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An evaluation formula for the Wiener integrals via the bounded linear operator

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Abstract. When we evaluate the Wiener integral involving the stochastic inner product via a bounded linear operator, there are some problems to calculate the Wiener integral as seen Section 2 below. In this paper, we give an idea to calculate the Wiener integral involving the stochastic inner product via a bounded linear operator. We then establish a theorem regarding this process. Our research result is a form of solving a problem that could not be solved before. Therefore, the previous research results corollaries of our research results.

1. Introduction

Let H be denote a separable infinite-dimensional Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle_H$ and the associated norm $|\cdot|_H = \sqrt{\langle\cdot,\cdot\rangle_H}$. Let B denote the completion of H with respect to $\|\cdot\|_0$, where $\|\cdot\|_0$ is a measurable norm on H with respect to the Gaussian cylinder measure ν_0 on H. Let i denote the inclusion map of H into B. Then we have that the adjoint operator i^* of i is one to one and maps B^* continuously onto a dense subset H^*, where B^* and H^* are topological duals of B and H, respectively. And we have a relationship $B^* \subset H^* \approx H \subset B$ with $\langle x, y\rangle_H = (x, y)$ for all x in H and y in B^*, where $(\cdot,\cdot)$ denotes the natural dual pairing between B and B^*. In [6], Gross showed that the measure $\nu_0 \circ i^{-1}$ has a unique countably additive extension ν to the Borel σ-algebra $\mathcal{B}(B)$ of B. We say that (B, H, ν) is called an abstract Wiener space, see [1–3, 5–7, 9, 10].

Let $\{\alpha_j\}_{j=1}^{\infty}$ be a complete orthonormal set in H with α_j's are in B^*. For each $h \in H$ and for $x \in B$, we define a stochastic inner product $(h, x)^\sim$ by

$$(h,x)^\sim = \begin{cases} \lim\limits_{n\to\infty} \sum_{j=1}^{n} \langle h, \alpha_j\rangle_H (x, \alpha_j), & \text{if the limit exists} \\ 0, & \text{otherwise} \end{cases}.$$

Then for $h(\neq 0)$ in H, the stochastic inner product $(h, x)^\sim$ exists for all $x \in B$, $(h, \cdot)^\sim$ is a Gaussian random variable on B with mean zero and variance $|h|_H^2$, and is essentially independent of the choice of the complete orthonormal set. If h and x are elements of H, then $(h, x)^\sim = \langle h, x\rangle_H$. Furthermore, $(h, \lambda x)^\sim = (\lambda h, x)^\sim =$

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$\lambda(h,x)^\sim$ for all $\lambda \in \mathbb{R}$, $h \in H$ and $x \in B$. One can see that if $\{h_1, \ldots, h_n\}$ is an orthonormal set in H, then the random variables $(h_j, x)^\sim$'s are independent, see [1–4, 6, 8–11].

There are various research papers on the properties and structure of the Wiener integrals involving the stochastic inner product $(h,x)^\sim$. Some of them are the analytic Feynman integral (analytic Fourier-Feynman transform), the integral transform and the Gauss transform. Furthermore, various results and formulas with related topics were also established in previous papers. However, because of the mathematical difficulty when the operator is used, there are no papers about the results.

In this paper, we introduce a method to solve the problem that the Wiener integral involving the stochastic inner product with a bounded linear operator on B. We then obtain an evaluation formula as a theorem regarding this process.

2. Motivation

In this section, we shall try to explain why this study is necessary and important.

Let X and Y be Banach spaces and let $\mathcal{L}(X:Y)$ be set of all bounded operators from X to Y. We note that for each $h \in H$, the functional

$$F(x) = \exp\{i(h,x)^\sim\} \tag{1}$$

is important in physics and mathematical applications. Furthermore, the set of all functionals of the form (1) forms a dense set in $L_2(B)$. Thus, there are many research results for the functional F of the form (1). From equation (4), we have

$$\int_B F(x)d\nu(x) = \int_B \exp\{i(h,x)^\sim\}d\nu(x) = \exp\Big\{-\frac{1}{2}|h|_H^2\Big\}$$

for all $h \in H$. For $T \in \mathcal{L}(B:B)$ and $h \in H$, it naturally arise a question that how to calculate the following Wiener integral

$$\int_B F(Tx)d\nu(x) = \int_B \exp\{i(h,Tx)^\sim\}d\nu(x). \tag{2}$$

If $(h,Tx)^\sim = (T^*h,x)^\sim$, then we have

$$\int_B \exp\{i(T^*h,x)^\sim\}d\nu(x) = \exp\Big\{-\frac{1}{2}|T^*h|_H^2\Big\}.$$

But it is not impossible. Since $T \in \mathcal{L}(B:B)$, $T^* \in \mathcal{L}(B^*:B^*)$ and so T^*h might not defined for some $h \in H$. For this reason, we have to restrict the domain of the stochastic inner product $(\cdot,x)^\sim$ to the space B^* instead of H whenever we use a bounded linear operator. In fact, if $h \in B^*$, then $(h,Tx)^\sim = (h,Tx) = (T^*h,x)$ and

$$\int_B \exp\{i(h,Tx)\}d\nu(x) = \int_B \exp\{i(T^*h,x)\}d\nu(x) = \exp\Big\{-\frac{1}{2}|T^*h|_H^2\Big\}.$$

Hence for any $h \in H$ and $T \in \mathcal{L}(B:B)$, how to obtain the Wiener integral of the form equation (1) above.

3. Definitions and Preliminaries

In this section we list some definitions and notations which are needed in this paper.

We first give an integration formula for the abstract Weiner integrals used in this paper. Let $\{g_1, \ldots, g_n\}$ be an orthogonal set in H and $F: B \to \mathbb{C}$ a functional defined by the formula

$$F(x) = f((g_1,x)^\sim, \ldots, (g_n,x)^\sim) \tag{3}$$

where $f : \mathbb{R}^n \to \mathbb{C}$ is a Lebesgue measurable function. Then

$$\int_B f((g_1, x)^\sim, \dots, (g_n, x)^\sim) d\nu(x) \doteq \Big(\prod_{j=1}^n \frac{1}{2\pi |g_j|_H^2}\Big)^{\frac{1}{2}} \int_{\mathbb{R}^n} f(\vec{u}) \exp\Big\{-\sum_{j=1}^n \frac{u_j^2}{2|g_j|_H^2}\Big\} d\vec{u} \tag{4}$$

in the sense that if either side of (4) exists, both sides exist and equality holds, where $\vec{u} = (u_1, \dots, u_n) \in \mathbb{R}^n$ and $d\vec{u} = du_1 \cdots du_n$.

In our first lemma in this paper, we give a condition that the angle preserving property holds between non-zero elements in B^*. The proof was established in [4].

Lemma 3.1. *Let $T : B^* \to B^*$ be a bounded linear operator. Suppose that there is a $\lambda > 0$ such that*

$$\langle Tg_1, Tg_2\rangle_H = \lambda^2 \langle g_1, g_2\rangle_H \tag{5}$$

for all g_1 and g_2 in B^*. Then T preserves angles between non-zero elements in B^*.

Let $\mathcal{L}_0(B^* : B^*)$ be the space of all operators in $\mathcal{L}(B^* : B^*)$ which satisfy the condition (5) above, namely,

$$\mathcal{L}_0(B^* : B^*) = \{T \in \mathcal{L}(B^* : B^*) | \, T \text{ satisfies the condition (5)}\}.$$

Then we have the following assertions.

(i) Given an operator $A \in \mathcal{L}(B : B)$, there is a bounded linear operator $A^* : B^* \to B^*$ so that for all $g \in B^*$ and $x \in B$,

$$(A^* g, x) = (g, Ax).$$

(ii) Since $(B^*)^* = B$, for $T \in \mathcal{L}_0(B^* : B^*)$, we have

$$T^* : B \to B$$

and $T^* \in \mathcal{L}(B : B)$.

(iii) Let

$$\mathcal{L}_{AP}(B : B) = \{A \in \mathcal{L}(B : B) | A = T^*, T \in \mathcal{L}_0(B^* : B^*)\}.$$

Then for each $A \in \mathcal{L}_{AP}(B : B)$, A^* preserves angles between non-zero elements in B^*.

4. Main theorem

In this section, we give an idea to calculate the Wiener integral given by equation (2). We then establish an explicit result for the Wiener integral.

We give a lemma to establish the main theorem in this paper. The proof of Lemma 4.1 below was established in [4].

Lemma 4.1. *Let $\{g_1, \dots, g_n\}$ be an orthogonal set in H with g_j in B^* for $j = 1, \dots, n$ and $F : B \to \mathbb{C}$ a functional defined by the formula*

$$F(x) = f((g_1, x)^\sim, \dots, (g_n, x)^\sim) \tag{6}$$

where $f : \mathbb{R}^n \to \mathbb{C}$ is a Lebesgue measurable function. Then for any $T \in \mathcal{T}_{AP}(B : B)$,

$$\int_B F(Tx) d\nu(x) \doteq \Big(\prod_{j=1}^n \frac{1}{2\pi |T^* g_j|_H^2}\Big)^{\frac{1}{2}} \int_{\mathbb{R}^n} f(\vec{u}) \exp\Big\{-\sum_{j=1}^n \frac{u_j^2}{2|T^* g_j|_H^2}\Big\} d\vec{u} \tag{7}$$

if it exists.

The following theorem is the main theorem in this paper.

Theorem 4.2. (Main Theorem) *Let* $\{g_j\}_{j=1}^{\infty}$ *be a complete orthonormal set in* H *with* g_j*'s are in* B^*. *Then for* $T \in \mathcal{L}_{AP}(B:B)$, *we have*

$$\int_B \exp\{i(h,Tx)^\sim\}d\nu(x) = \exp\Big\{-\frac{1}{2}\sum_{j=1}^{\infty}|T^*g_j|_H^2\langle h,g_j\rangle_H^2\Big\}. \tag{8}$$

Proof. First, for each $h \in H$, we have

$$h = \sum_{j=1}^{\infty}\langle h,g_j\rangle_H g_j = \lim_{n\to\infty}\sum_{j=1}^{n}\langle h,g_j\rangle_H g_j.$$

For each $n = 1,2,\ldots$, let

$$h_n = \sum_{j=1}^{n}\langle h,g_j\rangle_H g_j.$$

Then $h_n \in B^*$ for all $n = 1,2,\ldots$ and $|h_n - h|_H \to 0$ as $n \to \infty$. Hence $(h_n,x) \to (h,x)^\sim$ for s-a.e. $x \in B$. Next, using equation (7) with $f(\vec{u}) = \exp\Big\{i\sum_{j=1}^{n}\langle h,g_j\rangle_H u_j\Big\}$, we have

$$\begin{aligned}\int_B \exp\{i(h_n,Tx)\}d\nu(x) &= \int_B \exp\Big\{i\sum_{j=1}^{n}\langle h,g_j\rangle_H(g_j,Tx)\Big\}d\nu(x)\\ &= \Big(\prod_{j=1}^{n}\frac{1}{2\pi|T^*g_j|_H^2}\Big)^{\frac{1}{2}}\int_{\mathbb{R}^n}\exp\Big\{i\sum_{j=1}^{n}\langle h,g_j\rangle_H u_j - \sum_{j=1}^{n}\frac{u_j^2}{2|T^*g_j|_H}\Big\}d\vec{u}.\end{aligned}$$

Form the following integration formula

$$\int_{\mathbb{R}^n}\exp\Big\{-au^2+bu\Big\}du = \sqrt{\frac{\pi}{a}}\exp\Big\{\frac{b^2}{4a}\Big\}$$

for all complex numbers a and b with $Re(a) > 0$, we obtain that

$$\int_B \exp\{i(h_n,Tx)\}d\nu(x) = \exp\Big\{-\frac{1}{2}\sum_{j=1}^{n}|T^*g_j|_H^2\langle h,g_j\rangle_H^2\Big\}.$$

Finally, by using the dominated convergence theorem, we have

$$\int_B \exp\{i(h,Tx)^\sim\}d\nu(x) = \exp\Big\{-\frac{1}{2}\sum_{j=1}^{\infty}|T^*g_j|_H^2\langle h,g_j\rangle_H^2\Big\}.$$

Hence we have the desired result. □

Form Theorem 4.2, we obtain the following corollary, see [5, 6, 8, 10].

Corollary 4.3. *When* $T = I$ *on* B, *where* I *denotes the identity operator, we have*

$$\int_B \exp\{i(h,x)^\sim\}d\nu(x) = \exp\Big\{-\frac{|h|_H^2}{2}\Big\}. \tag{9}$$

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