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Complex symmetric difference of weighted composition operators on Fock space

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Abstract. The aim of the present paper is to completely characterize complex symmetric difference of weighted composition operators $W_{e^{\bar{p}z},az+b}$ with the conjugations J and $J_{r,s,t}$ defined by $Jf(z) = \overline{f(\bar{z})}$ and $J_{r,s,t}f(z) = te^{sz}\overline{f(\overline{rz+s})}$ on Fock space by building the relations between the parameters a, b, p, r and s. As an application, an interesting phenomenon that each operator $W_{e^{\bar{p}z},az+b}$ is not complex symmetric on Fock space with $J_{s,t,r}$ but their difference is complex symmetric on Fock space has been discovered.

1. Introduction

The aim of the present paper is to characterize the complex symmetric difference of weighted composition operators on Fock space. To proceed, we need to introduce some notations and terminology.

Throughout this paper, H will always denote a separable complex Hilbert space and $\mathcal{B}(H)$ the set of all bounded linear operators on H. For an operator $T \in \mathcal{B}(H)$, we let T^* denote the adjoint operator of T.

In this section, one of the definitions is the complex symmetric operators. It is well known that many problems in analysis require much research to non-Hermitian operators. Among them, we see from [14] that the complex symmetric operators have become particularly important in both theoretic and application aspects.

Definition 1.1. *Let C be an operator on H. The operator C is said to be a conjugation on H if it satisfies the following conditions:*

(a) conjugate-linear: $C(\alpha x + \beta y) = \bar{\alpha}C(x) + \bar{\beta}C(y)$, for $\alpha, \beta \in \mathbb{C}$ and $x, y \in H$;
(b) isometric: $\|C(x)\| = \|x\|$, for all $x \in H$;
(c) involutive: $C^2 = I_d$, where I_d is an identity operator.

It follows from [10] that for any conjugation C on H, there is an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ for H such that $Ce_n = e_n$ for each positive integer n. There are many conjugations on some holomorphic function spaces. For example, the common conjugation of complex numbers $(Jf)(z) = \overline{f(\bar{z})}$ defines a conjugation on Fock space, which will be defined later.

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By using Definition 1.1, we give the following definition.

Definition 1.2. *Let C be a conjugation on H. An operator $T \in \mathcal{B}(H)$ is said to be complex symmetric with C if $CTC = T^*$.*

Also from [10], it follows that if an operator $T \in \mathcal{B}(H)$ is complex symmetric, then T can be written as a symmetric matrix relative to some orthonormal basis. Because of this reason, the complex symmetric operators can be regarded as a generalization of the symmetric matrices. The class of complex symmetric operators includes all normal operators, Hankel matrices, finite Toeplitz matrices, all truncated Toeplitz operators, and some Volterra integration operators. The study of complex symmetric operators has a long story, dated back to the work of Glazman [15]. The study has many motivations in function theory, matrix analysis and many other disciplines.

The renewed interests in complex symmetric operators are inspired by many meaningful results obtained by Garcia et al. in [10–13]. Recently, there have been many studies about this operator on holomorphic function spaces (see [8, 16, 23, 26, 29, 30, 37]).

In addition to studying the complex symmetric operators, people also try to extend this definition. To this end, Helton in [20] studied the operators $T \in \mathcal{B}(H)$ which satisfy the following form

$$\sum_{j=0}^{m}(-1)^{m-j}\begin{pmatrix} m \\ j \end{pmatrix}T^{*j}T^{m-j} = 0. \tag{1.1}$$

As we expected, in light of complex symmetric operators, Chō et al. in [5] introduced the definition of m-complex symmetric operators by using the form (1.1). Such operators were continuously studied in [4] and [7].

Definition 1.3. *Let m be a positive integer and C a conjugation on H. An operator $T \in \mathcal{B}(H)$ is said to be m-complex symmetric with C if*

$$\sum_{j=0}^{m}(-1)^{m-j}\begin{pmatrix} m \\ j \end{pmatrix}T^{*j}CT^{m-j}C = 0.$$

Clearly, 1-complex symmetric operator is just the complex symmetric operator. From the definition, we see that an operator $T \in \mathcal{B}(H)$ is 2-complex symmetric with the conjugation C if and only if

$$CT^2 - 2T^*CT + T^{*2}C = 0.$$

Interestingly, it follows from a direct calculation that all complex symmetric operators are 2-complex symmetric operators. This shows that the set of all 2-complex symmetric operators is maybe larger than the set of all complex symmetric operators.

Recently, Hai et al. in [18] proved that the operator $Jf(z) = \overline{f(\bar{z})}$ is a conjugation on Fock space. They also proved that the operator $J_{r,s,t}f(z) = te^{sz}\overline{f(\overline{rz+s})}$ is a conjugation on Fock space if and only if

$$|r| = 1, \;\; \bar{r}s + \bar{s} = 0, \;\; |t|^2e^{|s|^2} = 1. \tag{1.2}$$

We can give a lot of examples satisfying (1.2) such as

$$r = e^{ix}, \;\; t = \frac{1}{2}e^{iy}, \;\; s = \sqrt{2\ln 2}e^{i\frac{x}{2}}i,$$

where x and y are real numbers. Actually, Hai et al. in [18] also studied some properties of complex symmetric weighted composition operators on Fock space. Hu et al. in [22] characterized 2-complex symmetric weighted composition operators on Hardy space. Most recently, Bai et al. in [2] characterized some special 2-complex symmetric weighted composition operators on Fock space. Of course, one can find the reasons why Bai et al. in [2] just studied some special weighted composition operators.

In the next time, we will provide the main motivations of this paper. With the basic questions such as boundedness and compactness settled, more attention has been paid to the study of the topological structure of the composition operators or weighted composition operators in the operator norm topology. In this research background, Shapiro and Sundberg in [33] posed a question on whether two composition operators belong to the same connected component, when their difference is compact. Motivated by this question, people initiated the study of difference of composition operators or weighted composition operators, which has become a very active topic (see [21, 34]). In the study of difference of composition operators, people found some interesting phenomena. For example, there is no compact composition operators on weighted Bergman space on the half-plane (see [27]), but there is compact difference of composition operators on this space (see [5]); two noncompact composition operators can induce a compact difference of composition operators on weighted Bergman space on the unit disk (see [28]).

Motivated by the above-mentioned interesting phenomena, it is natural to study complex symmetric difference of composition operators or weighted composition operators on holomorphic function spaces. Here, we consider such problems on Fock space. But, since the proper description of the adjoint $W_{u,\varphi}^*$ of the operator $W_{u,\varphi}$ for the general symbols on Fock space is too difficult, we just consider difference of the operators $W_{u,\varphi}$ on Fock space with the symbols $\varphi(z) = az + b$ and the weight functions $u(z) = e^{\overline{p}z}$. Our this consideration is also based on the next mentioned study results. Le in [25] showed that the operator $W_{u,\varphi}$ is bounded on Fock space if and only if the weight function $u(z)$ belongs to Fock space, $\varphi(z) = az + b$ with $|a| \leq 1$ and

$$M(u, \varphi) = \sup\left\{|u(z)|^2 e^{(|\varphi(z)|^2 - |z|^2)} : z \in \mathbb{C}\right\} < +\infty. \tag{1.3}$$

Zhao et al. in [41] continuously studied the condition (1.3) and proved that if $u(z) = e^{\overline{p}z}$ and $\varphi(z) = az + b$, then the operator $W_{u,\varphi}$ is bounded on Fock space if and only if one of the conditions holds: (i) $|a| < 1$; (ii) $|a| = 1$ and $p + \overline{a}b = 0$. Here, it is a light digression that it follows from this result that, if $\varphi(z) = az + b$, then the operator C_φ is bounded on Fock space if and only if one of the conditions holds: (i) $|a| < 1$; (ii) $|a| = 1$ and $b = 0$.

In this paper, we sometimes write the operator $W_{u,\varphi}$ as $W_{e^{\overline{p}z},az+b}$ for such special symbol and the weight function. Our work can be regarded as a continuous study of the weighted composition operators on Fock space.

2. Preliminaries

Throughout the paper, $\mathbb{N}$ always denotes the set of integers, $\mathbb{C}$ the complex plane and $H(\mathbb{C})$ the set of holomorphic functions on $\mathbb{C}$.

For a given holomorphic mapping $\varphi : \mathbb{C} \to \mathbb{C}$ and $u \in H(\mathbb{C})$, the weighted composition operator usually denoted by $W_{u,\varphi}$ between some subspaces of $H(\mathbb{C})$ is defined by

$$W_{u,\varphi} f(z) = u(z) f(\varphi(z)).$$

When $u \equiv 1$, it is the composition operator usually denoted by C_φ. While $\varphi(z) = z$, it is the multiplication operator usually denoted by M_u. It is well known that Forelli in [9] proved that the isometries on the Hardy space H^p defined on the open unit disk (for $p \neq 2$) are certain weighted composition operators, which can be regarded as the earliest presence of the weighted composition operators. Weighted composition operators have also been used in descriptions of adjoints of composition operators (see [6]). It is important to provide function-theoretic characterizations of the symbols u and φ which induce a bounded or compact weighted composition operator on various holomorphic function spaces (see, for example, [1, 17, 24, 32, 35]).

The desired Fock space $\mathcal{F}^2(\mathbb{C})$ is the Hilbert space of all holomorphic functions $f \in H(\mathbb{C})$ with the inner product

$$\langle f, g \rangle = \frac{1}{\pi} \int_{\mathbb{C}} f(z) \overline{g(z)} e^{-|z|^2} dv(z),$$

where $\nu(z)$ denotes the Lebesgue area measure on $\mathbb{C}$. To simplify notation we will often use $\mathcal{F}^2$ instead of $\mathcal{F}^2(\mathbb{C})$, and we will denote by $\|f\|$ the corresponding norm of f. The reproducing kernel functions of Fock space are obtained by

$$K_w(z) = e^{\overline{w}z}, \quad z \in \mathbb{C}.$$

That is,

$$f(z) = \langle f, K_z \rangle = \int_{\mathbb{C}} f(w)\overline{K_z(w)}d\nu(w)$$

for any $f \in \mathcal{F}^2$ and $z \in \mathbb{C}$. One can see [42] for more information on Fock space.

Recently, several authors have worked on the composition operators and weighted composition operators on Fock space. For the one-variable case, Ueki in [36] characterized the boundedness and compactness of weighted composition operators on Fock space. In [19], the author considered unbounded weighted composition operators on Fock space. Extending the corresponding result of [25], the author obtained a characterization of unbounded weighted composition operators which are normal and also those which are cohyponormal. In addition, Bhuia in [3] studied C-normality of the weighted composition operators on Fock space. Zhao in [38–40] studied the unitary, invertible and normal weighted composition operators $W_{u,\varphi}$ with the symbol $\varphi(z) = az + b$ and the weight function $u(z) = e^{\overline{p}z}$ on Fock space.

3. Complex symmetric difference of weighted composition operators

Since the linear span of the reproducing kernel functions $\{K_w : w \in \mathbb{C}\}$ is dense in $\mathcal{F}^2$, we have the following direct result.

Lemma 3.1. *Let $T \in \mathcal{B}(\mathcal{F}^2)$ and C be a conjugation on $\mathcal{F}^2$. Then the operator T is m-complex symmetric on $\mathcal{F}^2$ with the conjugation C if and only if*

$$\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}T^{*j}CT^{m-j}CK_w(z) = 0$$

for all $z, w \in \mathbb{C}$.

Remark 3.1. From Lemma 3.1, we see that, if $m = 1$, then $T \in \mathcal{B}(\mathcal{F}^2)$ is complex symmetric on $\mathcal{F}^2$ with the conjugation C if and only if $(CT - T^*C)K_w(z) = 0$ for all $z, w \in \mathbb{C}$; if $m = 2$, then $T \in \mathcal{B}(\mathcal{F}^2)$ is 2-complex symmetric on $\mathcal{F}^2$ with the conjugation C if and only if $(CT^2 - 2T^*CT + T^{*2}C)K_w(z) = 0$ for all $z, w \in \mathbb{C}$.

To study complex symmetry of difference of the weighted composition operators on Fock space, we need the following formula for the operator $W_{e^{\overline{p}z},az+b}$.

Lemma 3.2. *Let $u(z) = e^{\overline{p}z}$ and $\varphi(z) = az + b$ such that the operator $W_{u,\varphi}$ is bounded on $\mathcal{F}^2$. Then it follows that*

$$W_{u,\varphi}^* = W_{e^{\overline{b}z},\overline{a}z+p}. \tag{3.1}$$

Proof. For each $f \in \mathcal{F}^2$, we have

$$\begin{aligned}
W_{u,\varphi}^* f(z) &= \langle W_{u,\varphi}^* f, K_z \rangle = \langle f, W_{u,\varphi} K_z \rangle \\
&= \frac{1}{\pi} \int_{\mathbb{C}} f(w) \overline{W_{u,\varphi} K_z(w)} e^{-|w|^2} d\nu(w) \\
&= \frac{1}{\pi} \int_{\mathbb{C}} f(w) \overline{u(w) K_z(\varphi(w))} e^{-|w|^2} d\nu(w) \\
&= \frac{1}{\pi} \int_{\mathbb{C}} f(w) \overline{e^{\overline{p}w} K_z(aw+b)} e^{-|w|^2} d\nu(w) \\
&= \frac{1}{\pi} \int_{\mathbb{C}} f(w) e^{p\overline{w}} e^{z(\overline{aw}+\bar{b})} e^{-|w|^2} d\nu(w) \\
&= e^{\bar{b}z} \frac{1}{\pi} \int_{\mathbb{C}} f(w) e^{(\bar{a}z+p)\overline{w}} e^{-|w|^2} d\nu(w) \\
&= e^{\bar{b}z} \langle f, K_{\bar{a}z+p} \rangle = e^{\bar{b}z} f(\bar{a}z+p) = W_{e^{\bar{b}z}, \bar{a}z+p} f(z),
\end{aligned}$$

from which the desired result follows. □

The authors in [31] also obtained the formula (3.1) by using a complicated proof. Here, our proof is simple and clear. If $p = 0$ in Lemma 3.2, we have the following interesting result, which shows that the adjoint of the composition operator C_{az+b} on $\mathcal{F}^2$ becomes a weighted composition operator.

Corollary 3.1. *Let $\varphi(z) = az + b$ such that the operator C_φ is bounded on $\mathcal{F}^2$. Then it follows that*

$$C_\varphi^* = W_{e^{\bar{b}z}, \bar{a}z}.$$

We have the following obvious result.

Lemma 3.3. *Let $u(z) = e^{\overline{p}z}$ and $\varphi(z) = az + b$ such that the operator $W_{u,\varphi}$ is bounded on $\mathcal{F}^2$. Then the operator $W_{u,\varphi}$ is complex symmetric on $\mathcal{F}^2$ with the conjugation J if and only if $\overline{p} = b$.*

Proof. For each $w, z \in \mathbb{C}$, we have

$$W_{u,\varphi} J K_w(z) = W_{u,\varphi} e^{wz} = e^{\overline{p}z} e^{w(az+b)} = e^{awz+\overline{p}z+bw} \tag{3.2}$$

and

$$J W_{u,\varphi}^* K_w(z) = J W_{e^{\bar{b}z}, \bar{a}z+p} K_w(z) = J\Big(e^{\bar{b}z} e^{\overline{w}(\bar{a}z+p)}\Big) = e^{bz} e^{w(az+\overline{p})} = e^{awz+bz+\overline{p}w}. \tag{3.3}$$

From (3.2) and (3.3), it follows that the operator $W_{u,\varphi}$ is complex symmetric on $\mathcal{F}^2$ with the conjugation J if and only if

$$e^{awz+\overline{p}z+bw} = e^{awz+bz+\overline{p}w} \tag{3.4}$$

for all $z, w \in \mathbb{C}$. If $W_{u,\varphi}$ is complex symmetric on $\mathcal{F}^2$, then from (3.4), we obtain

$$\overline{p}z + bw = bz + \overline{p}w,$$

which shows that $\overline{p} = b$.

Conversely, if $\overline{p} = b$, then it is clear that (3.4) holds. Hence, the operator $W_{u,\varphi}$ is complex symmetric on $\mathcal{F}^2$ with the conjugation J. □

We first give the characterization of complex symmetric difference of the weighted composition operators on $\mathcal{F}^2$ with the conjugation J.

Theorem 3.1. *Let $u_j(z) = e^{\overline{p}_j z}$ and $\varphi_j(z) = a_j z + b_j$ such that the operator W_{u_j,φ_j} is bounded on $\mathcal{F}^2$ for $j = 1, 2$. Then the operator $W_{u_1,\varphi_1} - W_{u_2,\varphi_2}$ is complex symmetric on $\mathcal{F}^2$ with the conjugation J if and only if $\overline{p}_1 = b_1$ and $\overline{p}_2 = b_2$.*

Proof. For each $w, z \in \mathbb{C}$, we have the following calculations:

$$\begin{aligned}(W_{u_1,\varphi_1} - W_{u_2,\varphi_2})JK_w(z) &= (W_{u_1,\varphi_1} - W_{u_2,\varphi_2})\Big(e^{wz}\Big)\\ &= e^{\overline{p}_1 z}e^{w(a_1 z+b_1)} - e^{\overline{p}_2 z}e^{w(a_2 z+b_2)}\\ &= e^{a_1 wz+\overline{p}_1 z+b_1 w} - e^{a_2 wz+\overline{p}_2 z+b_2 w}\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}J(W^*_{u_1,\varphi_1} - W^*_{u_2,\varphi_2})K_w(z) &= C\Big(W_{e^{\bar{b}_1 z},\bar{a}_1 z+p_1} - W_{e^{\bar{b}_2 z},\bar{a}_2 z+p_2}\Big)K_w(z)\\ &= J\Big(e^{\bar{b}_1 z}e^{\overline{w}(\bar{a}_1 z+p_1)} - e^{\bar{b}_2 z}e^{\overline{w}(\bar{a}_2 z+p_2)}\Big)\\ &= e^{b_1 z}e^{w(a_1 z+\overline{p}_1)} - e^{b_2 z}e^{w(a_2 z+\overline{p}_2)}\\ &= e^{a_1 wz+b_1 z+\overline{p}_1 w} - e^{a_2 wz+b_2 z+\overline{p}_2 w}.\end{aligned} \tag{3.6}$$

From (3.5) and (3.6), it follows that the operator $W_{u_1,\varphi_1} - W_{u_2,\varphi_2}$ is complex symmetric on $\mathcal{F}^2$ with the conjugation J if and only if

$$e^{a_1 wz+\overline{p}_1 z+b_1 w} - e^{a_2 wz+\overline{p}_2 z+b_2 w} = e^{a_1 wz+b_1 z+\overline{p}_1 w} - e^{a_2 wz+b_2 z+\overline{p}_2 w} \tag{3.7}$$

for all $w, z \in \mathbb{C}$.

Now, assume that $W_{u_1,\varphi_1} - W_{u_2,\varphi_2}$ is complex symmetric on $\mathcal{F}^2$ with the conjugation J. Then, letting $w = 0$ in (3.7), we obtain

$$e^{\overline{p}_1 z} - e^{\overline{p}_2 z} = e^{b_1 z} - e^{b_2 z}. \tag{3.8}$$

Then, from (3.8), it follows that

$$\sum_{k=0}^{\infty}\frac{b_1^k - b_2^k}{k!}z^k = e^{b_1 z} - e^{b_2 z} = e^{\overline{p}_1 z} - e^{\overline{p}_2 z} = \sum_{k=0}^{\infty}\frac{\overline{p}_1^k - \overline{p}_2^k}{k!}z^k. \tag{3.9}$$

From (3.9), we obtain

$$b_1^k - b_2^k = \overline{p}_1^k - \overline{p}_2^k$$

for each $k \in \mathbb{N}$. Specially, we have

$$b_1 - b_2 = \overline{p}_1 - \overline{p}_2 \quad\text{and}\quad b_1^2 - b_2^2 = \overline{p}_1^2 - \overline{p}_2^2. \tag{3.10}$$

We will divide into two cases.

Case 1. Assume that $b_1 = b_2$. Then, we obtain $p_1 = p_2$.

Case 2. Assume that $b_1 \neq b_2$. Then, from (3.10), we obtain $\overline{p}_1 = b_1$ and $\overline{p}_2 = b_2$.

Combining Case 1 and Case 2, we see that if the operator $W_{u_1,\varphi_1} - W_{u_2,\varphi_2}$ is complex symmetric on $\mathcal{F}^2$ with the conjugation J, then $\overline{p}_1 = b_1$ and $\overline{p}_2 = b_2$.

Conversely, if $\overline{p}_1 = b_1$ and $\overline{p}_2 = b_2$, then it is easy to see that (3.7) holds. This shows that the operator $W_{u_1,\varphi_1} - W_{u_2,\varphi_2}$ is complex symmetric on $\mathcal{F}^2$ with the conjugation J. □

Remark 3.2. Interestingly, from Lemma 3.3 and Theorem 3.1, we see that the operator $W_{u_1,\varphi_1} - W_{u_2,\varphi_2}$ is complex symmetric on $\mathcal{F}^2$ with the conjugation J if and only if each operator W_{u_j,φ_j} is complex symmetric on $\mathcal{F}^2$ with the conjugation J for $j = 1, 2$.

In order to prove Theorem 3.2, we also need the following elementary result.

Lemma 3.4. *Let $z, w \in \mathbb{C}$. Then $e^z + e^w = 0$ if and only if $z = w + (2k+1)\pi i$ for some $k \in \mathbb{N}$.*

Proof. Let $z = x_1 + y_1 i$ and $w = x_2 + y_2 i$, where x_1, x_2, y_1, y_2 are real numbers and i is the imaginary unit satisfying $i^2 = -1$. If $e^z + e^w = 0$, then

$$\begin{aligned} e^z + e^w &= e^{x_1+y_1 i} + e^{x_2+y_2 i} = e^{x_1}(\cos y_1 + i\sin y_1) + e^{x_2}(\cos y_2 + i\sin y_2) \\ &= \Big(e^{x_1}\cos y_1 + e^{x_2}\cos y_2\Big) + \Big(e^{x_1}\sin y_1 + e^{x_2}\sin y_2\Big)i \\ &= 0. \end{aligned}$$

This is equivalent to

$$\begin{cases} e^{x_1}\cos y_1 + e^{x_2}\cos y_2 = 0 \\ e^{x_1}\sin y_1 + e^{x_2}\sin y_2 = 0. \end{cases} \tag{3.11}$$

From (3.11) and the fact $\sin^2 x + \cos^2 y \equiv 1$ for all real numbers x and y, we have

$$0 = e^{2x_1} + e^{2x_2} + 2e^{x_1}e^{x_2}\cos(y_1 - y_2) \geq 2e^{x_1}e^{x_2}[1 + \cos(y_1 - y_2)]. \tag{3.12}$$

On the other hand, since $\cos x \in [-1, 1]$, we have

$$2e^{x_1}e^{x_2}[1 + \cos(y_1 - y_2)] \geq 0. \tag{3.13}$$

From (3.12) and (3.13), it follows that

$$2e^{x_1}e^{x_2}[1 + \cos(y_1 - y_2)] = 0,$$

which forces that $\cos(y_1 - y_2) = -1$. From this, we deduce that

$$y_1 = y_2 + (2k+1)\pi$$

for some $k \in \mathbb{N}$. Hence, we have

$$e^{2x_1} + e^{2x_2} - 2e^{x_1}e^{x_2} = 0,$$

which shows that $x_1 = x_2$. We therefore prove that $z = w + (2k+1)\pi i$ for some $k \in \mathbb{N}$.

Conversely, if $z = w + (2k+1)\pi i$ for some $k \in \mathbb{N}$, then it is clear that $e^z + e^w = 0$. □

Now, we can characterize complex symmetric operator $W_{u_1,\varphi_1} - W_{u_2,\varphi_2}$ on $\mathcal{F}^2$ with the conjugation $J_{r,s,t}$. In order to clarify this problem, we will divide into two cases to consider. The first case is $s \neq 0$.

Theorem 3.2. *Let $u_j(z) = e^{\overline{p}_j z}$ and $\varphi_j(z) = a_j z + b_j$ such that the operator W_{u_j,φ_j} is bounded on $\mathcal{F}^2$ for $j = 1, 2$. Then the operator $W_{u_1,\varphi_1} - W_{u_2,\varphi_2}$ is complex symmetric on $\mathcal{F}^2$ with the conjugation $J_{r,s,t}$ if and only if one of the conditions holds: (i) $\overline{p}_1 + a_1 s = \overline{p}_2 + a_2 s$ and $b_1 = b_2$; (ii) $\overline{p}_1 + a_1 s = s + b_1 r$ and $\overline{p}_2 + a_2 s = s + b_2 r$; (iii) $b_1 s = b_2 s + (2k+1)\pi i$ for some $k \in \mathbb{N}$, $\overline{p}_1 + a_1 s = s + b_2 r$ and $\overline{p}_2 + a_2 s = s + b_1 r$.*

Proof. For each $w, z \in \mathbb{C}$, we have the following calculations:

$$\begin{aligned} (W_{u_1,\varphi_1} - W_{u_2,\varphi_2})J_{r,s,t}K_w(z) &= (W_{u_1,\varphi_1} - W_{u_2,\varphi_2})\Big(te^{sz}e^{w(rz+s)}\Big) \\ &= t\Big\{e^{\overline{p}_1 z}e^{s(a_1 z+b_1)}e^{w[r(a_1 z+b_1)+s]} - e^{\overline{p}_2 z}e^{s(a_2 z+b_2)}e^{w[r(a_2 z+b_2)+s]}\Big\} \\ &= t\Big[e^{(\overline{p}_1+sa_1)z+ra_1wz+(rb_1+s)w+b_1 s} - e^{(\overline{p}_2+sa_2)z+ra_2wz+(rb_2+s)w+b_2 s}\Big] \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} J_{r,s,t}(W^*_{u_1,\varphi_1} - W^*_{u_2,\varphi_2})K_w(z) &= J_{r,s,t}\Big(W_{e^{\bar{b}_1 z},\bar{a}_1 z+p_1} - W_{e^{\bar{b}_2 z},\bar{a}_2 z+p_2}\Big)K_w(z) \\ &= J_{r,s,t}\Big(e^{\bar{b}_1 z}e^{\overline{w}(\bar{a}_1 z+p_1)} - e^{\bar{b}_2 z}e^{\overline{w}(\bar{a}_2 z+p_2)}\Big) \\ &= t\Big\{e^{sz}e^{b_1(rz+s)}e^{w[a_1(rz+s)+\overline{p}_1]} - e^{sz}e^{b_2(rz+s)}e^{w[a_2(rz+s)+\overline{p}_2]}\Big\} \\ &= t\Big[e^{(s+b_1 r)z+a_1 rwz+(a_1 s+\overline{p}_1)w+b_1 s} - e^{(s+b_2 r)z+a_2 rwz+(a_2 s+\overline{p}_2)w+b_2 s}\Big]. \end{aligned} \tag{3.15}$$

From (3.14) and (3.15), it follows that the operator $W_{u_1,\varphi_1} - W_{u_2,\varphi_2}$ is complex symmetric on $\mathcal{F}^2$ with the conjugation $J_{r,s,t}$ if and only if

$$\begin{aligned} & e^{(\overline{p}_1+a_1s)z+ra_1wz+(rb_1+s)w+b_1s} - e^{(\overline{p}_2+a_2s)z+ra_2wz+(rb_2+s)w+b_2s} \\ & = e^{(s+b_1r)z+a_1rwz+(a_1s+\overline{p}_1)w+b_1s} - e^{(s+b_2r)z+a_2rwz+(a_2s+\overline{p}_2)w+b_2s} \end{aligned} \tag{3.16}$$

for all $w, z \in \mathbb{C}$.

We assume that $W_{u_1,\varphi_1} - W_{u_2,\varphi_2}$ is complex symmetric on $\mathcal{F}^2$ with the conjugation $J_{r,s,t}$. Letting $w = 0$ in (3.16), we obtain

$$e^{b_1s}\Big[e^{(\overline{p}_1+a_1s)z} - e^{(s+b_1r)z}\Big] = e^{b_2s}\Big[e^{(\overline{p}_2+a_2s)z} - e^{(s+b_2r)z}\Big] \tag{3.17}$$

for all $z \in \mathbb{C}$. Then, it follows from (3.17) that

$$e^{b_1s}\sum_{k=0}^{\infty}\frac{(\overline{p}_1+a_1s)^k-(s+b_1r)^k}{k!}z^k = e^{b_2s}\sum_{k=0}^{\infty}\frac{(\overline{p}_2+a_2s)^k-(s+b_2r)^k}{k!}z^k \tag{3.18}$$

for all $z \in \mathbb{C}$. From (3.18), we obtain

$$e^{b_1s}\Big[(\overline{p}_1+a_1s)^k-(s+b_1r)^k\Big] = e^{b_2s}\Big[(\overline{p}_2+a_2s)^k-(s+b_2r)^k\Big] \tag{3.19}$$

for each $k \in \mathbb{N}$.

For the convenience, we write

$$x_j = \overline{p}_j + a_js \quad \text{and} \quad y_j = s + b_jr$$

for $j = 1, 2$. Then, (3.19) is expressed as

$$e^{b_1s}\Big(x_1^k - y_1^k\Big) = e^{b_2s}\Big(x_2^k - y_2^k\Big) \tag{3.20}$$

for each $k \in \mathbb{N}$. We will divide into two cases.

Case 1. Assume that $x_1 \neq y_1$ in (3.20). Then, by choosing $k = 1, 2, 3$ successively in (3.20) and from a direct calculation, we have

$$\begin{cases} e^{b_1s}(x_1 - y_1) = e^{b_2s}(x_2 - y_2) \\ x_1 + y_1 = x_2 + y_2 \\ x_1y_1 = x_2y_2. \end{cases} \tag{3.21}$$

From (3.21), we deduce that

$$x_1^2 + y_1^2 = x_2^2 + y_2^2. \tag{3.22}$$

Choosing $k = 5$ in (3.20), we obtain

$$e^{b_1s}(x_1^5 - y_1^5) = e^{b_2s}(x_2^5 - y_2^5),$$

that is,

$$e^{b_1s}(x_1 - y_1)(x_1^4 + x_1^3y_1 + x_1^2y_1^2 + x_1y_1^3 + y_1^4) = e^{b_2s}(x_2 - y_2)(x_2^4 + x_2^3y_2 + x_2^2y_2^2 + x_2y_2^3 + y_2^4). \tag{3.23}$$

From (3.21), (3.22) and a calculation, (3.23) is reduced to

$$\Big(x_1 - x_2\Big)\Big(x_1 + x_2\Big)\Big(x_1^2 + x_2^2 - y_1^2 - y_2^2\Big) = 0. \tag{3.24}$$

We will divide into two subcases.

Subcase 1. If $x_1 - x_2 = 0$, then from (3.21) we obtain $y_1 = y_2$.

Subcase 2. If $x_1 + x_2 = 0$, then from (3.21) we obtain $x_2(y_1 + y_2) = 0$. We continuously discuss as follows:

If $x_2 = 0$, then $x_1 = 0$. From this and (3.21), we obtain $y_1 = y_2 = 0$.

If $x_2 \neq 0$, then $y_1 = -y_2$. From this and (3.21), we obtain

$$(x_2 - y_2)(e^{b_1 s} + e^{b_2 s}) = 0. \tag{3.25}$$

If $x_2 = y_2$, then $x_1 = y_1$. If $e^{b_1 s} + e^{b_2 s} = 0$, then it follows from Lemma 3.4 that $b_1 s = b_2 s + (2k+1)\pi i$ for some $k \in \mathbb{N}$. Since $e^{b_1 s} + e^{b_2 s} = 0$, from (3.21) we obtain that $x_1 = y_2$ and $x_2 = y_1$.

Combining Subcase 1 and Subcase 2, we see that if $x_1 \neq y_1$, then (3.21) holds if and only if one of the conditions holds: (i) $x_1 = x_2$ and $y_1 = y_2$; (ii) $x_1 = y_1$ and $x_2 = y_2$; (iii) $b_1 s = b_2 s + (2k+1)\pi i$ for some $k \in \mathbb{N}$, $x_1 = y_2$ and $x_2 = y_1$.

Case 2. Assume that $x_1 = y_1$ in (3.20). Then, we have that $x_2 = y_2$.

Combining Case 1 and Case 2, we prove that (3.20) holds if and only if one of the conditions holds: (i) $x_1 = x_2$ and $y_1 = y_2$; (ii) $x_1 = y_1$ and $x_2 = y_2$; (iii) $b_1 s = b_2 s + (2k+1)\pi i$ for some $k \in \mathbb{N}$, $x_1 = y_2$ and $x_2 = y_1$. From replacing x_j and y_j by their concrete expressions in (i), (ii) and (iii), the desired result follows.

Conversely, we first assume that (i) holds. Then, from Lemma 3.4, it follows that each W_{u_j,φ_j} is complex symmetric on $\mathcal{F}^2$ with the conjugation $J_{r,s,t}$. Therefore, $W_{u_1,\varphi_1} - W_{u_2,\varphi_2}$ is also.

Finally, assume that (ii) or (iii) holds. It is easy to see that (3.16) holds. So, $W_{u_1,\varphi_1} - W_{u_2,\varphi_2}$ is complex symmetric on $\mathcal{F}^2$ with the conjugation $J_{r,s,t}$. $\square$

For the second case $s = 0$, we have the following characterization.

Theorem 3.3. *Let $u_j(z) = e^{\overline{p}_j z}$ and $\varphi_j(z) = a_j z + b_j$ such that the operator W_{u_j,φ_j} is bounded on $\mathcal{F}^2$ for $j = 1, 2$. Then the operator $W_{u_1,\varphi_1} - W_{u_2,\varphi_2}$ is complex symmetric on $\mathcal{F}^2$ with the conjugation $J_{r,0,t}$ if and only if $\overline{p}_1 = b_1 r$ and $\overline{p}_2 = b_2 r$.*

Proof. From the proof of Theorem 3.2, we see that the operator $W_{u_1,\varphi_1} - W_{u_2,\varphi_2}$ is complex symmetric on $\mathcal{F}^2$ with the conjugation $J_{r,0,t}$ if and only if

$$e^{\overline{p}_1 z + r a_1 w z + r b_1 w} - e^{\overline{p}_2 z + r a_2 w z + r b_2 w} = e^{b_1 r z + a_1 r w z + \overline{p}_1 w} - e^{b_2 r z + a_2 r w z + \overline{p}_2 w} \tag{3.26}$$

for all $w, z \in \mathbb{C}$.

Assume that $W_{u_1,\varphi_1} - W_{u_2,\varphi_2}$ is complex symmetric on $\mathcal{F}^2$ with the conjugation $J_{r,0,t}$. Letting $w = 0$ in (3.26), we obtain

$$e^{\overline{p}_1 z} - e^{\overline{p}_2 z} = e^{b_1 r z} - e^{b_2 r z} \tag{3.27}$$

for all $z \in \mathbb{C}$. Then, it follows from (3.27) that

$$\sum_{k=0}^{\infty} \frac{\overline{p}_1^k - \overline{p}_2^k}{k!} z^k = \sum_{k=0}^{\infty} \frac{(b_1 r)^k - (b_2 r)^k}{k!} z^k \tag{3.28}$$

for all $z \in \mathbb{C}$. From (3.28), we obtain

$$\overline{p}_1^k - \overline{p}_2^k = (b_1 r)^k - (b_2 r)^k \tag{3.29}$$

for each $k \in \mathbb{N}$. Choosing $k = 1$ and $k = 2$ in (3.29) respectively, we have

$$\overline{p}_1 - \overline{p}_2 = b_1 r - b_2 r$$

and

$$(\overline{p}_1 - \overline{p}_2)(\overline{p}_1 + \overline{p}_2) = (b_1 r - b_2 r)(b_1 r + b_2 r).$$

Hence, we have

$$(\overline{p}_1 - \overline{p}_2)(\overline{p}_1 + \overline{p}_2) = (\overline{p}_1 - \overline{p}_2)(b_1 r + b_2 r). \tag{3.30}$$

We will divide into two cases.

Case 1. Assume that $p_1 \neq p_1$. Then, we have

$$\begin{cases} \overline{p}_1 - \overline{p}_2 = b_1 r - b_2 r \\ \overline{p}_1 + \overline{p}_2 = b_1 r + b_2 r. \end{cases} \tag{3.31}$$

From (3.31), we obtain that $\overline{p}_1 = b_1 r$ and $\overline{p}_2 = b_2 r$.

Case 2. Assume that $p_1 = p_2$. Then, it follows that $b_1 = b_2$ since $r \neq 0$.

Combining Case 1 and Case 2, we obtain that if the operator $W_{u_1,\varphi_1} - W_{u_2,\varphi_2}$ is complex symmetric on $\mathcal{F}^2$ with the conjugation $J_{r,0,t}$, then $\overline{p}_1 = b_1 r$ and $\overline{p}_2 = b_2 r$.

Conversely, assume that $\overline{p}_1 = b_1 r$ and $\overline{p}_2 = b_2 r$. It is clear that (3.26) holds. Hence, $W_{u_1,\varphi_1} - W_{u_2,\varphi_2}$ is complex symmetric on $\mathcal{F}^2$ with the conjugation $J_{r,0,t}$. $\square$

Lemma 3.5. *Let $u(z) = e^{\overline{p}z}$ and $\varphi(z) = az + b$ such that the operator $W_{u,\varphi}$ is bounded on $\mathcal{F}^2$. Then the operator $W_{u,\varphi}$ is complex symmetric on $\mathcal{F}^2$ with the conjugation $J_{r,s,t}$ if and only if $\overline{p} + as = s + br$.*

Proof. For each $w, z \in \mathbb{C}$, we have the following calculations:

$$\begin{aligned} W_{u,\varphi} J_{r,s,t} K_w(z) &= W_{u,\varphi}\Big(te^{sz} e^{w(rz+s)}\Big) = te^{\overline{p}z} e^{s(az+b)} e^{w[r(az+b)+s]} \\ &= te^{(\overline{p}+sa)z + arwz + (br+s)w + bs} \end{aligned}$$

and

$$\begin{aligned} J_{r,s,t} W^*_{u,\varphi} K_w(z) &= J_{r,s,t} W_{e^{\bar{b}z}, \bar{a}z+p} K_w(z) = J_{r,s,t}\Big(e^{\bar{b}z} e^{\overline{w}(\bar{a}z+p)}\Big) \\ &= te^{sz} e^{b(rz+s)} e^{w[a(rz+s)+\overline{p}]} \\ &= te^{(s+br)z + arwz + (sa+\overline{p})w + bs}. \end{aligned}$$

Since $t \neq 0$, the operator $W_{u,\varphi}$ is complex symmetric on $\mathcal{F}^2$ with the conjugation $J_{r,s,t}$ if and only if

$$e^{(\overline{p}+as)z + arwz + (br+s)w} = e^{(s+br)z + arwz + (as+\overline{p})w}. \tag{3.32}$$

Hence, if $W_{u,\varphi}$ is complex symmetric on $\mathcal{F}^2$ with the conjugation $J_{s,t,r}$, then from letting $w = 0$ in (3.32), we obtain

$$e^{(\overline{p}+sa)z} = e^{(s+br)z}$$

for all $z \in \mathbb{C}$, which shows that $\overline{p} + as = s + br$.

Conversely, assume that $\overline{p} + as = s + br$. Then, it is easy to see that (3.32) holds, which shows that $W_{u,\varphi}$ is complex symmetric on $\mathcal{F}^2$ with the conjugation $J_{r,s,t}$. $\square$

If $s = 0$ in Lemma 3.5, we obtain the next corollary since $|r| = 1$.

Corollary 3.1. *Let $u(z) = e^{\overline{p}z}$ and $\varphi(z) = az + b$ such that the operator $W_{u,\varphi}$ is bounded on $\mathcal{F}^2$. Then the operator $W_{u,\varphi}$ is complex symmetric on $\mathcal{F}^2$ with the conjugation $J_{r,0,t}$ if and only if $\overline{p} = br$.*

Remark 3.3. By Theorem 3.3 and Corollary 3.1, we see that the operator $W_{u_1,\varphi_1} - W_{u_2,\varphi_2}$ is complex symmetric on $\mathcal{F}^2$ with the conjugation $J_{r,0,t}$ if and only if each operator W_{u_j,φ_j} is complex symmetric on $\mathcal{F}^2$ with the conjugation $J_{r,0,t}$ for $j = 1, 2$. But for $s \neq 0$, that's not the case. Actually, we can give examples such that $W_{u_1,\varphi_1} - W_{u_2,\varphi_2}$ is complex symmetric on $\mathcal{F}^2$ with the conjugation $J_{r,s,t}$, but each W_{u_j,φ_j} is not complex symmetric on $\mathcal{F}^2$ with the conjugation $J_{r,s,t}$ for $j = 1, 2$. One can see the following concrete example.

Example 3.1. (a) Let $u_1(z) = e^{iz}$, $\varphi_1(z) = \frac{1}{2}z + i$, $u_2(z) = e^{(1+\frac{\sqrt{2\ln 2}}{4})iz}$, $\varphi_2(z) = \frac{1}{4}z + i$, $s = \sqrt{2\ln 2}i$. Then the operator $W_{u_1,\varphi_1} - W_{u_2,\varphi_2}$ is complex symmetric on $\mathcal{F}^2$ with the conjugation $J_{r,s,t}$, but each W_{u_j,φ_j} is not complex symmetric on $\mathcal{F}^2$ with the conjugation $J_{r,s,t}$.

(b) Let $u_1(z) = e^{-iz}$, $\varphi_1(z) = \frac{1}{3}z + \left(\frac{2\ln 2}{3} + \frac{\pi}{\sqrt{2\ln 2}}i\right)e^{-i} + e^{-2i}i$, $u_2(z) = e^{[(\frac{2\sqrt{2\ln 2}}{15} - \frac{\pi}{\sqrt{2\ln 2}}i)e^{i} - i]z}$, $\varphi_2(z) = \frac{1}{5}z + e^{-2i}i + \frac{2\sqrt{2\ln 2}}{3}e^{-i}$, $k = 0$, $t = \frac{1}{2}$, $r = -e^{2i}$, $s = \sqrt{2\ln 2}e^{i}$. Then the operator $W_{u_1,\varphi_1} - W_{u_2,\varphi_2}$ is complex symmetric on $\mathcal{F}^2$ with the conjugation $J_{r,s,t}$, but each W_{u_j,φ_j} is not complex symmetric on $\mathcal{F}^2$ with the conjugation $J_{r,s,t}$.

Proof. (a) Since $|a_1| = \frac{1}{2} < 1$ and $|a_2| = \frac{1}{4} < 1$, W_{u_1,φ_1} and W_{u_2,φ_2} are bounded on $\mathcal{F}^2$. From a calculation, we have

$$\overline{p}_1 + a_1 s = \left(1 + \frac{\sqrt{2\ln 2}}{2}\right)i \quad \text{and} \quad s + b_1 r = (\sqrt{2\ln 2} + r)i.$$

If $\overline{p}_1 + a_1 s = s + b_1 r$, then $r = 1 - \frac{\sqrt{2\ln 2}}{2}$. This is a contradiction since $|r| = 1$. From Lemma 3.5, it follows that W_{u_1,φ_1} is not complex symmetric on $\mathcal{F}^2$ the conjugation $J_{r,s,t}$. Also, we have that $\overline{p}_2 + a_2 s \neq s + b_2 r$, which shows that W_{u_2,φ_2} is not complex symmetric on $\mathcal{F}^2$ with the conjugation $J_{r,s,t}$.

It is clear that $b_1 = b_2$. We also have

$$\overline{p}_2 + a_2 s = \left(1 + \frac{\sqrt{2\ln 2}}{2}\right)i.$$

We therefore obtain that $\overline{p}_1 + a_1 s = \overline{p}_2 + a_2 s$. From Theorem 3.2, it follows that $W_{u_1,\varphi_1} - W_{u_2,\varphi_2}$ is complex symmetric on $\mathcal{F}^2$ with the conjugation $J_{r,s,t}$.

(b) Since $|a_1| = \frac{1}{3} < 1$ and $|a_2| = \frac{1}{5} < 1$, W_{u_1,φ_1} and W_{u_2,φ_2} are bounded on $\mathcal{F}^2$. As what Theorem 3.2 says, we will calculate one by one.

We first have

$$b_1 s = \left[\left(\frac{2\sqrt{2\ln 2}}{3} + \frac{\pi}{\sqrt{2\ln 2}}i\right)e^{-i} + e^{-2i}i\right]\sqrt{2\ln 2}e^{i} = \frac{4\ln 2}{3} + \sqrt{2\ln 2}e^{-i}i + \pi i \tag{3.33}$$

and

$$b_2 s = \left(e^{-2i}i + \frac{2\sqrt{2\ln 2}}{3}e^{-i}\right)\sqrt{2\ln 2}e^{i} = \frac{4\ln 2}{3} + \sqrt{2\ln 2}e^{-i}i. \tag{3.34}$$

From (3.33) and (3.34), it follows that $b_1 s = b_2 s + \pi i$.

We also have

$$\overline{p}_1 + a_1 s = -i + \frac{\sqrt{2\ln 2}}{3}e^{i} \tag{3.35}$$

and

$$s + b_2 r = \sqrt{2\ln 2}e^{i} - \left(e^{-2i}i + \frac{2\sqrt{2\ln 2}}{3}e^{-i}\right)e^{2i} = -i + \frac{\sqrt{2\ln 2}}{3}e^{i}. \tag{3.36}$$

From (3.35) and (3.36), we see that $\overline{p}_1 + a_1 s = s + b_2 r$.

Finally, we have

$$\begin{aligned}\overline{p}_2 + a_2 s &= \left(\frac{2\sqrt{2\ln 2}}{15} - \frac{\pi}{\sqrt{2\ln 2}}i\right)e^{i} - i + \frac{\sqrt{2\ln 2}}{5}e^{i}\\ &= \left(\frac{\sqrt{2\ln 2}}{3} - \frac{\pi}{\sqrt{2\ln 2}}i\right)e^{i} - i \end{aligned} \tag{3.37}$$

and

$$s+b_1r=\sqrt{2\ln 2}e^{i}-\Big[\Big(\frac{2\sqrt{2\ln 2}}{3}+\frac{\pi}{\sqrt{2\ln 2}}i\Big)e^{-i}+e^{-2i}i\Big]e^{2i}$$
$$=\Big(\frac{\sqrt{2\ln 2}}{3}-\frac{\pi}{\sqrt{2\ln 2}}i\Big)e^{i}-i. \tag{3.38}$$

So, (3.37) and (3.38) show that $\overline{p}_2+a_2s=s+b_1r$. From Theorem 3.2, it follows that $W_{u_1,\varphi_1}-W_{u_2,\varphi_2}$ is complex symmetric on $\mathcal{F}^2$ with the conjugation $J_{r,s,t}$.

On the other hand, from above calculations, it is easy to see that $\overline{p}_1+a_1\neq s+b_1r$ and $\overline{p}_2+a_2\neq s+b_2r$. From Lemma 3.5, it follows that W_{u_1,φ_1} and W_{u_2,φ_2} are not complex symmetric on $\mathcal{F}^2$ with the conjugation $J_{r,s,t}$. $\square$

4. Conclusion

In the preparation of this study, we find that the proper description of the adjoint $W^*_{u,\varphi}$ of the weighted composition operator $W_{u,\varphi}$ with the general symbols u and φ on Fock space is very difficult. So, in this paper, we just obtain a description for the operator $W_{u,\varphi}$ with the special symbols $u(z)=e^{\overline{p}z}$ and $\varphi(z)=az+b$ on Fock space. By using the description, we completely characterize complex symmetric difference of operators $W_{u,\varphi}$ with $u(z)=e^{\overline{p}z}$ and $\varphi(z)=az+b$ with the conjugations J and $J_{r,s,t}$ defined by $Jf(z)=\overline{f(\overline{z})}$ and $J_{r,s,t}f(z)=te^{sz}\overline{f(\overline{rz+s})}$ on Fock space in terms of the relations of the parameters a, b, p, r and s. We find that each operator $W_{e^{\overline{p}z},az+b}$ is not complex symmetric on Fock space with $J_{s,t,r}$ but their difference is complex symmetric on Fock space with $J_{r,s,t}$. Motivated by this interesting phenomenon, we hope that the study can attract people's more attention for such a topic.

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