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A newly developed study into fractional Boole's inequalities through twice differentiable functions and application

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Abstract. This study develops fractional Boole inequalities for h-convex functions utilizing Riemann-Liouville integral operators. This represents a novel variant of the established fractional Boole inequalities applicable to twice differentiable functions, derived through basic computations involving the B-function. Furthermore, new results regarding Boole inequalities related to s-convex functions and P-functions are introduced. Finally, an application for special means using different positive real numbers is given.

1. Introduction

In uncertain applications, optimization is very dependent on convexity to guarantee that the optimum solution is unaffected by minor changes in the issue or the data. Mathematically controlled and well-behaved convex functions help simplify analysis. A mathematical inequality shows the difference in quantity by comparing two numbers; hence, it expresses and examines their size or relationship. Mathematical inequalities have recently become more significant in several disciplines, including information theory, game theory, integral operator theory, error analysis, and approximation theory. Notable inequalities like those of Hermite-Hadamard in [6], Simpson, Boole, and Weddle set limits for quadrature methods. The study of integral inequalities is primarily to identify the accuracy of the inaccuracy in numerical quadrature formulas. The remarkable efficiency of Simpson-type inequalities sets them apart. Still, the third Simpson's rule, commonly known as the 2/45 Simpson's rule or Boole's rule (see [7, 10–12, 15]), is an enhancement of these kinds of inequalities.

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Boole inequality is estimated for functions with convex first derivative absolute values.

$$\left|\frac{1}{90}\left[7f(a)+32f\left(\frac{3a+b}{4}\right)+12f\left(\frac{a+b}{2}\right)+32f\left(\frac{a+3b}{4}\right)+7f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \frac{239(b-a)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]}{6480}. \tag{1}$$

Assume that $f \in L_1[a,b]$. The left- and right-sided Riemann-Liouville fractional operators with order $\alpha > 0$ are defined as follows:

$$\Im^{\alpha}_{a^+}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \quad x>a,$$

$$\Im^{\alpha}_{b^-}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \quad x<b.$$

In [15], the authors demonstrate the following fractional identity:

Lemma 1.1. *For a function $f:[a,b]\to\mathbb{R}$ that is twice differentiable on (a,b), the subsequent equality holds:*

$$\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\Im^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+\Im^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]$$
$$-\frac{1}{90}\left\{7\left(f(a)+f(b)\right)+12f\left(\frac{a+b}{2}\right)+32\left(f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right)\right\}$$
$$=\frac{(b-a)^2}{8(\alpha+1)}\left(I_1+I_2\right),$$

where

$$\begin{cases} I_1 := \int_0^{\frac{1}{2}}\left(t^{\alpha+1}-\frac{7(\alpha+1)}{45}t\right)\left[f''\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b\right)+f''\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b\right)\right]dt, \\ I_2 := \int_1^{\frac{1}{2}}\left(t^{\alpha+1}-\frac{13(\alpha+1)}{15}t+\frac{16(\alpha+1)}{45}\right)\left[f''\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b\right)+f''\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b\right)\right]dt. \end{cases}$$

The Boole inequality is determined for functions with convex absolute values of their twice derivatives [15, Remark.1].

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{90}\left[7f(a)+32f\left(\frac{3a+b}{4}\right)+12f\left(\frac{a+b}{2}\right)+32f\left(\frac{a+3b}{4}\right)+7f(b)\right]\right| \le \frac{509(b-a)^2}{273375}\left[\left|f''(a)\right|+\left|f''(b)\right|\right]. \tag{2}$$

The analysis of fractional computations is an extension of classical analysis that grew quickly owing to the intriguing idea of convexity. Its many applications in functional analysis and optimization theory have made it an attractive study topic. In [16], the author develops a new class of functions known as h-convex functions.

Definition 1.2. *Let $h:[0,1]\to\mathbb{R}$ be a positive function. We define a function $f: I\subseteq\mathbb{R}\to\mathbb{R}$ as h-convex if for any $x,y\in I$ and $\lambda\in[0,1]$, the following condition holds:*

$$f(\lambda x+(1-\lambda)y)\le h(\lambda)f(x)+h(1-\lambda)f(y). \tag{3}$$

In the case when the inequality (3) is reversed, we say that f is h-concave.

- By making $h(\lambda) = \lambda$, Definition 1.2 simplifies to a convex function [13].
- By setting $h(\lambda) = 1$, Definition 1.2 simplifies to P-functions [5, 14].
- By defining $h(\lambda) = \lambda^s$, Definition 1.2 simplifies to the concept of s-convex functions [4].

In recent work [1, 2], the authors introduce a novel class of functions termed B-functions, defined as follows:

Definition 1.3. *Let $g : [0, \infty) \to \mathbb{R}$ be a non-negative function. The function g is called a B-function, if*

$$g(x - a) + g(b - x) \leq 2g\left(\frac{a + b}{2}\right), \tag{4}$$

where $a < x < b$ with $a, b \in [0, \infty)$.
If the inequality (4) is reversed, g is called A-function, or that g belongs to the class $A(a, b)$.
If we have equality in (4), g is called AB-function, or that g belongs to the class $AB(a, b)$.

Corollary 1.4. *Let $h : [0, 1] \to \mathbb{R}$ be a non-negative function. The function h is a B-function if for all $\lambda \in [0, 1]$, we have*

$$h(\lambda) + h(1 - \lambda) \leq 2\,h\left(\frac{1}{2}\right). \tag{5}$$

- *The functions $h(\lambda) = \lambda$ and $h(\lambda) = 1$ are AB-function, B-function and A-function.*
- *The function $h(\lambda) = \lambda^s,\ s \in (0, 1]$ is B-function.*

Theorem 1.5 (Hölder's inequality). *Let $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If Ψ and Φ are real functions defined on $[\lambda_1, \lambda_2]$ and if $|\Psi|^p$, $|\Phi|^q$ are integrable functions on $[\lambda_1, \lambda_2]$, then*

$$\int_{\lambda_1}^{\lambda_2} |\Psi(t)\Phi(t)|\, dt \leq \left(\int_{\lambda_1}^{\lambda_2} |\Psi(t)|^p\, dt\right)^{\frac{1}{p}} \left(\int_{\lambda_1}^{\lambda_2} |\Phi(t)|^q\, dt\right)^{\frac{1}{q}}.$$

The power-mean integral inequality, derived from the Hölder's inequality, can be expressed as follows:

Theorem 1.6 (Power-mean integral inequality). *Let $p \geq 1$ and W, Φ be two real functions defined on $[\lambda_1, \lambda_2]$. If $|W|$ and $|W||\Phi|^q$ are integrable functions on $[\lambda_1, \lambda_2]$, then*

$$\int_{\lambda_1}^{\lambda_2} |W(t)\Phi(t)|\, dt \leq \left(\int_{\lambda_1}^{\lambda_2} |W(t)|\, dt\right)^{1-\frac{1}{p}} \left(\int_{\lambda_1}^{\lambda_2} |W(t)|\, |\Phi(t)|^p\, dt\right)^{\frac{1}{p}}.$$

For further information and clarification of the power-mean integral inequality, go to references [9] and [3].

Inspired by the aforementioned, this paper employs an exploratory approach to formulate a novel fractional Boole inequality using Riemann-Liouville fractional operators on functions that are twice differentiable and involve h-convex functions, with the help of the new category function known as the B-function.

2. Basic Identity

Entirety of the paper, with I° functioning as an open interval, let us define $L_1([a, b]) := \left\{ f : \int_a^b \left| f(t) \right| dt < \infty \right\}$.

Lemma 2.1. *Let $\alpha > 0$ and $f : I^\circ \to \mathbb{R}$ be twice differentiable function such that $f'' \in L_1([a, b])$ where $[a, b] \subset I^\circ \subseteq \mathbb{R}$, then the following equality holds:*

$$\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[\Im_{b^-}^\alpha f\left(\frac{a+b}{2}\right) + \Im_{a^+}^\alpha f\left(\frac{a+b}{2}\right)\right]$$

$$-\frac{1}{90}\left\{7(f(a)+f(b)) + 12f\left(\frac{a+b}{2}\right) + 32\left(f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right)\right\} \tag{6}$$

$$= \frac{(b-a)^2}{8(\alpha+1)}\int_0^1 N_\alpha(t)\left[f''\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right) + f''\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b\right)\right]dt,$$

where

$$N_\alpha(t) := \begin{cases} t^{\alpha+1} - \frac{6(\alpha+1)}{45}t + \frac{22\alpha-23}{45}, & if\ 0 \le t \le \frac{1}{2}; \\ t^{\alpha+1} - \frac{38(\alpha+1)}{45}t + \frac{38\alpha-7}{45}, & if\ \frac{1}{2} \le t \le 1. \end{cases} \tag{7}$$

Proof. By using the integration by parts, we deduce

$$H_1 = \int_0^{\frac{1}{2}}\left(t^{\alpha+1} - \frac{6(\alpha+1)}{45}t + \frac{22\alpha-23}{45}\right)\left[f''\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right) + f''\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b\right)\right]dt$$

$$= \left(\frac{2}{b-a}\right)\left(t^{\alpha+1} - \frac{6(\alpha+1)}{45}t + \frac{12\alpha-23}{45}\right)\left[f'\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right) - f'\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b\right)\right]\Bigg|_0^{\frac{1}{2}}$$

$$-\left(\frac{2}{b-a}\right)\int_0^{\frac{1}{2}}\left((\alpha+1)t^\alpha - \frac{6(\alpha+1)}{45}\right)\left[f'\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right) - f'\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b\right)\right]dt$$

$$= \frac{2}{b-a}\left\{\left(\left(\frac{1}{2}\right)^{\alpha+1} + \frac{19\alpha-26}{45}\right)\left(f'\left(\frac{a+3b}{4}\right) - f'\left(\frac{3a+b}{4}\right)\right)\right\}$$

$$-\left(\frac{4}{(b-a)^2}\right)\left((\alpha+1)t^\alpha - \frac{6(\alpha+1)}{45}\right)\left[f\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right) + f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b\right)\right]\Bigg|_0^{\frac{1}{2}}$$

$$+\left(\frac{4\alpha(\alpha+1)}{(b-a)^2}\right)\int_0^{\frac{1}{2}} t^{\alpha-1}\left[f\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right) + f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b\right)\right]dt.$$

Then

$$H_1 = \frac{2}{b-a}\left\{\left(\left(\frac{1}{2}\right)^{\alpha+1} + \frac{9\alpha-70}{45}\right)\left(f'\left(\frac{a+3b}{4}\right) - f'\left(\frac{3a+b}{4}\right)\right)\right\}$$

$$-\left(\frac{4}{(b-a)^2}\right)\left\{\left((\alpha+1)\left(\frac{1}{2}\right)^\alpha - \frac{6(\alpha+1)}{45}\right)\left(f\left(\frac{a+3b}{4}\right) + f\left(\frac{3a+b}{4}\right)\right) + \frac{12(\alpha+1)}{45}f\left(\frac{a+b}{2}\right)\right\}$$

$$+\left(\frac{4\,\alpha(\alpha+1)}{(b-a)^2}\right)\int_0^{\frac{1}{2}} t^{\alpha-1}\left[f\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)+f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\right]dt.$$

Similarly,

$$H_2=\int_{\frac{1}{2}}^1\left(t^{\alpha+1}-\frac{38(\alpha+1)}{45}t+\frac{38\alpha-7}{45}\right)\left[f''\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)+f''\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\right]dt$$

$$=\left(\frac{2}{b-a}\right)\left(t^{\alpha+1}-\frac{38(\alpha+1)}{45}t+\frac{38\alpha-7}{45}\right)\left[f'\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)-f'\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\right]\Bigg|_{\frac{1}{2}}^1$$

$$-\left(\frac{2}{b-a}\right)\int_{\frac{1}{2}}^1\left((\alpha+1)\,t^{\alpha}-\frac{38(\alpha+1)}{45}\right)\left[f'\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)-f'\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\right]dt$$

$$=\frac{2}{b-a}\left\{-\left(\left(\frac{1}{2}\right)^{\alpha+1}+\frac{19\alpha-26}{45}\right)\left(f'\left(\frac{a+3b}{4}\right)-f'\left(\frac{3a+b}{4}\right)\right)\right\}$$

$$-\left(\frac{4}{(b-a)^2}\right)\left((\alpha+1)\,t^{\alpha}-\frac{38(\alpha+1)}{45}\right)\left[f\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)+f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\right]\Bigg|_{\frac{1}{2}}^1$$

$$+\left(\frac{4\,\alpha(\alpha+1)}{(b-a)^2}\right)\int_{\frac{1}{2}}^1 t^{\alpha-1}\left[f\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)+f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\right]dt$$

$$=\frac{2}{b-a}\left\{-\left(\left(\frac{1}{2}\right)^{\alpha+1}+\frac{19\alpha-26}{45}\right)\left(f'\left(\frac{a+3b}{4}\right)-f'\left(\frac{3a+b}{4}\right)\right)\right\}$$

$$-\left(\frac{4}{(b-a)^2}\right)\left\{\left(\frac{7(\alpha+1)}{45}\right)(f(a)+f(b))-\left((\alpha+1)\left(\frac{1}{2}\right)^{\alpha}-\frac{38(\alpha+1)}{45}\right)\left(f\left(\frac{a+3b}{4}\right)+f\left(\frac{3a+b}{4}\right)\right)\right\}$$

$$+\left(\frac{4\,\alpha(\alpha+1)}{(b-a)^2}\right)\int_{\frac{1}{2}}^1 t^{\alpha-1}\left[f\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)+f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\right]dt.$$

Consequently,

$$I_1+I_2=-\left(\frac{4(\alpha+1)}{(b-a)^2}\right)\left\{\frac{7}{45}(f(a)+f(b))+\frac{12}{45}f\left(\frac{a+b}{2}\right)+\frac{32}{45}\left(f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right)\right\}$$

$$+\left(\frac{4\,\alpha(\alpha+1)}{(b-a)^2}\right)\int_0^1 t^{\alpha-1}\left[f\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)+f\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)\right]dt.$$

Given that

$$\int_0^1 t^{\alpha-1}f\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)dt=\frac{2^{\alpha}}{(b-a)^{\alpha}}\int_{\frac{a+b}{2}}^b\left(t-\frac{a+b}{2}\right)^{\alpha-1}f(t)\,dt,$$

and

$$\int_0^1 t^{\alpha-1} f\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right)dt = \frac{2^\alpha}{(b-a)^\alpha}\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)^{\alpha-1} f(t)\,dt.$$

Thus

$$I_1 + I_2 = \frac{2^{\alpha+2}(\alpha+1)\Gamma(\alpha+1)}{(b-a)^{\alpha+2}}\left[\Im^\alpha_{b^-} f\left(\frac{a+b}{2}\right) + \Im^\alpha_{a^+} f\left(\frac{a+b}{2}\right)\right]$$

$$-\left(\frac{4(\alpha+1)}{(b-a)^2}\right)\left\{\frac{7}{45}\left(f(a)+f(b)\right) + \frac{12}{45} f\left(\frac{a+b}{2}\right) + \frac{32}{45}\left(f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right)\right\}.$$

The desired result (6) is obtained by multiplying the previous equality by $\frac{(b-a)^2}{8(\alpha+1)}$. □

Putting $\alpha = 1$ in Lemma 2.1, we obtain the following corollary.

Corollary 2.2. *Let $f : I^\circ \to \mathbb{R}$ be twice differentiable function such that $f'' \in L_1([a,b])$ where $[a,b] \subset I^\circ \subseteq \mathbb{R}$, then the following equality holds:*

$$\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{90}\left[7f(a) + 32f\left(\frac{3a+b}{4}\right) + 12f\left(\frac{a+b}{2}\right) + 32f\left(\frac{a+3b}{4}\right) + 7f(b)\right]$$

$$= \frac{(b-a)^2}{16}\int_0^1 N_1(t)\left[f''\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right) + f''\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b\right)\right]dt,$$

where

$$N_1(t) := \begin{cases} t^2 - \frac{12}{45}t + \frac{1}{45}, & if\ 0 \le t \le \frac{1}{2}; \\ t^2 - \frac{76}{45}t + \frac{31}{45}, & if\ \frac{1}{2} \le t \le 1. \end{cases}$$

3. Boole's inequality involving power-mean integral inequality

The first estimation of the Boole inequalities is provided.

Theorem 3.1. *Let $p \ge 1$, h be a B-function on $[0,1]$ and assume that the assumptions of Lemma 6 are justified. If $|f''|^p$ is an h-convex function on $[a,b]$, then the next Boole's inequality holds:*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[\Im^\alpha_{b^-} f\left(\frac{a+b}{2}\right) + \Im^\alpha_{a^+} f\left(\frac{a+b}{2}\right)\right] - \frac{1}{90}\left\{7\left(f(a)+f(b)\right) + 12f\left(\frac{a+b}{2}\right) + 32\left(f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right)\right\}\right|$$

$$\le \frac{(b-a)^2}{4(\alpha+1)}\left(\int_0^1 |N_\alpha(t)|\,dt\right)\left(h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}}\left(\left|f''(a)\right|^p + \left|f''(b)\right|^p\right)^{\frac{1}{p}}, \tag{8}$$

where N_α is defined by (7).

Proof. The following inequality is required to establish the next results. Let $A, C \geq 0$ and $\mu > 0$, then

$$A^{\mu} + C^{\mu} \leq \max(1, 2^{1-\mu})\,(A + C)^{\mu}. \tag{9}$$

By applying the absolute value of identity (6), we conclude

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\mathfrak{I}^{\alpha}_{b^-}f\left(\frac{a+b}{2}\right) + \mathfrak{I}^{\alpha}_{a^+}f\left(\frac{a+b}{2}\right)\right] - \frac{1}{90}\left\{7\left(f(a)+f(b)\right)\right.\right.\\ &\left.\left.+12f\left(\frac{a+b}{2}\right) + 32\left(f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right)\right\}\right|\\ &\leq \frac{(b-a)^2}{8(\alpha+1)}\int_0^1 |N_{\alpha}(t)|\left[\left|f''\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right)\right| + \left|f''\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b\right)\right|\right]dt.\end{aligned} \tag{10}$$

For $p \geq 1$, the application of the power-mean integral inequality results in the following:

$$\begin{aligned}&\int_0^1 |N_{\alpha}(t)|\left[\left|f''\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right)\right| + \left|f''\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b\right)\right|\right]dt\\ &\leq \left(\int_0^1 |N_{\alpha}(t)|\,dt\right)^{1-\frac{1}{p}}\left[\left(\int_0^1 |B_{\alpha}(t)|\left|f''\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right)\right|^p dt\right)^{\frac{1}{p}}\right.\\ &\qquad \left.+\left(\int_0^1 |N_{\alpha}(t)|\left|f''\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b\right)\right|^p dt\right)^{\frac{1}{p}}\right].\end{aligned}$$

The inequality (9) yields $A^{\frac{1}{p}} + C^{\frac{1}{p}} \leq 2^{1-\frac{1}{p}}(A + C)^{\frac{1}{p}}$, thus

$$\begin{aligned}&\int_0^1 |N_{\alpha}(t)|\left[\left|f''\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right)\right| + \left|f''\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b\right)\right|\right]dt\\ &\leq \left(\int_0^1 |N_{\alpha}(t)|\,dt\right)^{1-\frac{1}{p}} 2^{1-\frac{1}{p}}\left[\int_0^1 |N_{\alpha}(t)|\left(\left|f''\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right)\right|^p + \left|f''\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b\right)\right|^p\right)dt\right]^{\frac{1}{p}}.\end{aligned}$$

Given that $|f''|^p$ is a h-convex function, the application of inequality (5) to $\lambda = \frac{1}{2}$ yields the following:

$$\left|f''\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right)\right|^p \leq h\left(\frac{1-t}{2}\right)\left|f''(a)\right|^p + h\left(\frac{1+t}{2}\right)\left|f''(b)\right|^p.$$

Then

$$\begin{aligned}\left|f''\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right)\right|^p + \left|f''\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b\right)\right|^p &\leq \left[h\left(\frac{1-t}{2}\right) + h\left(\frac{1+t}{2}\right)\right]\left(\left|f''(a)\right|^p + \left|f''(b)\right|^p\right)\\ &\leq 2\,h\left(\frac{1}{2}\right)\left(\left|f''(a)\right|^p + \left|f''(b)\right|^p\right).\end{aligned}$$

Hence

$$\int_0^1 |N_\alpha(t)| \left[\left| f''\left(\left(\frac{1-t}{2}\right) a + \left(\frac{1+t}{2}\right) b \right) \right| + \left| f''\left(\left(\frac{1+t}{2}\right) a + \left(\frac{1-t}{2}\right) b \right) \right| \right] dt$$

$$\leq \left(\int_0^1 |N_\alpha(t)| \, dt \right)^{1-\frac{1}{p}} 2^{1-\frac{1}{p}} \left[\int_0^1 |N_\alpha(t)| \, 2\, h\left(\frac{1}{2}\right) \left(\left|f''(a)\right|^p + \left|f''(b)\right|^p \right) dt \right]^{\frac{1}{p}}.$$

As a result

$$\begin{aligned} &\int_0^1 |N_\alpha(t)| \left[\left| f''\left(\left(\frac{1-t}{2}\right) a + \left(\frac{1+t}{2}\right) b \right) \right| + \left| f''\left(\left(\frac{1+t}{2}\right) a + \left(\frac{1-t}{2}\right) b \right) \right| \right] dt \\ &\leq \left(\int_0^1 |N_\alpha(t)| \, dt \right) 2 \left(h\left(\frac{1}{2}\right) \right)^{\frac{1}{p}} \left(\left|f''(a)\right|^p + \left|f''(b)\right|^p \right)^{\frac{1}{p}}. \end{aligned} \tag{11}$$

Combine the inequality (10) with the inequality (11), we get

$$\left| \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[\mathfrak{I}_{b^-}^\alpha f\left(\frac{a+b}{2}\right) + \mathfrak{I}_{a^+}^\alpha f\left(\frac{a+b}{2}\right) \right] - \frac{1}{90} \left\{ 7\left(f(a)+f(b)\right) \right. \right.$$

$$\left. \left. +12 f\left(\frac{a+b}{2}\right) + 32\left(f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right) \right\} \right|$$

$$\leq \frac{(b-a)^2}{4(\alpha+1)} \left(\int_0^1 |N_\alpha(t)| \, dt \right) \left(h\left(\frac{1}{2}\right) \right)^{\frac{1}{p}} \left(\left|f''(a)\right|^p + \left|f''(b)\right|^p \right)^{\frac{1}{p}}.$$

This concludes the demonstration of the inequality (8). □

Replacing $p = 1$ in Theorem 3.1 yields the next results.

Corollary 3.2. *Let h be a B-function on $[0,1]$ and assume that the assumptions of Lemma 6 hold. If $|f''|$ is an h-convex function on $[a,b]$, then the following Boole inequality holds:*

$$\begin{aligned} &\left| \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[\mathfrak{I}_{b^-}^\alpha f\left(\frac{a+b}{2}\right) + \mathfrak{I}_{a^+}^\alpha f\left(\frac{a+b}{2}\right) \right] - \frac{1}{90} \left\{ 7\left(f(a)+f(b)\right) \right. \right. \\ &\left. \left. +12 f\left(\frac{a+b}{2}\right) + 32\left(f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right) \right\} \right| \\ &\leq \frac{(b-a)^2}{4(\alpha+1)} \left(\int_0^1 |N_\alpha(t)| \, dt \right) \left(h\left(\frac{1}{2}\right) \right) \left(\left|f''(a)\right| + \left|f''(b)\right| \right), \end{aligned} \tag{12}$$

where N_α is defined by (7).

By putting $\alpha = 1$ in Theorem 3.1, we obtain the following results pertaining to the Riemann integral.

Corollary 3.3. *Let $p \geq 1$, h be a B-function on $[0,1]$ and assume that the assumptions of Lemma 6 hold. If $|f'|^p$ is an h-convex function on $[a,b]$, then the following Boole inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{90}\left[7f(a) + 32f\left(\frac{3a+b}{4}\right) + 12f\left(\frac{a+b}{2}\right) + 32f\left(\frac{a+3b}{4}\right) + 7f(b)\right]\right|$$
$$\leq \frac{1018(b-a)^2}{273375}\left(h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}}\left[\left|f''(a)\right|^p + \left|f''(b)\right|^p\right]^{\frac{1}{p}}. \tag{13}$$

Here

$$\int_0^{\frac{1}{2}}\left|t^2 - \frac{12}{45}t - \frac{1}{45}\right|dt = \frac{11}{648} \quad \text{and} \quad \int_{\frac{1}{2}}^1\left|t^2 - \frac{76}{45}t + \frac{31}{45}\right|dt = \frac{28027}{2187000}.$$

Next, examine specific instances of h-convexity.

1. Putting $h(t) = t^s$ with $s \in (0, 1]$ in Theorem 3.1, Corollary 3.2, and Corollary 3.3, we get the following results.

Corollary 3.4. *Assume p, α, and f are defined according to Theorem 3.1. If $|f''|^p$ is an s-convex function on $[a,b]$, then*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[\mathfrak{I}_{b^-}^{\alpha} f\left(\frac{a+b}{2}\right) + \mathfrak{I}_{a^+}^{\alpha} f\left(\frac{a+b}{2}\right)\right] - \frac{1}{90}\left\{7\left(f(a)+f(b)\right)\right.\right.$$
$$\left.\left. + 12f\left(\frac{a+b}{2}\right) + 32\left(f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right)\right\}\right| \tag{14}$$
$$\leq \frac{(b-a)^2}{4(\alpha+1)}\left(\int_0^1 |N_\alpha(t)|\,dt\right)\left(\frac{1}{2}\right)^{\frac{s}{p}}\left(\left|f''(a)\right|^p + \left|f''(b)\right|^p\right)^{\frac{1}{p}}.$$

For $p = 1$, we get

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[\mathfrak{I}_{b^-}^{\alpha} f\left(\frac{a+b}{2}\right) + \mathfrak{I}_{a^+}^{\alpha} f\left(\frac{a+b}{2}\right)\right] - \frac{1}{90}\left\{7\left(f(a)+f(b)\right)\right.\right.$$
$$\left.\left. + 12f\left(\frac{a+b}{2}\right) + 32\left(f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right)\right\}\right| \tag{15}$$
$$\leq \frac{(b-a)^2}{4(\alpha+1)}\left(\int_0^1 |N_\alpha(t)|\,dt\right)\left(\frac{1}{2}\right)^{s}\left(\left|f''(a)\right| + \left|f''(b)\right|\right).$$

For $\alpha = 1$, we get

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{90}\left[7f(a) + 32f\left(\frac{3a+b}{4}\right) + 12f\left(\frac{a+b}{2}\right) + 32f\left(\frac{a+3b}{4}\right) + 7f(b)\right]\right|$$
$$\leq \frac{1018(b-a)^2}{273375}\left(\frac{1}{2}\right)^{\frac{s}{p}}\left[\left|f''(a)\right|^p + \left|f''(b)\right|^p\right]^{\frac{1}{p}}. \tag{16}$$

2. Setting $h(t) = t$ in Theorem 3.1, Corollary 3.2, and Corollary 3.3 gives the next corollary.

Corollary 3.5. *Assume p, α, and f are defined according to Theorem 3.1. If $|f''|^p$ is a convex function on $[a,b]$, then*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\Im_{b^-}^{\alpha}f\left(\frac{a+b}{2}\right)+\Im_{a^+}^{\alpha}f\left(\frac{a+b}{2}\right)\right]-\frac{1}{90}\left\{7\left(f(a)+f(b)\right)\right.\right.$$
$$\left.\left.+12f\left(\frac{a+b}{2}\right)+32\left(f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right)\right\}\right| \tag{17}$$
$$\leq\frac{(b-a)^2}{4(\alpha+1)}\left(\int_0^1|N_\alpha(t)|\,dt\right)\left[\frac{\left|f''(a)\right|^p+\left|f''(b)\right|^p}{2}\right]^{\frac{1}{p}}.$$

For $p=1$, we get

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\Im_{b^-}^{\alpha}f\left(\frac{a+b}{2}\right)+\Im_{a^+}^{\alpha}f\left(\frac{a+b}{2}\right)\right]-\frac{1}{90}\left\{7\left(f(a)+f(b)\right)\right.\right.$$
$$\left.\left.+12f\left(\frac{a+b}{2}\right)+32\left(f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right)\right\}\right| \tag{18}$$
$$\leq\frac{(b-a)^2}{8(\alpha+1)}\left(\int_0^1|N_\alpha(t)|\,dt\right)\left(\left|f''(a)\right|+\left|f''(b)\right|\right).$$

For $\alpha=1$, we get

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{90}\left[7f(a)+32f\left(\frac{3a+b}{4}\right)+12f\left(\frac{a+b}{2}\right)+32f\left(\frac{a+3b}{4}\right)+7f(b)\right]\right|$$
$$\leq\frac{1018(b-a)^2}{273375}\left[\frac{\left|f''(a)\right|^p+\left|f''(b)\right|^p}{2}\right]^{\frac{1}{p}}. \tag{19}$$

Remark 3.6. *This inequality (19) is a generalization of the inequality (2), and it requires just that $p=1$.*

3. Assuming $h(t)=1$ in Theorem 3.1, Corollary 3.2, and Corollary 3.3 yields the following new result for the class of P-functions. It is also equivalent to the situation $s\to 0^+$ in inequality (14).

Corollary 3.7. *Assume p and f are defined according to Theorem 3.1. If $|f''|^p$ is a P-function on $[a,b]$, then*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\Im_{b^-}^{\alpha}f\left(\frac{a+b}{2}\right)+\Im_{a^+}^{\alpha}f\left(\frac{a+b}{2}\right)\right]-\frac{1}{90}\left\{7\left(f(a)+f(b)\right)\right.\right.$$
$$\left.\left.+12f\left(\frac{a+b}{2}\right)+32\left(f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right)\right\}\right| \tag{20}$$
$$\leq\frac{(b-a)^2}{4(\alpha+1)}\left(\int_0^1|N_\alpha(t)|\,dt\right)\left(\left|f''(a)\right|^p+\left|f''(b)\right|^p\right)^{\frac{1}{p}}.$$

For $p = 1$, we get

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\Im_{b^-}^{\alpha} f\left(\frac{a+b}{2}\right)+\Im_{a^+}^{\alpha} f\left(\frac{a+b}{2}\right)\right]-\frac{1}{90}\left\{7\left(f(a)+f(b)\right)\right.\right.$$
$$\left.\left.+12f\left(\frac{a+b}{2}\right)+32\left(f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right)\right\}\right| \tag{21}$$
$$\leq \frac{(b-a)^2}{4(\alpha+1)}\left(\int_0^1 |N_\alpha(t)|\,dt\right)\left(\left|f''(a)\right|+\left|f''(b)\right|\right).$$

For $\alpha = 1$, we get

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{90}\left[7f(a)+32f\left(\frac{3a+b}{4}\right)+12f\left(\frac{a+b}{2}\right)+32f\left(\frac{a+3b}{4}\right)+7f(b)\right]\right|$$
$$\leq \frac{1018(b-a)^2}{273375}\left[\left|f''(a)\right|^p+\left|f''(b)\right|^p\right]^{\frac{1}{p}}. \tag{22}$$

4. Boole's inequality involving Hölder's inequality

Theorem 4.1. *Let h be a B-function on $[0,1]$, p, $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$, and assume that f is defined as in Lemma 2.1. If $\left|f''\right|^p$ is an h-convex function on $[a,b]$, then the following Boole inequality holds:*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\Im_{b^-}^{\alpha} f\left(\frac{a+b}{2}\right)+\Im_{a^+}^{\alpha} f\left(\frac{a+b}{2}\right)\right]-\frac{1}{90}\left\{7\left(f(a)+f(b)\right)\right.\right.$$
$$\left.\left.+12f\left(\frac{a+b}{2}\right)+32\left(f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right)\right\}\right| \tag{23}$$
$$\leq \frac{(b-a)^2}{4(\alpha+1)}\left(\int_0^1 |N_\alpha(t)|^q\,dt\right)^{\frac{1}{q}}\left(h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}}\left[\left|f''(a)\right|^p+\left|f''(b)\right|^p\right]^{\frac{1}{p}},$$

where N_α is defined by (7).

Proof. By employing the absolute value of the identity (6), we obtain

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\Im_{b^-}^{\alpha} f\left(\frac{a+b}{2}\right)+\Im_{a^+}^{\alpha} f\left(\frac{a+b}{2}\right)\right]-\frac{1}{90}\left\{7\left(f(a)+f(b)\right)\right.\right.$$
$$\left.\left.+12f\left(\frac{a+b}{2}\right)+32\left(f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right)\right\}\right|$$

$$\leq \frac{(b-a)^2}{8(\alpha+1)}\int_0^1 |N_\alpha(t)|\left|f''\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)\right|dt+\frac{(b-a)^2}{8(\alpha+1)}\int_0^1 |N_\alpha(t)|\left|f''\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\right|dt.$$

By using Hölder's inequality and $A^{\frac{1}{p}}+C^{\frac{1}{p}}\leq 2^{1-\frac{1}{p}}(A+C)^{\frac{1}{p}}$ yields

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\Im_{b^-}^{\alpha} f\left(\frac{a+b}{2}\right)+\Im_{a^+}^{\alpha} f\left(\frac{a+b}{2}\right)\right]-\frac{1}{90}\left\{7\left(f(a)+f(b)\right)\right.\right.$$
$$\left.\left.+12f\left(\frac{a+b}{2}\right)+32\left(f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right)\right\}\right|$$

$$\leq \frac{(b-a)^2}{8(\alpha+1)}\left(\int_0^1 |N_\alpha(t)|^q\,dt\right)^{\frac{1}{q}}\left(\int_0^1 \left|f''\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)\right|^p dt\right)^{\frac{1}{p}}$$
$$+\frac{(b-a)^2}{8(\alpha+1)}\left(\int_0^1 |N_\alpha(t)|^q\,dt\right)^{\frac{1}{q}}\left(\int_0^1 \left|f''\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\right|^p dt\right)^{\frac{1}{p}}$$
$$\leq \frac{(b-a)^2}{8(\alpha+1)}\left(\int_0^1 |N_\alpha(t)|^q\,dt\right)^{\frac{1}{q}} 2^{1-\frac{1}{p}}$$
$$\times\left[\int_0^1 \left|f''\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)\right|^p dt+\int_0^1 \left|f''\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\right|^p dt\right]^{\frac{1}{p}}.$$

The fact that $|f''|^p$ is an h-convex function allows us to obtain the following:

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[\Im^\alpha_{b^-}f\left(\frac{a+b}{2}\right)+\Im^\alpha_{a^+}f\left(\frac{a+b}{2}\right)\right]-\frac{1}{90}\{7(f(a)+f(b))\right.$$
$$\left.+12f\left(\frac{a+b}{2}\right)+32\left(f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right)\right\}\Bigg|$$

$$\leq \frac{(b-a)^2}{8(\alpha+1)}\left(\int_0^1 |N_\alpha(t)|^q\,dt\right)^{\frac{1}{q}} 2^{\frac{1}{q}}\left[\int_0^1\left(h\left(\frac{1-t}{2}\right)\left|f''(a)\right|^p+h\left(\frac{1+t}{2}\right)\left|f''(b)\right|^p\right)dt\right.$$
$$\left.+\int_0^1\left(h\left(\frac{1+t}{2}\right)\left|f''(a)\right|^p+h\left(\frac{1-t}{2}\right)\left|f''(b)\right|^p\right)dt\right]^{\frac{1}{p}}$$

$$\leq \frac{(b-a)^2}{8(\alpha+1)}\left(\int_0^1 |N_\alpha(t)|^q\,dt\right)^{\frac{1}{q}} 2^{\frac{1}{q}}\left(\int_0^1\left[h\left(\frac{1-t}{2}\right)+h\left(\frac{1+t}{2}\right)\right]dt\right)^{\frac{1}{p}}\left[\left|f''(a)\right|^p+\left|f''(b)\right|^p\right]^{\frac{1}{p}}.$$

Under the assumption of inequality (5) for the expression $\lambda=\frac{1}{2}$, we obtain

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[\Im^\alpha_{b^-}f\left(\frac{a+b}{2}\right)+\Im^\alpha_{a^+}f\left(\frac{a+b}{2}\right)\right]-\frac{1}{90}\{7(f(a)+f(b))\right.$$
$$\left.+12f\left(\frac{a+b}{2}\right)+32\left(f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right)\right\}\Bigg|$$
$$\leq \frac{(b-a)^2}{4(\alpha+1)}\left(\int_0^1 |N_\alpha(t)|^q\,dt\right)^{\frac{1}{q}}\left(h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}}\left[\left|f''(a)\right|^p+\left|f''(b)\right|^p\right]^{\frac{1}{p}}.$$

The proof of the inequality (23) has been completed using this evidence. □

The next step is to investigate particular instances involving h-convexity.

Corollary 4.2. *Assume p and f are defined according to Theorem 4.1.*

- *Suppose that the function $|f''|^p$ is an s-convex function on $[a,b]$, then*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[\Im^\alpha_{b^-}f\left(\frac{a+b}{2}\right)+\Im^\alpha_{a^+}f\left(\frac{a+b}{2}\right)\right]-\frac{1}{90}\{7(f(a)+f(b))\right.$$

$$+12f\left(\frac{a+b}{2}\right)+32\left(f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right)\Big\}\Bigg|$$

$$\leq \frac{(b-a)^2}{4(\alpha+1)}\left(\int_0^1 |N_\alpha(t)|^q\,dt\right)^{\frac{1}{q}}\left[\frac{\left|f''(a)\right|^p+\left|f''(b)\right|^p}{2^s}\right]^{\frac{1}{p}}.$$

- *In the case that $|f''|^p$ is a convex function on $[a,b]$, then*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[\mathfrak{I}^\alpha_{b^-}f\left(\frac{a+b}{2}\right)+\mathfrak{I}^\alpha_{a^+}f\left(\frac{a+b}{2}\right)\right]-\frac{1}{90}\Big\{7\left(f(a)+f(b)\right)\right.$$
$$+12f\left(\frac{a+b}{2}\right)+32\left(f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right)\Big\}\Bigg|$$
$$\leq \frac{(b-a)^2}{4(\alpha+1)}\left(\int_0^1 |N_\alpha(t)|^q\,dt\right)^{\frac{1}{q}}\left[\frac{\left|f''(a)\right|^p+\left|f''(b)\right|^p}{2}\right]^{\frac{1}{p}}.$$

- *Consider the case where $|f''|^p$ is a P-function on $[a,b]$, then*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[\mathfrak{I}^\alpha_{b^-}f\left(\frac{a+b}{2}\right)+\mathfrak{I}^\alpha_{a^+}f\left(\frac{a+b}{2}\right)\right]-\frac{1}{90}\Big\{7\left(f(a)+f(b)\right)\right.$$
$$+12f\left(\frac{a+b}{2}\right)+32\left(f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right)\Big\}\Bigg|$$
$$\leq \frac{(b-a)^2}{4(\alpha+1)}\left(\int_0^1 |N_\alpha(t)|^q\,dt\right)^{\frac{1}{q}}\left[\left|f''(a)\right|^p+\left|f''(b)\right|^p\right]^{\frac{1}{p}}.$$

5. Application

For any different positive real values $a, b > 0$, we consider the following special means:

- The weighted arithmetic mean:

$$W(\eta_1,\eta_2,a,b) := \frac{\eta_1 a+\eta_2 b}{\eta_1+\eta_2}.$$

- The arithmetic mean:

$$A(a,b) := \frac{a+b}{2}.$$

- The n-logarithmic mean:

$$L_n(a,b) := \left(\frac{b^{n+1}-a^{n+1}}{(b-a)(n+1)}\right)^{\frac{1}{n}}, \quad n\in\mathbb{Z}\setminus\{-1,0\},\ b>a.$$

Consider the following example, which can be observed in [8]: Define a function $f : [0,+\infty) \to \mathbb{R}$, where s belongs to the interval $[0,1]$ and d, k, c belong to the set of real numbers defined by $\mathbb{R}$.

$$f(t)=\begin{cases} d & , t=0,\\ k\,t^s+c & , t>0.\end{cases}$$

If $k \geq 0$ and $0 \leq c \leq d$, then f is an s-convex function.

Example 5.1. *Let $t > 0$, $0 < s < 1$ and consider the function $f(t) = t^{s+2}$, then*

$$f''(t) = (s+2)(s+1)\ t^{s}.$$

With regard to the equations $d = c = 0$ and $k = (s+2)\ (s+1)$, it can be shown that the function $|f''(t)|$ is an s-convex.

For the inequality (16) with a value of $p = 1$, the following conclusion is derived by using the prior example.

Proposition 5.2. *Let $b > a > 0$, $0 < s < 1$ and $n = s + 2$. Then the following inequality holds:*

$$\left|\frac{7}{90}A(a^n, b^n) + \frac{16}{45}\left[W^n(1,3,a,b) + W^n(3,1,a,b)\right] + \frac{2}{15}A^n(a,b) - L_n^n(a,b)\right|$$
$$\leq \frac{1018}{273375 \cdot 2^s}(b-a)^2\,(s+2)(s+3)\,A\left(a^s,\, b^s\right).$$

6. Conclusion

Using Riemann-Liouville fractional operators on functions that are twice differentiable and include h-convex functions, we have presented a method for formulating a novel fractional Boole inequality. Additionally, we have constructed s-convex functions and P-functions, which serve as the foundation for some special instances. When it comes to new study fields that investigate various concepts of convexity and other fractional operators, the primary objective is to ensure that these inequalities may be utilized.

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