S-p(z))^*\bar{\partial}^{2k-(r+1-t)}f_n\right]*(-\frac{s_{r-t+2}}{\pi z})*\cdots*(-\frac{s_k}{\pi z}) \\
- & \sum_{t=0}^{r}(-1)^t\binom{r}{t}\left[(S-p(z))^*\bar{\partial}^{2k-(r-t)}f_n\right]*(-\frac{s_{r-t+1}}{\pi z})*\cdots*(-\frac{s_k}{\pi z}) \\
= & \sum_{t=0}^{r+1}(-1)^t\binom{r+1}{t}\left[(S-p(z))^*\bar{\partial}^{2k-(r+1-t)}f_n\right]*(-\frac{s_{r-t+2}}{\pi z})*\cdots*(-\frac{s_k}{\pi z}).
\end{aligned}$$

Hence we complete the proof of Claim II, by induction.

From Claim II and (3), it follows that

$$\begin{aligned}
& f_n - g_n + \frac{1}{k!}(S-p(z))^*\bar{\partial}^k f_n \\
= & \frac{1}{k!}\sum_{j=1}^{k-1}\sum_{t=0}^{j}\binom{k}{j}\binom{j}{t}(-1)^t\left[(S-p(z))^*\bar{\partial}^{2k-(j-t)}f_n\right]*(-\frac{s_{j-t+1}}{\pi z})*\cdots*(-\frac{s_k}{\pi z}).
\end{aligned}$$

If R is the radius of D, then $u*(-\frac{s_j}{\pi z})$ is a L^2-function with

$$\|u*(-\frac{s_j}{\pi z})\|_{2,D} \le 4R\|u\|_{2,D}$$

for any $u \in L^2(D,\mathcal{H})$ and $j = 1, 2, \cdots, k$, which ensures that there is a constant C_D, depending only on D, such that

$$\|f_n - g_n\|_{2,D} \leq C_D \sum_{i=k}^{2k} \|(S - p(z))^* \bar{\partial}^i f_n\|_{2,D}.$$

Since $g_n \in A^2(D,\mathcal{H})$, it holds that

$$\begin{aligned} \|(I-P)f\|_{2,D} &\leq \|f - g_n\| \\ &\leq \|f - f_n\|_{2,D} + \|f_n - g_n\|_{2,D} \\ &\leq \|f - f_n\|_{2,D} + C_D \sum_{i=k}^{2k} \|(S - p(z))^* \bar{\partial}^i f_n\|_{2,D}. \end{aligned} \tag{7}$$

Letting $n \to \infty$ in (7), we get what we desired. □

Corollary 3.2. *Let $S \in \mathcal{L}(\mathcal{H})$ be a hyponormal operator and D any bounded disk in $\mathbb{C}$. If p is a polynomial of degree k, then there exists a constant C_D, depending only on D, such that for all $f \in W^{2k}(D,\mathcal{H})$ we have*

$$\|(I-P)f\|_{2,D} \leq C_D \sum_{i=k}^{2k} \|(S - p(z))\bar{\partial}^i f\|_{2,D}$$

where P denotes the orthogonal projection of $L^2(D,\mathcal{H})$ onto $A^2(D,\mathcal{H})$.

Proof. If S is hyponormal, then

$$\|(S - p(z))^* g\|_{2,D} \leq \|(S - p(z))g\|_{2,D}$$

for all $g \in L^2(D,\mathcal{H})$. Hence, the proof follows from Proposition 3.1. □

Lemma 3.3. *Let $T \in \mathcal{L}(\mathcal{H})$ be a polynomial root of a class A operator such that $p(T)$ belongs to class A for some nonconstant polynomial p of degree k. For any bounded disk D in $\mathbb{C}$ containing $\sigma(T)$, define the map $V : \mathcal{H} \to H(D)$ by*

$$Vh = \widetilde{1 \otimes h} \left(\equiv 1 \otimes h + \overline{(T - z)W^{12k}(D,\mathcal{H})} \right)$$

where $H(D) := W^{12k}(D,\mathcal{H})/\overline{(T - z)W^{12k}(D,\mathcal{H})}$ and $1 \otimes h$ denotes the constant function sending any $z \in D$ to h. Then V is one-to-one and has closed range.

Proof. Let $f_n \in W^{12k}(D,\mathcal{H})$ and $h_n \in \mathcal{H}$ be sequences such that

$$\lim_{n\to\infty} \|(T - z)f_n + 1 \otimes h_n\|_{W^{12k}} = 0. \tag{8}$$

Equation (8) implies that

$$\lim_{n\to\infty} \|(T - z)\bar{\partial}^i f_n\|_{2,D} = 0$$

for $i = 1, 2, \cdots, 12k$. Since $T - z$ divides $p(T) - p(z)$, we see that

$$\lim_{n\to\infty} \|(p(T) - p(z))\bar{\partial}^i f_n\|_{2,D} = 0$$

for $i = 1, 2, \cdots, 12k$. Set $q(z) = p(z)^2$. Since $\widetilde{R}|R|^{\frac{1}{2}} = |R|^{\frac{1}{2}}R$ for any $R \in \mathcal{L}(\mathcal{H})$, we obtain that

$$\begin{cases} \lim_{n\to\infty} \|(q(T) - q(z))\bar{\partial}^i f_n\|_{2,D} = 0 \\ \lim_{n\to\infty} \|(\widetilde{q(T)} - q(z))|q(T)|^{\frac{1}{2}}\bar{\partial}^i f_n\|_{2,D} = 0 \\ \lim_{n\to\infty} \|(\widetilde{q(T)}^{(2)} - q(z))|\widetilde{q(T)}|^{\frac{1}{2}}|q(T)|^{\frac{1}{2}}\bar{\partial}^i f_n\|_{2,D} = 0 \end{cases} \tag{9}$$

for $i = 1, 2 \cdots, 12k$. Since $q(T)$ is a w-hyponormal operator by [14, Corollary 5], the operator $\widetilde{q(T)}^{(2)}$ is hyponormal. It follows from Corollary 3.2 and (9) that

$$\lim_{n\to\infty} \|(I_{\mathcal{H}} - P)|\widetilde{q(T)}|^{\frac{1}{2}}|q(T)|^{\frac{1}{2}}\bar{\partial}^i f_n\|_{2,D} = 0 \tag{10}$$

for $i = 0, 1, 2, \cdots, 8k$, where $I_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$ and P denotes the orthogonal projection of $L^2(D, \mathcal{H})$ onto $A^2(D, \mathcal{H})$. Therefore, (9) and (10) imply that

$$\lim_{n\to\infty} \|(\widetilde{q(T)}^{(2)} - q(z))P|\widetilde{q(T)}|^{\frac{1}{2}}|q(T)|^{\frac{1}{2}}\bar{\partial}^i f_n\|_{2,D} = 0$$

for $i = 1, 2, \cdots, 8k$. Let $q(T) = U|q(T)|$ and $\widetilde{q(T)} = V|\widetilde{q(T)}|$ be the polar decompositions of $q(T)$ and $\widetilde{q(T)}$, respectively. Since $U|q(T)|^{\frac{1}{2}}\widetilde{q(T)} = q(T)U|q(T)|^{\frac{1}{2}}$ and $V|\widetilde{q(T)}|^{\frac{1}{2}}\widetilde{q(T)}^{(2)} = \widetilde{q(T)}V|\widetilde{q(T)}|^{\frac{1}{2}}$, we obtain that

$$\begin{cases} \lim_{n\to\infty} \|(\widetilde{q(T)} - q(z))V|\widetilde{q(T)}|^{\frac{1}{2}}P|\widetilde{q(T)}|^{\frac{1}{2}}|q(T)|^{\frac{1}{2}}\bar{\partial}^i f_n\|_{2,D} = 0 \\ \lim_{n\to\infty} \|(q(T) - q(z))U|q(T)|^{\frac{1}{2}}V|\widetilde{q(T)}|^{\frac{1}{2}}P|\widetilde{q(T)}|^{\frac{1}{2}}|q(T)|^{\frac{1}{2}}\bar{\partial}^i f_n\|_{2,D} = 0 \end{cases}$$

for $i = 1, 2, \cdots, 8k$, which yields that

$$\begin{cases} \lim_{n\to\infty} \|(\widetilde{q(T)} - q(z))\widetilde{q(T)}P|q(T)|^{\frac{1}{2}}\bar{\partial}^i f_n\|_{2,D} = 0 \\ \lim_{n\to\infty} \|(q(T) - q(z))U|q(T)|^{\frac{1}{2}}\widetilde{q(T)}P|q(T)|^{\frac{1}{2}}\bar{\partial}^i f_n\|_{2,D} = 0 \end{cases} \tag{11}$$

for $i = 1, 2, \cdots, 8k$. Write

$$q(\lambda) - q(z) = (\lambda - c_1 z)(\lambda - c_2 z)\cdots(\lambda - c_{2k} z)$$

for some constants $c_1, c_2, \cdots, c_{2k}$; here, each c_j is nonzero since $q(\lambda) - q(z)$ is a polynomial of degree $2k$ in z. Then, it holds that

$$\lim_{n\to\infty} \|(T - c_1 z)(T - c_2 z)\cdots(T - c_{2k} z)U|q(T)|^{\frac{1}{2}}\widetilde{q(T)}P|q(T)|^{\frac{1}{2}}\bar{\partial}^i f_n\|_{2,D} = 0$$

for $i = 1, 2, \cdots, 8k$. Dividing both sides by c_1, we have

$$\lim_{n\to\infty} \|(\frac{1}{c_1}T - z)(T - c_2 z)\cdots(T - c_{2k} z)U|q(T)|^{\frac{1}{2}}\widetilde{q(T)}P|q(T)|^{\frac{1}{2}}\bar{\partial}^i f_n\|_{2,D} = 0$$

for $i = 1, 2, \cdots, 8k$. Since every class A operator has property (β) by [16, Corollary 3.4], we know from [20, Theorem 3.3.9] that every polynomial root of a class A operator has property (β). Thus $\frac{1}{c_1}T$ has property (β), and so

$$\lim_{n\to\infty} \|(T - c_2 z)\cdots(T - c_{2k} z)U|q(T)|^{\frac{1}{2}}\widetilde{q(T)}P|q(T)|^{\frac{1}{2}}\bar{\partial}^i f_n\|_{2,D} = 0$$

for $i = 1, 2, \cdots, 8k$. After repeating this procedure $2k$ times, we finally obtain that

$$\lim_{n\to\infty} \|U|q(T)|^{\frac{1}{2}}\widetilde{q(T)}P|q(T)|^{\frac{1}{2}}\bar{\partial}^i f_n\|_{2,D} = 0$$

for $i = 1, 2, \cdots, 8k$, which implies that

$$\lim_{n\to\infty} \|(\widetilde{q(T)})^2 P|q(T)|^{\frac{1}{2}} \bar{\partial}^i f_n\|_{2,D} = 0 \tag{12}$$

for $i = 1, 2, \cdots, 8k$. Using (11) and (12), we see that

$$\lim_{n\to\infty} \|q(z)\widetilde{q(T)} P|q(T)|^{\frac{1}{2}} \bar{\partial}^i f_n\|_{2,D} = 0$$

for $i = 1, 2, \cdots, 8k$. If $w_1, w_2 \cdots, w_{2k}$ are the zeros of $q(z)$, then

$$\lim_{n\to\infty} \|(w_1 I_{\mathcal{H}} - z)(w_2 I_{\mathcal{H}} - z) \cdots (w_{2k} I_{\mathcal{H}} - z)\widetilde{q(T)} P|q(T)|^{\frac{1}{2}} \bar{\partial}^i f_n\|_{2,D} = 0$$

for $i = 1, 2, \cdots, 8k$. Since every scalar multiple of $I_{\mathcal{H}}$ has property (β), it follows that

$$\lim_{n\to\infty} \|\widetilde{q(T)} P|q(T)|^{\frac{1}{2}} \bar{\partial}^i f_n\|_{2,D} = 0 \tag{13}$$

for $i = 1, 2, \cdots, 8k$. From (10) and (13), we derive that

$$\lim_{n\to\infty} \|\widetilde{q(T)}|q(T)|^{\frac{1}{2}} \bar{\partial}^i f_n\|_{2,D} = 0$$

for $i = 1, 2, \cdots, 8k$. Multiplying both sides by $U|q(T)|^{\frac{1}{2}}$, we obtain that

$$\lim_{n\to\infty} \|q(T)^2 \bar{\partial}^i f_n\|_{2,D} = 0$$

for $i = 1, 2, \cdots, 8k$. Then, the first equation in (9) ensures that

$$\lim_{n\to\infty} \|q(z)^2 \bar{\partial}^i f_n\|_{2,D} = 0$$

for $i = 1, 2, \cdots, 8k$. Since $q(z)^2$ is a polynomial of degree $4k$, we have

$$\lim_{n\to\infty} \|(I_{\mathcal{H}} - P) f_n\|_{2,D} = 0 \tag{14}$$

due to Corollary 3.2. Combining (8) and (14), we see that

$$\lim_{n\to\infty} \|(T - z) P f_n + 1 \otimes h_n\|_{2,D} = 0. \tag{15}$$

Let Γ be a closed curve in D surrounding $\sigma(T)$. Then

$$\lim_{n\to\infty} \|P f_n(z) + (T - z)^{-1} h_n\| = 0$$

uniformly for all $z \in \Gamma$. Applying the Riesz-Dunford functional calculus, we obtain that

$$\lim_{n\to\infty} \|\frac{1}{2\pi i} \int_\Gamma P f_n(z)\, dz + h_n\| = 0.$$

But $\frac{1}{2\pi i} \int_\Gamma P f_n(z)\, dz = 0$ by Cauchy's theorem, and so $\lim_{n\to\infty} \|h_n\| = 0$. Therefore V is one-to-one and has closed range. □

Theorem 3.4. *Every polynomial root of a class A operator is subscalar. More precisely, if $p(T)$ belongs to class A for some nonconstant polynomial p of degree k, then T is subscalar of order* $12k$.

Proof. Suppose that $T \in \mathcal{L}(\mathcal{H})$ is a polynomial root of a class A operator such that $p(T)$ belongs to class A for some nonconstant polynomial p of degree k. Let D be an arbitrary bounded open disk in $\mathbb{C}$ that contains $\sigma(T)$ and consider the quotient space

$$H(D) = W^{12k}(D, \mathcal{H})/\overline{(T - z)W^{12k}(D, \mathcal{H})}$$

endowed with the Hilbert space norm. The class of a vector f or an operator S on $H(D)$ will be denoted by $\widetilde{f}$ or $\widetilde{S}$, respectively. Let M be the operator of multiplication by z on $W^{12k}(D, \mathcal{H})$. As noted at the end of section 2, the operator M is scalar of order $12k$ and has a spectral distribution Φ. Observe that $\widetilde{M}$ is well-defined, since the range of $T - z$ is M-invariant. Consider the spectral distribution $\Phi : C_0^{12k}(\mathbb{C}) \to \mathcal{L}(W^{12k}(D, \mathcal{H}))$ defined by the following relation: $\Phi(\varphi)f = \varphi f$ for $\varphi \in C_0^{12k}(\mathbb{C})$ and $f \in W^{12k}(D, \mathcal{H})$. Then the spectral distribution Φ of M commutes with $T - z$, and so $\widetilde{M}$ is still a scalar operator of order $12k$ with $\widetilde{\Phi}$ as a spectral distribution. Consider the operator $V : \mathcal{H} \to H(D)$ given by $Vh = \widetilde{1 \otimes h}$. Since

$$VTh = \widetilde{1 \otimes Th} = \widetilde{z \otimes h} = \widetilde{M}(\widetilde{1 \otimes h}) = \widetilde{M}Vh$$

for all $h \in \mathcal{H}$, we have $VT = \widetilde{M}V$. Furthermore, ran(V) is closed by Lemma 3.3. Hence, ran(V) is an $\widetilde{M}$-invariant subspace. Since T is similar to the restriction $\widetilde{M}|_{\text{ran}(V)}$ and $\widetilde{M}$ is scalar of order $12k$, we conclude that T is subscalar of order $12k$. □

In the following corollary, we provide a partial solution to the invariant subspace problem.

Corollary 3.5. *If $T \in \mathcal{L}(\mathcal{H})$ is a polynomial root of a class A operator and $\sigma(T)$ has nonempty interior in $\mathbb{C}$, then T has a nontrivial invariant subspace.*

Proof. The proof follows from Theorem 3.4 and [8]. □

Corollary 3.6. *Let $T \in \mathcal{L}(\mathcal{H})$ be a polynomial root of a class A operator. For any function f analytic on a neighborhood of $\sigma(T)$, the following assertions hold.*
(i) $f(T)$ is subscalar.
(ii) $f(T)$ has Bishop's property (β), Dunford's property (C), and SVEP.
(iii) $\sigma_{f(T)}(h) = f(\sigma_T(h))$ for any $h \in \mathcal{H}$.
(iv) Both $f(T)$ and $f(T)^*$ satisfy Weyl's theorem.

Proof. (i) With the same notations as in the proof of Theorem 3.4, it holds that $Vf(T) = f(\widetilde{M})V$. Thus $f(T)$ is subscalar.

(ii) It suffices to prove that $f(T)$ has property (β). Every scalar operator has property (β) (see [27]). Since $f(T)$ is subscalar by Theorem 3.4 and property (β) is transmitted from an operator to its restrictions to closed invariant subspaces, we conclude that $f(T)$ has property (β).

(iii) Since $f(T)$ has SVEP by (ii), the proof follows from [20, Theorem 3.3.8].

(iv) Combining (i) with [1, Theorem 3.99 and page 175], we see that Weyl's theorem holds for $f(T)$ and $f(T)^*$. □

Recall that an operator $X \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is called a *quasiaffinity* if it has trivial kernel and dense range. We say that two operators $S \in \mathcal{L}(\mathcal{H})$ and $T \in \mathcal{L}(\mathcal{K})$ are *quasisimilar* if there are quasiaffinities $X \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ and $Y \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ such that $XS = TX$ and $SY = YT$.

Corollary 3.7. *Let $T, S \in \mathcal{L}(\mathcal{H})$ be polynomial roots of class A operators. If T and S are quasisimilar, then $\sigma(T) = \sigma(S)$ and $\sigma_e(T) = \sigma_e(S)$.*

Proof. We obtain this assertion from Corollary 3.6 (ii) and [28]. □

If $T \in \mathcal{L}(\mathcal{H})$ and $x \in \mathcal{H}$, then $\{T^n x\}_{n=0}^{\infty}$ is called the *orbit of x under T*, denoted by $O(x, T)$. A vector $x \in \mathcal{H}$ is said to be *cyclic* for an operator $T \in \mathcal{L}(\mathcal{H})$ if the linear span of the orbit $O(x, T)$ of x under T is dense in $\mathcal{H}$.

Corollary 3.8. *If $T \in \mathcal{L}(\mathcal{H})$ is a polynomial root of a class A operator, then the following statements hold.*
(i) $r_T(h) = \lim_{n\to\infty} \|T^n h\|^{\frac{1}{n}}$ for all $h \in \mathcal{H}$, where $r_T(h) := \limsup_{n\to\infty} \|T^n h\|^{\frac{1}{n}}$ is the local spectral radius of T at h.
(ii) $\sigma_T(h) = \sigma(T)$ and $r_T(h) = r(T)$ for every cyclic vector h for T.

Proof. Since T has properties (β) and (C) by Corollary 3.6, we obtain (i) and (ii) from [20, Proposition 3.3.17 and page 238]. □

We say that $x \in \mathcal{H}$ is a *hypercyclic vector* for an operator $T \in \mathcal{L}(\mathcal{H})$ if $O(x, T)$ is dense in $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *hypercyclic* if there is at least one hypercyclic vector for T.

Corollary 3.9. *Let $T \in \mathcal{L}(\mathcal{H})$ be a polynomial root of a class A operator. Assume that $\sigma_T(x) \cap \mathbb{D} \neq \emptyset$ and $\sigma_T(x) \cap (\mathbb{C} \setminus \overline{\mathbb{D}}) \neq \emptyset$ for all nonzero $x \in \mathcal{H}$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Then T^* is hypercyclic.*

Proof. We observe that $H_T(\mathbb{C} \setminus \mathbb{D}) = \{0\}$ and $H_T(\overline{\mathbb{D}}) = \{0\}$. Since T has property (β) by Corollary 3.6, its adjoint T^* has property (δ). From [20, Proposition 2.5.14], both $H_{T^*}(\mathbb{D})$ and $H_{T^*}(\mathbb{C} \setminus \overline{\mathbb{D}})$ are dense in $\mathcal{H}$. Thus T^* is hypercyclic by [9, Theorem 3.2]. □

An operator $T \in \mathcal{L}(\mathcal{H})$ is called *hypertransitive* if every nonzero vector in $\mathcal{H}$ is hypercyclic for T. The hypertransitive operator problem is the open question whether (NHT) $= \mathcal{L}(\mathcal{H})$, where (NHT) denotes the set of all nonhypertransitive operators in $\mathcal{L}(\mathcal{H})$. In the following corollary, we prove that every kth root of a class A operator belongs to (NHT).

Corollary 3.10. *Let T be an operator in $\mathcal{L}(\mathcal{H})$ such that T^k belongs to class A for some positive integer k. Then T is nonhypertransitive.*

Proof. If T is not a quasiaffinity, then $0 \in \sigma_p(T) \cup \sigma_p(T^*)$. Hence T has a nontrivial invariant subspace, and so $T \in$ (NHT). On the other hand, if T is a quasiaffinity, then so is T^{2k}. Since T^k is a class A operator, T^{2k} is w-hyponormal from [14, Corollary 5]. Set $S = \widetilde{T^{2k}}$. Since $\widetilde{S} = \widetilde{T^{2k}}^{(2)}$ is hyponormal, it is not hypercyclic from [18]. Let $x \in \mathcal{H}$ be any nonzero vector such that $O(x, \widetilde{S})$ is not dense in $\mathcal{H}$. Since $U|S|^{\frac{1}{2}}\widetilde{S} = SU|S|^{\frac{1}{2}}$ where $S = U|S|$ is the polar decomposition of S, we obtain that

$$S(U|S|^{\frac{1}{2}}O(x, \widetilde{S})) = U|S|^{\frac{1}{2}}(\widetilde{S}O(x, \widetilde{S})) \subseteq U|S|^{\frac{1}{2}}O(x, \widetilde{S}).$$

Observe that S is a quasiaffinity, so that $|S|$ is a quasiaffinity and U is unitary. Then, $U|S|^{\frac{1}{2}}O(x, \widetilde{S})$ is not dense in $\mathcal{H}$, and so $S = \widetilde{T^{2k}} \in$ (NHT). We also derive that $T^{2k} \in$ (NHT) in a similar fashion. Therefore $T \in$ (NHT) by [3, Theorem 1]. □

In [17], the authors provided some structures for an analytic root of a hyponormal operator. We now extend these results to a polynomial root of a class A operator.

Theorem 3.11. *Let $T \in \mathcal{L}(\mathcal{H})$ be a polynomial root of a class A operator. If $\lambda_1, \lambda_2, \cdots, \lambda_m$ are isolated points of $\sigma(T)$, then T is representable as the direct sum*

$$T = \Big(\bigoplus_{j=1}^{m}(N_j + \lambda_j)\Big)\oplus B$$

where N_j is nilpotent for $j = 1, 2, \cdots, m$ and B is a polynomial root of a class A operator with $\sigma(B) = \sigma(T)\backslash\{\lambda_1, \cdots, \lambda_m\}$.

Proof. It suffices to consider the case $m = 1$ by induction. If λ_1 is an isolated point of $\sigma(T)$, consider the Riesz idempotent $E = \frac{1}{2\pi i}\int_{\partial D}(\lambda - T)^{-1}d\lambda$ for λ_1, where D is a closed disk centered at λ_1 such that $D \cap \sigma(T) = \{\lambda_1\}$. Put

$$T = T_1\oplus B \text{ on } \mathcal{H} = \text{ran}(E)\oplus\text{ran}(I_{\mathcal{H}} - E),$$

where $\sigma(T_1) = \{\lambda_1\}$ and $\sigma(B) = \sigma(T) \setminus \{\lambda_1\}$. Let p be a nonconstant polynomial such that $p(T)$ belongs to class A. Then, both $p(T_1)$ and $p(B)$ belong to class A. In addition, since $\sigma(p(T_1)) = p(\sigma(T_1)) = \{p(\lambda_1)\}$, it follows from [5, Lemma 3.1] that $q(T_1) = 0$ where $q(z) := p(z) - p(\lambda_1)$, meaning that T_1 is algebraic. Since $T_1 - \lambda_1$ is quasinilpotent and algebraic, it is nilpotent. Thus, $T = (N_1 + \lambda_1)\oplus B$ where $N_1 := T_1 - \lambda_1$ is nilpotent and B is a polynomial root of a class A operator. □

Corollary 3.12. *Let $T \in \mathcal{L}(\mathcal{H})$ be a polynomial root of a class A operator. If T is algebraic, then $T = D + N$ where D is diagonal, N is nilpotent, and $DN = ND$.*

Proof. If T is algebraic of order k, set $\sigma(T) = \{\lambda_1, \cdots, \lambda_m\}$ for some positive integer $m \le k$. Then each λ_j is an isolated points of $\sigma(T)$ for $j = 1, 2, \cdots, m$. Let $E_j := \frac{1}{2\pi i}\int_{\partial D_j}(\lambda - T)^{-1}d\lambda$ be the Riesz idempotent of T for λ_j, where D_j is a closed disk centered at λ_j such that $D_j \cap \sigma(T) = \{\lambda_j\}$. As in the proof of Theorem 3.11, one can express T as

$$T = (N_1 + \lambda_1)\oplus B \text{ on } \mathcal{H} = \text{ran}(E_1)\oplus\text{ran}(I_{\mathcal{H}} - E_1)$$

where N_1 is nilpotent and B is a polynomial root of a class A operator with $\sigma(B) = \{\lambda_2, \cdots, \lambda_m\}$. Repeating this procedure, we obtain that

$$T = \bigoplus_{j=1}^{m}(N_j + \lambda_j) \text{ on } \mathcal{H} = \bigoplus_{j=1}^{m}\text{ran}(E_j).$$

Taking $D = \bigoplus_{j=1}^{m}\lambda_j$ and $N = \bigoplus_{j=1}^{m}N_j$, we complete the proof. □

In the following theorem, we consider compactness of polynomial roots of class A operators.

Theorem 3.13. *Let $T \in \mathcal{L}(\mathcal{H})$ be an operator such that $p(T)$ belongs to class A for some nonconstant polynomial p with $p(0) = 0$. If T is compact, then T is decomposed into the direct sum*

$$T = A\oplus\Big(\bigoplus_{n=1}^{\infty}R_n\Big)$$

where $p(A) = 0$, R_n has finite rank for $n = 1, 2, 3, \cdots$, and $p(R_n) = \lambda_n \to 0$ as $n \to \infty$.

Proof. The Putnam's type inequality for class A operators, given in [25, Corollary 3.2], implies that

$$\| \, |p(T)^2| - |p(T)|^2 \, \| \leq \frac{1}{\pi}\mu(\sigma(p(T))) = 0$$

where μ denotes the planar Lebesgue measure, and $p(T)$ is normal by [29]. Hence, we can write $p(T) = 0 \oplus S$ on the decomposition $\ker(p(T)) \oplus \overline{\operatorname{ran}(p(T))}$, where S is an injective compact normal operator. Set $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ on $\ker(p(T)) \oplus \overline{\operatorname{ran}(p(T))}$. By using $TP(T) = P(T)T$, it holds that $BS = SC = 0$ and $DS = SD$. Since S has trivial kernel and dense range, we have $B = C = 0$, and then $T = A \oplus D$. This yields that $p(T) = p(A) \oplus p(D) = 0 \oplus S$, so that $p(A) = 0$, which means that A is algebraic. In addition, since S is a compact operator, we can express S as $S = \sum_{n=1}^{\infty} \lambda_n Q_n$ where $\{\lambda_n\}$ is the set of distinct nonzero eigenvalues of $p(T)$ and Q_n is the orthogonal projection of $\mathcal{H}$ onto $\ker(p(T) - \lambda_n)$ for every positive integer n. Here, Q_n has finite rank and $\lim_{n\to\infty} \lambda_n = 0$. Since $DS = SD$, we see that $\ker(p(T) - \lambda_n)$ is D-invariant and the restriction of D to this invariant subspace is of finite rank. Thus $T = A \oplus D$ where A is algebraic, $D = \bigoplus_{n=1}^{\infty} R_n$, and R_n is a finite rank operator on $\ker(p(T) - \lambda_n)$ such that $p(R_n) = \lambda_n \to 0$ as $n \to \infty$. □

Since every algebraic operator is a polynomial root of a class A operator, we obtain the following corollary by combining Theorem 3.13 with Corollary 3.12.

Corollary 3.14. *Let $T \in \mathcal{L}(\mathcal{H})$ be an operator such that $p(T)$ belongs to class A for some nonconstant polynomial p with $p(0) = 0$. If T is compact, then T is decomposed into the direct sum*

$$T = (D + N) \oplus \Big(\bigoplus_{n=1}^{\infty} R_n \Big)$$

where D is a diagonal operator and N is a nilpotent operator with $DN = ND$ and $p(D + N) = 0$, R_n has finite rank for $n = 1, 2, 3, \cdots$, and $p(R_n) = \lambda_n \to 0$ as $n \to \infty$.

4. Algebraic extensions

In this section, we deal with the algebraic extension of a polynomial root of a class A operators, i.e., an operator matrix of the form $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{K})$ where T_1 is a polynomial root of a class A operator and T_3 is algebraic. We will use the notation $\mathcal{AP}(\mathcal{H} \oplus \mathcal{K})$ for the collection of such operator matrices in $\mathcal{L}(\mathcal{H} \oplus \mathcal{K})$. We first prove that every operator matrix in $\mathcal{AP}(\mathcal{H} \oplus \mathcal{K})$ has a scalar extension.

Theorem 4.1. *Every $T \in \mathcal{AP}(\mathcal{H} \oplus \mathcal{K})$ is subscalar.*

Proof. Let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ where T_1 is a polynomial root of a class A operator and T_3 is algebraic. Choose nonconstant polynomials p and q of degree k and ℓ, respectively, for which $p(T_1)$ belongs to class A and $q(T_3) = 0$. For any fixed bounded disk D in $\mathbb{C}$ containing $\sigma(T)$, define the map $V : \mathcal{H} \oplus \mathcal{K} \to H(D)$ by

$$Vh = \widetilde{1 \otimes h} \; (\equiv 1 \otimes h + \overline{(T - z)W^{12k+2\ell}(D, \mathcal{H}) \oplus W^{12k+2\ell}(D, \mathcal{K})} \,)$$

where

$$H(D) := W^{12k+2\ell}(D, \mathcal{H}) \oplus W^{12k+2\ell}(D, \mathcal{K}) / \overline{(T - z)W^{12k+2\ell}(D, \mathcal{H}) \oplus W^{12k+2\ell}(D, \mathcal{K})}$$

and $1\otimes h$ denotes the constant function sending any $z \in D$ to h. As in the proof of Theorem 3.4, it is sufficient to prove that V is one-to-one and has closed range. Let $f_n = f_n^1 \oplus f_n^2 \in W^{12k+2\ell}(D,\mathcal{H})\oplus W^{12k+2\ell}(D,\mathcal{K})$ and $h_n = h_n^1 \oplus h_n^2 \in \mathcal{H} \oplus \mathcal{K}$ be sequences such that

$$\lim_{n\to\infty} \|(T-z)f_n + 1\otimes h_n\|_{W^{12k+2\ell}(D,\mathcal{H})\oplus W^{12k+2\ell}(D,\mathcal{K})} = 0.$$

Then, it follows that

$$\begin{cases}\lim_{n\to\infty} \|(T_1-z)f_n^1 + T_2 f_n^2 + 1\otimes h_n^1\|_{W^{12k+2\ell}} = 0\\ \lim_{n\to\infty} \|(T_3-z)f_n^2 + 1\otimes h_n^2\|_{W^{12k+2\ell}} = 0.\end{cases} \tag{16}$$

By the definition of the norm for the Sobolev space, (16) implies that

$$\begin{cases}\lim_{n\to\infty} \|(T_1-z)\bar{\partial}^i f_n^1 + T_2\bar{\partial}^i f_n^2\|_{2,D} = 0\\ \lim_{n\to\infty} \|(T_3-z)\bar{\partial}^i f_n^2\|_{2,D} = 0\end{cases} \tag{17}$$

for $i = 1,2,\cdots,12k+2\ell$. Suppose that $\lambda_1,\lambda_2,\cdots,\lambda_\ell$ are the zeros of q. Put $q_j(z) = (z-\lambda_{j+1})\cdots(z-\lambda_\ell)$ for $j = 0,1,2,\cdots,\ell-1$ and $q_\ell(z) = 1$.

Claim. It holds for every $j = 0,1,2,\cdots,\ell$ that

$$\lim_{n\to\infty} \|q_j(T_3)\bar{\partial}^i f_n^2\|_{2,D} = 0$$

for $i = 1,2,\cdots,12k+2\ell-2j$.

To show this claim, we will use the induction on j. The claim obviously holds when $j = 0$. Suppose that the claim is true for some $j = r$ where $0 \le r < \ell$, that is,

$$\lim_{n\to\infty} \|q_r(T_3)\bar{\partial}^i f_n^2\|_{2,D} = 0 \tag{18}$$

for $i = 1,2,\cdots,12k+2\ell-2r$. By (17) and (18), we see that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \|q_{r+1}(T_3)(T_3-z)\bar{\partial}^i f_n^2\|_{2,D}\\ &= \lim_{n\to\infty} \|q_{r+1}(T_3)(T_3-\lambda_{r+1}+\lambda_{r+1}-z)\bar{\partial}^i f_n^2\|_{2,D}\\ &= \lim_{n\to\infty} \|(\lambda_{r+1}I_{\mathcal{K}}-z)q_{r+1}(T_3)\bar{\partial}^i f_n^2\|_{2,D}\end{aligned}$$

for $i = 1,2,\cdots,12k+2\ell-2r$, where $I_{\mathcal{K}}$ is written for the identity operator in $\mathcal{L}(\mathcal{K})$. Since $\lambda_{r+1}I_{\mathcal{K}}$ is hyponormal, we obtain from [27, Corollary 2.2] that

$$\lim_{n\to\infty} \|(I_{\mathcal{K}}-P_2)q_{r+1}(T_3)\bar{\partial}^i f_n^2\|_{2,D} = 0$$

for $i = 1,2,\cdots,12k+2\ell-2r-2$, where P_2 denotes the orthogonal projection of $L^2(D,\mathcal{K})$ onto $A^2(D,\mathcal{K})$. Hence

$$\lim_{n\to\infty} \|(\lambda_{r+1}I_{\mathcal{K}}-z)P_2 q_{r+1}(T_3)\bar{\partial}^i f_n^2\|_{2,D} = 0$$

for $i = 1,2,\cdots,12k+2\ell-2r-2$. The fact that every hyponormal operator has property (β) ensures that

$$\lim_{n\to\infty} \|P_2 q_{r+1}(T_3)\bar{\partial}^i f_n^2\|_{2,D} = 0$$

for $i = 1,2,\cdots,12k+2\ell-2r-2$. Thus

$$\lim_{n\to\infty} \|q_{r+1}(T_3)\bar{\partial}^i f_n^2\|_{2,D} = 0$$

for $i = 1,2,\cdots,12k+2\ell-2r-2$, completing the proof of our claim.

From the claim with $j = \ell$, we see that

$$\lim_{n\to\infty} \|\bar{\partial}^i f_n^2\|_{2,D} = 0 \tag{19}$$

for $i = 1, 2, \cdots, 12k$. Then, [27, Corollary 2.2] implies that

$$\lim_{n\to\infty} \|(I_{\mathcal{K}} - P_2) f_n^2\|_{2,D} = 0. \tag{20}$$

From (19) and the first equation of (17), it follows that

$$\lim_{n\to\infty} \|(T_1 - z)\bar{\partial}^i f_n^1\|_{2,D} = 0$$

for $i = 1, 2, \cdots, 12k$. Applying the proof of Lemma 3.3, we obtain that

$$\lim_{n\to\infty} \|(I_{\mathcal{H}} - P_1) f_n^1\|_{2,D} = 0 \tag{21}$$

where $I_{\mathcal{H}}$ stands for the identity operator in $\mathcal{L}(\mathcal{H})$ and P_1 denotes the orthogonal projection of $L^2(D, \mathcal{H})$ onto $A^2(D, \mathcal{H})$. Now, set $Pf_n := \binom{P_1 f_n^1}{P_2 f_n^2}$. Combining (20) and (21) with (16), we have

$$\lim_{n\to\infty} \|(T - z)Pf_n + 1 \otimes h_n\|_{2,D} = 0.$$

From the proof of Lemma 3.3, we infer that $\lim_{n\to\infty} \|h_n\| = 0$. Hence V is one-to-one and has closed range. □

As an application of Theorem 4.1, we obtain the following corollary, whose proof is similar to that of Corollary 3.6.

Corollary 4.2. *Every $T \in \mathcal{AP}(\mathcal{H}\oplus\mathcal{K})$ has Bishop's property (β), Dunford's property (C), and SVEP.*

We next prove that each $T \in \mathcal{AP}(\mathcal{H}\oplus\mathcal{K})$ satisfying that $\sigma_p(T_3) \not\subset \sigma(T_1)$ must have a nontrivial hyperinvariant subspace.

Lemma 4.3. *If $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \in \mathcal{AP}(\mathcal{L}(\oplus\mathcal{K})$, then $\sigma(T) = \sigma(T_1) \cup \sigma(T_3)$.*

Proof. Since $\sigma(T_3)$ is a finite set, the intersection $\sigma(T_1) \cap \sigma(T_3)$ has no interior point, and so $\sigma(T) = \sigma(T_1) \cup \sigma(T_3)$ by [13, Corollary 8]. □

Theorem 4.4. *Let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \in \mathcal{AP}(\mathcal{H}\oplus\mathcal{K})$, and suppose that q is a minimal polynomial such that $q(T_3) = 0$. If $\sigma(T_1)$ does not contain all zeros of q, then $H_T(\sigma_T(x\oplus 0))$ is a nontrivial T-hyperinvariant subspace for every nonzero vector $x \in \mathcal{H}$.*

Proof. Take a zero λ_0 of q such that $\lambda_0 \notin \sigma(T_1)$. Observe that

$$\sigma_{T_1}(x) \subset \sigma(T_1) \subsetneq \sigma(T_1) \cup \{\lambda_0\} \subset \sigma(T_1) \cup \sigma(T_3) = \sigma(T)$$

for any $x \in \mathcal{H}$, where Lemma 4.3 is used for the last equality. Then, we have

$$\sigma_T(x\oplus 0) \subset \sigma_{T_1}(x) \subsetneq \sigma(T)$$

for any $x \in \mathcal{H}$. Fix any nonzero $x \in \mathcal{H}$. Consider the local spectral subspace $\mathcal{M} := H_T(\sigma_T(x\oplus 0))$ for T. Since T has Dunford's property (C) by Corollary 4.2, we know that $\mathcal{M}$ is a T-hyperinvariant subspace (see [20, Proposition 1.2.16] for more details). It is trivial that $x\oplus 0 \in \mathcal{M}$, and so $\mathcal{M} \neq \{0\}$. Assume that $\mathcal{M} = \mathcal{H}\oplus\mathcal{K}$. Since T has SVEP by Corollary 4.2, it follows from [20, Proposition 1.3.2] that

$$\sigma(T) = \bigcup\{\sigma_T(y) : y \in \mathcal{H}\oplus\mathcal{K}\} \subset \sigma_T(x\oplus 0) \subsetneq \sigma(T),$$

which is a contradiction. Hence $\mathcal{M}$ is a nontrivial T-hyperinvariant subspace. □

We find a concrete example for Theorem 4.4, as follows.

Example 4.5. *Let $T_1 = N\oplus S$ where $N \in \mathcal{L}(\mathcal{H})$ is a nilpotent operator of order k and $S \in \mathcal{L}(\mathcal{H})$ is a class A operator. Since $T_1^k = 0\oplus S^k$ belongs to class A by [14, Corollary 5], T_1 is a polynomial root of a class A operator. Set $T_3 = \begin{pmatrix} \lambda_1 I_{\mathcal{H}} & B \\ 0 & \lambda_2 I_{\mathcal{H}} \end{pmatrix}$ where λ_1 and λ_2 are complex constants with $|\lambda_1| > \|S\|$ and $B \in \mathcal{L}(\mathcal{H})$ is a nonzero operator. Since $q(z) = (z-\lambda_1)(z-\lambda_2)$ is a minimal polynomial such that $q(T_3) = 0$, the operator T_3 is algebraic and $\sigma(T_3) = \{\lambda_1, \lambda_2\}$. For any fixed $T_2 \in \mathcal{L}(\mathcal{H}\oplus\mathcal{H})$, consider $T := \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$. Since T is an algebraic extension of a polynomial root of a class A operator and $\lambda_1 \notin \sigma(T_1)$, we conclude from Theorem 4.4 that T has a nontrivial hyperinvariant subspace.*

Corollary 4.6. *Assume that $T \in \mathcal{L}(\mathcal{H})$ satisfies that*

$$T^{n*}\Big[|p(T)^2| - |p(T)|^2\Big]T^n \geq 0$$

for some nonconstant polynomial p and some positive integer n. Then T is subscalar. Moreover, if $ran(T^n)$ is not dense in $\mathcal{H}$ and $T|_{\overline{ran(T^n)}}$ is invertible, then $H_T(\sigma_T(x))$ is a nontrivial T-hyperinvariant subspace for each nonzero $x \in \overline{ran(T^n)}$.

Proof. Set $\mathcal{M} = \overline{\text{ran}(T^n)}$. If $\mathcal{M} = \mathcal{H}$, then $p(T)$ is a class A operator. Hence T is subscalar by Theorem 3.4. Now, consider the case when $\mathcal{M} \neq \mathcal{H}$. Since $\mathcal{M}$ is a T-invariant subspace, we can represent T as $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\mathcal{H} = \mathcal{M}\oplus\mathcal{M}^\perp$, where $T_1 = T|_{\mathcal{M}}$, $T_3 = (I-P)T(I-P)|_{\mathcal{M}^\perp}$, and P denotes the projection of $\mathcal{H}$ onto $\mathcal{M}$. Since $T_3^n = (I-P)T^n(I-P)|_{\mathcal{M}^\perp}$, we obtain that

$$\langle T_3^n x, x\rangle = \langle T^n x, x\rangle = \langle x, T^{n*}x\rangle = 0$$

for each $x \in \mathcal{M}^\perp = \ker(T^{n*})$. Hence $T_3^n = 0$ and so T_3 is algebraic. It follows from [10] that

$$|p(T)^2| = \begin{pmatrix} B & C \\ C^* & D \end{pmatrix} \text{ on } \mathcal{H} = \mathcal{M}\oplus\mathcal{M}^\perp,$$

where $B \geq 0$, $D \geq 0$, and $C = B^{\frac{1}{2}}SD^{\frac{1}{2}}$ for some contraction $S : \mathcal{M}^\perp \to \mathcal{M}$. Then

$$|p(T)^2|^2 = \begin{pmatrix} B^2 + CC^* & BC + CD \\ C^*B + DC^* & C^*C + D^2 \end{pmatrix}.$$

and

$$|p(T)^2|^2 = \begin{pmatrix} |p(T_1)^2|^2 & * \\ * & * \end{pmatrix},$$

implying that $|p(T_1)^2|^2 = B^2 + CC^*$ and

$$|p(T_1)^2| = (B^2 + CC^*)^{\frac{1}{2}} \geq B.$$

Since $P\Big[|p(T)^2| - |p(T)|^2\Big]P \geq 0$, we have

$$|p(T_1)^2| - |p(T_1)|^2 \geq B - |p(T_1)|^2 \geq 0.$$

Accordingly, $p(T_1)$ is a class A operator and $T \in \mathcal{AP}(\mathcal{M}\oplus\mathcal{M}^{\perp})$. The remainder of the proof follows from Theorems 4.1 and 4.4. □

We say that $T \in \mathcal{L}(\mathcal{H})$ is *isoloid* if each isolated point of the spectrum $\sigma(T)$ is an eigenvalue of T. It is known that every polynomial root of a class A operator is isoloid (see [5, Lemma 3.3] or [15, Lemma 4.2]).

Proposition 4.7. *If $T \in \mathcal{AP}(\mathcal{H}\oplus\mathcal{K})$, then the following statements hold.*
(i) T is isoloid.
(ii) If T is quasinilpotent, then it is nilpotent.

Proof. Write $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \in \mathcal{L}(\mathcal{H}\oplus\mathcal{K})$ where T_1 is a polynomial root of a class A operator and T_3 is algebraic.

(i) Let $\lambda \in \mathbb{C}$ be an isolated point of $\sigma(T)$. Since $\sigma(T) = \sigma(T_1) \cup \sigma(T_3)$ by Lemma 4.3 and $\sigma(T_3)$ is a finite set, either $\lambda \in \text{iso}(\sigma(T_1))$ or $\lambda \in \sigma(T_3) = \sigma_p(T_3)$. If $\lambda \in \sigma_p(T_3) \setminus \sigma(T_1)$, then

$$(T - \lambda)[-(T_1 - \lambda)^{-1}T_2 x]\oplus x = 0$$

for any $x \in \ker(T_3 - \lambda)$, and so $\lambda \in \sigma_p(T)$. If $\lambda \in \text{iso}(\sigma(T_1))$, we have $\lambda \in \sigma_p(T_1) \subset \sigma_p(T)$ since T_1 is isoloid as remarked above. Consequently, T is isoloid.

(ii) Since $\{0\} = \sigma(T) = \sigma(T_1) \cup \sigma(T_3)$ from Lemma 4.3, it follows that $\sigma(T_1) = \{0\}$ and T_3 is nilpotent. According to [5, Lemma 3.3], T_1 is nilpotent. Hence T is also nilpotent. □

The following theorem shows that Weyl's theorem holds for every $T \in \mathcal{AP}(\mathcal{L}\oplus\mathcal{K})$.

Theorem 4.8. *For every $T \in \mathcal{AP}(\mathcal{L}\oplus\mathcal{K})$, the following assertions hold.*
(i) T satisfies Weyl's theorem.
(ii) $f(\sigma_w(T)) = \sigma_w(f(T))$ for any analytic function f on some neighborhood of $\sigma(T)$.

Proof. Suppose that $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ where T_1 is a polynomial root of a class A operator and $p(T_3) = 0$ for some nonconstant polynomial p.

(i) Note that every polynomial root of a class A operator is isoloid and satisfies Weyl's theorem by [5]. Furthermore, T_3 is isoloid and satisfies Weyl's theorem by [24]. Since $\sigma_w(T_1) \cap \sigma_w(T_3)$ is a finite set, it has no interior points, and so Weyl's theorem holds for $T_1 \oplus T_3$ from [23, Corollary 11]. If there exists a point $\lambda_0 \in \sigma_e(T_3) \setminus \Big[\sigma_{le}(T_3) \cap \sigma_{re}(T_3)\Big]$, then $T_3 - \lambda_0$ is semi-Fredholm and $\lambda_0 \in \sigma(T_3)$. Since T_3 is algebraic, λ_0 is an isolated point of $\sigma(T_3)$. By [7], $T_3 - \lambda_0$ is Fredholm and $\text{ind}(T_3 - \lambda_0) = 0$, which is a contradiction. Thus $\sigma_e(T_3) = \sigma_{le}(T_3) \cap \sigma_{re}(T_3)$, which implies that $\sigma_e(T_3) = \sigma_{le}(T_3) = \sigma_{re}(T_3)$. Therefore, $SP(T_3)$ has no pseudoholes, and so from [22, Theorem 2.4] we can draw the conclusion that Weyl's theorem holds for T.

(ii) If f is analytic on some neighborhood of $\sigma(T)$, then $\sigma_w(f(T_1)) = f(\sigma_w(T_1))$ by the proof of [5, Corollary 3.5]. Moreover, $\sigma_w(f(T_3)) = f(\sigma_w(T_3))$ since T_3 is algebraic. Since $\sigma_w(T_3)$ is finite, $\sigma_w(f(T_1)) \cap \sigma_w(f(T_3)) = f(\sigma_w(T_1)) \cap f(\sigma_w(T_3))$ has no interior points. Hence, we obtain from [23, Corollary 7] that

$$\begin{aligned} \sigma_w(f(T)) &= \sigma_w(f(T_1)) \cup \sigma_w(f(T_3)) = f(\sigma_w(T_1)) \cup f(\sigma_w(T_3)) \\ &= f(\sigma_w(T_1) \cup \sigma_w(T_3)) = f(\sigma_w(T)), \end{aligned}$$

which completes our proof. □

Corollary 4.9. *Let $T \in \mathcal{AP}(\mathcal{H} \oplus \mathcal{K})$, and let f be a function analytic on some neighborhood of $\sigma(T)$. Then Weyl's theorem holds for $f(T)$.*

Proof. Since T is isoloid from Proposition 4.7, we have

$$\sigma(f(T)) \setminus \pi_{00}(f(T)) = f(\sigma(T) \setminus \pi_{00}(T))$$

by [21, Lemma]. Thus

$$\sigma(f(T)) \setminus \pi_{00}(f(T)) = f(\sigma(T) \setminus \pi_{00}(T)) = f(\sigma_w(T)) = \sigma_w(f(T))$$

by Theorem 4.8, which means that Weyl's theorem holds for $f(T)$. □

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