Filomat 39:32 (2025), 11451–11460
https://doi.org/10.2298/FIL2532451A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Refinements of recent inequalities via the numerical radii of matrices

Ahmad Al-Natoor[a]

[a] *Department of Mathematics, Faculty of Sciences, Isra University, Amman 11622, Jordan*

Abstract. In this paper, we employ the numerical radii of matrices as a tool to refine and strengthen recent results in matrix inequalities. Our results not only refine previous findings but also provide generalizations of several known results. Among other results, we prove that if A and B are complex matrices of size $n \times n$, then

$$\|A + B\| \leq \sqrt{\|A^*A + B^*B\| + 2\sqrt{\frac{1}{2}\|A^*B\|^2 + \frac{1}{2}w(A^*B)\|A^*B\|}},$$

which presents a refinement of the triangle inequality, where $\|\cdot\|$ and $w(\cdot)$ are the spectral norm and the numerical radius of matrices, respectively.

1. Introduction

We start this paper by presenting some known norms defined on the space of all $n \times n$ complex matrices $M_n(\mathbb{C})$. The spectral norm of $A \in M_n(\mathbb{C})$ is defined by

$$\|A\| = \max\left\{\frac{\|Ax\|}{\|x\|} : 0 \neq x \in \mathbb{C}^n\right\} = \max_{\|x\|=1}\{\|Ax\| : x \in \mathbb{C}^n\}.$$

The spectral norm is a special case of a general class called the unitarily invariant norms which are denoted by $|||\cdot|||$ and satisfy the invariance property, that is, $|||UAV||| = |||A|||$ for all $A \in M_n(\mathbb{C})$ and all unitary matrices $U, V \in M_n(\mathbb{C})$. The numerical radius of a matrix $A \in M_n(\mathbb{C})$ is defined by

$$w(A) = \max_{\|x\|=1}\{\langle Ax, x\rangle, x \in \mathbb{C}^n\}.$$

Also, the numerical radius of a matrix $A \in M_n(\mathbb{C})$ can be presented [10] by

$$w(A) = \max_{\theta\in\mathbb{R}}\left\|\mathrm{Re}\left(e^{i\theta}A\right)\right\|, \tag{1}$$

where $\mathrm{Re}\left(e^{i\theta}A\right)$ is the real part of the matrix $e^{i\theta}A$. A generalization of equation (1) can be presented [1] by

$$w_{|||\cdot|||}(A) = \max_{\theta\in\mathbb{R}}\left|\left|\left|\mathrm{Re}\left(e^{i\theta}A\right)\right|\right|\right|.$$

2020 *Mathematics Subject Classification*. Primary 15A60; Secondary 47A12.
Keywords. Positive semidefinite matrix, Numerical radius, Norm inequality.
Received: 24 July 2025; Revised: 01 September 2025; Accepted: 22 September 2025
Communicated by Dragan S. Djordjević
Email address: ahmad.alnatoor@iu.edu.jo (Ahmad Al-Natoor)
ORCID iD: https://orcid.org/0000-0003-1046-9104 (Ahmad Al-Natoor)

It is clear that, for $A \in M_n(\mathbb{C})$, we always have

$$\|\operatorname{Re} A\| \le \|A\|, \tag{2}$$

$$w(A) \le \|A\|, \tag{3}$$

and

$$w_{|||\cdot|||}(A) \le |||A|||. \tag{4}$$

In this paper, we use the relations between the numerical radius and norm of matrices that are shown in the inequalities (3) and (4) to present refinements of recent inequalities. For recent results concerning norms and numerical radii of matrices, we refer the reader to [2], [5], and [8].

2. Main results

To state our first main result, we need the following lemma [9].

Lemma 2.1. *Let $x, y \in \mathbb{C}^n$, and let $f : (0,1) \to \mathbb{R}^+$ be a well-defined function. Then*

$$\left|\langle x, y\rangle\right|^2 \le \frac{f(t)}{1+f(t)} \|x\|^2 \left\|y\right\|^2 + \frac{1}{1+f(t)} \left|\langle x, y\rangle\right| \|x\| \left\|y\right\|.$$

Theorem 2.2. *Let $A, B \in M_n(\mathbb{C})$, and let $f : (0,1) \to \mathbb{R}^+$ be a well-defined function. Then, for $x \in \mathbb{C}^n$, we have*

$$\|(A+B)x\|^2 \le \langle (A^*A + B^*B)x, x\rangle + 2\sqrt{\begin{array}{c} \frac{f(t)}{1+f(t)} \|(\operatorname{Re}(A^*B))x\|^2 \|x\|^2 \\ + \frac{1}{1+f(t)} |\langle (\operatorname{Re}(A^*B))x, x\rangle| \, \|(\operatorname{Re}(A^*B))x\| \, \|x\| \end{array}}.$$

Proof. For $x \in \mathbb{C}^n$, we have

$$\begin{aligned} \|(A+B)x\|^2 &= \|Ax + Bx\|^2 \\ &= \langle Ax + Bx, Ax + Bx\rangle \\ &= \langle Ax, Ax\rangle + \langle Bx, Bx\rangle + \langle Ax, Bx\rangle + \langle Bx, Ax\rangle \\ &= \langle A^*Ax, x\rangle + \langle B^*Bx, x\rangle + \langle x, A^*Bx\rangle + \langle A^*Bx, x\rangle \\ &= \langle (A^*A + B^*B)x, x\rangle + \overline{\langle A^*Bx, x\rangle} + \langle A^*Bx, x\rangle \\ &= \langle (A^*A + B^*B)x, x\rangle + 2\operatorname{Re}\langle A^*Bx, x\rangle \\ &= \langle (A^*A + B^*B)x, x\rangle + 2\langle (\operatorname{Re}(A^*B))x, x\rangle \\ &\le \langle (A^*A + B^*B)x, x\rangle + 2|\langle (\operatorname{Re}(A^*B))x, x\rangle| \\ &\le \langle (A^*A + B^*B)x, x\rangle \\ &\quad + 2\sqrt{\begin{array}{c} \frac{f(t)}{1+f(t)} \|(\operatorname{Re}(A^*B))x\|^2 \|x\|^2 \\ + \frac{1}{1+f(t)} |\langle (\operatorname{Re}(A^*B))x, x\rangle| \, \|(\operatorname{Re}(A^*B))x\| \, \|x\| \end{array}} \\ &\qquad \text{(by Lemma 2.1)}, \end{aligned}$$

as required. □

Corollary 2.3. *Let $A, B \in M_n(\mathbb{C})$, and let $f : (0,1) \to \mathbb{R}^+$ be a well-defined function. Then*

$$\|A+B\| \le \sqrt{\|A^*A + B^*B\| + 2\sqrt{\frac{f(t)}{1+f(t)} \|A^*B\|^2 + \frac{1}{1+f(t)} w(A^*B) \|A^*B\|}}.$$

In particular, if $f(t) = 1$, we have

$$\|A + B\| \leq \sqrt{\|A^*A + B^*B\| + 2\sqrt{\frac{1}{2}\|A^*B\|^2 + \frac{1}{2}w(A^*B)\|A^*B\|}}. \tag{5}$$

Proof. By Theorem 2.2, for $x \in \mathbb{C}^n$, we have

$$\begin{aligned}
\|(A + B)x\|^2 &\leq \langle (A^*A + B^*B)x, x\rangle \\
&\quad +2\sqrt{\begin{array}{c}\frac{f(t)}{1+f(t)}\|(\mathrm{Re}(A^*B))x\|^2\|x\|^2 \\ +\frac{1}{1+f(t)}|\langle(\mathrm{Re}(A^*B))x, x\rangle|\,\|(\mathrm{Re}(A^*B))x\|\,\|x\|\end{array}} \\
&\leq \langle (A^*A + B^*B)x, x\rangle \\
&\quad +2\sqrt{\frac{f(t)}{1+f(t)}\|\mathrm{Re}(A^*B)\|^2\|x\|^4 + \frac{1}{1+f(t)}|\langle(\mathrm{Re}(A^*B))x, x\rangle|\,\|\mathrm{Re}(A^*B)\|\,\|x\|^2} \\
&\leq \langle (A^*A + B^*B)x, x\rangle \\
&\quad +2\sqrt{\frac{f(t)}{1+f(t)}\|A^*B\|^2\|x\|^4 + \frac{1}{1+f(t)}|\langle(\mathrm{Re}(A^*B))x, x\rangle|\,\|A^*B\|\,\|x\|^2} \\
&\qquad\qquad \text{(by the inequality (2)).}
\end{aligned}$$

Now, by taking the maximum over $\|x\| = 1$, we have

$$\begin{aligned}
\|A + B\|^2 &\leq w(A^*A + B^*B) + 2\sqrt{\frac{f(t)}{1+f(t)}\|A^*B\|^2 + \frac{1}{1+f(t)}w(\mathrm{Re}(A^*B))\|A^*B\|}, \\
&= \|A^*A + B^*B\| + 2\sqrt{\frac{f(t)}{1+f(t)}\|A^*B\|^2 + \frac{1}{1+f(t)}\|\mathrm{Re}(A^*B)\|\,\|A^*B\|},
\end{aligned}$$

or equivalently,

$$\begin{aligned}
\|A + B\| &\leq \sqrt{\|A^*A + B^*B\| + 2\sqrt{\frac{f(t)}{1+f(t)}\|A^*B\|^2 + \frac{1}{1+f(t)}\|\mathrm{Re}(A^*B)\|\,\|A^*B\|}} \\
&\leq \sqrt{\|A^*A + B^*B\| + 2\sqrt{\frac{f(t)}{1+f(t)}\|A^*B\|^2 + \frac{1}{1+f(t)}\max_{\theta\in\mathbb{R}}\left\|\mathrm{Re}(e^{i\theta}A^*B)\right\|\,\|A^*B\|}} \\
&= \sqrt{\|A^*A + B^*B\| + 2\sqrt{\frac{f(t)}{1+f(t)}\|A^*B\|^2 + \frac{1}{1+f(t)}w(A^*B)\|A^*B\|}},
\end{aligned}$$

as required. □

Remark 2.4. *Corollary 2.3 contains an interesting interpolating inequality involving the numerical radii of matrices in which a special case of Corollary 2.3 leads to a refinement of the triangle inequality. In fact, by the inequality (5), we have*

$$\begin{aligned}
\|A + B\| &\leq \sqrt{\|A^*A + B^*B\| + 2\sqrt{\frac{1}{2}\|A^*B\|^2 + \frac{1}{2}w(A^*B)\|A^*B\|}} \\
&\leq \sqrt{\|A^*A + B^*B\| + 2\sqrt{\frac{1}{2}\|A^*B\|^2 + \frac{1}{2}\|A^*B\|^2}}
\end{aligned}$$

$$
\begin{aligned}
& \quad \text{(by the inequality (3))} \\
&= \sqrt{\|A^*A + B^*B\| + 2\|A^*B\|} \\
&\leq \sqrt{\|A^*A\| + \|B^*B\| + 2\|A\|\|B\|} \\
&= \sqrt{\|A\|^2 + \|B\|^2 + 2\|A\|\|B\|} \\
&= \sqrt{(\|A\| + \|B\|)^2} \\
&= \|A\| + \|B\| .
\end{aligned}
$$

To state our next result, we need the well-known arithmetic-geometric mean inequality for matrices which was proved in [7] and also we need the following lemma which contains two inequalities, the first inequality was proved in [4] and the second inequality, which is a general version of the first inequality, was proved in [3] using different argument.

Lemma 2.5. *Let $A, B \in M_n(\mathbb{C})$ be such that AB is Hermitian. Then*

$$w(AB) \leq w(BA) \tag{6}$$

and

$$w_{|||\cdot|||}(AB) \leq w_{|||\cdot|||}(BA). \tag{7}$$

Theorem 2.6. *Let $A, B, X, Y \in M_n(\mathbb{C})$ be such that A and B are positive semidefinite matrices. Then for $0 \leq p, q \leq 1$, we have*

$$
\begin{aligned}
2\,|||AX - YB||| \leq\;& w_{|||\cdot|||}\left(\left(A^{2(1-p)}\,|X^*|^2\right) \oplus \left(B^{2q}\,|Y|^2\right)\right) + \frac{1}{2} w_{|||\cdot|||}\left(A^{2p} \oplus B^{2(1-q)}\right) \\
&+ \left|\left|\left|\left(B^q Y^* A^p - B^{1-q} X^* A^{1-p}\right) \oplus \left(A^p Y B^q - A^{1-p} X B^{1-q}\right)\right|\right|\right|
\end{aligned}
\tag{8}
$$

and

$$
\begin{aligned}
2\,|||AX + YB||| \leq\;& w_{|||\cdot|||}\left(\left(A^{2(1-p)}\,|X^*|^2\right) \oplus \left(B^{2q}\,|Y|^2\right)\right) + \frac{1}{2} w_{|||\cdot|||}\left(A^{2p} \oplus B^{2(1-q)}\right) \\
&+ \left|\left|\left|\left(B^q Y^* A^p + B^{1-q} X^* A^{1-p}\right) \oplus \left(A^p Y B^q + A^{1-p} X B^{1-q}\right)\right|\right|\right| .
\end{aligned}
\tag{9}
$$

Proof. We have

$$
\begin{aligned}
|||AX - YB||| &= \left|\left|\left|\begin{bmatrix} A^p & YB^q \end{bmatrix}\begin{bmatrix} A^{1-p}X \\ -B^{1-q} \end{bmatrix}\right|\right|\right| \\
&\leq \frac{1}{2}\left|\left|\left|\begin{bmatrix} A^p \\ B^q Y^* \end{bmatrix}\begin{bmatrix} A^p & YB^q \end{bmatrix} + \begin{bmatrix} A^{1-p}X \\ -B^{1-q} \end{bmatrix}\begin{bmatrix} X^*A^{1-p} & -B^{1-q} \end{bmatrix}\right|\right|\right| \\
&\quad \text{(by the arithmetic-geometric mean inequality for matrices)} \\
&= \frac{1}{2}\left|\left|\left|\begin{bmatrix} A^{2p} & A^p Y B^q \\ B^q Y^* A^p & B^q\,|Y|^2\,B^q \end{bmatrix} + \begin{bmatrix} A^{1-p}\,|X^*|^2\,A^{1-p} & -A^{1-p} X B^{1-q} \\ -B^{1-q} X^* A^{1-p} & B^{2(1-q)} \end{bmatrix}\right|\right|\right| \\
&= \frac{1}{2}\left|\left|\left|\begin{bmatrix} A^{2p} + A^{1-p}\,|X^*|^2\,A^{1-p} & A^p Y B^q - A^{1-p} X B^{1-q} \\ B^q Y^* A^p - B^{1-q} X^* A^{1-p} & B^q\,|Y|^2\,B^q + B^{2(1-q)} \end{bmatrix}\right|\right|\right| \qquad (10) \\
&= \frac{1}{2}\left|\left|\left| \begin{bmatrix} A^{1-p}\,|X^*|^2\,A^{1-p} & O \\ O & B^q\,|Y|^2\,B^q \end{bmatrix} + \begin{bmatrix} A^{2p} & O \\ O & B^{2(1-q)} \end{bmatrix} + \begin{bmatrix} O & A^p Y B^q - A^{1-p} X B^{1-q} \\ B^q Y^* A^p - B^{1-q} X^* A^{1-p} & O \end{bmatrix} \right|\right|\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{2}\left|\left|\left|\begin{bmatrix} A^{1-p}|X^*|^2A^{1-p} & O \\ O & B^q|Y|^2B^q \end{bmatrix}\right|\right|\right| + \frac{1}{2}\left|\left|\left|\begin{bmatrix} A^{2p} & O \\ O & B^{2(1-q)} \end{bmatrix}\right|\right|\right| \\
&\quad + \frac{1}{2}\left|\left|\left|\begin{bmatrix} O & A^pYB^q - A^{1-p}XB^{1-q} \\ B^qY^*A^p - B^{1-q}X^*A^{1-p} & O \end{bmatrix}\right|\right|\right| \\
&= \frac{1}{2}w_{|||\cdot|||}\left(\begin{bmatrix} A^{1-p}|X^*|^2A^{1-p} & O \\ O & B^q|Y|^2B^q \end{bmatrix}\right) + \frac{1}{2}w_{|||\cdot|||}(A^{2p}\oplus B^{2(1-q)}) \\
&\quad + \frac{1}{2}\left|\left|\left|\left(B^qY^*A^p - B^{1-q}X^*A^{1-p}\right)\oplus\left(A^pYB^q - A^{1-p}XB^{1-q}\right)\right|\right|\right| \\
&= \frac{1}{2}w_{|||\cdot|||}\left(\begin{bmatrix} A^{1-p} & O \\ O & B^q \end{bmatrix}\begin{bmatrix} |X^*|^2 & O \\ O & |Y|^2 \end{bmatrix}\begin{bmatrix} A^{1-p} & O \\ O & B^q \end{bmatrix}\right) + \frac{1}{2}w_{|||\cdot|||}\left(A^{2p}\oplus B^{2(1-q)}\right) \\
&\quad + \frac{1}{2}\left|\left|\left|\left(B^qY^*A^p - B^{1-q}X^*A^{1-p}\right)\oplus\left(A^pYB^q - A^{1-p}XB^{1-q}\right)\right|\right|\right| \\
&\leq \frac{1}{2}w_{|||\cdot|||}\left(\begin{bmatrix} A^{1-p} & O \\ O & B^q \end{bmatrix}\begin{bmatrix} A^{1-p} & O \\ O & B^q \end{bmatrix}\begin{bmatrix} |X^*|^2 & O \\ O & |Y|^2 \end{bmatrix}\right) + \frac{1}{2}w_{|||\cdot|||}\left(A^{2p}\oplus B^{2(1-q)}\right) \\
&\quad + \frac{1}{2}\left|\left|\left|\left(B^qY^*A^p - B^{1-q}X^*A^{1-p}\right)\oplus\left(A^pYB^q - A^{1-p}XB^{1-q}\right)\right|\right|\right| \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by the inequality (7))} \\
&= \frac{1}{2}w_{|||\cdot|||}\left(\begin{bmatrix} A^{2(1-p)} & O \\ O & B^{2q} \end{bmatrix}\begin{bmatrix} |X^*|^2 & O \\ O & |Y|^2 \end{bmatrix}\right) + \frac{1}{2}w_{|||\cdot|||}\left(A^{2p}\oplus B^{2(1-q)}\right) \\
&\quad + \frac{1}{2}\left|\left|\left|\left(B^qY^*A^p - B^{1-q}X^*A^{1-p}\right)\oplus\left(A^pYB^q - A^{1-p}XB^{1-q}\right)\right|\right|\right| \\
&= \frac{1}{2}w_{|||\cdot|||}\left(\begin{bmatrix} A^{2(1-p)}|X^*|^2 & O \\ O & B^{2q}|Y|^2 \end{bmatrix}\right) + \frac{1}{2}w_{|||\cdot|||}\left(A^{2p}\oplus B^{2(1-q)}\right) \\
&\quad + \frac{1}{2}\left|\left|\left|\left(B^qY^*A^p - B^{1-q}X^*A^{1-p}\right)\oplus\left(A^pYB^q - A^{1-p}XB^{1-q}\right)\right|\right|\right| \\
&= \frac{1}{2}w_{|||\cdot|||}\left(\left(A^{2(1-p)}|X^*|^2\right)\oplus\left(B^{2q}|Y|^2\right)\right) + \frac{1}{2}w_{|||\cdot|||}\left(A^{2p}\oplus B^{2(1-q)}\right) \\
&\quad + \frac{1}{2}\left|\left|\left|\left(B^qY^*A^p - B^{1-q}X^*A^{1-p}\right)\oplus\left(A^pYB^q - A^{1-p}XB^{1-q}\right)\right|\right|\right|, \qquad (11)
\end{aligned}
$$

which proves the inequality (8). The inequality (9) follows from the inequality (8) by replacing X by $-X$(or Y by $-Y$). □

Corollary 2.7. *Let $A, B, X, Y \in M_n(\mathbb{C})$ be such that A and B are positive semidefinite matrices. Then*

$$
\begin{aligned}
2\,|||AX - YB||| &\leq w_{|||\cdot|||}\left(\left(A(I + |X^*|^2)\right)\oplus\left(B(|Y|^2 + I)\right)\right) \\
&\quad + \left|\left|\left|\left(A^{1/2}(Y - X)B^{1/2}\right)\oplus\left(B^{1/2}(Y^* - X^*)A^{1/2}\right)\right|\right|\right| \qquad (12)
\end{aligned}
$$

and

$$
\begin{aligned}
2\,|||AX + YB||| &\leq w_{|||\cdot|||}\left(\left(A(I + |X^*|^2)\right)\oplus\left(B(|Y|^2 + I)\right)\right) \\
&\quad + \left|\left|\left|\left(A^{1/2}(Y + X)B^{1/2}\right)\oplus\left(B^{1/2}(Y^* + X^*)A^{1/2}\right)\right|\right|\right|, \qquad (13)
\end{aligned}
$$

where I denotes the identity matrix in $M_n(\mathbb{C})$.

Proof. Letting $p = q = \frac{1}{2}$ in the inequality (10), we have

$$|||AX - YB|||$$

$$\begin{aligned}
&\leq \frac{1}{2}\left|\left|\left|\begin{bmatrix} A + A^{1/2}|X^*|^2 A^{1/2} & A^{1/2}YB^{1/2} - A^{1/2}XB^{1/2} \\ B^{1/2}Y^*A^{1/2} - B^{1/2}X^*A^{1/2} & B^{1/2}|Y|^2 B^{1/2} + B \end{bmatrix}\right|\right|\right| \\
&= \frac{1}{2}\left|\left|\left|\begin{bmatrix} A^{1/2}(I + |X^*|^2)A^{1/2} & A^{1/2}(Y - X)B^{1/2} \\ B^{1/2}(Y^* - X^*)A^{1/2} & B^{1/2}(|Y|^2 + I)B^{1/2} \end{bmatrix}\right|\right|\right| \qquad (14)\\
&= \frac{1}{2}w_{|||\cdot|||}\left(\begin{bmatrix} A^{1/2}(I + |X^*|^2)A^{1/2} & A^{1/2}(Y - X)B^{1/2} \\ B^{1/2}(Y^* - X^*)A^{1/2} & B^{1/2}(|Y|^2 + I)B^{1/2} \end{bmatrix}\right) \\
&\leq \frac{1}{2}w_{|||\cdot|||}\left(\begin{bmatrix} A^{1/2}(I + |X^*|^2)A^{1/2} & O \\ O & B^{1/2}(|Y|^2 + I)B^{1/2} \end{bmatrix}\right) \\
&\quad + \frac{1}{2}\left(\left|\left|\left|\begin{bmatrix} O & A^{1/2}(Y - X)B^{1/2} \\ B^{1/2}(Y^* - X^*)A^{1/2} & O \end{bmatrix}\right|\right|\right|\right) \\
&= \frac{1}{2}w_{|||\cdot|||}\left(\begin{bmatrix} A^{1/2} & O \\ O & B^{1/2} \end{bmatrix}\begin{bmatrix} I + |X^*|^2 & O \\ O & |Y|^2 + I \end{bmatrix}\begin{bmatrix} A^{1/2} & O \\ O & B^{1/2} \end{bmatrix}\right) \\
&\quad + \frac{1}{2}\left|\left|\left|\left(A^{1/2}(Y - X)B^{1/2}\right) \oplus \left(B^{1/2}(Y^* - X^*)A^{1/2}\right)\right|\right|\right| \\
&\leq \frac{1}{2}w_{|||\cdot|||}\left(\begin{bmatrix} A & O \\ O & B \end{bmatrix}\begin{bmatrix} I + |X^*|^2 & O \\ O & |Y|^2 + I \end{bmatrix}\right) \\
&\quad + \frac{1}{2}\left|\left|\left|\left(A^{1/2}(Y - X)B^{1/2}\right) \oplus \left(B^{1/2}(Y^* - X^*)A^{1/2}\right)\right|\right|\right| \\
&\qquad\qquad \text{(by the inequality (7))} \\
&= \frac{1}{2}w_{|||\cdot|||}\left(\begin{bmatrix} A(I + |X^*|^2) & O \\ O & B(|Y|^2 + I) \end{bmatrix}\right) \\
&\quad + \frac{1}{2}\left|\left|\left|\left(A^{1/2}(Y - X)B^{1/2}\right) \oplus \left(B^{1/2}(Y^* - X^*)A^{1/2}\right)\right|\right|\right| \\
&= \frac{1}{2}w_{|||\cdot|||}\left(\left(A(I + |X^*|^2)\right) \oplus \left(B(|Y|^2 + I)\right)\right) \\
&\quad + \frac{1}{2}\left|\left|\left|\left(A^{1/2}(Y - X)B^{1/2}\right) \oplus \left(B^{1/2}(Y^* - X^*)A^{1/2}\right)\right|\right|\right|,
\end{aligned}$$

which proves the inequality (12). The inequality (13) follows from the inequality (12) by replacing X by $-X$(or Y by $-Y$). □

Remark 2.8. *The authors in [4] proved that if $A, B, X, Y \in M_n(\mathbb{C})$ are such that A and B are positive semidefinite. Then*

$$w(AX - YB) \leq \frac{1}{2}\max(w(A\left(I + |X^*|^2\right), w\left(B\left(I + |Y|^2\right)\right)) + \frac{1}{2}\left\|A^{1/2}(Y - X)B^{1/2}\right\|. \qquad (15)$$

In particular, if $Y = X$, then

$$w(AX - XB) \leq \frac{1}{2}\max(w(A\left(I + |X^*|^2\right), w\left(B\left(I + |X|^2\right)\right)). \qquad (16)$$

The inequality (12) presents a generalization and a refinement of the inequality (15). In fact, by specifying the inequality (12) to the spectral norm, we have

$$\begin{aligned}
\|AX - YB\| &\leq \frac{1}{2}w\left(\left(A(I + |X^*|^2)\right) \oplus \left(B(|Y|^2 + I)\right)\right) + \frac{1}{2}\left\|\left(A^{1/2}(Y - X)B^{1/2}\right) \oplus \left(B^{1/2}(Y^* - X^*)A^{1/2}\right)\right\| \\
&= \frac{1}{2}\max\left(w\left(A(I + |X^*|^2)\right), w\left(B(|Y|^2 + I)\right)\right) + \frac{1}{2}\left\|A^{1/2}(Y - X)B^{1/2}\right\|. \qquad (17)
\end{aligned}$$

Now, in view of the inequality (3), the inequality (17) refines the inequality (15).

In the next corollary, we give a refinement and a generalization of the inequality (16).

Corollary 2.9. *Let $A, B, X, Y \in M_n(\mathbb{C})$ be such that A and B are positive semidefinite matrices. Then*

$$|||AX - YB||| \leq \frac{1}{2} w_{|||\cdot|||}\left(\left(A\left(I + |X^*|^2\right)\right) \oplus \left(B\left(I + |Y|^2\right)\right)\right) + |||(A(Y - X)) \oplus (B(Y^* - X^*))||| .$$

In particular, if $|||\cdot||| = \|\cdot\|$, then

$$\|AX - YB\| \leq \frac{1}{2}\max(w(A\left(I + |X^*|^2\right), w\left(B\left(I + |Y|^2\right)\right)) + \max\left(\|A(Y - X)\|, \|B(Y^* - X^*)\|\right). \tag{18}$$

Proof. By the inequality (14), we have

$$\begin{aligned}
|||AX - YB||| &\leq \frac{1}{2}\left|\left|\left|\begin{bmatrix} A^{1/2} & O \\ O & B^{1/2} \end{bmatrix}\begin{bmatrix} I + |X^*|^2 & Y - X \\ Y^* - X^* & |Y|^2 + I \end{bmatrix}\begin{bmatrix} A^{1/2} & O \\ O & B^{1/2} \end{bmatrix}\right|\right|\right| \\
&= \frac{1}{2} w_{|||\cdot|||}\left(\begin{bmatrix} A^{1/2} & O \\ O & B^{1/2} \end{bmatrix}\begin{bmatrix} I + |X^*|^2 & Y - X \\ Y^* - X^* & |Y|^2 + I \end{bmatrix}\begin{bmatrix} A^{1/2} & O \\ O & B^{1/2} \end{bmatrix}\right) \\
&\leq \frac{1}{2} w_{|||\cdot|||}\left(\begin{bmatrix} A^{1/2} & O \\ O & B^{1/2} \end{bmatrix}\begin{bmatrix} A^{1/2} & O \\ O & B^{1/2} \end{bmatrix}\begin{bmatrix} I + |X^*|^2 & Y - X \\ Y^* - X^* & |Y|^2 + I \end{bmatrix}\right) \\
&\qquad \text{(by the inequality (7))} \\
&= \frac{1}{2} w_{|||\cdot|||}\left(\begin{bmatrix} A & O \\ O & B \end{bmatrix}\begin{bmatrix} I + |X^*|^2 & Y - X \\ Y^* - X^* & |Y|^2 + I \end{bmatrix}\right) \\
&= \frac{1}{2} w_{|||\cdot|||}\left(\begin{bmatrix} A\left(I + |X^*|^2\right) & A(Y - X) \\ B(Y^* - X^*) & B\left(I + |Y|^2\right) \end{bmatrix}\right) \\
&\leq \frac{1}{2} w_{|||\cdot|||}\left(\begin{bmatrix} A\left(I + |X^*|^2\right) & O \\ O & B\left(I + |Y|^2\right) \end{bmatrix}\right) + \frac{1}{2} w_{|||\cdot|||}\left(\begin{bmatrix} O & A(Y - X) \\ B(Y^* - X^*) & O \end{bmatrix}\right) \\
&\leq \frac{1}{2} w_{|||\cdot|||}\left(\begin{bmatrix} A\left(I + |X^*|^2\right) & O \\ O & B\left(I + |Y|^2\right) \end{bmatrix}\right) + \frac{1}{2}\left|\left|\left|\begin{bmatrix} O & A(Y - X) \\ B(Y^* - X^*) & O \end{bmatrix}\right|\right|\right| \\
&= \frac{1}{2} w_{|||\cdot|||}\left(\left(A\left(I + |X^*|^2\right)\right) \oplus \left(B\left(I + |Y|^2\right)\right)\right) + \frac{1}{2}|||(A(Y - X)) \oplus (B(Y^* - X^*))|||,
\end{aligned}$$

as required. □

Letting $Y = X$ in the inequality (18), we have

$$\|AX - XB\| \leq \frac{1}{2}\max(w(A\left(I + |X^*|^2\right), w\left(B\left(I + |X|^2\right)\right)),$$

which is a refinement of the inequality (16).

Theorem 2.10. *Let $A, B, X \in M_n(\mathbb{C})$ be such that A and B are positive semidefinite. Then, for $0 \leq p, q \leq 1$, we have*

$$w(AX^* + XB) \leq \frac{w(A^{2p} + B^{2(1-q)}) + w(|X|^2\left(A^{2(1-p)} + B^{2q}\right))}{2}. \tag{19}$$

Proof. Let $E = \begin{bmatrix} A^p & XB^q \end{bmatrix}$ and $F = \begin{bmatrix} XA^{1-p} & B^{1-q} \end{bmatrix}$. Then for $x \in \mathbb{C}^n$, we have

$$\begin{aligned}
&|\langle (AX^* + XB)x, x\rangle| \\
&\quad = |\langle EF^* x, x\rangle|
\end{aligned}$$

$$\begin{aligned}
&= |\langle F^*x, E^*x\rangle| \\
&\leq \|F^*x\|\,\|E^*x\| \text{ (by the Cauchy-Schwarz inequality)} \\
&\leq \frac{\|F^*x\|^2 + \|E^*x\|^2}{2} \text{ (by the arithmetic-geometric mean inequality for scalars)} \\
&= \frac{\langle F^*x, F^*x\rangle + \langle E^*x, E^*x\rangle}{2} \\
&= \frac{\langle FF^*x, x\rangle + \langle EE^*x, x\rangle}{2} \\
&= \frac{\left\langle \begin{bmatrix} XA^{1-p} & B^{1-q} \end{bmatrix} \begin{bmatrix} A^{1-p}X^* \\ B^{1-q} \end{bmatrix} x, x \right\rangle + \left\langle \begin{bmatrix} A^p & XB^q \end{bmatrix} \begin{bmatrix} A^p \\ B^qX^* \end{bmatrix} x, x \right\rangle}{2} \\
&= \frac{\left\langle \left(XA^{2(1-p)}X^* + B^{2(1-q)}\right) x, x \right\rangle + \left\langle \left(A^{2p} + XB^{2q}X^*\right) x, x \right\rangle}{2} \\
&= \frac{\left\langle \left(XA^{2(1-p)}X^* + B^{2(1-q)} + A^{2p} + XB^{2q}X^*\right) x, x \right\rangle}{2}.
\end{aligned}$$

So, by taking the maximum over $\|x\| = 1$, we have

$$\begin{aligned}
w(AX^* + XB) &\leq \frac{w(XA^{2(1-p)}X^* + B^{2(1-q)} + A^{2p} + XB^{2q}X^*)}{2} \\
&\leq \frac{w(A^{2p} + B^{2(1-q)}) + w(XA^{2(1-p)}X^* + XB^{2q}X^*)}{2} \\
&= \frac{w(A^{2p} + B^{2(1-q)}) + w(X\left(A^{2(1-p)} + B^{2q}\right)X^*)}{2} \\
&\leq \frac{w(A^{2p} + B^{2(1-q)}) + w(|X|^2\left(A^{2(1-p)} + B^{2q}\right))}{2} \\
&\qquad \text{(by the inequality (6))},
\end{aligned}$$

which is the desired inequality (19). □

In the proof of Theorem 2.10, by replacing the matrix $F = \begin{bmatrix} XA^{1-p} & B^{1-q} \end{bmatrix}$ by the matrix $\begin{bmatrix} XA^{1-p} & -B^{1-q} \end{bmatrix}$, and by following the same steps that are used in the rest of the proof, we have

$$w(AX^* - XB) \leq \frac{w(A^{2p} + B^{2(1-q)}) + w(|X|^2\left(A^{2(1-p)} + B^{2q}\right))}{2}. \tag{20}$$

Remark 2.11. *It follows from Theorem 2.15 in [6] that if $A, B, X \in \mathbb{M}_n(\mathbb{C})$ are such that A and B are positive semidefinite, then*

$$\|AX + XB\| \leq 2\,\|X\|\,\|A \oplus B\|. \tag{21}$$

Using the inequality (21) and some basic known facts, then for positive semidefinite matrices A and B, we have

$$\begin{aligned}
w(AX + XB) &\leq \|AX + XB\| \\
&\leq 2\,\|X\|\,\|A \oplus B\| \\
&\leq 2\,\|X\|\,\|A + B\|.
\end{aligned} \tag{22}$$

The inequality (19) generalizes and refines the inequality (22) for the case of Hermitian matrix X. In fact, by specifying the inequality (19) for positive semidefinite matrices A and B and Hermitian matrix X and for $p = q = \frac{1}{2}$, we have

$$w(AX + XB) \leq \frac{w(A + B) + w(X^2\,(A + B))}{2}$$

$$\leq \frac{\|A+B\| + \left\|X^2 (A+B)\right\|}{2}$$
$$\leq \frac{\|A+B\| + \|X\|^2 \|A+B\|}{2}.$$

Now, by replacing X by tX and taking the minimum over $t > 0$, we have

$$\begin{aligned} w(AX+XB) &\leq \sqrt{\|A+B\|^2 \|X\|^2} \\ &= \|X\| \|A+B\| . \end{aligned} \tag{23}$$

Remark 2.12. *It follows from Theorem 2.2 in [6] that if $A, B, X \in \mathbb{M}_n(\mathbb{C})$ are such that A and B are positive semidefinite, then*

$$\|AX - XB\| \leq \|X\| \|A \oplus B\| . \tag{24}$$

Using the inequality (24) and some basic known facts, then for positive semidefinite matrices A and B, we have

$$\begin{aligned} w(AX-XB) &\leq \|AX - XB\| \\ &\leq \|X\| \|A \oplus B\| \\ &\leq \|X\| \|A+B\| . \end{aligned} \tag{25}$$

The inequality (20) refines the inequality (25). In fact, by specifying the inequality (20) for positive semidefinite matrices A and B and Hermitian matrix X and for $p = q = \frac{1}{2}$, then, by using the same steps that are used to get the inequality (23), we have

$$\begin{aligned} w(AX-XB) &\leq \frac{w(A+B) + w(X^2 (A+B))}{2} \\ &\leq \|X\| \|A+B\| . \end{aligned}$$

Remark 2.13. *The inequality (19) can be considered as a generalization of the inequality (2). In fact, by letting $A = B = I$ and $p = q = \frac{1}{2}$ in the inequality (19), we get the inequality (2) as follows:*

$$\begin{aligned} \|\operatorname{Re} X\| &= w(\operatorname{Re} X) \\ &= \frac{1}{2} w(X^* + X) \\ &\leq \frac{1}{2}\left(\frac{2 + 2w(|X|^2)}{2}\right) \\ &\quad \text{(by a special case of the inequality (19))} \\ &= \frac{1}{2}\left(1 + w(|X|^2)\right) \\ &= \frac{1}{2}\left(1 + \left\||X|^2\right\|\right). \end{aligned}$$

Now, by replacing X by tX and taking the minimum over $t > 0$, we have

$$\begin{aligned} \|\operatorname{Re} X\| &\leq \sqrt{\left\||X|^2\right\|} \\ &= \sqrt{\|X^* X\|} \\ &= \sqrt{\|X\|^2} \\ &= \|X\| . \end{aligned}$$

References

[1] A. Abu-Omar and F. Kittaneh, *A generalization of the numerical radius*, Linear Algebra Appl. **569** (2019), 323–334.
[2] A. Al-Natoor, *Norm inequalities for product of matrices*, Acta Sci. Math. (Szeged) **91** (2025), 153–160. https://doi.org/10.1007/s44146-024-00121-1
[3] A. Al-Natoor, O. Hirzallah, and F. Kittaneh, *Norm inequalities for matrices via the numerical radius*, Period. Math. Hung. (2025). https://doi.org/10.1007/s10998-025-00669-7
[4] A. Al-Natoor, O. Hirzallah, and F. Kittaneh, *Interpolating numerical radius inequalities for matrices*, Adv. Oper. Theory **9** (2024), 23.
[5] A. Al-Natoor, O. Hirzallah, and F. Kittaneh, *Singular value and norm inequalities involving the numerical radii of matrices*, Ann. Funct. Anal. **15** (2024), 10.
[6] A. Al-Natoor and F. Kittaneh, *Singular value and norm inequalities for positive semidefinite matrices*, Linear Multilinear Algebra **70** (2022), 4498–4507.
[7] R. Bhatia and F. Kittaneh, *On the singular values of a product of operators*, SIAM J. Matrix Anal. Appl. **11**(2) (1990), 272–277.
[8] F. Kittaneh, H. R. Moradi, and M. Sababheh, *Complementary bounds for the numerical and spectral radii*, Quaest. Math. (2025). https://doi.org/10.2989/16073606.2025.2458781
[9] R. K. Nayak, *Enhancement of the Cauchy–Schwarz inequality and its implications for numerical radius inequalities*, arXiv:2405.19698v1.
[10] T. Yamazaki, *On upper and lower bounds of the numerical radius and an equality condition*, Studia Math. **178** (2007), 83–89.