Filomat 39:32 (2025), 11461–11472
https://doi.org/10.2298/FIL2532461Y

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Strong converse inequality for Bernstein operators in weighted uniform norm

Dansheng Yu[a]

[a]*Department of Mathematics, Hangzhou Normal University, Hangzhou, Zhejiang 310036, China*

Abstract. In the present paper, we introduce a new weighted K-functional and uses it to obtain strong inverse inequalities in the weighted norm for a class of modified Bernstein operators that can approximate functions with singularities.

1. Introduction

Denote by $C(I)$ the set of all continuous functions defined on the interval I. For any $f \in C([0,1])$, the corresponding Bernstein operators are defined as follows([17]):

$$B_n(f,x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{nk}(x),$$

where

$$p_{nk}(x) := \binom{n}{k} x^k (1-x)^{n-k},\ k = 0,1,\cdots,n,\ x \in [0,1].$$

The Bernstein operator, introduced by Sergei Bernstein in 1912, is a cornerstone in approximation theory. It provides a constructive proof of the Weierstrass approximation theorem by approximating continuous functions on $[0,1]$ using polynomials. Its properties, such as positivity, linearity, and shape preservation, have made it a fundamental tool in various fields, including computer-aided geometric design (CAGD), numerical analysis, and probability theory. Over the past century, numerous generalizations and modifications of the Bernstein operator have been proposed to enhance its convergence rate, extend its applicability to broader function spaces, and address specific challenges in applications ([1]-[6], [8],[10],[12]-[21], [24]).

In order to approximate the functions with singularities, Della Vecchia, Mastroianni and Szabados [7], Yu [20] and Yu-Zhao [22] introduced several modified Bernstein operators. Let

$$w(x) = x^{\alpha}(1-x)^{\beta},\ \alpha,\beta \geq 0,\ \alpha+\beta > 0,\ 0 \leq x \leq 1,$$

2020 *Mathematics Subject Classification*. Primary 41A35, 41A25.
Keywords. Bernstein operators, strong converse inequality, weighted approximation, functions with singularities.
Received: 30 July 2025; Accepted: 28 September 2025
Communicated by Dragan S. Djordjević
Research supported by NNSFC(12271133).
Email address: dsyu@hznu.edu.cn (Dansheng Yu)
ORCID iD: https://orcid.org/0000-0003-3378-3251 (Dansheng Yu)

and

$$C_w := \left\{ f \in C\left((0,1)\right) : \lim_{x\to 1} (wf)(x) = \lim_{x\to 0} (wf)(x) = 0 \right\}.$$

The norm in C_w is defined as

$$\left\| wf \right\|_{C_w} := \left\| wf \right\| = \sup_{0\le x\le 1} \left| (wf)(x) \right|.$$

Define

$$W_w^r := \left\{ f \in C_w : f^{(r-1)} \in A.C.\left((0,1)\right), \left\| w\varphi^r f^{(r)} \right\| < \infty \right\}.$$

For $f \in C_w$, define the weighted modulus of smoothness by

$$\omega_\varphi^r (f,t)_w := \sup_{0<h\le t} \left\{ \left\| w\Delta_{h\varphi}^r f \right\|_{\left[16h^2, 1-16h^2\right]} + \left\| w\overrightarrow{\Delta}_h^r f \right\|_{\left[0,16h^2\right]} + \left\| w\overleftarrow{\Delta}_h^r f \right\|_{\left[1-16h^2,1\right]} \right\},$$

where

$$\Delta_{h\varphi}^r f(x) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} f\left(x + \left(\frac{r}{2} - k \right) h\varphi(x) \right),$$

$$\overrightarrow{\Delta}_h^r f(x) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} f\left(x + (r-k)h \right),$$

$$\overleftarrow{\Delta}_h^r f(x) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} f\left(x + -kh \right),$$

and

$$\varphi(x) = \sqrt{x(1-x)}.$$

The weighted K−functional is given by

$$K_{r,\varphi}(f,t^r)_w := \inf\left\{ w\left\| f-g \right\| + t^r \left\| w\varphi^r g^{(r)} \right\| : g \in W_w^r \right\}. \tag{1}$$

It was shown in [9] that

$$K_{r,\varphi}(f,t^r)_w \sim \omega_\varphi^r (f,t)_w. \tag{2}$$

For every $f \in C_w$, define

$$B_n^*(f,x) := (1-x)^n \left(2f\left(\frac{1}{n}\right) - f\left(\frac{2}{n}\right) \right) + \sum_{k=1}^{n-1} f\left(\frac{k}{n}\right) p_{nk}(x)$$
$$+ x^n \left(2f\left(1-\frac{1}{n}\right) - f\left(1-\frac{2}{n}\right) \right).$$

$B_n^*(f)$ is first introduced by Della Vecchia, Mastroianni and Szabados [7], who also studied its properties. Among others, they proved that for $\alpha, \beta > 0$,

$$\left\| w\left(f - B_n^*(f)\right) \right\| \le C\omega_\varphi^2 \left(f, n^{-1/2}\right), \ f \in C_w, \tag{3}$$

$$\left\| w\left(f - B_n^*(f)\right) \right\| = O\left(n^{-\delta/2}\right) \Longleftrightarrow \omega_\varphi^2 (f,h)_w = O\left(h^\delta\right), \ \delta < 2.$$

In this paper, we consider the strong converse inequalities of this operator in C_w. To state our results, we need some notions and notations.

Let $0 \leq \lambda \leq 1$, $0 < \delta < 2$, define

$$\left\|wf\right\|_0 := \sup_{x\in(0,1)} \left|\left(w\varphi^{\delta(\lambda-1)}\right)(x) f(x)\right|,$$

$$\left\|wf\right\|_2 := \sup_{x\in(0,1)} \left|\left(w\varphi^{2+\delta(\lambda-1)}\right)(x) f''(x)\right|,$$

$$C^0_{\lambda,\delta,w} := \left\{f \in C_w : \left\|wf\right\|_0 < \infty\right\},$$

$$C^2_{\lambda,\delta,w} := \left\{f \in C_w : f'' \in C_w, \left\|wf\right\|_2 < \infty\right\},$$

and $K-$functional $K^\delta_\lambda\left(f, t^2\right)_w$ as follows:

$$K^\delta_\lambda\left(f, t^2\right)_w := \inf_{g\in C^2_{\lambda,\delta,w}} \left\{\left\|w(f-g)\right\|_0 + t^2\left\|wg\right\|_2\right\}.$$

Our main result is the following

Theorem 1.1. *Let $0 \leq \lambda \leq 1$, $0 < \delta < 2, \min(\alpha,\beta) > \frac{\delta(1-\lambda)}{2}$. If $f \in C^0_{\lambda,\delta,w}$, then there exists a positive constant K such that*

$$K^\delta_\lambda\left(f, \frac{1}{n}\right)_w \leq C\frac{l}{n}\left(\left\|w\left(B^*_n f - f\right)\right\|_0 + \left\|w\left(B^*_l f - f\right)\right\|_0\right)$$

for $l \geq Kn$.

Taking $\lambda = 1$, by Theorem 1, (2) and (3), we have

Corollary 1.2. *Let $f \in C^0_{\lambda,\delta,w}$. There exists a positive constant K such that*

$$\omega^2_\varphi\left(f, \frac{1}{\sqrt{n}}\right)_w \leq C\frac{l}{n}\left(\left\|w\left(B^*_n f - f\right)\right\|_0 + \left\|w\left(B^*_l f - f\right)\right\|_0\right).$$

for $l \geq Kn$, and

$$\omega^2_\varphi\left(f, \frac{1}{\sqrt{n}}\right)_w \sim \left\|w\left(B^*_n f - f\right)\right\|_0 + \left\|w\left(B^*_{Kn} f - f\right)\right\|_0.$$

2. Auxiliary Lemmas

Let

$$\psi(x) := \begin{cases} 10x^3 - 15x^4 + 6x^5, & 0 \leq x \leq 1, \\ 0, & x \leq 0, \\ 1, & x \geq 1. \end{cases}$$

Define

$$\begin{aligned} F_n(f,x) &:= F_n(x) \\ &= (1-\psi(nx-1))P_1(x) + (1-\psi(nx-n+2))\psi(nx-1)f(x) + \psi(nx-n+2)P_2(x) \\ &= \begin{cases} P_1(x), & 0 \leq x < \frac{1}{n}, \\ (1-\psi(nx-1))P_1(x) + \psi(nx-1)f(x), & \frac{1}{n} \leq x < \frac{2}{n}, \\ f(x), & \frac{2}{n} \leq x < 1-\frac{2}{n}, \\ (1-\psi(nx-n+2))f(x) + \psi(nx-n+2)P_2(x), & 1-\frac{2}{n} \leq x < 1-\frac{1}{n}, \\ P_2(x), & 1-\frac{1}{n} \leq x \leq 1, \end{cases} \end{aligned}$$

where

$$P_1(f,x) := P_1(x) = (2-nx) f\left(\frac{1}{n}\right) + (nx-1) f\left(\frac{2}{n}\right),$$

$$P_2(f,x) := P_2(x) = (2-(1-x)n) f\left(1-\frac{1}{n}\right) + (n(1-x)-1) f\left(1-\frac{2}{n}\right).$$

Then

$$B_n^*(f,x) = B_n(F_n,x).$$

Since both $F_n(f,x)$ and $B_n(f,x)$ reproduce constants and linear functions, we have

$$B_n^*(1,x) = 1,\ B_n^*(t,x) = x.$$

Write $h(x) = (t-x)^2$. It is well known that

$$B_n(h,x) = \frac{\varphi^2(x)}{n}.$$

Thus

$$\begin{aligned} B_n^*(h,x) &= B_n(h,x) + \left(2h\left(\frac{1}{n}\right) - h\left(\frac{2}{n}\right) - x^2\right)(1-x)^n + \left(2h\left(1-\frac{1}{n}\right) - h\left(1-\frac{2}{n}\right) - (1-x)^2\right)x^n \\ &= \frac{\varphi^2(x)}{n} + \frac{2}{n^2}(1-x)^n + \frac{2}{n^2}x^n \\ &=: \frac{\Delta_n^2(x)}{n} \end{aligned} \tag{4}$$

where

$$\Delta_n(x) := \sqrt{\varphi^2(x) + \frac{2}{n}(1-x)^n + \frac{2}{n}x^n}.$$

Write

$$\delta_n(x) := \frac{1}{\sqrt{n}} + \varphi(x).$$

Then

$$\Delta_n(x) \sim \delta_n(x) \geq \varphi(x).$$

Lemma 2.1. *([25]) For any $u, v \geq 0$, we have*

$$\sum_{k=1}^{n-1} \left(\frac{k}{n}\right)^{-u} \left(1-\frac{k}{n}\right)^{-v} p_{nk}(x) \leq C x^{-u} (1-x)^{-v}.$$

Lemma 2.2. *([24]) Let $\beta > 0$. Then,*

$$\sum_{k=0}^{n} \left|x - \frac{k}{n}\right|^{\beta} p_{nk}(x) \leq \left(\frac{\varphi(x)}{\sqrt{n}}\right)^{\beta} \tag{5}$$

$$\sum_{k=0}^{n} \left|x - \frac{k}{n}\right|^{\beta+1} p_{nk}(x) \leq C_\beta \left(\frac{\varphi(x)}{\sqrt{n}}\right)^{\beta} \frac{\delta_n(x)}{\sqrt{n}} \tag{6}$$

for $0<\beta\leq 1$, *and*

$$\sum_{k=0}^{n}\left|x-\frac{k}{n}\right|^{\beta}p_{nk}(x)\leq C_{\beta}\frac{\varphi(x)}{\sqrt{n}}\left(\frac{\delta_n(x)}{\sqrt{n}}\right)^{\beta-1} \tag{7}$$

for $\beta>1$.

Lemma 2.3. *Let* $0\leq\lambda\leq 1, 0<\delta<2, \min(\alpha,\beta)>\frac{\delta(1-\lambda)}{2}, f^{(i)}(x)\in C^0_{\lambda,\delta,w}, i=0,1,2,3, \varphi^3 f'''\in C^*_w$. *Then*

$$\left\|w\left(B^*_n(f)-f-\frac{\Delta^2_n(x)}{2n}f''\right)\right\|_0\leq C\left(n^{-\frac{3}{2}}\left\|w\varphi^3 f'''\right\|_0+n^{-2}\left\|w\varphi^2 f'''\right\|_0\right).$$

Proof. Set

$$G_{n,x}(t):=\int_x^t (t-u)^2 f'''(u)\,du.$$

By Taylor's expansion

$$f(t)=f(x)+f'(x)(t-x)+\frac{1}{2}f''(x)(t-x)^2+\frac{1}{2}G_{n,x}(t),$$

we have

$$F_n(G_{n,x},t)=\begin{cases}(2-nt)\,G_{n,x}\left(\frac{1}{n}\right)+(nt-1)\,G_{n,x}\left(\frac{2}{n}\right), & 0\leq t\leq\frac{1}{n},\\ (1-\psi(nt-1))\,P_1(G_{n,x},t)+\psi(nt-1)\,G_{n,x}(t), & \frac{1}{n}\leq t\leq\frac{2}{n},\\ G_{n,x}(t), & \frac{2}{n}\leq t\leq 1-\frac{2}{n},\\ (1-\psi(nt-n+2))\,G_{n,x}(t)+\psi(nt-n+2)\,P_2(G_{n,x},t), & 1-\frac{2}{n}\leq t\leq 1-\frac{1}{n},\\ (2-(1-t)\,n)\,G_{n,x}\left(1-\frac{1}{n}\right)+(n(1-t)-1)\,G_{n,x}\left(1-\frac{2}{n}\right), & 1-\frac{1}{n}\leq t\leq 1.\end{cases}$$

By the definition of $B^*_n(f,x)$ and (4),

$$\begin{aligned} w(x)\,\varphi^{\delta(\lambda-1)}(x)\left|B^*_n(f,x)-f(x)-\frac{\Delta^2_n(x)}{2n}f''(x)\right| &= w(x)\,\varphi^{\delta(\lambda-1)}(x)\left|B^*_n(G_{n,x},x)\right| \\ &\leq\sum_{i=1}^{5}I_i(x), \end{aligned} \tag{8}$$

where

$$I_1(x):=2w(x)\,\varphi^{\delta(\lambda-1)}(x)\,(1-x)^n\left|G_{n,x}\left(\frac{1}{n}\right)\right|,$$

$$I_2(x):=w(x)\,\varphi^{\delta(\lambda-1)}(x)\,(1-x)^n\left|G_{n,x}\left(\frac{2}{n}\right)\right|,$$

$$I_3(x):=w(x)\,\varphi^{\delta(\lambda-1)}(x)\sum_{k=1}^{n-1}\left|G_{n,x}\left(\frac{k}{n}\right)\right|p_{nk}(x),$$

$$I_4(x):=2w(x)\,\varphi^{\delta(\lambda-1)}(x)\,x^n\left|G_{n,x}\left(1-\frac{1}{n}\right)\right|,$$

$$I_5(x):=w(x)\,\varphi^{\delta(\lambda-1)}(x)\,x^n\left|G_{n,x}\left(1-\frac{2}{n}\right)\right|.$$

Now, we only need to verify that

$$I_i(x) \le C\left(n^{-\frac{3}{2}}\left\|w\varphi^3 f'''\right\|_0 + n^{-2}\left\|w\varphi^2 f'''\right\|_0\right),\ i = 1,2,3,4,5. \tag{9}$$

We divide the proof to the following four steps.

Step 1. Estimate of $I_1(x)$.

If $x \in E_n = \left[\frac{1}{n}, 1-\frac{1}{n}\right]$, without loss of generality, assume that $x \in \left[\frac{1}{n}, \frac{1}{2}\right]$, then by using the inequality ([9]):

$$\frac{|t-v|}{\varphi^2(v)} \le \frac{|t-x|}{\varphi^2(x)}$$

for any v between x and t, we have

$$\begin{aligned}
I_1(x) &\le 2w(x)\varphi^{\delta(\lambda-1)}(x)(1-x)^n \int_{\frac{1}{n}}^{x}\left(u-\frac{1}{n}\right)^2\left|f'''(u)\right|du \\
&\le 2\left\|w\varphi^3 f'''\right\|_0 w(x)\varphi^{\delta(\lambda-1)}(x)(1-x)^n \int_{\frac{1}{n}}^{x}\frac{\left(u-\frac{1}{n}\right)^2}{(w\varphi^{3+\delta(\lambda-1)})(u)}du \\
&\le C\left\|w\varphi^3 f'''\right\|_0 w(x)(1-x)^n \int_{\frac{1}{n}}^{x}\frac{\left(u-\frac{1}{n}\right)^2}{(w\varphi^3)(u)}du \\
&\le C\left\|w\varphi^3 f'''\right\|_0 w(x)\varphi^{-3}(x)(1-x)^n\left(x-\frac{1}{n}\right)^{3/2}\int_{\frac{1}{n}}^{x}\frac{\left(u-\frac{1}{n}\right)^{1/2}}{w(u)}du \\
&\le C\left\|w\varphi^3 f'''\right\|_0 w(x)\varphi^{-3}(x)(1-x)^n x^3/w(1/n) \\
&\le Cn^{-\frac{3}{2}}\left\|w\varphi^3 f'''\right\|_0 (nx)^{\alpha+3/2}(1-x)^n \\
&\le Cn^{-\frac{3}{2}}\left\|w\varphi^3 f'''\right\|_0 (nx_0)^{\alpha+3/2}(1-x_0)^n \\
&\le Cn^{-\frac{3}{2}}\left\|w\varphi^3 f'''\right\|,
\end{aligned}$$

where

$$x_0 := \frac{1}{1+\frac{n}{\alpha+\frac{3}{2}}}.$$

If $x \in E_n^c := \left(0, \frac{1}{n}\right] \cup \left[1-\frac{1}{n}, 1\right)$, say $x \in \left(0, \frac{1}{n}\right]$ (when $x \in \left[1-\frac{1}{n}, 1\right)$, it can be proved similarly), then

$$\begin{aligned}
I_1(x) &\le C\left\|w\varphi^2 f'''\right\|_0 w(x)\varphi^{\delta(\lambda-1)}(x)(1-x)^n \int_{x}^{\frac{1}{n}}\frac{\left(\frac{1}{n}-u\right)^2}{(w\varphi^{2+\delta(\lambda-1)})(u)}du \\
&\le Cn^{-2}\left\|w\varphi^2 f''' w(x)\varphi^{\delta(\lambda-1)}(x)\right\|_0 \int_{x}^{\frac{1}{n}}\frac{1}{u^{\alpha+1+\delta(\lambda-1)/2}}du \\
&\le Cn^{-2}\left\|w\varphi^2 f'''\right\|_0
\end{aligned}$$

by the fact that $\alpha > \delta(1-\lambda)/2$.

Step 2. Estimate of $I_2(x)$.

If $x \in \left[\frac{1}{n}, \frac{2}{n}\right]$, then

$$w(x) \sim w\left(\frac{1}{n}\right) \sim w(u),\ \varphi(x) \sim \varphi\left(\frac{1}{n}\right) \sim \varphi(u)$$

for any $u \in \left[\frac{1}{n}, \frac{2}{n}\right]$. Therefore

$$\begin{aligned} I_2(x) &\leq 2w(x)\varphi^{\delta(\lambda-1)}(x)(1-x)^n \int_x^{\frac{2}{n}} \left(u-\frac{1}{n}\right)^2 \left|f'''(u)\right| du \\ &\leq C\left\|w\varphi^3 f'''\right\|_0 (1-x)^n x^3 \varphi^{-3}(x) \\ &\leq Cn^{-\frac{3}{2}} \left\|w\varphi^3 f'''\right\|_0 . \end{aligned}$$

If $x \in \left[\frac{2}{n}, 1-\frac{1}{n}\right] \cup E_n^c$, then in a similar way to estimate $I_1(x)$, we can also deduce that

$$I_2(x) \leq Cn^{-\frac{3}{2}} \left\|w\varphi^3 f'''\right\|_0 .$$

Step 3. Estimate of $I_3(x)$.

If $x \in E_n$, we have

$$\frac{\left(\frac{k}{n}-u\right)^2}{\left(w\varphi^{3+\delta(\lambda-1)}\right)(u)} du \leq \frac{1}{\left(w\varphi^{3+\delta(\lambda-1)}\right)(x)} + \frac{1}{\left(w\varphi^{3+\delta(\lambda-1)}\right)\left(\frac{k}{n}\right)}$$

for any u between x and $\frac{k}{n}$, $k = 1, 2, \cdots, n-1$. Therefore,

$$\begin{aligned} I_3(x) &\leq 2\left\|w\varphi^3 f'''\right\|_0 w(x)\varphi^{\delta(\lambda-1)}(x) \sum_{k=1}^{n-1} \left| \int_x^{\frac{k}{n}} \frac{\left(\frac{k}{n}-u\right)^2}{\left(w\varphi^{3+\delta(\lambda-1)}\right)(u)} du \right| p_{nk}(x) \\ &\leq C\left\|w\varphi^3 f'''\right\|_0 w(x)\varphi^{\delta(\lambda-1)}(x) \sum_{k=1}^{n-1} \left|\frac{k}{n}-x\right|^3 \left(\frac{1}{\left(w\varphi^{3+\delta(\lambda-1)}\right)(x)} + \frac{1}{\left(w\varphi^{3+\delta(\lambda-1)}\right)\left(\frac{k}{n}\right)}\right) p_{nk}(x) \\ &:= I_{31}(x) + I_{32}(x). \end{aligned} \tag{10}$$

Applying (7) and the fact $\delta_n(x) \sim \varphi(x)$, $x \in E_n$, we have

$$\begin{aligned} I_{31}(x) &\leq C\left\|w\varphi^3 f'''\right\|_0 \varphi^{-3}(x) \sum_{k=1}^{n-1} \left|\frac{k}{n}-x\right|^3 p_{nk}(x) \\ &\leq Cn^{-\frac{3}{2}} \left\|w\varphi^3 f'''\right\|_0 . \end{aligned} \tag{11}$$

By Hölder's inequality, (7) and Lemma 1, we derive

$$\begin{aligned} I_{32}(x) &\leq Cw(x)\varphi^{\delta(\lambda-1)}(x)\left\|w\varphi^3 f'''\right\|_0 \left(\sum_{k=1}^{n-1} \left|\frac{k}{n}-x\right|^6 p_{nk}(x)\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n-1} \left(\frac{1}{\left(w\varphi^{3+\delta(\lambda-1)}\right)\left(\frac{k}{n}\right)} p_{nk}(x)\right)^2\right)^{\frac{1}{2}} \\ &\leq Cw(x)\varphi^{\delta(\lambda-1)}(x)\left\|w\varphi^3 f'''\right\|_0 \left(\frac{\varphi(x)}{\sqrt{n}}\right)^3 \frac{1}{\left(w\varphi^{3+\delta(\lambda-1)}\right)(x)} \\ &\leq Cn^{-\frac{3}{2}} \left\|w\varphi^3 f'''\right\|. \end{aligned} \tag{12}$$

Combining (10), (11) and (12), we get

$$|I_3(x)| \leq Cn^{-\frac{3}{2}} \left\|w\varphi^3 f'''\right\|, \; x \in E_n. \tag{13}$$

If $x \in E_n^c$, we have

$$\begin{aligned}
|I_3(x)| &\leq Cw(x)\varphi^{\delta(\lambda-1)}(x)\|w\varphi^2 f'''\|_0 \sum_{k=1}^{n-1} p_{nk}(x)\int_x^{\frac{k}{n}} \frac{\left(\frac{k}{n}-u\right)^2}{(w\varphi^{2+\delta(\lambda-1)})(u)}du \\
&\leq Cw(x)\varphi^{\delta(\lambda-1)}(x)\|w\varphi^2 f'''\|_0 \sum_{k=1}^{n-1}\left(\frac{k}{n}-x\right)^2 p_{nk}(x)\int_x^{\frac{k}{n}} \frac{1}{(w\varphi^{2+\delta(\lambda-1)})(u)}du \\
&\leq C\|w\varphi^2 f'''\|_0 \sum_{k=1}^{n-1}\left(\frac{k}{n}-x\right)^2 p_{nk}(x) \\
&\leq C\|w\varphi^2 f'''\|_0 \frac{\varphi^2(x)}{n} \\
&\leq Cn^{-2}\|w\varphi^2 f'''\|_0, \quad x \in E_n^c. \qquad (14)
\end{aligned}$$

By (13) and (14), we prove that (9) holds for $i = 3$.

Step 4. Estimate of $I_4(x)$ and $I_5(x)$.

We can be prove (9) holds for $i = 4$ and $i = 5$ by similar deductions as $I_1(x)$ and $I_2(x)$. □

Lemma 2.4. *Let* $0 \leq \lambda \leq 1, 0 < \delta < 2, f \in C^2_{\lambda,\delta,w}$. *Then*

$$\left\|w\varphi^3 B_n^{*'''} f\right\|_0 \leq C_1\sqrt{n}\left\|w\varphi^2 f''\right\|_0. \qquad (15)$$

$$\left\|w\varphi^2 B_n^{*'''} f\right\|_0 \leq C_2 n\left\|wf\right\|_2. \qquad (16)$$

Proof. Recall that [7]

$$\begin{aligned}
B_n^{*''}(f,x) &= n(n-1)\sum_{k=0}^{n-2} p_{n-2,k}(x)\overrightarrow{\Delta}^2_{1/n}F_n\left(\frac{k}{n}\right) \\
&= n(n-1)\sum_{k=1}^{n-3} p_{n-2,k}(x)\overrightarrow{\Delta}^2_{1/n}f\left(\frac{k}{n}\right). \qquad (17)
\end{aligned}$$

Then

$$\begin{aligned}
B_n^{*'''}(f,x) &= n(n-1)\sum_{k=1}^{n-3} p'_{n-2,k}(x)\overrightarrow{\Delta}^2_{1/n}f\left(\frac{k}{n}\right) \\
&= \frac{n(n-1)(n-2)}{\varphi^2(x)}\sum_{k=1}^{n-3}\left(\frac{k}{n-2}-x\right)p_{n-2,k}(x)\overrightarrow{\Delta}^2_{1/n}f\left(\frac{k}{n}\right).
\end{aligned}$$

By Lemma 1 and Hölder's inequality, we have

$$\begin{aligned}
&\left(w\varphi^{3+\delta(\lambda-1)}\right)(x)\left|B_n^{*'''}(f,x)\right| \\
&\leq Cn^3\left(w\varphi^{1+\delta(\lambda-1)}\right)(x)\left|\sum_{k=1}^{n-3}\left(\frac{k}{n-2}-x\right)p_{n-2,k}(x)\overrightarrow{\Delta}^2_{1/n}f\left(\frac{k}{n}\right)\right| \\
&\leq Cn\left(w\varphi^{1+\delta(\lambda-1)}\right)(x)\left\|\varphi^2 f''\right\|_0\left(\sum_{k=1}^{n-3}\left(\frac{k}{n-2}-x\right)^2 p_{n-2,k}(x)\right)^{1/2}\left(\sum_{k=1}^{n-3}\frac{1}{(w\varphi^{2+\delta(\lambda-1)})^2\left(\frac{k}{n}\right)}p_{n-2,k}(x)\right)^{1/2} \\
&\leq C\sqrt{n}\left\|w\varphi^2 f''\right\|_0,
\end{aligned}$$

which proves (15).

(16) can be proved similarly. □

Lemma 2.5. *Let* $0 \le \lambda \le 1, 0 < \delta < 2, f \in C^0_{\lambda,\delta,w}$. *Then*

$$\left\|wB_n^*(f)\right\|_2 \le Cn\left\|wf\right\|_0. \tag{18}$$

Proof. If $x \in E_n^c = \left(0, \frac{1}{n}\right] \cup \left[1-\frac{1}{n}, 1\right)$, then by (17), Lemma 1,

$$\begin{aligned}
&\left|\left(w\varphi^{2+\delta(\lambda-1)}\right)(x)\, B_n^{*\prime\prime}(f,x)\right| \\
&= \left(w\varphi^{2+\delta(\lambda-1)}\right)(x)\, n(n-1)\left|\sum_{k=1}^{n-3} p_{n-2,k}(x)\, \overrightarrow{\Delta}^2_{1/n} f\left(\frac{k}{n}\right)\right| \\
&\le Cn^2\left(w\varphi^{2+\delta(\lambda-1)}\right)(x)\left\|wf\right\|_0 \sum_{k=1}^{n-3}\frac{p_{n-2,k}(x)}{\left(w\varphi^{\delta(\lambda-1)}\right)\left(\frac{k}{n}\right)} \\
&\le Cn^2\left(w\varphi^{2+\delta(\lambda-1)}\right)(x)\left\|f\right\|_0\left(\sum_{k=1}^{n-3}\frac{p_{n-2,k}(x)}{\left(w\left(\frac{k}{n}\right)\right)^2}\right)^{1/2}\left(\sum_{k=1}^{n-3}\left(\delta_n^{\delta(1-\lambda)}\left(\frac{k}{n}\right)\right)^2 p_{n-2,k}(x)\right)^{1/2} \\
&\le C\delta_n^{\delta(1-\lambda)}(x)\,\varphi^{2+\delta(\lambda-1)}(x)\left\|wf\right\|_0 \\
&\le Cn\left\|wf\right\|_0,
\end{aligned}$$

where in the third inequality, we used the following inequality (see [23])

$$\sum_{k=0}^{n}\left(\delta_n\left(\frac{k}{n}\right)\right)^{\gamma} p_{nk}(x) = O\left((\delta_n(x))^{\gamma}\right),\ x \in [-1,1], \gamma \in \mathbf{R}. \tag{19}$$

Now, we consider the case when $x \in E_n$. Since ([7])

$$\begin{aligned}
B_n^{*\prime\prime}(f,x) = &-\frac{n}{\varphi^2(x)}B_n(F_n,x) + \left(\frac{n}{\varphi^2(x)}\right)^2\sum_{k=0}^{n}p_{nk}(x)\,F_n\left(\frac{k}{n}\right)\left(\frac{k}{n}-x\right)^2 \\
&-\frac{n(1-2x)}{x^2(1-x)^2}\sum_{k=0}^{n}p_{nk}(x)\,F_n\left(\frac{k}{n}\right)\left(\frac{k}{n}-x\right),
\end{aligned}$$

then

$$\begin{aligned}
\left|\left(w\varphi^{2+\delta(\lambda-1)}\right)(x)\,B_n^{*\prime\prime}(f,x)\right| &\le n\left(w\varphi^{\delta(\lambda-1)}\right)(x)\left|B_n(F_n,x)\right| \\
&\quad+\left(w\varphi^{-2+\delta(\lambda-1)}\right)(x)\sum_{k=0}^{n}p_{nk}(x)\left|F_n\left(\frac{k}{n}\right)\right||k-nx| \\
&\quad+\left(w\varphi^{-2+\delta(\lambda-1)}\right)(x)\sum_{k=0}^{n}p_{nk}(x)\left|F_n\left(\frac{k}{n}\right)\right||k-nx|^2 \\
&=: A_1(x)+A_2(x)+A_3(x).
\end{aligned} \tag{20}$$

Recall that ([20])

$$\sum_{k=0}^{n}\frac{1}{w^2\left(\frac{k^*}{n}\right)}p_{nk}(x) \le Cw^{-2}(x), \tag{21}$$

where $k^* = 1$ for $k = 0$, $k^* = n-1$ for $k = n$, and $k^* = k$ for $1 < k < n$. By (19) and the fact $\delta_n(x) \sim \varphi(x)$ for $x \in E_n$, we have

$$\begin{aligned}
A_1(x) &\le Cn\left(w\varphi^{\delta(\lambda-1)}\right)(x)\left\|wf\right\|_0 \sum_{k=0}^{n} \frac{1}{w\left(\frac{k^*}{n}\right)\delta_n^{\delta(\lambda-1)}\left(\frac{k^*}{n}\right)} p_{nk}(x) \\
&\le Cn\left(w\varphi^{\delta(\lambda-1)}\right)(x)\left\|wf\right\|_0 \left(\sum_{k=0}^{n} \frac{1}{w^2\left(\frac{k^*}{n}\right)} p_{nk}(x)\right)^{1/2} \left(\sum_{k=0}^{n} \frac{1}{\delta_n^{2\delta(\lambda-1)}\left(\frac{k}{n}\right)} p_{nk}(x)\right)^{1/2} \\
&\le Cn\left\|wf\right\|_0 .
\end{aligned} \tag{22}$$

By Hölder's inequality, (2.4) and (19) again,

$$\begin{aligned}
\sum_{k=0}^{n} p_{nk}(x) \frac{|k-nx|^2}{\varphi^{2\delta(\lambda-1)}\left(\frac{k^*}{n}\right)} &\le \left(\sum_{k=0}^{n} p_{nk}(x)|k-nx|^4\right)^{1/2} \left(\sum_{k=0}^{n} p_{nk}(x) \frac{1}{\varphi^{4\delta(\lambda-1)}\left(\frac{k^*}{n}\right)}\right)^{1/2} \\
&\le Cn\varphi^{2+2\delta(1-\lambda)}(x).
\end{aligned} \tag{23}$$

Now, by Hölder's inequality, (21) and (23), we have

$$\begin{aligned}
A_2(x) &\le C\left(w\varphi^{-2+\delta(\lambda-1)}\right)(x)\left\|wf\right\|_0 \left(\sum_{k=0}^{n} \frac{1}{w^2\left(\frac{k^*}{n}\right)} p_{nk}(x)\right)^{1/2} \left(\sum_{k=0}^{n} p_{nk}(x) \frac{|k-nx|^2}{\varphi^{2\delta(\lambda-1)}\left(\frac{k^*}{n}\right)}\right)^{1/2} \\
&\le C\sqrt{n}\varphi^{-1}(x)\left\|wf\right\|_0 \\
&\le Cn\left\|wf\right\|_0 .
\end{aligned} \tag{24}$$

Analogously,

$$A_3(x) \le Cn\left\|wf\right\|_0 . \tag{25}$$

Combining (20), (22), (24) and (25), we get (18) for $x \in E_n$. □

3. Proof of results

We follow the proof strategy of Guo and Qi [11].

By the proof of (22), we observe that

$$\left\|wB_n^* f\right\|_0 \le C\left\|wf\right\|_0 , \tag{26}$$

which implies that $B_n^* f - f \in C^0_{\lambda,\delta,w}$ if only $f \in C^0_{\lambda,\delta,w}$. Write $B_n^{*2} := B_n^*(B_n^* f, x)$. Then

$$\begin{aligned}
K_\lambda^\delta\left(f, \frac{1}{n}\right)_w &\le \left\|w\left(B_n^{*2} f - f\right)\right\|_0 + \frac{1}{n}\left\|B_n^{*2} f\right\|_2 \\
&\le C\left(\left\|w\left(B_n^* f - f\right)\right\|_0 + \frac{1}{n}\left\|B_n^{*2} f\right\|_2\right).
\end{aligned} \tag{27}$$

By Lemma 2,

$$\begin{aligned}
&\left\|w\left(B_l\left(B_n^{*2}(f)\right) - B_n^{*2}(f) - \frac{\Delta_l^2(x)}{2l}\left(B_n^{*2}\right)''(f)\right)\right\|_0 \\
&\le C_1\left(l^{-\frac{3}{2}}\left\|w\varphi^3\left(B_n^{*2}\right)'''(f)\right\|_0 + l^{-2}\left\|w\varphi^2\left(B_n^{*2}\right)'''(f)\right\|_0\right),
\end{aligned}$$

which, combining with the fact $\Delta_l(x) > \varphi(x)$, implies that

$$
\begin{aligned}
\frac{1}{2l}\left\|wB_n^{*2}f\right\|_2 &\le \left\|w\frac{\Delta_l^2}{2l}\varphi^{\delta(\lambda-1)}\left(B_n^{*2}\right)''(f)\right\|_0 \\
&\le \left\|w\left(B_l\left(B_n^{*2}(f)\right)-B_n^{*2}(f)\right)\right\|_0 \\
&\quad + C_1\left(l^{-\frac{3}{2}}\left\|w\varphi^3\left(B_n^{*2}\right)'''(f)\right\|_0 + l^{-2}\left\|w\varphi^2\left(B_n^{*2}\right)'''(f)\right\|_0\right) \\
&\le \left\|wB_l^*\left(B_n^{*2}f-B_n^*f\right)\right\|_0 + \left\|wB_l^*\left(B_n^*f-f\right)\right\|_0 + \left\|w\left(B_l^*f-f\right)\right\|_0 \\
&\quad + \left\|w\left(f-B_n^*f\right)\right\|_0 + \left\|w\left(B_n^*f-B_n^{*2}f\right)\right\|_0 \\
&\quad + C_1C_2l^{-\frac{3}{2}}\sqrt{n}\left(\left\|w\left(B_n^*f-B_n^{*2}f\right)\right\|_2 + \left\|wB_n^{*2}f\right\|_2\right) \\
&\quad + C_1l^{-2}\left\|w\varphi^2\left(B_n^{*2}\right)'''(f)\right\|_0 \quad \text{(by Lemma 2.4)} \\
&\le C_3\left(\left\|w\left(B_l^*f-f\right)\right\|_0 + \left\|w\left(B_n^*f-f\right)\right\|_0\right) \\
&\quad + C_1C_2C_4l^{-\frac{3}{2}}\sqrt{n}\left\|w\left(B_n^*f-f\right)\right\|_0 + \left(C_1C_5l^{-2}n + C_1C_2l^{-\frac{3}{2}}\sqrt{n}\right)\left\|wB_n^{*2}f\right\|_2 \\
&\le C_6\left(\left\|w\left(B_l^*f-f\right)\right\|_0 + \left\|w\left(B_n^*f-f\right)\right\|_0\right) + \frac{1}{4l}\left\|wB_n^{*2}f\right\|_2
\end{aligned}
$$

for $l \ge Kn$ with K a large enough constant. Therefore,

$$
\frac{1}{4l}\left\|wB_n^{*2}f\right\|_2 \le C_6\left(\left\|w\left(B_l^*f-f\right)\right\|_0 + \left\|w\left(B_n^*f-f\right)\right\|_0\right),
$$

which combining with (27), proves Theorem 1.

References

[1] T. Acar, A. Aral, V. Gupta, *On approximation properties of a new type of Bernstein-Durrmeyer operators*, Math. Slovaca, **65**(2015), 1107-1122.
[2] T. Acar, A. Aral, *Approximation properties of two dimensional Bernstein-Stancu-Chlodowsky operators*, Matematiche (Catania), **68**(2013), 15-31.
[3] A. Alotaibi, M. Nasiruzzaman, S. A. Mohiuddine, *On the convergence of Bernstein-Kantorovich-Stancu shifted knots operators involving Schur parameter*, Complex Anal. Oper. Theory, **18**(2024). Paper No. 4, 22 pp.
[4] A. Aral, *Weighted approximation: Korovkin and quantitative type theorems*, Modern Mathematical Methods, **1**(2023), 1-21.
[5] A. Aral, T. Acar, *Weighted approximation by new Bernstein-Chlodowsky-Gadjiev operators*, Filomat, **27**(2013), 371-380.
[6] Q. B. Cai, G. R. Zhou, *Approximation properties of (λ,μ)-Bernstein-Durrmeyer operators*, Math. Methods Appl. Sci., **48**(2025), 5946-5953.
[7] B. Della Vecchia, G. Mastroianni, J. Szabados, *Weighted approximation of functions with endpoints or inner singularities by Bernstein operators*, Acta Math. Hungar., **103**(2004), 19-41.
[8] Z. Ditzian and K. G. Ivanov, *Strong converse inequalities*, J. Anal. Math., **61**(1993), 61-111.
[9] Z. Ditzian and V. Totik, *Moduli of Smoothness*, Springer-Verlag, Berlin, 1987.
[10] Z. Finta, *On converse approximation theorems*, J. Math. Anal. Appl., **312**(2005), 159-180.
[11] S. S. Guo and Q. L. Qi, *Strong converse inequality for Bernstein operators*, Acta Math. Sinica, **46**(2003), 891-896. (in Chinese)
[12] V. Gupta, R. P. Agarwal, *Convergence Estimates in Approximation Theory*, Springer, Switzerland, 2014.
[13] V. Gupta, R. Gupta, *Convergence estimates for some composition operators*, Constr. Math. Anal., **7**(2024), 69-76.
[14] V. Gupta, *Approximation properties by Bernstein-Durrmeyer type operators*, Complex Anal. Op. Theory, **7**(2013), 363-374.
[15] A. Kgjla, T. Acar, *Blending type approximation by generalized Bernstein-Durrmeyer type operators*, Miskolc Mathematical Notes, **19**(2018), 319-326.
[16] A. Kgjla, T. Acar, *Bezier-Bernstein-Durrmeyer type operators*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, **114**(2020), Article 31.
[17] G. G. Lorentz, *Bernstein Polynomials*, 2nd edn, Chelsea, New York, 1986.
[18] N. I. Mahmudov, *Approximation by Bernstein-Durrmeyer-type operators in compact disks*, Appl. Math. Letter, **24**(2011), 1231-1238.
[19] F. F. Wang, D. S.Yu, *On approximation of Bernstein-Stancu operators in movable interval*, Appl. Math. J. Chinese Univ., **40**(2025),85-101.

[20] D. S. Yu, *Weighted approximation of functions with singularities by combinations of Bernstein operators*, Appl. Math. Comput., **206**(2008), 906-918.
[21] D. S. Yu, Z. J. Pang, *On Approximation by Stancu variant of Bernstein-Durrmyer-type operators in movable compact disks*, Demonstratio Math., **58**(2025),20240092
[22] D. S. Yu and D. J. Zhao, *Approximation of functions with singularities by truncated Bernstein operators*, Southeast Asian Bull. Math., **30**(2006), 1178-1189.
[23] D. S. Yu, S. P. Zhou, *Global and pointwise estimate for approximation by rational functions with polynomials of positive coefficients as the denominators*, Acta Math. Scientia, **31**(2011), 305-319.
[24] D. S. Yu, P. Zhou, *Fractal approximation by Bernstein operators*, Filomat, **38**(2024),5499-5506.
[25] D. X. Zhou, *Rate of convergence for Bernstein operators with Jacobi weights*, Acta Math. Sinica, **35**(1992), 331-338.