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Stability of some essential spectra of 2×2 block matrices of linear relations with non diagonal domain

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Abstract. Our study investigates the stability properties of specific essential spectra associated with 2×2 block matrices formed by linear relations with unbounded components. These matrices are characterized by a non-diagonal domain, meaning that the domain consists of vectors which satisfy certain relations between their components. The analysis focuses on linking these spectra to the union of the essential spectra of the restricted diagonal multivalued operator entries.

1. Introduction

In the last years, the concept of a linear relation, (or multivalued linear operator) in a linear space generalizes the notion of a (single valued) linear operator to that of a mutlivalued operator. This notion was introduced by R. Arens who provided a scientific treatment in [5]. At present, the study of linear relations in Banach spaces is of extreme significance since it offers multiple applications in many problems in physics and other areas of applied mathematics. We would like to mention that the spectral proprieties of block matrices of linear relations are the vital importance as they govern for instance the solvability and stability of the underlying physical systems.
As a basis for spectral analysis, the characterization and the investigation of some essential spectra of block matrices of linear relations with unbounded entries associated with the following form:

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1}$$

defined on $X \times Y$ product of Banach spaces, have drawn the attention of several authors involve the corresponding Schur-complements with diagonal domain (i.e dom $(\mathcal{A}) = (\mathrm{dom}\,(A) \cap \mathrm{dom}\,(C)) \times (\mathrm{dom}\,(B) \cap \mathrm{dom}\,(D))$, see [2, 3]. Later, in paper [4], Ammar et al. they improved the previous results by considering the case of matrix of linear relation with domain consisting of vectors satisfying one relation between their components expressed as:
$\Gamma_X x = \Gamma_Y Y$ for $\begin{pmatrix} x \\ y \end{pmatrix} \in (\mathrm{dom}\,(A) \cap \mathrm{dom}\,(C)) \times (\mathrm{dom}\,(B) \cap \mathrm{dom}\,(D))$ where Γ_X and Γ_Y are two linear relations.

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Furthermore, they supposed that the operator $C(A_0-\lambda)^{-1}$ is compact for some (and hence for all) $\lambda \in \rho(A_0)$, where $A_0 := A|_{\mathrm{dom}\,(A)\cap\ker\Gamma_X}$. Along these lines, the common tool in their investigation is based on the Frobenuis-Schur factorization in order to localize some essential spectra of 2×2 block matrix of multivalued linear operator in terms of its Schur-complement.
Besides, it appears that a mathematical treatment of this problem has not yet been undertaken in the case which the domain of matrix (1) consists of vectors satisfying more than one relation between their components.
To treat this problem, we would assert that our alternative approach starts from the theory of block operator matrices which opens us a new line of attack. During the last years, the papers [6, 10–13, 15–17] were explored the spectrum (essential) of operator matrices as in the form (1).
As reported in [15], R. Nagel has paid attention to the research of the problem related to spectral properties of 2×2 operator matrices $\mathcal{A}$ with non-diagonal domain defined by two relations between their components, that is:

$$\mathrm{dom}\,(\mathcal{A}) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathrm{dom}\,(A_m)\times \mathrm{dom}\,(D_m) \text{ such that } \begin{array}{c} \phi_1(x)=\psi_2(y) \\ \text{and} \\ \phi_2(y)=\psi_1(x) \end{array} \right\}, \tag{2}$$

for continuous linear operators ϕ_i and ψ_i, $i = 1,2$, A_m and D_m are two maximal linear operators. Particularly, the author presented a necessary and sufficient conditions for the matrix $\mathcal{A}$ in order to maintain its invertibility and to compute its spectrum $\sigma(\mathcal{A})$ in order to provide the expression of its resolvent as

$$\sigma(\mathcal{A}) = \sigma(A_0)\cup\sigma(D_0),$$

where $A_0 = A_m|_{\ker\phi_1}$, $D_0 = D_m|_{\ker\phi_2}$. Recently, in [7, 14], The authors generalized the results of R. Nagel mentioned above for computation several types of essential spectra of unbounded operators matrices with non diagonal domain. So, In [7], Belguith et. al characterized many essential spectra of the operator matrix defined by (1) in the form:

$$\sigma_{e,k}(\mathcal{A}) = \sigma_{e,k}(A_0)\cup\sigma_{e,k}(D_0), \quad k\in\{1,2,3,4,5\}. \tag{3}$$

The purpose of this work is to extend the results recorded in [7] to linear relation. The study involves an elegant use of the notion of Fredholm-type properties of 2×2 block matrices of linear relations in order to deduce the stability of spectral theory problems. We recall that the techniques employed by the authors in [2–4] are based on the Frobenuis-Schur factorization, but, our alternative approach is inspired on the techniques used by R. Nagel. For this, To fulfill this goal, first, we determine the expression of the resolvent $(\mathcal{A}-\lambda)^{-1}$ for some convenable λ. More precisely, the idea is to associate to the matrix $\mathcal{A}$ a diagonal matrix $\mathcal{A}_0 = \begin{pmatrix} A_0 & 0 \\ 0 & D_0 \end{pmatrix}$, which is more easier to deal with and we prove the characterization (3) for linear relations without knowing the totality of the essential spectra of its diagonal linear relations entries, but only the essential spectra of its restrictions of its diagonal entries on some subspace.(For more details see Theorems 3.1 and 3.2).

Our paper is organized as follow: In the next section, we give some preliminary results and notations used in the sequel of the paper. The third Section consists of two subsections: On the one hand, we establish a decomposition of the matrices of linear relations (see Theorem 3.9) and the other consider an intermediary that one needs to use in the sequel of the second subsection which we will give the resolvent of this kind matrix as product between two matrices of linear relations one diagonal matrix. Section 4 focuses on the characterization of some essential spectra of unbounded matrices of linear relations with non-diagonal domain.

2. Preliminary results

In this section, we gather some auxiliary notations and definitions that we will need in the rest of the paper. let X and Y be two Banach spaces.

A linear relation T is a subspace of $X\times Y$ and the set of all linear relations is denoted by $LR(X,Y)$. Moreover, $CR(X,Y)$ is the set of all closed linear relations from X into Y and $KR(X,Y)$ will denote the class of compact linear relation from X to Y . Also here, we set $LR(X) = LR(X,X)$, $CR(X) = CR(X,X)$ and $KR(X) = KR(X,X)$. Each $T \in LR(X,Y)$ is uniquely determined via its graph, $G(T)$, which is defined by

$$G(T) := \{(x,y) \in X \times Y \ : \ x \in \operatorname{dom}(T), \ y \in Tx\}.$$

For the basic notions and properties of linear relations we refer to [1, 8] and for the more advanced notions see [2, 4, 9] Given a linear relation $T \in LR(X,Y)$, we introduce the following sets

$$\begin{aligned} \operatorname{dom} T &= \{x \in X : \{x,y\} \in G(T)\},\\ \ker T &= \{x \in X : \{x,0\} \in G(T)\},\\ \operatorname{ran} T &= \{y \in Y : \{x,y\} \in G(T)\},\\ \operatorname{mul} T &= \{y \in Y : \{0,y\} \in G(T)\} \end{aligned}$$

which are called the *domain*, the *kernel*, the *range* and the *multivalued* part of T, respectively. We note that T is single valued (resp. injective, surjective) if and only if the subspace $T(0) = 0$. (resp $\ker(T) = 0$, $\operatorname{ran}(T) = Y$). When T is injective and surjective, we say that T is bijective.
The *inverse* of the linear relation T is given by

$$G(T^{-1}) := \{\{y,x\} \in Y \times X : \{x,y\} \in G(T)\}.$$

The linear relation αT with $\alpha \in \mathbb{C}$ is defined by

$$G(\alpha T) := \{\{x,\alpha y\} \in X \times Y : \{x,y\} \in G(T)\}. \tag{4}$$

The (operator-like) sum of two linear relations $T,S \in LR(X,Y)$ is defined as

$$G(T+S) := \{\{x,y+y'\} \in X \times Y : \{x,y\} \in G(T), \{x,y'\} \in G(S)\}. \tag{5}$$

If we assume that $X = Y$ then in view of (4) and (7) we have

$$G(T-\lambda) = G(T-\lambda I) = \{\{x,y-\lambda x\} : \{x,y\} \in G(T)\}. \tag{6}$$

Let M be a subspace of X such that $M \cap \operatorname{dom}(T) \neq \emptyset$. Then, the restriction $T|_M$ is defined by

$$G(T|_M) := \{\{x,y\} \in G(T) : x \in M\}$$

Lemma 2.1. [8] *Let X and Y be two vector spaces and let $T \in LR(X,Y)$. Then*

(*i*) $\operatorname{dom}(T^{-1}) = \operatorname{ran}(T)$ *and* $\operatorname{dom}(T) = \operatorname{ran}(T^{-1})$.

(*ii*) *T injective, if and only if, $T^{-1}T = I_{\operatorname{dom}(T)}$.*

(*iii*) *T is single valued, if and only if, $T(0) = \{0\}$.*

(*iv*) *If* $\operatorname{dom}(T) = \operatorname{dom}(S)$ *and $T(0) = S(0)$. Then, $T = S$ or the graphs of T and S are incomparable.*

Lemma 2.2. [2, Lemma 2.5]
Let $T,S \in LR(X,Y)$. If $S(0) \subset T(0)$ and $\operatorname{dom}(T) \subset \operatorname{dom}(S)$. *Then*

$$T - S + S = T$$

We recall some basic notions from Fredholm theory for linear relations, see [8].

Definition 2.3. *Let $T \in LR(X,Y)$. The nullity and the deficiency of T are defined as follows*

$$\operatorname{nul} T := \dim \ker T, \quad \textit{and}$$

$$\operatorname{def} T := \operatorname{codim} \operatorname{ran} T := \dim Y/\operatorname{ran} T.$$

If either $\operatorname{nul} T < \infty$ *or* $\operatorname{def} T < \infty$*, we define the index of a linear relation as follows*

$$\operatorname{ind} T := \operatorname{nul} T - \operatorname{def} T,$$

where the value of the difference is taken to be $\operatorname{ind} T := \infty$ *if* $\operatorname{nul} T$ *is infinite and* $\operatorname{ind} T := -\infty$ *if* $\operatorname{def} T$ *is infinite.*

In what follows, we need to introduce some important classes of linear relations, see e.g.[8]. The set of *upper* (*lower*) *semi Fredholm* relations from X into Y is defined by:

$$\Phi_+(X,Y) := \{T \in CR(X,Y) : \operatorname{nul} T < \infty \text{ and } \operatorname{ran} T \text{ is closed in } Y\},$$
$$\Phi_-(X,Y) := \{T \in CR(X,Y) : \operatorname{def} T < \infty \text{ and } \operatorname{ran} T \text{ is closed in } Y\},$$

and the set of *Fredholm relations* as

$$\Phi(X,Y) := \Phi_+(X,Y) \cap \Phi_-(X,Y).$$

If $X = Y$, we write briefly $\Phi_+(X)$, $\Phi_-(X)$, and $\Phi(X)$, respectively.
Next, we define the spectrum of a linear relation and introduce different types of essential spectra.

Definition 2.4. *[8] Let $T \in LR(X)$. The* resolvent set *of T is defined by*

$$\rho(T) := \{\lambda \in \mathbb{C} : (L-\lambda)^{-1} \textit{ is bijective, open with dense range}\}.$$

Referring back to [8, Exercice V.I.1.2], we have that the $\rho(T)$ when T is closed coincides with the set

$$\rho(T) := \{\lambda \in \mathbb{C} : (L-\lambda)^{-1} \textit{ is everywhere defined and single valued}\}.$$

For $\lambda \in \rho(T)$, $\mathcal{R}(\lambda, T)$ is the resolvent of $T - \lambda$. The spectrum $\sigma(T)$ is the complement of $\rho(T)$ in the complex plane,

Our concern in this paper is about the following definitions of some essential spectra of linear relations. For $T \in LR(X,Y)$, we consider

$$\begin{aligned}
\sigma_{e,1}(T) &:= \{\lambda \in \mathbb{C} \;:\; T-\lambda \notin \Phi_+(X,Y)\},\\
\sigma_{e,2}(T) &:= \{\lambda \in \mathbb{C} \;:\; T-\lambda \notin \Phi_-(X,Y)\},\\
\sigma_{e,3}(T) &:= \{\lambda \in \mathbb{C} \;:\; T-\lambda \notin \Phi(X,Y)\},\\
\sigma_{e,4}(T) &:= \{\lambda \in \mathbb{C} \;:\; T-\lambda \notin \Phi(X,Y)\},\\
\sigma_{e,5}(T) &:= \mathbb{C} \setminus \rho_{e5}(T),
\end{aligned}$$

where

$$\rho_{e5}(T) := \{\lambda \in \mathbb{C} \;:\; T-\lambda \in \Phi(X,Y)\,,\ \operatorname{ind}(T-\lambda) = 0\}.$$

These sets can be ordered as

$$\sigma_{e,1}(T) \cap \sigma_{e,2}(T) := \sigma_{e,3}(T) \subseteq \sigma_{e,4}(T) \subseteq \sigma_{e,5}(T). \tag{7}$$

In order to state our main results, let us introduce some definitions on Fredholm perturbations and then continue with some theorems:

Definition 2.5. *Let X and Y be two Banach spaces and $F \in LR(X,Y)$ be continuous.*
(i) *F is called a Fredholm perturbation if $T + F \in \Phi(X,Y)$ whenever $T \in \Phi(X,Y)$ with* $\mathrm{dom}\,(T) \subset \mathrm{dom}\,F$ *and* $F(0) \subset T(0)$.
(ii) *F is called an upper semi-Fredholm perturbation if $T + F \in \Phi_+(X,Y)$ whenever $T \in \Phi_+(X,Y)$ with* $\mathrm{dom}\,(T) \subset \mathrm{dom}\,F$ *and* $F(0) \subset T(0)$.
(iii) *Y is called a lower semi-Fredholm perturbation if $T+F \in \Phi_-(X,Y)$ whenever $T \in \Phi_-(X,Y)$ with* $\mathrm{dom}\,(T) \subset \mathrm{dom}\,F$ *and* $F(0) \subset T(0)$.

We denote by $\mathcal{P}(X,Y)$ the set of Fredholm perturbations, by $\mathcal{P}_+(X,Y)$ (resp. $\mathcal{P}_-(X,Y)$) the set of upper semi-Fredholm (resp. lower semi-Fredholm) perturbations.
If $X = Y$ we write $\mathcal{P}(X)$, $\mathcal{P}_+(X)$ and $\mathcal{F}_-(X)$ for $\mathcal{F}(X,X)$, $\mathcal{P}_+(X,X)$ and $\mathcal{P}_-(X,X)$, respectively and we have:

$$KR(X,Y) \subseteq \mathcal{P}_+(X,Y) \subseteq \mathcal{P}(X,Y) \tag{8}$$

and

$$KR(X,Y) \subseteq \mathcal{P}_-(X,Y) \subseteq \mathcal{P}(X,Y). \tag{9}$$

Let us recall the following useful results on Fredholm perturbations theory of 2×2 block matrix of multivalued linear operator which are established in [2].

Theorem 2.6. [2, Remark 5.3] *Let X_1 and X_2 be two Banach spaces and*

$$F := \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}$$

where $F_{ij} \in L\mathcal{R}(X_i, X_j)$ $\forall\, i,j = 1,2.$ and $F_{ij}(0) = 0$ Then :

$$F \in \mathcal{P}(X_1 \times X_2) \Leftrightarrow F_{ij} \in \mathcal{P}(X_i, X_j) \;\; \forall\, i,j = 1,2.$$

We recall some stability results of the essential spectra of multivalued operator subjected to Fredholm perturbations which is essential to provide the main purpose of this paper.

Theorem 2.7. [2] *Let $T, S \in L\mathcal{R}(X)$.*

(i) *If for some $\lambda \in \rho(T) \cap \rho(S)$, we have $(T-\lambda)^{-1} - (S-\lambda)^{-1} \in \mathcal{P}(X)$, then*

$$\sigma_{e,i}(T) = \sigma_{e,i}(S), \;\; i = 4,5.$$

(ii) *If for some $\lambda \in \rho(T) \cap \rho(S)$, we have $(T-\lambda)^{-1} - (S-\lambda)^{-1} \in \mathcal{P}_+(X)$, then*

$$\sigma_{e,1}(T) = \sigma_{e,1}(S).$$

(iii) *If for some $\lambda \in \rho(T) \cap \rho(S)$, we have $(T-\lambda)^{-1} - (S-\lambda)^{-1} \in \mathcal{P}_-(X)$, then*

$$\sigma_{e,2}(T) = \sigma_{e,2}(S).$$

(v) *If for some $\lambda \in \rho(T) \cap \rho(S)$, we have $(T-\lambda)^{-1} - (S-\lambda)^{-1} \in \mathcal{P}_+(X) \cap \mathcal{P}_-(X)$, then*

$$\sigma_{e,3}(T) = \sigma_{e,3}(S).$$

3. The virtual matrix of linear relations $\mathcal{A}$ and its Resolvent

This section deals with the spectral theory of the block matrix of linear relation $\mathcal{A}$ on the product of two Banach space X and Y. First, we start by giving a simplified decomposition of $\mathcal{A}$ which are needed to describe the resolvent of this kind matrix.

3.1. Decomposition of 2×2 *block matrix of linear relations with non diagonal domain*

To treat this problem in a functional analytic setting, we consider the following assumptions introduced in Nagel [15].
(H_1) A_m and D_m two closed, densely defined linear relations with domains $\mathrm{dom}\,(A_m)$ in X and $\mathrm{dom}\,(D_m)$ in Y, respectively.
(H_2) Let X_1 and Y_1 be two Banach spaces (called "spaces of boundary conditions"), endow $\mathrm{dom}\,(A_m)$ and $\mathrm{dom}\,(D_m)$ with the graph norm and define continuous linear relations ϕ_i and ψ_i for $i = 1, 2$ as in the following diagram:

$$\begin{tikzcd}
X \supset \mathrm{dom}\,(A_m) \arrow[r, "\phi_1"] \arrow[dr, "\psi_1" near end] & X_1 \\
Y \supset \mathrm{dom}\,(D_m) \arrow[r, "\phi_2"'] \arrow[ur, "\psi_2"' near end] & Y_1
\end{tikzcd}$$

(H_3) ϕ_1 and ϕ_2 are surjective.

Definition 3.1. *On the Banach space* $X\times Y$ *we consider the non diagonal domain*

$$\mathrm{dom}\,(\mathcal{A}) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathrm{dom}\,(A_m)\times\mathrm{dom}\,(D_m) \text{ such that } \begin{array}{c} \phi_1(x) = \psi_2(y) \\ \text{and} \\ \phi_2(y) = \psi_1(x) \end{array} \right\}, \tag{10}$$

to define the matrix of linear relation $\mathcal{A}$ *by*

$$\mathcal{A}\begin{pmatrix} x \\ y \end{pmatrix} = \mathcal{A}_m\begin{pmatrix} x \\ y \end{pmatrix}, \text{ for } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathrm{dom}\,(\mathcal{A}).$$

where $\mathcal{A}_m$ *is of the form*

$$\mathcal{A}_m = \begin{pmatrix} A_m & B \\ C & D_m \end{pmatrix}$$

where B, C *are bounded linear relations such that* $B \in LR(\mathrm{dom}\,(D_m), X)$ *and* $C \in LR(\mathrm{dom}\,(A_m), Y)$.

Remark 3.2. *In view of the continuity assumption on the linear relations* ϕ_1, ϕ_2, ψ_1 *and* ψ_2, *the domain* $\mathrm{dom}\,(\mathcal{A})$ *is closed in* $\mathrm{dom}\,(A_m)\times\mathrm{dom}\,(D_m)$ *with respect to the graph norm. Hence* $(\mathcal{A}, \mathrm{dom}\,(\mathcal{A}))$ *is a closed linear relation on* $X\times Y$

As a first towards, the description of the virtual matrix of linear relation $\mathcal{A}$ with non diagonal domain will be investigated by associating $\mathcal{A}$ with an matrix of linear relation $\mathcal{A}_0$ with the form $\begin{pmatrix} A_0 & 0 \\ 0 & D_0 \end{pmatrix}$ which is easier to deal with since it has diagonal domain, where $A_0 = A_m|_{\ker\phi_1}$ and $D_0 = D_m|_{\ker\phi_2}$ represent the restriction of the linear relation A_m to $\ker\phi_1$ and D_m to $\ker\phi_2$ respectively.

Remark 3.3. (i) *From the definition of the linear relations* A_0 *and* D_0, *one can easily check that* A_0 *and* D_0 *are closed linear relations. Hence the matrix of linear relation* $\mathcal{A}_0 = \begin{pmatrix} A_0 & 0 \\ 0 & D_0 \end{pmatrix}$ *is closed on* $\mathrm{dom}\,(\mathcal{A}_0) = \mathrm{dom}\,(A_0)\times\mathrm{dom}\,(D_0)$.
(ii) *By the restriction of* A_0 *and* D_0, *we can see that* $\ker(A_0) \subset \ker(A_m)$ *and* $\ker(D_0) \subset \ker(D_m)$ *and thus* $\ker(\mathcal{A}_0) \subset \ker(\mathcal{A})$. *More precisely, in our case, we just use the restriction of the entry* A_m *and* D_m *of the matrix* $\mathcal{A}$ *without knowing the totality of the essential spectra of the entry component* A *and* D.

In series of lemmas, we now explain the relation between $\mathcal{A}$ and $\mathcal{A}_0$.

Lemma 3.4. (*i*) *For $\lambda \in \rho(A_0)$(resp. $\lambda \in \rho(D_0)$), the following decomposition holds:*

$$\operatorname{dom}(A_m) = \operatorname{dom}(A_0) \oplus \ker(A_m - \lambda) \tag{11}$$

$$(resp.\ \operatorname{dom}(D_m) = \operatorname{dom}(D_0) \oplus \ker(D_m - \lambda)). \tag{12}$$

(*ii*) *For $\lambda \in \rho(A_0)$ (resp. $\lambda \in \rho(D_0)$), the following linear relations*

$$\phi_{1,\lambda} := \phi_1|_{\ker(A_m-\lambda)} \ \ and\ \ \phi_{2,\lambda} := \phi_2|_{\ker(D_m-\lambda)}.$$

is a continuous bijection from $\ker(A_m - \lambda)$ *onto* X_1 *(resp. from* $\ker(D_m - \lambda)$ *onto* Y_1*).*

Solution 3.5. (*i*) *The sum in* (11) *is contained in* $\operatorname{dom}(A_m)$ *and is direct since, by assumption,*

$$\operatorname{dom}(A_0) \cap \ker(A_m - \lambda) = \ker(A_0 - \lambda) = (A_0 - \lambda)^{-1}(0) = \{0\}.\ (since \lambda \in \rho(A_0)).$$

Take any $x \in \operatorname{dom}(A_m)$ *and set*

$$y := (A_0 - \lambda)^{-1}(A_m - \lambda)x \in \operatorname{dom}(A_0).$$

Then,

$$\begin{array}{rcl}(A_m - \lambda)(x - y) & = & (A_m - \lambda)(x) - (A_m - \lambda)(y) \\ & = & (A_m - \lambda)(x) - (A_0 - \lambda)(y),\ \ since\ y \in \operatorname{dom}(A_0) \\ & = & (A_m - \lambda)(x) - (A_m - \lambda)(x) \\ & = & (A_m - \lambda)(0).\end{array}$$

Therefore, $x - y \in \ker(A_m - \lambda)$ *and*

$$x = y + (x - y) \in \operatorname{dom}(A_0) + \ker(A_m - \lambda).$$

A same reasoning as helps us to reach the result of $\operatorname{dom}(D_m) = \operatorname{dom}(D_0) \oplus \ker(D_m - \lambda M_4)$, *which ends the proof of this assertion.*
(*ii*) *We know that ϕ_1 et ϕ_2 are bounded then $\phi_{1,\lambda}$ and $\phi_{2,\lambda}$ there are.*
The injectivity of the linear relation $\phi_{1,\lambda}$ follows from the following fact:

$$x \in \ker(\phi_\lambda)\ if\ and\ only\ if\ x \in \ker\phi_1 \cap \ker(A_m - \lambda) = \ker(A_0 - \lambda) = (A_0 - \lambda)^{-1}(0) = \{0\}.$$

Now Using the item (*i*) *and the linearity of $\phi_{1,\lambda}$, we get*

$$Im(\phi_{1,\lambda}) = \phi_1(\ker(A_m - \lambda)) = \phi_1(\operatorname{dom}(A_m)) = Im(\phi_1)$$

As long as ϕ_1 is surjective, then $Im(\phi_{1,\lambda}) = X_1$. Hence $\phi_{1,\lambda}$ is surjective.
A same reasoning to reach the result of $\phi_{2,\lambda}$.

Lemma 3.6. *For $\lambda \in \rho(A_0) \cap \rho(D_0)$ define linear relations*

$$\mathrm{K}_\lambda := \phi_{1,\lambda}^{-1} \circ \psi_2 \ \ and\ \ \mathrm{L}_\lambda := \phi_{2,\lambda}^{-1} \circ \psi_1.$$

Note that K_λ and L_λ are single valued and

$$\phi_1(\mathrm{K}_\lambda y) = \psi_2(y)\ \ for\ \ y \in \operatorname{dom}(D_m), \tag{13}$$

and

$$\phi_2(\mathrm{L}_\lambda x) = \psi_1(x)\ \ for\ \ x \in \operatorname{dom}(A_m). \tag{14}$$

Lemma 3.7. *If $\lambda \in \rho(A_0) \cap \rho(D_0)$, $x \in \operatorname{dom}(A_m)$ and $y \in \operatorname{dom}(D_m)$ then*

$$(A_m - \lambda)x = (A_0 - \lambda)(x - \mathrm{K}_\lambda y)$$

and

$$(D_m - \lambda)y = (D_0 - \lambda)(y - \mathrm{L}_\lambda x)$$

Solution 3.8. *We have $(A_m - \lambda)\mathrm{K}_\lambda(0) = (A_m - \lambda)(0)$, since K_λ is single valued. Then, from Lemma 2.2, we get*

$$(A_m - \lambda)x = (A_m - \lambda)(x - \mathrm{K}_\lambda y) + (A_m - \lambda)\mathrm{K}_\lambda y. \tag{15}$$

In addition, for $\lambda \in \rho(A_0) \cap \rho(D_0)$, the operator K_λ is the unique linear relation from $\operatorname{dom}(D_m)$ into $\operatorname{dom}(A_m)$ satisfying (13) and $Im(\mathrm{K}_\lambda) \subset \ker(A_m - \lambda)$. Then, for $y \in dom(D_m)$, we have

$$(A_m - \lambda)(\mathrm{K}_\lambda y) = \{0\}. \tag{16}$$

Let $x = (x - K_\lambda y) + K_\lambda y \in \mathcal{D}(A_0) + \ker(A_m - \lambda)$ and according to $A_0 = A_m|_{\ker \phi_1}$, we obtain the following result:

$$(A_m - \lambda)(x - K_\lambda y) = (A_0 - \lambda)(x - K_\lambda y). \tag{17}$$

Using Eqs 15, 16 and 17, we get:

$$\begin{aligned} (A_m - \lambda)x &= (A_m - \lambda)(x - K_\lambda y) \\ &= (A_0 - \lambda)(x - K_\lambda y). \end{aligned}$$

A same reasoning as helps us to reach the result of

$$(D_m - \lambda)y = (D_0 - \lambda)(y - \mathrm{L}_\lambda x).$$

The following factorizations may be used to formulate the key tool for our investigations.

Theorem 3.9. *Let $\lambda \in \rho(A_0) \cap \rho(D_0)$. Then,*

$$(\mathcal{A} - \lambda) = (\mathcal{A}_0 - \lambda)\mathcal{Q}_\lambda \quad \text{on } \operatorname{dom}(\mathcal{A}) \tag{18}$$

where

$$\mathcal{Q}_\lambda = \begin{pmatrix} Id & -\mathrm{K}_\lambda + (A_0 - \lambda)^{-1}B \\ -\mathrm{L}_\lambda + (D_0 - \lambda)^{-1}C & Id \end{pmatrix} \tag{19}$$

Solution 3.10. *Let $\left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right] \in G[(\mathcal{A}_0 - \lambda)\mathcal{Q}_\lambda]$. Then there exists $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in X \times Y$, such that $\left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right] \in G(\mathcal{Q}_\lambda)$ and $\left[\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right] \in G(\mathcal{A}_0 - \lambda)$.*
We obtain $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \begin{pmatrix} Id & -\mathrm{K}_\lambda + (A_0 - \lambda)^{-1}B \\ -\mathrm{L}_\lambda + (D_0 - \lambda)^{-1}C & Id \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and
$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \begin{pmatrix} A_0 - \lambda & 0 \\ 0 & D_0 - \lambda \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$.

This implies that $\begin{cases} z_1 \in x_1 + [-\mathrm{K}_\lambda + (A_0 - \lambda)^{-1}B]x_2, \\ z_2 \in [-\mathrm{L}_\lambda + (D_0 - \lambda)^{-1}C]x_1 + x_2, \\ y_1 = (A_0 - \lambda)z_1, \\ y_2 = (D_0 - \lambda)z_2. \end{cases}$

Thus, $\begin{cases} y_1 = (A_0-\lambda)\Big(x_1 + [-\mathrm{K}_\lambda + (A_0-\lambda)^{-1}B]x_2\Big), \\ \\ y_2 = (D_0-\lambda)\Big([-\mathrm{L}_\lambda + (D_0-\lambda)^{-1}\mathrm{C}]x_1 + x_2\Big). \end{cases}$

We can get

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} (A_0-\lambda)\Big(x_1 - \mathrm{K}_\lambda x_2\Big) + Bx_2 \\ (D_0-\lambda)\Big(x_2 - \mathrm{K}_\lambda x_1\Big) + Cx_1 \end{pmatrix}.$$

Using Lemma 3.7*, it follows that*

$$\left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right] \in G\left[\begin{pmatrix} A_m-\lambda & B \\ C & D_m-\lambda \end{pmatrix}\right] = G\left[\begin{pmatrix} A-\lambda & B \\ C & D-\lambda \end{pmatrix}\right].$$

We end up with the following inclusion:

$$G[(\mathcal{A}_0-\lambda)\mathcal{Q}_\lambda] \subseteq G(\mathcal{A}-\lambda). \tag{20}$$

It remains to show that $(\mathcal{A}-\lambda)\begin{pmatrix} 0 \\ 0 \end{pmatrix} = (\mathcal{A}_0-\lambda)\mathcal{Q}_\lambda\begin{pmatrix} 0 \\ 0 \end{pmatrix}$*. Indeed,*

$$\begin{aligned} (\mathcal{A}_0-\lambda)\mathcal{Q}_\lambda\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} A_0-\lambda & 0 \\ & \\ 0 & D_0-\lambda \end{pmatrix}\begin{pmatrix} Id & -\mathrm{K}_\lambda + (A_0-\lambda)^{-1}B \\ & \\ -\mathrm{L}_\lambda + (D_0-\lambda)^{-1}C & Id \end{pmatrix}\begin{pmatrix} 0 \\ \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} (A_0-\lambda) & -(A_0-\lambda)\mathrm{K}_\lambda + B \\ & \\ (D_0-\lambda)\mathrm{L}_\lambda + C & (D_0-\lambda) \end{pmatrix}\begin{pmatrix} 0 \\ \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} (A_0-\lambda)(I-\mathrm{K}_\lambda)(0) + B(0) \\ \\ (D_0-\lambda)(I-\mathrm{L}_\lambda)(0) + C(0) \end{pmatrix} \\ &= \begin{pmatrix} (A-\lambda)(0) & B(0) \\ & \\ C(0) & (D-\lambda)(0) \end{pmatrix}. \end{aligned}$$

Thus

$$(\mathcal{A}_0-\lambda)\mathcal{Q}_\lambda\begin{pmatrix} 0 \\ 0 \end{pmatrix} = (\mathcal{A}-\lambda)\begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{21}$$

Moreover,
$\operatorname{dom}\Big((\mathcal{A}_0-\lambda)\mathcal{Q}_\lambda\Big) = \left\{\begin{pmatrix} x \\ y \end{pmatrix} \in X \times Y : \Big((\mathcal{A}_0-\lambda)\mathcal{Q}_\lambda\Big)\begin{pmatrix} x \\ y \end{pmatrix} \neq \emptyset\right\}.$
Let $\begin{pmatrix} x \\ y \end{pmatrix} \in \operatorname{dom}\Big((\mathcal{A}_0-\lambda)\mathcal{Q}_\lambda\Big)$*. This means that there exists*

$$\begin{pmatrix} x' \\ y' \end{pmatrix} \in \operatorname{dom}(\mathcal{A}_0-\lambda) = \operatorname{dom}(A_0) \times \operatorname{dom}(D_0) \textit{ such that } \begin{pmatrix} x' \\ y' \end{pmatrix} \in \mathcal{Q}_\lambda\begin{pmatrix} x \\ y \end{pmatrix}.$$

This is equivalent to

$$\mathcal{Q}_\lambda\begin{pmatrix} x \\ \\ y \end{pmatrix} = \begin{pmatrix} x' \\ \\ y' \end{pmatrix} + \mathcal{Q}_\lambda\begin{pmatrix} 0 \\ \\ 0 \end{pmatrix} = \begin{pmatrix} x' - \mathrm{K}_\lambda(0) + [(A_0-\lambda)^{-1}B](0) \\ \\ -\mathrm{L}_\lambda(0) + [(D_0-\lambda)^{-1}C](0) + y' \end{pmatrix}.$$

which implies that

$$\begin{pmatrix} x - \mathrm{K}_\lambda(y) + [(A_0-\lambda)^{-1}B](y) \\ \\ -\mathrm{L}_\lambda(x) + [(D_0-\lambda)^{-1}C](x) + y \end{pmatrix} = \begin{pmatrix} x' - \mathrm{K}_\lambda(0) + [(A_0-\lambda)^{-1}B](0) \\ \\ -\mathrm{L}_\lambda(0) + [(D_0-\lambda)^{-1}C](0) + y' \end{pmatrix}.$$

It follows that

$$\begin{cases} x - \mathrm{K}_\lambda(y) + [(A_0-\lambda)^{-1}B](y) \in \operatorname{dom}(A_0) = \ker\phi_1 \cap \operatorname{dom}(A_m), \\ \\ -\mathrm{L}_\lambda(x) + [(D_0-\lambda)^{-1}C](x) + y \in \operatorname{dom}(D_0) = \ker\phi_2 \cap \operatorname{dom}(D_m). \end{cases}$$

Thus

$$x \in \operatorname{dom}(A_m) \text{ and } y \in \operatorname{dom}(D_m).$$

It is enough to demonstrate that

$$\begin{cases} \phi_1(x) = \psi_2(y), \\ \quad and \\ \phi_2(y) = \psi_1(x). \end{cases} \tag{22}$$

Indeed, we have $\begin{cases} x - \mathrm{K}_\lambda(y) + [(A_0-\lambda)^{-1}B](y) \in \ker\phi_1, \\ \\ -\mathrm{L}_\lambda(x) + [(D_0-\lambda)^{-1}C](x) + y \in \ker\phi_2. \end{cases}$

Then $\begin{cases} \phi_1\Big(x - \mathrm{K}_\lambda y + [(A_0-\lambda)^{-1}B]y\Big) = \phi_1(x) - \phi_1(\mathrm{K}_\lambda y) + \phi_1([(A_0-\lambda)^{-1}B]y) = \phi_1(0), \\ \\ \phi_2\Big(y - \mathrm{L}_\lambda x + [(D_0-\lambda)^{-1}C]x\Big) = \phi_2(y) - \phi_2(\mathrm{L}_\lambda x) + \phi_2([(D_0-\lambda)^{-1}C]x) = \phi_2(0). \end{cases}$

Besides, we have $[(A_0-\lambda)^{-1}B]y \in \operatorname{dom}(A_0)$ *and* $[(D_0-\lambda)^{-1}C]x \in \operatorname{dom}(D_0)$*, then,* $\phi_1\Big([(A_0-\lambda)^{-1}B]y\Big) = \phi_1(0)$ *and* $\phi_2\Big([(D_0-\lambda)^{-1}C]x\Big) = \phi_2(0)$*, respectively. Hence, by using Eqs.* (13), (14), *we find* (22). *Therefore*

$$\operatorname{dom}\Big((\mathcal{A}_0-\lambda)\mathcal{Q}_\lambda\Big) = \operatorname{dom}(\mathcal{A}-\lambda). \tag{23}$$

Finally, using estimation (20), *Eqs* (21),(23) *and Lemma* 2.1, (*iv*), *we find the virtual matrix of multivalued operator* (18).

3.2. *Resolvent of $\mathcal{A}$ with non diagonal domain*

In view of the above decomposition of $(\mathcal{A}-\lambda)$, we impose some conditions on the components of the two matrix of linear relations $\mathcal{Q}_\lambda$ and $\mathcal{A}_0$ to describe the resolvent of $(\mathcal{A}-\lambda)$. First, in the next Lemma, we shall determine the Frobenuis-Schur decomposition for 2×2 matrix of linear relation $\mathcal{Q}_\lambda$. For this, we consider the following notation

$$\begin{cases} G_\lambda = -\mathrm{K}_\lambda + (A_0-\lambda)^{-1}B \in BR(\operatorname{dom}(D_m), \operatorname{dom}(A_m)), \\ F_\lambda = -\mathrm{L}_\lambda + (D_0-\lambda)^{-1}C \in BR(\operatorname{dom}(A_m), \operatorname{dom}(D_m)). \end{cases} \tag{24}$$

Lemma 3.11. *For $\lambda \in \rho(A_0)\cap\rho(D_0)$, if $(A_0-\lambda)^{-1}B$ and $(D_0-\lambda)^{-1}C$ are bounded operators, then we have*

$$\mathcal{Q}_\lambda = UVW,$$

where U and W are the lower- and upper-triangular linear relations matrices defined by:

$$U = \begin{pmatrix} Id & 0 \\ F_\lambda & Id \end{pmatrix}, \quad W = \begin{pmatrix} Id & G_\lambda \\ 0 & Id \end{pmatrix},$$

and V is the diagonal multivalued linear relation matrix

$$V = \begin{pmatrix} Id & 0 \\ 0 & Id - F_\lambda G_\lambda \end{pmatrix}.$$

Solution 3.12. *For $\lambda \in \rho(A_0) \cap \rho(D_0)$, we supposed that the hypotheses of Frobenuis-Schur factorization introduced in [3] are satisfied. (case $\mu = 0$).*
Let $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \operatorname{dom}(UVW)$ *such that*

$$\begin{aligned} (UVW)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} Id & 0 \\ F_\lambda & Id \end{pmatrix}\begin{pmatrix} Id & 0 \\ 0 & Id - F_\lambda G_\lambda \end{pmatrix}\begin{pmatrix} Id & G_\lambda \\ 0 & Id \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \\ &= \begin{pmatrix} x_1 + G_\lambda x_2 \\ F_\lambda x_1 + F_\lambda G_\lambda x_2 + (Id - F_\lambda G_\lambda)x_2 \end{pmatrix} \end{aligned}$$

Then, $F_\lambda G_\lambda$ is bounded. Since, $(A_0 - \lambda)^{-1}B$, $(D_0 - \lambda)^{-1}C$, K_λ and L_λ are bounded operators. Hence, we get $F_\lambda x_1 + F_\lambda G_\lambda x_2 + (Id - F_\lambda G_\lambda)x_2 = F_\lambda x_1 + x_2$.
Therefore,

$$\begin{aligned} (UVW)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} x_1 + G_\lambda x_2 \\ F_\lambda x_1 + x_2 \end{pmatrix} \\ &= \mathcal{Q}_\lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \end{aligned}$$

However,

$$\begin{aligned} \operatorname{dom}(UVW) &= \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X \times Y \ : \ x_1 \in \operatorname{dom}(F_\lambda) \text{ and } x_2 \in \operatorname{dom}(G_\lambda) \right\}, \\ &= \operatorname{dom}(A_m) \times \operatorname{dom}(D_m), \\ &= \operatorname{dom} \mathcal{Q}_\lambda. \end{aligned}$$

We deduce that $\mathcal{Q}_\lambda = UVW$.

By the decomposition $(\mathcal{A} - \lambda) = (\mathcal{A}_0 - \lambda)\mathcal{Q}_\lambda$ and Lemma 3.11, we deduce the expression of the resolvent $\mathcal{R}(\lambda, \mathcal{A}) := (\mathcal{A} - \lambda)^{-1}$.

Theorem 3.13. *Let $\lambda \in \rho(A_0) \cap \rho(D_0)$ such that $1 \in \rho(F_\lambda G_\lambda)$ and we supposed that the condition of Lemma 3.11 are satisfied. We obtain*

$$\mathcal{R}(\lambda, \mathcal{A}) = \mathcal{R}(\lambda, \mathcal{A}_0) + \mathcal{M}(\lambda) \tag{25}$$

Here,

$$\mathcal{M}(\lambda) = \begin{pmatrix} G_\lambda(Id - F_\lambda G_\lambda)^{-1}F_\lambda \mathcal{R}(\lambda, A_0) & G_\lambda[Id + (Id - F_\lambda G_\lambda)^{-1}F_\lambda G_\lambda]\mathcal{R}(\lambda, D_0) \\ \\ (Id - F_\lambda G_\lambda)^{-1}F_\lambda \mathcal{R}(\lambda, A_0) & (Id - F_\lambda G_\lambda)^{-1}F_\lambda G_\lambda \mathcal{R}(\lambda, D_0) \end{pmatrix}. \tag{26}$$

Solution 3.14. *For $\lambda \in \rho(A_0) \cap \rho(D_0)$, we start by determining the properties of the matrices*

$$U = \begin{pmatrix} Id & 0 \\ F_\lambda & Id \end{pmatrix} \text{ and } W = \begin{pmatrix} Id & G_\lambda \\ 0 & Id \end{pmatrix}.$$

We note that, U and W are single valued bounded, invertible and

$$U^{-1} = \begin{pmatrix} Id & 0 \\ -F_\lambda & Id \end{pmatrix} \text{ and } W^{-1} = \begin{pmatrix} Id & -G_\lambda \\ 0 & Id \end{pmatrix}.$$

Hence, we get

$$\begin{aligned} Q_\lambda^{-1} &= \begin{pmatrix} Id & -G_\lambda \\ 0 & Id \end{pmatrix}\begin{pmatrix} Id & 0 \\ 0 & (Id - F_\lambda G_\lambda)^{-1} \end{pmatrix}\begin{pmatrix} Id & -F_\lambda \\ 0 & Id \end{pmatrix}, \\ &= \begin{pmatrix} Id + G_\lambda(Id - F_\lambda G_\lambda)^{-1}F_\lambda & -G_\lambda(Id - F_\lambda G_\lambda)^{-1} \\ -(Id - F_\lambda G_\lambda)^{-1}F_\lambda & (Id - F_\lambda G_\lambda)^{-1} \end{pmatrix}. \end{aligned}$$

Therefore

$$\mathcal{R}_\lambda(\mathcal{A}) = \begin{pmatrix} [Id + G_\lambda(Id - F_\lambda G_\lambda)^{-1}F_\lambda]\mathcal{R}(\lambda, A_0) & -G_\lambda(Id - F_\lambda G_\lambda)^{-1}\mathcal{R}(\lambda, D_0) \\ -(Id - F_\lambda G_\lambda)^{-1}F_\lambda\mathcal{R}(\lambda, A_0) & (Id - F_\lambda G_\lambda)^{-1}\mathcal{R}(\lambda, D_0) \end{pmatrix}. \tag{27}$$

In addition, we have $1 \in \rho(F_\lambda G_\lambda)$ implies that $Id - F_\lambda G_\lambda$ is injective. Then, in the rest of proof, applying Lemma 2.1 (ii), we can write

$$(Id - F_\lambda G_\lambda)^{-1}(Id - F_\lambda G_\lambda) = Id$$

Thus, we get

$$(Id - F_\lambda G_\lambda)^{-1} = Id + (Id - F_\lambda G_\lambda)^{-1}F_\lambda G_\lambda \text{ on } \operatorname{dom}(D_m).$$

Consequently,

$$(Id - F_\lambda G_\lambda)^{-1}\mathcal{R}(\lambda, D_0) = \mathcal{R}(\lambda, D_0) + (Id - F_\lambda G_\lambda)^{-1}F_\lambda G_\lambda\mathcal{R}(\lambda, D_0).$$

$$G_\lambda(Id - F_\lambda G_\lambda)^{-1}\mathcal{R}(\lambda, D_0) = G_\lambda\mathcal{R}(\lambda, D_0) + G_\lambda(Id - F_\lambda G_\lambda)^{-1}F_\lambda G_\lambda\mathcal{R}(\lambda, D_0).$$

Then, thanks to its above expressions, we can rewrite the new entries of the resolvent (27) to obtain our result.

4. Essential spectra of 2 × 2 matrix of linear relations with non diagonal domain

Now, we are in the position to express the first main results of this section.

Theorem 4.1. *Let $\lambda \in \rho(\mathcal{A}) \cap \rho(A_0) \cap \rho(D_0)$, we assume that*

i) The conditions of Lemma 3.11 are satisfied.

ii) $1 \in \rho(F_\lambda G_\lambda)$.

iii) G_λ and F_λ are compact linear relations.

Then,

$$\sigma_{e,4}(\mathcal{A}) = \sigma_{e,4}(A_0) \cup \sigma_{e,4}(D_0), \tag{28}$$

and

$$\sigma_{e,5}(\mathcal{A}) \subseteq \sigma_{e,5}(A_0) \cup \sigma_{e,5}(D_0) \tag{29}$$

Moreover, if the set $\mathbb{C} \setminus \sigma_{e,4}(A_0)$ is connected, then

$$\sigma_{e,5}(\mathcal{A}) = \sigma_{e,5}(A_0) \cup \sigma_{e,5}(E_0) \cup \sigma_{e,5}(L_0). \tag{30}$$

Solution 4.2. *According to i) and ii), we note that $\mathcal{M}(\lambda)$ is single valued. Thus, based on Theorem 2.6, it is sufficient to show that all the entries of this block matrix in the form (26) are compact linear relations.*
Indeed, Consider $\lambda \in \rho(A_0) \cap \rho(D_0)$ and from the assumptions $F_\lambda \mathcal{R}(\lambda, A_0) \in KR(X, \mathrm{dom}\,(D_m))$ and $G_\lambda \mathcal{R}(\lambda, D_0) \in KR(Y, \mathrm{dom}\,(A_m))$, we get that $\mathcal{M}(\lambda)$ is compact because their entries are the product between bounded and compact linear relations.
Consequently, for $\lambda \in \rho(A_0) \cap \rho(D_0)$, we obtained

$$\mathcal{R}(\lambda, \mathcal{A}) - \mathcal{R}(\lambda, \mathcal{A}_0) \in KR(X \times Y) \subseteq \mathcal{P}(X \times Y).$$

Hence, according to Theorem 2.7 and (8), one get $\sigma_{e,4}(\mathcal{A}) = \sigma_{e,4}(\mathcal{A}_0)$. Therefore

$$\sigma_{e,4}(\mathcal{A}) = \sigma_{e,4}(A_0) \cup \sigma_{e,4}(D_0).$$

The second result stems from

$$\mathrm{ind}\,(\mathcal{A} - \lambda) = \mathrm{ind}\,(A_0 - \lambda) + \mathrm{ind}\,(D_0 - \lambda). \tag{31}$$

If $\mathrm{ind}\,(\mathcal{A} - \lambda) \neq 0$, then one of the terms in (31) is non-zero, hence

$$\sigma_{e,5}(\mathcal{A}) = \sigma_{e,5}(\mathcal{A}_0) \subseteq \sigma_{e,5}(A_0) \cup \sigma_{e,5}(D_0).$$

According to the previous result (29), it is sufficient to check the opposite inclusion. Since $\mathbb{C} \setminus \sigma_{e,4}(A_0)$ is connected, by Proposition 4.2 in [2], $\sigma_{e,4}(A_0) = \sigma_{e,5}(A_0)$ and $\mathrm{ind}\,(A_0 - \lambda) = 0$ for each $\lambda \in \mathbb{C} \setminus \sigma_{e,4}(A_0)$. If $\lambda \in \mathbb{C} \setminus \sigma_{e,5}(\mathcal{A})$, then $\lambda \in \mathbb{C} \setminus \sigma_{e,4}(A_0)$ and $\lambda \in \mathbb{C} \setminus \sigma_{e,4}(D_0)$. Further, $\mathrm{ind}\,(\mathcal{A} - \lambda) = \mathrm{ind}\,(D_0 - \lambda)$, hence $\lambda \in \mathbb{C} \setminus \sigma_{e,5}(D_0)$ and (30) is proved.

We can translate the results of the above Theorem in terms of some essential spectra of type $\sigma_{e,k}(.)$ for $k \in \{1, 2, 3\}$.

Theorem 4.3. *Let $\lambda \in \rho(\mathcal{A}) \cap \rho(A_0) \cap \rho(D_0)$, we assume that $1 \in \rho(F_\lambda G_\lambda)$ and $(A_0 - \lambda)^{-1}B$, $(D_0 - \lambda)^{-1}C$ are bounded operators. Then*
(i) If $\mathcal{R}(\lambda, \mathcal{A}) - \mathcal{R}(\lambda, \mathcal{A}_0) \in \mathcal{P}_+(X \times Y)$, then

$$\sigma_{e,1}(\mathcal{A}) = \sigma_{e,1}(A_0) \cup \sigma_{e,1}(D_0).$$

(ii) If $\mathcal{R}_\lambda(\mathcal{A}) - \mathcal{R}_\lambda(\mathcal{A}_0) \in \mathcal{P}_-(X \times Y)$ then

$$\sigma_{e,2}(\mathcal{A}) = \sigma_{e,2}(A_0) \cup \sigma_{e,2}(D_0).$$

(iii) If $\mathcal{R}(\lambda, \mathcal{A}) - \mathcal{R}(\lambda, \mathcal{A}_0) \in \mathcal{P}_+(X \times Y) \cap \mathcal{P}_-(X \times Y)$ then

$$\sigma_{e,3}(\mathcal{A}) = \sigma_{e,3}(A_0) \cup \sigma_{e,3}(D_0).$$

Solution 4.4. *(i) For $\lambda \in \rho(\mathcal{A}) \cap \rho(A_0) \cap \rho(D_0)$, we infer that $\lambda \in \rho(\mathcal{A}) \cap \rho(\mathcal{A}_0)$ and if adding the assymptions that $1 \in \rho(F_\lambda G_\lambda)$ and $(A_0 - \lambda)^{-1}B$, $(D_0 - \lambda)^{-1}C$ are bounded operators, we obtain that $\mathcal{M}(\lambda)$ is single valued. This together with the fact that $\mathcal{R}_\lambda(\mathcal{A}) - \mathcal{R}_\lambda(\mathcal{A}_0) \in \mathcal{P}_+(X \times Y)$, leads from Theorem 2.7 (ii), to $\sigma_{e,1}(\mathcal{A}) = \sigma_{e,1}(\mathcal{A}_0)$. As $\mathcal{A}_0 - \lambda$ is a diagonal matrix of linear relation, this shows that*

$$\sigma_{e,1}(\mathcal{A}_0) = \sigma_{e,1}(A_0) \cup \sigma_{e,1}(D_0).$$

So, we get

$$\sigma_{e,1}(\mathcal{A}) = \sigma_{e,1}(A_0) \cup \sigma_{e,1}(D_0).$$

(ii) A same reasoning as helps us to reach the result of item (ii).

(iii) According to Eq (7) we see that this assertion is a consequence of the items (i) and (ii).

Conclusion 4.5. *It is noted that in the paper* [2], *Álvarez et al. They supposed that the matrix of multivalued linear operator* $\begin{pmatrix} 0 & \overline{(A-\lambda)^{-1}B} \\ & \\ C(A-\lambda)^{-1} & C(A-\lambda)^{-1}\overline{(A-\lambda)^{-1}B} \end{pmatrix} \in KR(X\times Y)$, *to characterize the essential spectra of* 2×2 *matrix of linear relations with diagonal domain in terms of its Schur complement. But in our case we suppose only that* F_λ *and* G_λ *are compact linear relations and by using the difference between the resolvent of two block matrices of linear relations, (see Theorems* 4.1 *and* 4.3*) to investigate the essential spectra of the matrix* $\mathcal{A}$ *in terms of the essential spectra of the restriction of its diagonal multivalued operators entries. More precisely, the additional relations* $\phi_1(x) = \psi_2(y)$ *and* $\phi_2(y) = \psi_1(x)$ *between the two components of the elements of the domain of the matrix of multivalued linear operator* $\mathcal{A}$ *enables us to determine the essential spectra of this kind of matrix* $\mathcal{A}$ *without knowing the totality of the entry component A and D. So, the results obtained in the papers of* [2–4] *are covered in this work.*

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