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Compact weighted composition operators on uniformly closed subspaces of $\mathbf{C}(X)$

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Abstract. This paper studies weighted composition operators and their finite sums on the disk algebra and on a uniformly closed subspace $\mathcal{A}$ of $\mathbf{C}(X)$. We investigate the closedness of their ranges and their compactness properties. Furthermore, we provide some applications of our results.

1. Introduction and preliminaries

Let X be a compact Hausdorff space, and let $\mathbf{C}(X)$ denote the Banach space of all continuous complex-valued functions on X equipped with the supremum norm. A point $x_0 \in X$ is called a strong boundary point for a uniformly closed subspace $\mathcal{A}$ of $\mathbf{C}(X)$ if there exists a sequence $\{f_n\} \subseteq \mathcal{A}$ such that $\|f_n\| = f_n(x_0) = 1$ and $f_n(x) \to 0$ uniformly on the complement of every neighborhood of x_0. The set of all strong boundary points is denoted by Γ. Let $G = X \setminus \Gamma$. For every $x \in X$, the evaluation functional $\delta_x : \mathcal{A} \to \mathbb{C}$ is given by $f \mapsto f(x)$. This functional lies in the unit ball of the dual space $\mathcal{A}^*$. Let $\Delta = \{\delta_x : x \in X\}$. We denote by $(X, \sigma(X, \Delta))$ the set X endowed with the weakest topology that makes every functional in Δ continuous. Let ϱ be the original topology on X. We shall always assume that the topologies coincide on G, i.e., $(G, \varrho) = (G, \sigma(X, \Delta))$, that G is dense in X, and that $\Gamma \neq \emptyset$. A typical example is the disk algebra $A(\bar{\mathbb{D}})$, which consists of all functions continuous on the closed unit disk $\bar{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ and analytic on its interior. Here, $X = \bar{\mathbb{D}}$, $\Gamma = \partial\mathbb{D}$, and $G = \mathbb{D}$. Given $u \in \mathbf{C}(X)$ and a continuous map $\varphi : X \to X$, the operator $uC_\varphi : \mathcal{A} \to \mathbf{C}(X)$ defined by

$$(uC_\varphi f)(x) = u(x) f(\varphi(x)), \quad \text{for } f \in \mathcal{A},\ x \in X,$$

is called a weighted composition operator. Throughout this paper, we assume that uC_φ is nontrivial; that is, u is not the zero function and φ is not constant. Weighted composition operators on continuous and analytic function spaces have been studied extensively. Important sources include works by Nordgren [9], Caughran and Schwartz [1], Kamowitz [5, 6], Swanton [10], and Shahbazov [11], among others. The books [2, 13, 14] also provide comprehensive information on these operators across various function spaces.

The compactness of weighted composition operators on the disk algebra and on $\mathbf{C}(X)$ was studied by

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Kamowitz in [5, 6]. In the subsequent sections, we investigate the compactness of uC_φ on $A(\bar{\mathbb{D}})$ and $\mathbf{C}(X)$ using an alternative method.

2. Compact sums of weighted composition operators on $A(\bar{\mathbb{D}})$

Let T be a bounded linear operator on a Banach space $\mathcal{B}$. We say that T is (weakly) compact if it maps every bounded set to a relatively (weakly) compact set. The symbols $\mathcal{N}(T)$ and $\mathcal{R}(T)$ denote the kernel and range of T, respectively. Now, fix a function $u \in A(\bar{\mathbb{D}})$ and a self-map φ of $\bar{\mathbb{D}}$ that is analytic on $\mathbb{D}$. Define the multiplication operator M_u, the composition operator C_φ, and the weighted composition operator uC_φ on $A(\bar{\mathbb{D}})$ by

$$M_u f = u \cdot f, \quad C_\varphi f = f \circ \varphi, \quad uC_\varphi f = u \cdot (f \circ \varphi).$$

By the closed graph theorem, M_u and C_φ are bounded on $A(\bar{\mathbb{D}})$. Since $uC_\varphi = M_u C_\varphi$, it follows that uC_φ is also bounded, with $\|uC_\varphi\| = \|u\|$. It is straightforward to verify that $\mathcal{N}(uC_\varphi) = \{0\}$. Note that if uC_φ is trivial, then $\mathcal{R}(uC_\varphi) \subseteq \operatorname{span}\{u\}$; hence, in this case, uC_φ is compact and has closed range.

In the sequel, we will use the following two lemmas, which are needed for the proofs of the subsequent results.

Lemma 2.1. *Let X be a compact Hausdorff space. The bounded linear operator $T : \mathcal{B} \to \mathbf{C}(X)$ from a Banach space $\mathcal{B}$ into $\mathbf{C}(X)$ is compact if and only if the mapping $\tau : x \to T^*\delta_x$ from X with original topology into $\mathcal{B}^*$ with norm topology is continuous (see Theorem VI.7.1 in [3]).*

There is a more precise result connecting the Euclidean distance in the unit disk and the norm distance in the dual of the disk algebra by means of the Poincaré distance due to König [7], namely:

$$\frac{4\|\delta_z - \delta_w\|}{4 + \|\delta_z - \delta_w\|^2} = \frac{|z - w|}{|1 - \bar{z}w|}.$$

We state his result in our setting as follows:

Lemma 2.2. *In case of disk algebra for $a \neq b$ we have*

$$\|\delta_a - \delta_b\| = \begin{cases} d(a,b) & \text{if } |a| < 1, |b| < 1 \\ 2 & \text{for other cases,} \end{cases}$$

in which

$$\left|\frac{a - b}{1 - \bar{a}b}\right| = \frac{d(a,b)}{1 + \frac{1}{4}d^2(a,b)} .$$

In particular, if $|a| < 1$ and $|b| < 1$, then $\|\delta_a - \delta_b\| \sim |\frac{a-b}{1-\bar{a}b}|$.

Proposition 2.3. *A necessary and sufficient condition for compactness of the operator uC_φ on $A(\bar{\mathbb{D}})$ is that at each point $z_0 \in \bar{\mathbb{D}}$,*

$$|u(z_0)|\, \|\delta_{\varphi(z_n)} - \delta_{\varphi(z_0)}\| \to 0 \tag{1}$$

in the $A(\bar{\mathbb{D}})^*$-norm, when $(z_n) \to z_0$.

Proof. By Lemma 2.1, uC_φ is compact if and only if

$$\|u(z_n)\delta_{\varphi(z_n)} - u(z_0)\delta_{\varphi(z_0)}\| \to 0 \tag{2}$$

in the $A(\bar{\mathbb{D}})^*$-norm, when $(z_n) \to z_0$. Since

$$\begin{aligned} |u(z_0)|\|\delta_{\varphi(z_n)} - \delta_{\varphi(z_0)}\| &\leq |u(z_n) - u(z_0)|\|\delta_{\varphi(z_n)}\| + \|u(z_n)\delta_{\varphi(z_n)} - u(z_0)\delta_{\varphi(z_0)}\| \\ &\leq 2|u(z_n) - u(z_0)|\|\delta_{\varphi(z_n)}\| + |u(z_0)|\|\delta_{\varphi(z_n)} - \delta_{\varphi(z_0)}\|, \end{aligned}$$

from the continuity of u and (2) the result follows. □

Let $|z_0| < 1$ and $(z_n) \to z_0$. Since φ is not a constant function, the maximum modulus principle implies that $|\varphi(z_0)| < 1$ and $|\varphi(z_n)| < 1$ for sufficiently large n. By Lemma 2.2 and from the continuity of φ the relation (1) holds. Thus for studying compactness of the operator T, it is sufficient to investigate the continuity of τ only on $\partial\mathbb{D}$.

Corollary 2.4. *Let $T = uC_\varphi$ be a weighted composition operator on $A(\bar{\mathbb{D}})$. Then the following statements are equivalent:*

(a) T is compact.

(b) T is weakly compact.

(c) $|u(z_0)|\,\|\delta_{\varphi(z_n)} - \delta_{\varphi(z_0)}\|_{A(\bar{\mathbb{D}})^*} \to 0$, whenever $(z_n) \to z_0$ for all $z_0 \in \partial\mathbb{D}$.

(d) $u(z) = 0$, whenever $|\varphi(z)| = 1$ for all $z \in \partial\mathbb{D}$.

Proof. (a) $\Rightarrow$ (b) is trivial. As mentioned above, the equivalence of (a) and (d) is proved by Kamovitz in [5]. To complete the proof, we show that (b) $\Rightarrow$ (c) $\Rightarrow$ (d).

(b) $\Rightarrow$ (c). The statement in the case where $\varphi(z_0) \in \mathbb{D}$ for some $z_0 \in \partial\mathbb{D}$ follows from Lemma 2.2. Now, let $z_0 \in \partial\mathbb{D}$ with $|\varphi(z_0)| = 1$. According to Lemma 2.2, we have $\|\delta_{\varphi(z_n)} - \delta_{\varphi(z_0)}\| = 2$ for all $z_n \in \bar{\mathbb{D}}$ with $\varphi(z_n) \neq \varphi(z_0)$. Thus, it suffices to prove that $u(z_0) = 0$. The fact that $\Gamma = \partial\mathbb{D}$ implies that there exists a sequence $\{f_n\} \subseteq A(\bar{\mathbb{D}})$ such that $\|f_n\| = f_n(\varphi(z_0)) = 1$ and, on the complement of any neighborhood of $\varphi(z_0)$, the sequence $\{f_n\}$ converges uniformly to 0 as $n \to \infty$. By assumption, there exists a subsequence $\{Tf_{n_k}\}$ that converges in the weak topology to some $f \in A(\bar{\mathbb{D}})$. In particular, since the evaluation functionals are bounded on $A(\bar{\mathbb{D}})$, we have $Tf_{n_k}(z) \to f(z)$ as $k \to \infty$ for all $z \in \mathbb{D}$. Given that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$ and $\mathbb{D}$ is dense in $\bar{\mathbb{D}}$, it follows that $f \equiv 0$. Therefore, we conclude

$$u(z_0) = \lim_{k\to\infty} u(z_0)f_{n_k}(\varphi(z_0)) = \lim_{k\to\infty} Tf_{n_k}(z_0) = 0.$$

(c) $\Rightarrow$ (d). Let $z_0 \in \partial\mathbb{D}$ such that $|\varphi(z_0)| = 1$. We choose the sequence $(z_n) \subseteq \mathbb{D}$ such that $(z_n) \to z_0$. Then by the Lemma 2.2 we have

$$\|\delta_{\varphi(z_n)} - \delta_{\varphi(z_0)}\| = 2, \quad \text{for all } n \in N,$$

therefore, $u(z_0) = 0$ if the relation (1) holds. □

Now we consider finite sums of weighted composition operators on $A(\bar{\mathbb{D}})$ of the form

$$T_m f(z) = \sum_{i=1}^{m} u_i(z) f(\varphi_i(z)) \quad (z \in \bar{\mathbb{D}}),$$

where $u_i \in A(\bar{\mathbb{D}})$, $\varphi_i \in A(\bar{\mathbb{D}})$, $\|\varphi_i\| \leq 1$ and m is a positive integer. The following result follows immediately from Lemma 2.1.

Corollary 2.5. *The operator $T_m f = \Sigma_{i=1}^{m} u_i.(f \circ \varphi_i)$ on $A(\bar{\mathbb{D}})$ is compact if and only if at each point $z \in \partial\mathbb{D}$ the mapping*

$$\tau : z \to T_m^*\delta_z = \sum_{i=1}^{m} u_i(z)\delta_{\varphi_i(z)}$$

from $\bar{D}$ with original topology into $A(\bar{\mathbb{D}})^*$ with norm topology is continuous; equivalently for each $z_0 \in \partial\mathbb{D}$

$$\sum_{i=1}^{m} u_i(z_0)[\delta_{\varphi_i(z)} - \delta_{\varphi_i(z_0)}] \to 0$$

in the $A(\bar{\mathbb{D}})^*$-norm, as $z \to z_0$.

Now we consider sums of two weighted composition operators on $A(\bar{\mathbb{D}})$. Using Lemma 2.2 and Corollary 2.5, we have the following corollary.

Corollary 2.6. *The operator $T_2 f(z) = u_1(z)f(\varphi_1(z)) + u_2(z)f(\varphi_2(z))$ on $A(\bar{\mathbb{D}})$ is compact if and only if, for each $z_0 \in \partial\mathbb{D}$, the followings hold:*

(a) $u_i(z_0) = 0$, whenever $|\varphi_i(z_0)| = 1$ and $\varphi_1(z_0) \neq \varphi_2(z_0)$ $(i = 1, 2)$.

(b) $u_1(z_0) + u_2(z_0) = 0$ and

$$u_i(z_0)\left|\frac{\varphi_1(z) - \varphi_2(z)}{1 - \varphi_1(z)\overline{\varphi_2(z)}}\right| \longrightarrow 0$$

as $z \to z_0$, whenever $\varphi_1(z_0) = \varphi_2(z_0) \in \partial\mathbb{D}$.

In this stage our aim is to characterize the nontrivial weighted composition operators on $\mathcal{A}$ with closed range. The result gives a geometrical view of the problem.

Let F be a closed subset of X. A necessary and sufficient condition for $\mathcal{A}_{|_F}$, the restriction of $\mathcal{A}$ to F, to be closed in $\mathbf{C}(F)$ is as follows:

Lemma 2.7. *Let $\mathcal{A}$ be a closed subalgebra of $\mathbf{C}(X)$. Let F be a closed subset of X. Then $\mathcal{A}_{|_F}$ is closed in $\mathbf{C}(F)$ if and only if there exists $c > 0$ such that $c\|f + K_F\| \leq \|f_{|_F}\|$ for all $f \in \mathcal{A}$. Here $K_F = \{f \in \mathcal{A} : f_{|_F} = 0\}$, and $f + K_F \in \mathcal{A}/K_F$ (see p. 65 in [8]).*

The next result gives necessary and sufficient conditions for a weighted composition operator uC_φ on the disk algebra $A(\bar{\mathbb{D}})$ to have a closed range.

Proposition 2.8. *Let $uC_\varphi : A(\bar{\mathbb{D}}) \to A(\bar{\mathbb{D}})$ be a nontrivial weighted composition operator and $E_\delta = \{x \in \bar{\mathbb{D}} : |u(x)| \geq \delta\}$. Then uC_φ has closed range if and only if $A(\bar{\mathbb{D}})_{|_{\varphi(E_\delta)}}$ is closed in $\mathbf{C}(\varphi(E_\delta))$ for some $\delta > 0$.*

Proof. Let $F_\delta = \varphi(E_\delta)$. Suppose $A(\bar{\mathbb{D}})_{|_{F_\delta}}$ is closed in $\mathbf{C}(F_\delta)$ and let $(u.f_n \circ \varphi)$ be a sequence in $\mathcal{R}(T)$ such that $u.f_n \circ \varphi \to g$ for some $g \in A(\bar{\mathbb{D}})$. So $f_n \circ \varphi \to \frac{g}{u}$ on E_δ. Since $(f_{n_{|_{F_\delta}}})$ is a Cauchy sequence in $A(\bar{\mathbb{D}})_{|_{F_\delta}}$ we have $f_{n_{|_{F_\delta}}} \to f_{|_{F_\delta}}$ for some $f \in A(\bar{\mathbb{D}})$. Since u has at most finitely many zeros in $\mathbb{D}$, $g = u.f \circ \varphi$.

Conversely, assume T has closed range and let (f_n) be in $A(\bar{\mathbb{D}})$ such that for some $\delta > 0$ and some $g \in A(\bar{\mathbb{D}})$, $f_{n_{|_{F_\delta}}} \to g$. Then $(u.f_n \circ \varphi)$ is a Cauchy sequence in $\mathcal{R}(T)$, so $u.f_n \circ \varphi \to u.f \circ \varphi$ for some $f \in A(\bar{\mathbb{D}})$. If $y \in F_\delta$ then $y = \varphi(x)$ for some $x \in E_\delta$ and

$$g(y) = \lim_{n\to\infty} f_n(y) = \frac{1}{u(x)} \lim_{n\to\infty} u(x) f_n(\varphi(x)) = f(\varphi(x)) = f(y).$$

Hence $g = f_{|_{F_\delta}}$. It follows that $A(\bar{\mathbb{D}})_{|_{\varphi(E_\delta)}}$ is closed in $\mathbf{C}(\varphi(E_\delta))$. □

By using Lemma 2.7 and Proposition 2.8 we have the following corollary.

Corollary 2.9. *Let uC_φ be the nontrivial weighted composition operator on $A(\bar{\mathbb{D}})$. Then uC_φ has closed range if and only if there exists $c > 0$ such that $c\|f + K_{F_\delta}\| \leq \|f_{|_{F_\delta}}\|$ for all $f \in A(\bar{\mathbb{D}})$ and for some $\delta > 0$.*

Suppose uC_φ has closed range. Since $\mathcal{N}(uC_\varphi) = \{0\}$, it follows that $K_{F_\delta} = \{0\}$. So for all $f \in A(\bar{\mathbb{D}})$ we have

$$\begin{aligned} c\|f\| \leq \|f_{|_{F_\delta}}\| &= \sup\{|f(x)| : x \in F_\delta\} \\ &= \sup\{|f(\varphi(x))| : x \in E_\delta\} \\ &= \sup\{\frac{1}{|u(x)|}|u(x)f(\varphi(x))| : x \in E_\delta\} \\ &\leq \frac{1}{\delta}\|uC_\varphi f\|. \end{aligned}$$

Thus, there exists $\gamma = \delta c > 0$ such that $\gamma\|f\| \le \|Tf\|$ for all $f \in A(\bar{\mathbb{D}})$. The converse is trivial.

In [4] the following characterization of closedness of range of the weighted composition operator uC_φ was obtained on $A(\bar{\mathbb{D}})$ as follows:

Theorem 2.10. *Let uC_φ be a nontrivial weighted composition operator on $A(\bar{\mathbb{D}})$. Then the following statements are equivalent:*
(a) uC_φ has closed range.
(b) There exists $\delta > 0$ such that $\partial\mathbb{D} \subseteq \varphi(E_\delta)$ where $E_\delta = \{z \in \partial\mathbb{D} : |u(z)| \ge \delta\}$.
(c) There exists $\delta > 0$ such that $C^*_\varphi(\{\delta_z \in \Delta : |u(z)| \ge \delta\}) \supseteq \Delta$ where $\Delta = \{\delta_z : z \in \partial\mathbb{D}\}$.

Remark 2.11. *We note that ontoness of φ is not a necessary condition for the closedness of range of uC_φ. For example, take $u(z) = z$ and let $\varphi : \mathbb{D} \to G$ be a conformal mapping, where $G = \{(r, \theta) : 0 < a < r < 1, -\pi < \theta < \pi\}$. Then φ has a continuous extension to $\bar{\mathbb{D}}$. It is easy to see that $\varphi(E_\delta) = \bar{G} \supseteq \partial\mathbb{D}$ for each $\delta > 0$. Hence uC_φ has closed range. However, φ is not surjective.*

Theorem 2.12. *Suppose $\mathcal{D} \subseteq \mathbb{C}$ have a finite number of components $X_1, \cdots, X_n$ such that each component is the closure of it's interior and the interior of each component is connected. Let $A(\mathcal{D})$ be the algebra of all continuous functions on $\mathcal{D}$ and analytic inside. Assume $uC_\varphi : A(\mathcal{D}) \to A(\mathcal{D})$ is a weighted composition operator such that the corresponding φ is not constant on each component of $\mathcal{D}$ and $u \not\equiv 0$. Then uC_φ has closed range if and only if there exists a positive δ such that $\varphi(\{x \in \partial\mathcal{D} : |u(x)| \ge \delta\})$ contains the boundary of each component of $\mathcal{D}$ which intersects it.*

Proof. Put $F_\delta = \varphi(E_\delta)$, where $E_\delta = \{x \in \partial\mathcal{D} : |u(x)| \ge \delta\}$. Suppose there exists a $\delta > 0$ such that $X_i \cap F_\delta \neq \emptyset$, where $1 \le i \le k < n$ and $\partial X_1 \cup \ldots \partial X_k \subseteq F_\delta$. Then each $f \in K_{F_\delta}$ vanishes on $X_1 \cup \ldots X_k$. Consider the function $f_0 \in A(\mathcal{D})$ such that f_0 is equal to zero on $X_1, X_2, \ldots, X_k$ and it's value on $X_{k+1}, \ldots, X_n$ is -1. Then observing the fact that $f + ff_0 = f$ on $X_1, X_2, \ldots, X_k$ and $f + ff_0 = 0$ on $X_{k+1}, \ldots, X_n$ and since $ff_0 \in K_{F_\delta}$, we conclude that $\|f + K_{F_\delta}\| \le \|f + ff_0\| \le \sup\{2|f(x)| : x \in F_\delta\} = 2\|f_{|F_\delta}\|$. Hence by Corollary 2.9, uC_φ has closed range.

Conversely, suppose $x_0 \in \partial X_{i_0} \setminus F_\delta$ for some $1 \le i_0 \le n$ and $X_{i_0} \cap F_\delta \neq \emptyset$. Then, since x_0 is a peak point, it follows that there is a function $g \in A(\mathcal{D})$ such that $\|g\| = g(x_0) = 1$ and $|g(x)| < 1$ for every $x \in F_\delta$. For $c > 0$, if we put $f = g^n$ for a sufficiently large integer n, then $f \in A(\mathcal{D})$ satisfies $|f(x)| \le \frac{c}{2}$ for all $x \in F_\delta$. For such an f, we have $\|f_{|F_\delta}\| < c \le \|f + K_{F_\delta}\|$. Now using Corollary 2.9, the range of uC_φ is not closed. □

Example 2.13. *Let $\mathcal{D} = X_1 \cup X_2$ where $X_1 = \{z \in \mathbb{C} : |z| \le 1\}$ and $X_2 = \{z \in \mathbb{C} : |z - 3| \le 1\}$. Let $A(\mathcal{D})$ be the algebra of all continuous functions on $\mathcal{D}$ that are analytic inside. Define $u(z) = z$ for all $z \in \mathcal{D}$, and*

$$\varphi(z) = \begin{cases} z & |z| \le 1 \\ 2 & |z - 3| \le 1. \end{cases}$$

It is easy to see that $\varphi(E_\delta) = \varphi(\partial\mathcal{D}) = \partial X_1 \cup \{2\}$ for all $\delta > 0$ and $\partial X_2 \cap \varphi(E_\delta) = \{2\}$. Since $\partial X_2 \not\subseteq \varphi(E_\delta)$, the operator uC_φ does not have closed range on $A(\mathcal{D})$.

3. Compact sums of weighted composition operators on C(X)

In [6], Kamowitz showed that uC_φ on $\mathbf{C}(X)$ is compact if and only if for each connected component C of $\{x \in X : u(x) \neq 0\}$, there exists an open set $U \supset C$ such that φ is constant on U. As is mentioned in [12, 15], this condition is equivalent to the following:

$$\text{For each } r > 0,\ \varphi(\{x \in X : |u(x)| \ge r\}) \text{ is finite.} \tag{3}$$

Let $x_1, x_2 \in X$ and $x_1 \neq x_2$. It is clear that $\|\lambda_1\delta_{x_1} - \lambda_2\delta_{x_2}\| \le |\lambda_1| + |\lambda_2|$, for all λ_1 and λ_2 in $\mathbb{C}$. For the opposite inequality, if $\lambda_1 = \lambda_2 = 0$, there is nothing to prove and so we assume $\lambda_1 \neq 0$. Then there is a function $f \in \mathbf{C}(X)$ such that $0 \le f \le 1$, $f(x_1) = \frac{|\lambda_1| + |\lambda_2|}{|\lambda_1|}$ and $f(x_2) = 0$. So we have $\|\lambda_1\delta_{x_1} - \lambda_2\delta_{x_2}\| \ge |\lambda_1 f(x_1) - \lambda_2 f(x_2)| = |\lambda_1| + |\lambda_2|$.

Now, let $\mathcal{A}$ be a uniformly closed subspace of $\mathbf{C}(X)$. By using Lemma 2.1, $uC_\varphi : \mathcal{A} \to \mathbf{C}(X)$ is compact if and only if

$$\|u(x_n)\delta_{\varphi(x_n)} - u(x_0)\delta_{\varphi(x_0)}\| = \begin{cases} |u(x_n) - u(x_0)| & \text{if } \varphi(x_n) = \varphi(x_0) \\ |u(x_n)| + |u(x_0)| & \text{if } \varphi(x_n) \neq \varphi(x_0) \end{cases} \longrightarrow 0$$

in the $\mathbf{C}(X)^*$- norm, as $(x_n) \to x_0$. If $(x_n) \to x_0$ and $u(x_0) \neq 0$, then for compactness of uC_φ the condition $\varphi(x_n) \neq \varphi(x_0)$ for large n, is necessary. In other words, if uC_φ is compact, then for each point $x_0 \in X$ either $u(x_0) = 0$ or $\varphi(x) = \varphi(x_0)$ on some neighborhood of x_0. Since the condition is sufficient for compactness of uC_φ, then we arrived at the result of Kamowitz for compactness of uC_φ on $\mathbf{C}(X)$.

Now we consider finite sums of weighted composition operators from uniformly closed subspace $\mathcal{A} \subseteq \mathbf{C}(X)$ into $\mathbf{C}(X)$, i.e. the operator $T_m : \mathcal{A} \to \mathbf{C}(X)$ of the form

$$T_m f(x) = \sum_{i=1}^{m} u_i(x) f(\varphi_i(x)),$$

where each $\varphi_i : X \to X$ is continuous, $u_i \in \mathbf{C}(X)$ and m is a positive integer. The following result follows immediately from Lemma 2.1.

Corollary 3.1. *The operator $T_m : \mathcal{A} \to \mathbf{C}(X)$ defined by $T_m f = \Sigma_{i=1}^m u_i.(f \circ \varphi_i)$ is compact if and only if $\sum_{i=1}^m u_i(x_0)[\delta_{\varphi_i(x)} - \delta_{\varphi_i(x_0)}] \to 0$ in the $\mathbf{C}(X)^*$-norm, as $x \to x_0$.*

We note that if $x \in G = X \setminus \Gamma$ and $(x_n) \to x$, then $\|\delta_{\varphi_i(x_n)} - \delta_{\varphi_i(x)}\|_{\mathcal{A}^*} \to 0$, since $\varphi_i(x) \in G$ and on G the original topology coincides with $\mathcal{A}^*$-topology, i.e. $(G, \varrho) = (G, \sigma(X, \Delta))$. Thus for studying compactness of the operator uC_φ, it is sufficient to investigate the continuity of τ only on Γ.

Fix a point $x \in \Gamma$. Let $i, j \in M = \{1, 2, \ldots, n\}$. We say that i is equivalent with j at point x ($i \overset{x}{\sim} j$) if $\varphi_i(x) = \varphi_j(x) \in \Gamma$. The equivalence classes are denoted by $K_i = K_i(x)$. Put $K_0 = \{i \in M : \varphi_i(x) \in G\}$ and $\widetilde{M} = \{K_i : i \in M\} \cup \{K_0\}$.

Lemma 3.2. *Fix a point $x \in \Gamma$. If the operator T_m is compact, then for each $K \neq K_0$ in $\widetilde{M}$ we have $\sum_{i \in K} u_i(x) = 0$.*

Proof. Since $K \neq K_0$, then there is a point $y \in \Gamma$ such that $\varphi_i(x) = y$ for all $i \in K$. On the other hand by definition of strongly boundary points, there is such a sequence of functions $f_n \in \mathcal{A}$, that $\|f_n\| \leq 1$, $f_n(y) = 1$, and outside any neighborhood of point y the sequence f_n uniformly converges to 0 as $n \to \infty$. By assumption, the sequence $\{T_m f_n\}$ may be considered as uniformly convergent. Since $\varphi_i(G) \subseteq G$ and G is everywhere dense in X, then $\|T_m f_n\| \to 0$ as $n \to \infty$, whence the statement of the lemma follows. □

In the following examples we show that in Lemma 3.2, condition $\varphi_i(G) \subseteq G$ is essential and the converse of this lemma is not true.

Example 3.3. *(a) Let $\mathcal{D} = \{z \in C : |z| \leq 1\} \cup \{z \in C : |z + 2| \leq 1\}$, $A(\mathcal{D})$ be the algebra of all continuous functions on $\mathcal{D}$ that are analytic inside. Define $\varphi_1(z) = z$ for all $z \in \mathcal{D}$, and*

$$\varphi_2(z) = \begin{cases} z & |z| \leq 1 \\ -1 & |z + 2| \leq 1, \end{cases} \qquad \varphi_3(z) = \begin{cases} -1 & |z| \leq 1 \\ z & |z + 2| \leq 1. \end{cases}$$

Since $T_3 f := -f \circ \varphi_1 + f \circ \varphi_2 + f \circ \varphi_3 = f(-1)$ for all $z \in \mathcal{D}$, it follows that the operator T_3 is compact on $A(\mathcal{D})$. However $u_1 + u_2 + u_3 = 1$.

(b) Put $\varphi_1(z) = z$, $\varphi_2(z) = \frac{1+z}{2}$. If we consider $T_2 : A(\bar{\mathbb{D}}) \to A(\bar{\mathbb{D}})$ as $T_2 f = f \circ \varphi_1 - f \circ \varphi_2$, it follows that $\Sigma_{i \in K(1)} u_i(1) = u_1(1) + u_2(1) = 0$. However, it is easy to see that

$$\left| \frac{\varphi_1(z) - \varphi_2(z)}{1 - \varphi_1(z)\overline{\varphi_2(z)}} \right| \longrightarrow 1 \neq 0$$

as $z \to 1$, so by Corollary 2.6, T_2 is not compact. We note that if we replace φ_2 with $\varphi_2(z) = \frac{z}{2}$, then $\mathcal{R}(T_2)$ is closed.

Theorem 3.4. *The operator $T_m : \mathcal{A} \to \mathbf{C}(X)$ is compact, if and only if for an arbitrary point $x \in \Gamma$, $\sum_{i \notin K_0} u_i(x)\delta_{\varphi_i(z)} \to 0$ with respect to $\mathcal{A}^*$-norm as $z \to x$ in original topology of X and $\sum_{i \in K} u_i(x) = 0$ for any class $K \neq K_0$.*

Proof. Fix a point $x \in \Gamma$. Let $\widetilde{M} = \{K_1(x), \cdots, K_n(x)\} \cup \{K_0\}$ and $K_j \in \widetilde{M} \setminus \{K_0\}$. Then there exists a point $y_j \in \Gamma$ such that $\varphi_i(x) = y_j$ for all $i \in K_j$. If T is compact, then by Lemma 3.2 and Corollary 3.1 we have $\sum_{j=1}^{n} \sum_{i \in K_j} u_i(x) = 0$ and $\sum_{i \notin K_0} u_i(x)[\delta_{\varphi_i(z)} - \delta_{\varphi_i(x)}] \to 0$ in the $\mathcal{A}^*$-norm, as $z \to x$. Therefore we have

$$\begin{aligned}
\sum_{i \notin K_0} u_i(x)\delta_{\varphi_i(z)} &= \sum_{j=1}^{n} \sum_{i \in K_j} u_i(x)\delta_{\varphi_i(z)} = \sum_{j=1}^{n} \left(\sum_{i \in K_j} u_i(x)\delta_{\varphi_i(z)} - \delta_{y_j} \sum_{i \in K_j} u_i(x) \right) \\
&= \sum_{j=1}^{n} \left(\sum_{i \in K_j} u_i(x)\delta_{\varphi_i(z)} - \sum_{i \in K_j} u_i(x)\delta_{\varphi_i(x)} \right) \\
&= \sum_{j=1}^{n} \left(\sum_{i \in K_j} u_i(x) \left[\delta_{\varphi_i(z)} - \delta_{\varphi_i(x)} \right] \right) \\
&= \sum_{i \notin K_0} u_i(x) \left[\delta_{\varphi_i(z)} - \delta_{\varphi_i(x)} \right] \longrightarrow 0
\end{aligned}$$

in the $\mathcal{A}^*$-norm, as $z \to x$ in the original topology of X. Conversely, it is easy to see that

$$\begin{aligned}
\sum_{i \notin K_0} u_i(x)[\delta_{\varphi_i(z)} - \delta_{\varphi_i(x)}] &= \sum_{j=1}^{n} \left(\sum_{i \in K_j} u_i(x) \left[\delta_{\varphi_i(z)} - \delta_{\varphi_i(x)} \right] \right) \\
&= \sum_{j=1}^{n} \sum_{i \in K_j} u_i(x)\delta_{\varphi_i(z)} - \sum_{j=1}^{n} y_j \left(\sum_{i \in K_j} u_i(x) \right) \\
&= \sum_{i \notin K_0} u_i(x)\delta_{\varphi_i(z)} \to 0
\end{aligned}$$

in the $\mathcal{A}^*$-norm, as $z \to x$ in the original topology of X. Thus, T_m is compact. □

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