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Analysis on Milne-type inequalities through generalized convexity with applications

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Abstract. The development of precise fractional integral inequalities is crucial for advancing mathematical methods, with convexity theory offering enhanced insights into their scope and applications. In this study, we establish new identity involving the Caputo-Fabrizio fractional integral operator. Based on this identity, we derive several Milne-type integral inequalities for (s, m)-convex functions. Our results generalize numerous classical inequalities and extend the analysis to include inequalities for bounded and Lipschitzian functions. Additionally, we explore applications of these inequalities to special means and q-digamma functions. We also present graphical representations to demonstrate the behavior and significance of the derived inequalities.

1. Introduction

Integral inequalities are versatile tools with broad applications in mathematical analysis, spanning approximation theory, spectral analysis, statistical analysis, and distribution theory. Their utility lies in providing rigorous bounds and estimates, which are essential for understanding the behavior of mathematical models and solving a wide range of problems in diverse fields. Research on numerical integration and error bounds holds a significant position in mathematical literature. Investigations into inequalities aim to establish error bounds for different classes of functions, including bounded functions, Lipschitzian functions, functions of bounded variation, among others. These error bounds are crucial for understanding the accuracy and reliability of numerical integration methods. Furthermore, researchers have explored error bounds for functions with varying degrees of differentiability, ranging from differentiable to n-times differentiable mappings. By leveraging concepts from fractional calculus, authors have derived new bounds, expanding the scope of error analysis in numerical integration.

In recent years, there has been a notable focus on inequalities of the trapezoid, midpoint, Simpson types, and Milne type. Many researchers have contributed to extending and generalizing these integral

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inequalities, thereby enhancing our understanding of numerical integration methods and improving their applicability across diverse mathematical contexts. For instance, fractional integral inequalities have garnered significant attention from mathematicians and scholars, who are actively engaged in exploring and refining techniques to approximate fractional versions using a wide range of convexity principles. Desta et al. [1] established a Milne-type inequality for twice-differentiable convex functions. Mateen et al. [2] presented Milne's-rule-type inequalities for convex functions, accompanied by a computational analysis within the framework of quantum calculus. Ying et al. [3] investigated conformable fractional Milne-type inequalities. Román-Flores et al. [4] explored Milne-type inequalities in the context of interval orders. Demir et al. [5] introduced a new approach to Milne-type inequalities based on the proportional Caputo-Hybrid operator. Arslan et al. [6] derived refined error estimates for Milne-Mercer-type inequalities for three-times-differentiable functions, including a detailed error analysis and related applications. Dragomir et al. [7] presented error estimates for the trapezoidal formula and Cerone et al. [8] investigated trapezoidal-type rules and established explicit bounds using modern inequality theory. Alomari et al. [9] discussed Lipschitzian functions in the context of the generalized trapezoidal inequality. Dragomir et al. [10] studied functions of bounded variation in the context of the trapezoid formula. Sarikaya et al. [11] attained new inequalities of Simpson and trapezoid type for functions with convex second derivatives. M. U. D. Junjua et al. [12] obtained new general identity for twice differentiable functions after replace different parameters give different inequalities. Explored fractional analogs of established results [13, 14]. Established Simpson-type inequalities for differentiable convex mappings, s-convex functions, extended-(s,m) convex mappings, bounded functions, twice differentiable convex functions, and fractional integrals [15, 16, 17, 18, 19, 20]. A notable parallel exists between Milne's formula, an open-type Newton-Cotes formula, and Simpson's formula, a closed-type Newton-Cotes formula. This parallel arises from the fact that both formulas hold under similar conditions, despite their structural differences.

Theorem 1.1. *[21] Suppose that $F : [w,\Pi] \to \mathbb{R}$ is a four-times continuously differentiable function on (w,Π), and let $\left\|F^{(4)}\right\|_\infty := \sup_{x\in(w,\Pi)} \left|F^{(4)}(x)\right| < \infty$. Then, one has the inequality:*

$$\left|\frac{1}{3}\left[2F(w) - F\left(\frac{w+\Pi}{2}\right) + 2F(\Pi)\right] - \frac{1}{\Pi - w}\int_w^\Pi F(x)\,dx\right| \leq \frac{7(\Pi - w)^4}{23040}\left\|F^{(4)}\right\|_\infty. \tag{1}$$

In recent years, mathematicians have significantly expanded and refined the role of convex functions in the theory of inequalities. A well-known definition of convex functions and generalized is expressed as follows:

Definition 1.2. *[22] Let $F : I \subseteq \mathbb{R} \to \mathbb{R}$ be a real-valued function defined on a closed interval I in the real line. We say that F is convex on I, if for any $w,\Pi \in I$ and any $\Upsilon \in [0,1]$, the following inequality holds:*

$$F(\Upsilon w + (1-\Upsilon)\Pi) \leq \Upsilon F(w) + (1-\Upsilon)F(\Pi).$$

Definition 1.3. *[23] A function $F : I \subset [0,\infty) \to \mathbb{R}$ is said to be s-convex in the second sense on I, if*

$$F(\Upsilon w + (1-\Upsilon)\Pi) \leq \Upsilon^s F(w) + (1-\Upsilon)^s F(\Pi),$$

holds for all $w,\Pi \in I$ and $\Upsilon \in [0,1]$ for some fixed $s \in (0,1]$.

In the year 2011, Park et al. [24] introduced and investigated the term (s,m)-convexity.

Definition 1.4. *[24] A function $F : I \subset [0,\infty) \to \mathbb{R}$ is said to be (s,m)-convex in the second sense on I, if*

$$F(\Upsilon w + m(1-\Upsilon)\Pi) \leq \Upsilon^s F(w) + m(1-\Upsilon)^s F(\Pi),$$

holds for all $w,\Pi \in I$ and $\Upsilon \in [0,1]$ for some fixed $s \in (0,1]$, and $m \in [0,1]$.

In the field of fractional analysis, researchers have focused on developing new operators and modeling problems based on their unique characteristics. Singularity and locality are key features that differentiate these operators from each other. The kernel expression of these operators often involves functions such as power laws, exponential functions, or Mittag–Leffler functions. The Caputo-Fabrizio operator stands out due to its non-singular kernel. This operator has been applied in various studies addressing modeling problems and real-world issues. Its definition is particularly effective in explaining heterogeneity and systems with diverse scales and memory effects. We recall the well-known definitions of fractional operator as fellows.

Definition 1.5. *[25] Suppose $F \in L[w,\Pi]$ (the set of all Lebesgue integrable functions on closed interval $[w,\Pi]$). The Riemann-Liouville fractional integration operators of order $\rho > 0$ applicable to the left and right sides, are formulated as follows:*

$$I_{w^+}^{\rho} F(x) = \frac{1}{\Gamma(\rho)} \int_w^x (x-\Upsilon)^{\rho-1} F(\Upsilon)\, d\Upsilon, \quad x > w,$$

and

$$I_{\Pi^-}^{\rho} F(x) = \frac{1}{\Gamma(\rho)} \int_x^{\Pi} (\Upsilon - x)^{\rho-1} F(\Upsilon)\, d\Upsilon, \ x < \Pi.$$

Here $\Gamma(\rho)$ is the Gamma function.

Definition 1.6. *[26, 27] Let $F \in H'(w,\Pi)$ (Sobolev space), $0 < w < \Pi$, $\rho \in [0,1]$. The left and right Caputo-Fabrizio fractional integrals of order $\rho > 0$ are expressed as follows:*

$$\begin{aligned} \left({}^{CF}{}_{w}I^{\rho} F\right)(x) &= \frac{1-\rho}{\beta(\rho)} F(x) + \frac{\rho}{\beta(\rho)} \int_w^x F(u)\, du, \\ \left({}^{CF}I_{\Pi}^{\rho} F\right)(x) &= \frac{1-\rho}{\beta(\rho)} F(x) + \frac{\rho}{\beta(\rho)} \int_x^{\Pi} F(u)\, du, \end{aligned}$$

where $\beta(\rho) > 0$ is a normalization function that satisfies $\beta(0) = \beta(1) = 1$.

The main objective of this article is to introduce novel fractional versions of Milne-type inequalities. Specifically, we aim to establish Milne-type inequalities for functions that are (s,m)-convex, utilizing the Caputo-Fabrizio fractional integral operators. We have extended our analysis to include bounded functions, and Lipschitzian functions, developing Milne-type inequalities for each of these function classes. Additionally, we have presented generalizations of previous results [28, 29, 30]. Furthermore, we explore applications of these inequalities to special means and q-digamma functions

2. Milne-type inequalities for differentiable (s,m)-convex functions

In this section, we present a few inequalities of Milne-type for differentiable (s,m)-convex function.

Lemma 2.1. *Let $[w, m\Pi] \subset I \subset [0,+\infty)$, let $F : [w, m\Pi] \to \mathbb{R}$ be a differentiable function on $(w, m\Pi)$ such that $F' \in L[w, m\Pi]$, then the following equality holds:*

$$\frac{1}{3}\left[2F(w) - F\left(\frac{w+m\Pi}{2}\right) + 2F(m\Pi)\right] + \frac{2(1-\rho)}{\rho(m\Pi - w)} F(k)$$

$$-\frac{\beta(\rho)}{\rho(m\Pi - w)} \left[\left({}^{CF}{}_{w}I^{\rho} F\right)(k) + \left({}^{CF}I_{m\Pi}^{\rho} F\right)(k)\right]$$

$$= \frac{(m\Pi - w)}{4}\left[\int_0^1 \left(\Upsilon + \frac{1}{3}\right)F'\left(\frac{1-\Upsilon}{2}w + \frac{1+\Upsilon}{2}m\Pi\right)d\Upsilon - \int_0^1 \left(\Upsilon + \frac{1}{3}\right)F'\left(\frac{1+\Upsilon}{2}w + \frac{1-\Upsilon}{2}m\Pi\right)d\Upsilon\right],$$

for all $k \in (w, m\Pi)$, $\rho \in (0,1]$, *where* $\beta(\rho) > 0$ *is a normalization function.*

Proof. Let

$$\begin{aligned} I &= \int_0^1 \left(\Upsilon + \frac{1}{3}\right)F'\left(\frac{1-\Upsilon}{2}w + \frac{1+\Upsilon}{2}m\Pi\right)d\Upsilon \\ &\quad - \int_0^1 \left(\Upsilon + \frac{1}{3}\right)F'\left(\frac{1+\Upsilon}{2}w + \frac{1-\Upsilon}{2}m\Pi\right)d\Upsilon \\ &= I_2 - I_1. \end{aligned}$$

By using integration by parts, we get

$$\begin{aligned} I_1 &= \int_0^1 \left(\Upsilon + \frac{1}{3}\right)F'\left(\frac{1+\Upsilon}{2}w + \frac{1-\Upsilon}{2}m\Pi\right)d\Upsilon \\ &= \left.\frac{-2\left(\Upsilon + \frac{1}{3}\right)F\left(\frac{1+\Upsilon}{2}w + \frac{1-\Upsilon}{2}m\Pi\right)}{m\Pi - w}\right|_0^1 + \frac{2}{m\Pi - w}\int_0^1 F\left(\frac{1+\Upsilon}{2}w + \frac{1-\Upsilon}{2}m\Pi\right)d\Upsilon \\ &= \frac{-8}{3(m\Pi - w)}F(w) + \frac{2}{3(m\Pi - w)}F\left(\frac{w + m\Pi}{2}\right) + \frac{2}{m\Pi - w}\int_0^1 F\left(\frac{1+\Upsilon}{2}w + \frac{1-\Upsilon}{2}m\Pi\right)d\Upsilon \\ &= \frac{-8}{3(m\Pi - w)}F(w) + \frac{2}{3(m\Pi - w)}F\left(\frac{w + m\Pi}{2}\right) + \frac{4}{(m\Pi - w)^2}\int_w^{\frac{w+m\Pi}{2}} F(u)\,du. \end{aligned} \tag{2}$$

And similarly, we have

$$\begin{aligned} I_2 &= \int_0^1 \left(\Upsilon + \frac{1}{3}\right)F'\left(\frac{1-\Upsilon}{2}w + \frac{1+\Upsilon}{2}m\Pi\right)d\Upsilon \\ &= \left.\frac{2\left(\Upsilon + \frac{1}{3}\right)F\left(\frac{1-\Upsilon}{2}w + \frac{1+\Upsilon}{2}m\Pi\right)}{(m\Pi - w)}\right|_0^1 - \frac{2}{m\Pi - w}\int_0^1 F\left(\frac{1-\Upsilon}{2}w + \frac{1+\Upsilon}{2}m\Pi\right)d\Upsilon \\ &= \frac{8}{3(m\Pi - w)}F(m\Pi) - \frac{2}{3(m\Pi - w)}F\left(\frac{w + m\Pi}{2}\right) - \frac{2}{m\Pi - w}\int_0^1 F\left(\frac{1-\Upsilon}{2}w + \frac{1+\Upsilon}{2}m\Pi\right)d\Upsilon \\ &= \frac{8}{3(m\Pi - w)}F(m\Pi) - \frac{2}{3(m\Pi - w)}F\left(\frac{w + m\Pi}{2}\right) - \frac{4}{(m\Pi - w)^2}\int_{\frac{w+m\Pi}{2}}^{m\Pi} F(u)\,du. \end{aligned} \tag{3}$$

Subtracting above equalities (2) and (3) and multiplying both sides $\frac{m\Pi - w}{4}$, we have

$$\frac{(m\Pi - w)}{4}(I_2 - I_1) = \frac{1}{3}\left[2F(w) - F\left(\frac{w + m\Pi}{2}\right) + 2F(m\Pi)\right] - \frac{1}{(m\Pi - w)}\int_w^{m\Pi} F(u)\,du. \tag{4}$$

Subtracting $\frac{2(1-\rho)}{\rho(m\Pi - w)}F(k)$ from both sides of the equality (4), we get

$$\frac{(m\Pi - w)}{4}(I_2 - I_1) - \frac{2(1-\rho)}{\rho(m\Pi - w)}F(k)$$

$$= \frac{1}{3}\left[2F(w)-F\left(\frac{w+m\Pi}{2}\right)+2F(m\Pi)\right]-\frac{1}{(m\Pi-w)}\int_w^{m\Pi}F(u)\,du-\frac{2(1-\rho)}{\rho(m\Pi-w)}F(k)$$

$$= \frac{1}{3}\left[2F(w)-F\left(\frac{w+m\Pi}{2}\right)+2F(m\Pi)\right]$$

$$-\frac{\beta(\rho)}{\rho(m\Pi-w)}\left(\frac{\rho}{\beta(\rho)}\int_w^k F(u)\,du+\frac{(1-\rho)}{\beta(\rho)}F(k)+\frac{\rho}{\beta(\rho)}\int_k^{m\Pi}F(u)\,du+\frac{(1-\rho)}{\beta(\rho)}F(k)\right)$$

$$= \frac{1}{3}\left[2F(w)-F\left(\frac{w+m\Pi}{2}\right)+2F(m\Pi)\right]-\frac{\beta(\rho)}{\rho(m\Pi-w)}\left[\left({}^{CF}{}_{w}I^{\rho}F\right)(k)+\left({}^{CF}I^{\rho}_{m\Pi}F\right)(k)\right].$$

Thus, we have

$$\frac{1}{3}\left[2F(w)-F\left(\frac{w+m\Pi}{2}\right)+2F(m\Pi)\right]+\frac{2(1-\rho)}{\rho(m\Pi-w)}F(k)$$

$$-\frac{\beta(\rho)}{\rho(m\Pi-w)}\left[\left({}^{CF}{}_{w}I^{\rho}F\right)(k)+\left({}^{CF}I^{\rho}_{m\Pi}F\right)(k)\right]$$

$$= \frac{(m\Pi-w)}{4}\left[\int_0^1\left(\Upsilon+\frac{1}{3}\right)F'\left(\frac{1-\Upsilon}{2}w+\frac{1+\Upsilon}{2}m\Pi\right)d\Upsilon\right.$$

$$\left.-\int_0^1\left(\Upsilon+\frac{1}{3}\right)F'\left(\frac{1+\Upsilon}{2}w+\frac{1-\Upsilon}{2}m\Pi\right)d\Upsilon\right].$$

Hence, the proof is concluded. □

Remark 2.2. *If we take $m=1$ in Lemma 2.1, then we have*

$$\frac{1}{3}\left[2F(w)-F\left(\frac{w+\Pi}{2}\right)+2F(\Pi)\right]+\frac{2(1-\rho)}{\rho(\Pi-w)}F(k)$$

$$-\frac{\beta(\rho)}{\rho(\Pi-w)}\left[\left({}^{CF}{}_{w}I^{\rho}F\right)(k)+\left({}^{CF}I^{\rho}_{\Pi}F\right)(k)\right]$$

$$= \frac{(\Pi-w)}{4}\left[\int_0^1\left(\Upsilon+\frac{1}{3}\right)F'\left(\frac{1-\Upsilon}{2}w+\frac{1+\Upsilon}{2}\Pi\right)d\Upsilon\right.$$

$$\left.-\int_0^1\left(\Upsilon+\frac{1}{3}\right)F'\left(\frac{1+\Upsilon}{2}w+\frac{1-\Upsilon}{2}\Pi\right)d\Upsilon\right].$$

Theorem 2.3. *Let $F: I\subset[0,\infty)\to\mathbb{R}$ be a differentiable function on I^o (the interior of I) such that $F\in L[w,\Pi]$ where $w,\Pi\in I$. If $|F'|$ is (s,m)-convex on $[w,\Pi]$ for some fixed $s\in(0,1]$ and $m\in\left(\frac{w}{\Pi},1\right]$, then the following fractional inequality holds:*

$$\left|\frac{1}{3}\left[2F(w)-F\left(\frac{w+m\Pi}{2}\right)+2F(m\Pi)\right]+\frac{2(1-\rho)}{\rho(m\Pi-w)}F(k)\right.$$

$$\left.-\frac{\beta(\rho)}{\rho(m\Pi-w)}\left[\left({}^{CF}{}_{w}I^{\rho}F\right)(k)+\left({}^{CF}I^{\rho}_{m\Pi}F\right)(k)\right]\right|$$

$$\leq \frac{(m\Pi-w)}{2^{2+s}}\left[\frac{1+2^{2+s}-s+2^{3+s}s}{3(s^2+3s+2)}|F'(w)|+\frac{m(5+s)}{3(s^2+3s+2)}|F'(\Pi)|\right],$$

for all $k\in(w,m\Pi)$, $\rho\in(0,1]$, where $\beta(\rho)>0$ is a normalization function.

Proof. By taking the absolute value in Lemma 2.1, we have

$$\begin{aligned}
&\left|\frac{1}{3}\left[2F(w)-F\left(\frac{w+m\Pi}{2}\right)+2F(m\Pi)\right]+\frac{2(1-\rho)}{\rho(m\Pi-w)}F(k)\right.\\
&\left.-\frac{\beta(\rho)}{\rho(m\Pi-w)}\left[\left({}^{CF}{}_{w}I^{\rho}F\right)(k)+\left({}^{CF}I^{\rho}_{m\Pi}F\right)(k)\right]\right|\\
\leq\ &\frac{(m\Pi-w)}{4}\left[\int_0^1\left(\Upsilon+\frac{1}{3}\right)\left|F'\left(\frac{1+\Upsilon}{2}w+\frac{1-\Upsilon}{2}m\Pi\right)\right|d\Upsilon\right.\\
&\left.+\int_0^1\left(\Upsilon+\frac{1}{3}\right)\left|F'\left(\frac{1-\Upsilon}{2}w+\frac{1+\Upsilon}{2}m\Pi\right)\right|d\Upsilon\right].
\end{aligned}$$

By utilizing the (s,m)-convexity of $|F'|$, we have

$$\begin{aligned}
&\left|\frac{1}{3}\left[2F(w)-F\left(\frac{w+m\Pi}{2}\right)+2F(m\Pi)\right]+\frac{2(1-\rho)}{\rho(m\Pi-w)}F(k)\right.\\
&\left.-\frac{\beta(\rho)}{\rho(m\Pi-w)}\left[\left({}^{CF}{}_{w}I^{\rho}F\right)(k)+\left({}^{CF}I^{\rho}_{m\Pi}F\right)(k)\right]\right|\\
\leq\ &\frac{(m\Pi-w)}{4}\left[\int_0^1\left(\Upsilon+\frac{1}{3}\right)\left|F'\left(\frac{1+\Upsilon}{2}w+\frac{1-\Upsilon}{2}m\Pi\right)\right|d\Upsilon\right.\\
&\left.+\int_0^1\left(\Upsilon+\frac{1}{3}\right)\left|F'\left(\frac{1-\Upsilon}{2}w+\frac{1+\Upsilon}{2}m\Pi\right)\right|d\Upsilon\right]\\
\leq\ &\frac{(m\Pi-w)}{4}\left[\int_0^1\left(\Upsilon+\frac{1}{3}\right)\left(\left(\frac{1+\Upsilon}{2}\right)^s|F'(w)|+m\left(\frac{1-\Upsilon}{2}\right)^s|F'(\Pi)|\right)d\Upsilon\right.\\
&\left.+\int_0^1\left(\Upsilon+\frac{1}{3}\right)\left(\left(\frac{1-\Upsilon}{2}\right)^s|F'(w)|+m\left(\frac{1+\Upsilon}{2}\right)^s|F'(\Pi)|\right)d\Upsilon\right]\\
=\ &\frac{(m\Pi-w)}{2^{2+s}}\left[\frac{1+2^{2+s}-s+2^{3+s}s}{3(s^2+3s+2)}|F'(w)|+\frac{m(5+s)}{3(s^2+3s+2)}|F'(\Pi)|\right].
\end{aligned}$$

Hence, the proof is concluded. □

Corollary 2.4. *If we consider $m=1$ in Theorem 2.3, then it follows that*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2F(w)-F\left(\frac{w+\Pi}{2}\right)+2F(\Pi)\right]+\frac{2(1-\rho)}{\rho(\Pi-w)}F(k)-\frac{\beta(\rho)}{\rho(\Pi-w)}\left[\left({}^{CF}{}_{w}I^{\rho}F\right)(k)+\left({}^{CF}I^{\rho}_{\Pi}F\right)(k)\right]\right|\\
&\leq\frac{(\Pi-w)}{2^{2+s}}\left[\frac{1+2^{2+s}-s+2^{3+s}s}{3(s^2+3s+2)}|F'(w)|+\frac{5+s}{3(s^2+3s+2)}|F'(\Pi)|\right].
\end{aligned}$$

Corollary 2.5. *Applying again convexity on Corollary 2.4, then it follows that*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2F(w)-F\left(\frac{w+\Pi}{2}\right)+2F(\Pi)\right]+\frac{2(1-\rho)}{\rho(\Pi-w)}F(k)-\frac{\beta(\rho)}{\rho(\Pi-w)}\left[\left({}^{CF}{}_{w}I^{\rho}F\right)(k)+\left({}^{CF}I^{\rho}_{\Pi}F\right)(k)\right]\right|\\
&\leq\frac{(\Pi-w)}{2^{2+s}}\left[\frac{6+2^{2+s}+2^{3+s}s}{3s^2+9s+6}\left(|F'(w)|+|F'(\Pi)|\right)\right].
\end{aligned}$$

Corollary 2.6. *If we consider $s = 1$ in Corollary 2.5, then it follows that*

$$\left|\frac{1}{3}\left[2F(w) - F\left(\frac{w+\Pi}{2}\right) + 2F(\Pi)\right] + \frac{2(1-\rho)}{\rho(\Pi - w)}F(k) - \frac{\beta(\rho)}{\rho(\Pi - w)}\left[\left({}^{CF}{}_{w}I^{\rho} F\right)(k) + \left({}^{CF}I^{\rho}_{\Pi} F\right)(k)\right]\right|$$
$$\leq \frac{5(\Pi - w)}{24}\left(|F'(w)| + |F'(\Pi)|\right).$$

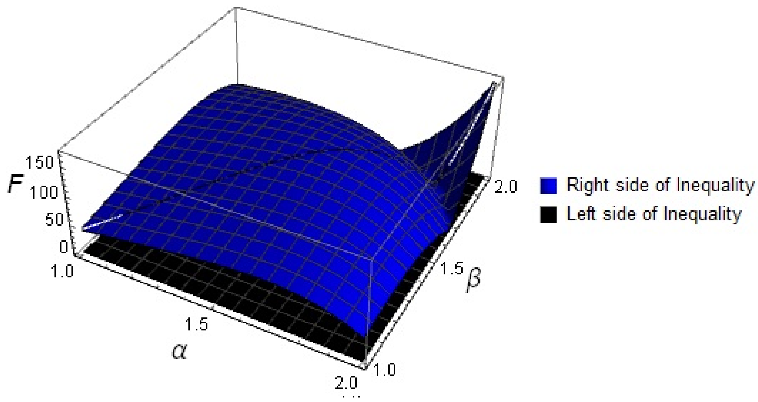

Figure 1: Graphical representation of the error bound for Corollary 2.6, where the left side of the inequality is depicted in black and the right side in blue.

Remark 2.7. *If we consider $\rho = 1$ and $\beta(0) = \beta(1) = 1$ in Corollary 2.6, then it follows that*

$$\left|\frac{1}{3}\left[2F(w) - F\left(\frac{w+\Pi}{2}\right) + 2F(\Pi)\right] - \frac{1}{\Pi - w}\int_{w}^{\Pi} F(x)\,dx\right|$$
$$\leq \frac{5(\Pi - w)}{24}\left(|F'(w)| + |F'(\Pi)|\right),$$

which is obtained by Djenaoui et al. in [28, Corollary 2.4].

Example 2.8. *Let $F(x) = x^2$ is a convex function over the interval $[w, \Pi] = \left[\frac{1}{5}, 1\right]$. Substituting $\rho = 1$, the Milne-type inequality in Remark 2.7 becomes:*

$$L.H.S = \left|\frac{1}{3}\left[2\left(\frac{1}{5}\right)^2 - \left(\frac{6}{10}\right)^2 + 2(1)^2\right] - \frac{1}{1-\frac{1}{5}}\int_{\frac{1}{5}}^{1} x^2 dx\right| = \frac{4}{25},$$

and

$$R.H.S = \frac{5\left(1-\frac{1}{5}\right)}{24}\left[\left|\frac{2}{5}\right| + |2|\right] = \frac{2}{5},$$
$$\frac{4}{25} < \frac{2}{5}.$$

Example 2.9. *Let $F(x) = x^{\frac{3}{2}}$ is a convex function over the interval $[w, \Pi] = \left[\frac{1}{2}, 1\right]$. Substituting $\rho = 1$, the Milne-type inequality in Remark 2.7 becomes:*

$$L.H.S = \left|\frac{1}{3}\left[2\left(\frac{1}{2}\right)^{\frac{3}{2}} - \left(\frac{3}{4}\right)^{\frac{3}{2}} + 2(1)^{\frac{3}{2}}\right] - \frac{1}{1-\frac{1}{2}}\int_{\frac{1}{2}}^{1} x^{\frac{3}{2}} dx\right|$$

$$= \quad 0.027,$$

and

$$R.H.S = \frac{5\left(1-\frac{1}{2}\right)}{24}\left[\frac{3}{2}\left|\left(\frac{1}{2}\right)^{\frac{1}{2}}\right|+\frac{3}{2}\left|(1)^{\frac{1}{2}}\right|\right]=0.266,$$

$$0.027 < 0.266.$$

Theorem 2.10. *Let F be defined as in Theorem 2.3 such that $F' \in L[w,\Pi]$. If $|F'|^q$ for $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$ is (s,m)-convex on $[w,\Pi]$ and $m \in \left(\frac{w}{\Pi},1\right]$ for some fixed $s \in (0,1]$, then the following fractional inequality holds:*

$$\left|\frac{1}{3}\left[2F(w)-F\left(\frac{w+m\Pi}{2}\right)+2F(m\Pi)\right]+\frac{2(1-\rho)}{\rho(m\Pi-w)}F(k)\right.$$
$$\left.-\frac{\beta(\rho)}{\rho(m\Pi-w)}\left[\left({}^{CF}{}_{w}I^{\rho}F\right)(k)+\left({}^{CF}I^{\rho}_{m\Pi}F\right)(k)\right]\right|$$
$$\leq \frac{(m\Pi-w)}{12}\left(\frac{4^{p+1}-1}{3(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{2^{s+1}-1}{2^s(s+1)}|F'(w)|^q+\frac{m}{2^s(s+1)}|F'(\Pi)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{1}{2^s(s+1)}|F'(w)|^q+m\frac{2^{s+1}-1}{2^s(s+1)}|F'(\Pi)|^q\right)^{\frac{1}{q}}\right],$$

for all $k \in (w,m\Pi)$, $\rho \in (0,1]$, where $\beta(\rho)>0$ is a normalization function.

Proof. By taking the absolute value in Lemma 2.1, we get

$$\left|\frac{1}{3}\left[2F(w)-F\left(\frac{w+m\Pi}{2}\right)+2F(m\Pi)\right]+\frac{2(1-\rho)}{\rho(m\Pi-w)}F(k)\right.$$
$$\left.-\frac{\beta(\rho)}{\rho(m\Pi-w)}\left[\left({}^{CF}{}_{w}I^{\rho}F\right)(k)+\left({}^{CF}I^{\rho}_{m\Pi}F\right)(k)\right]\right|$$
$$\leq \frac{(m\Pi-w)}{4}\left[\int_0^1\left(\Upsilon+\frac{1}{3}\right)\left|F'\left(\frac{1+\Upsilon}{2}w+\frac{1-\Upsilon}{2}m\Pi\right)\right|d\Upsilon\right.$$
$$\left.+\int_0^1\left(\Upsilon+\frac{1}{3}\right)\left|F'\left(\frac{1-\Upsilon}{2}w+\frac{1+\Upsilon}{2}m\Pi\right)\right|d\Upsilon\right]. \tag{5}$$

By using the Hölder's inequality in (5) and applying the (s,m)-convexity of $|F'|^q$, we get

$$\left|\frac{1}{3}\left[2F(w)-F\left(\frac{w+m\Pi}{2}\right)+2F(m\Pi)\right]+\frac{2(1-\rho)}{\rho(m\Pi-w)}F(k)\right.$$
$$\left.-\frac{\beta(\rho)}{\rho(m\Pi-w)}\left[\left({}^{CF}{}_{w}I^{\rho}F\right)(k)+\left({}^{CF}I^{\rho}_{m\Pi}F\right)(k)\right]\right|$$
$$\leq \frac{(m\Pi-w)}{4}\left[\int_0^1\left(\Upsilon+\frac{1}{3}\right)\left|F'\left(\frac{1+\Upsilon}{2}w+\frac{1-\Upsilon}{2}m\Pi\right)\right|d\Upsilon\right.$$
$$\left.+\int_0^1\left(\Upsilon+\frac{1}{3}\right)\left|F'\left(\frac{1-\Upsilon}{2}w+\frac{1+\Upsilon}{2}m\Pi\right)\right|d\Upsilon\right]$$

$$\leq \frac{(m\Pi - w)}{4}\left(\int_0^1 \left(\Upsilon + \frac{1}{3}\right)^p d\Upsilon\right)^{\frac{1}{p}}$$
$$\times\left[\left(\int_0^1 \left|F'\left(\frac{1+\Upsilon}{2}w + \frac{1-\Upsilon}{2}m\Pi\right)\right|^q d\Upsilon\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\int_0^1 \left|F'\left(\frac{1-\Upsilon}{2}w + \frac{1+\Upsilon}{2}m\Pi\right)\right|^q d\Upsilon\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{(m\Pi - w)}{4}\left(\int_0^1 \left(\Upsilon + \frac{1}{3}\right)^p d\Upsilon\right)^{\frac{1}{p}}$$
$$\times\left[\left(\int_0^1 \left(\left(\frac{1+\Upsilon}{2}\right)^s |F'(w)|^q + m\left(\frac{1-\Upsilon}{2}\right)^s |F'(\Pi)|^q\right) d\Upsilon\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\int_0^1 \left(\left(\frac{1-\Upsilon}{2}\right)^s |F'(w)|^q + m\left(\frac{1+\Upsilon}{2}\right)^s |F'(\Pi)|^q\right) d\Upsilon\right)^{\frac{1}{q}}\right]$$
$$= \frac{(m\Pi - w)}{12}\left(\frac{4^{p+1}-1}{3(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{2^{s+1}-1}{2^s(s+1)}|F'(w)|^q + \frac{m}{2^s(s+1)}|F'(\Pi)|^q\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\frac{1}{2^s(s+1)}|F'(w)|^q + m\frac{2^{s+1}-1}{2^s(s+1)}|F'(\Pi)|^q\right)^{\frac{1}{q}}\right].$$

Hence, we proved the Theorem 2.10. □

Corollary 2.11. *If we consider $m = 1$ in Theorem 2.10, then it follows that*

$$\left|\frac{1}{3}\left[2F(w) - F\left(\frac{w+\Pi}{2}\right) + 2F(\Pi)\right] + \frac{2(1-\rho)}{\rho(\Pi - w)}F(k)\right.$$
$$\left. - \frac{\beta(\rho)}{\rho(\Pi - w)}\left[\left({}^{CF}{}_{w}I^{\rho} F\right)(k) + \left({}^{CF}I^{\rho}_{\Pi} F\right)(k)\right]\right|$$
$$\leq \frac{(\Pi - w)}{12}\left(\frac{4^{p+1}-1}{3(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{2^{s+1}-1}{2^s(s+1)}|F'(w)|^q + \frac{1}{2^s(s+1)}|F'(\Pi)|^q\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\frac{1}{2^s(s+1)}|F'(w)|^q + \frac{2^{s+1}-1}{2^s(s+1)}|F'(\Pi)|^q\right)^{\frac{1}{q}}\right].$$

Corollary 2.12. *If we consider $s = 1$ in Corollary 2.11, then it follows that*

$$\left|\frac{1}{3}\left[2F(w) - F\left(\frac{w+\Pi}{2}\right) + 2F(\Pi)\right] + \frac{2(1-\rho)}{\rho(\Pi - w)}F(k)\right.$$
$$\left. - \frac{\beta(\rho)}{\rho(\Pi - w)}\left[\left({}^{CF}{}_{w}I^{\rho} F\right)(k) + \left({}^{CF}I^{\rho}_{\Pi} F\right)(k)\right]\right|$$

$$\leq \frac{(\Pi - w)}{12}\left(\frac{4^{p+1}-1}{3(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{3|F'(w)|^q + |F'(\Pi)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{|F'(w)|^q + 3|F'(\Pi)|^q}{4}\right)^{\frac{1}{q}}\right].$$

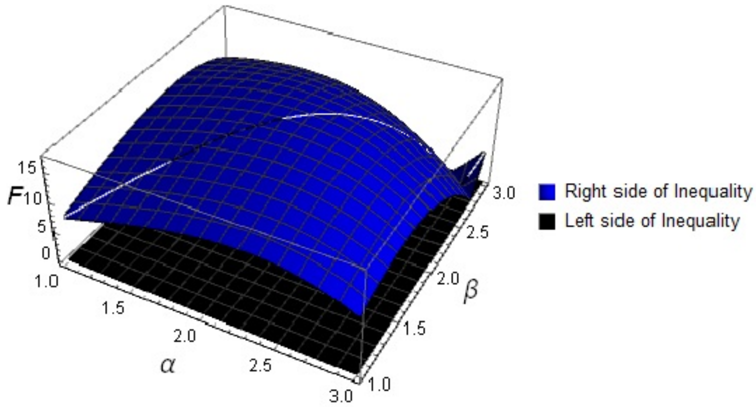

Figure 2: Graphical representation of the error bound for Corollary 2.12, where the left side of the inequality is depicted in black and the right side in blue.

Remark 2.13. *If we consider $\rho = 1$ and $\beta(0) = \beta(1) = 1$ in Corollary 2.12, then it follows that*

$$\left|\frac{1}{3}\left[2F(w) - F\left(\frac{w+\Pi}{2}\right) + 2F(\Pi)\right] - \frac{1}{\Pi - w}\int_w^{\Pi} F(x)\,dx\right|$$

$$\leq \frac{(\Pi - w)}{12}\left(\frac{4^{p+1}-1}{3(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{3|F'(w)|^q + |F'(\Pi)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{|F'(w)|^q + 3|F'(\Pi)|^q}{4}\right)^{\frac{1}{q}}\right],$$

which is obtained by Budak et al. in [29, Corollary 1].

Theorem 2.14. *Let F be defined as in Theorem 2.3 such that $F' \in L[w, \Pi]$. If $|F'|^q$ for $q \geq 1$ is (s, m)-convex on $[w, \Pi]$, then the following fractional inequality holds:*

$$\left|\frac{1}{3}\left[2F(w) - F\left(\frac{w+m\Pi}{2}\right) + 2F(m\Pi)\right] + \frac{2(1-\rho)}{\rho(m\Pi - w)}F(k)\right.$$

$$\left.-\frac{\beta(\rho)}{\rho(m\Pi - w)}\left[\left({}^{CF}{}_{w}I^{\rho} F\right)(k) + \left({}^{CF}I^{\rho}_{m\Pi} F\right)(k)\right]\right|$$

$$\leq \frac{(m\Pi - w)}{4}\left(\frac{5}{6}\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\left(\frac{2^{-s}\left(1 + 2^{2+s} - s + 2^{3+s}s\right)}{6 + 9s + 3s^2}\right)|F'(w)|^q + m\left(\frac{2^{-s}(5+s)}{6 + 9s + 3s^2}\right)|F'(\Pi)|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\left(\frac{2^{-s}(5+s)}{6 + 9s + 3s^2}\right)|F'(w)|^q + m\left(\frac{2^{-s}\left(1 + 2^{2+s} - s + 2^{3+s}s\right)}{6 + 9s + 3s^2}\right)|F'(\Pi)|^q\right)^{\frac{1}{q}}\right],$$

for all $k \in (w, m\Pi)$, $\rho \in (0, 1]$, where $\beta(\rho) > 0$ is a normalization function.

Proof. By taking the absolute value in Lemma 2.1, using the power-mean inequality and utilizing the (s,m)-convexity of $|F'|^q$, we get

$$\begin{aligned}
&\left|\frac{1}{3}\left[2F(w)-F\left(\frac{w+m\Pi}{2}\right)+2F(m\Pi)\right]+\frac{2(1-\rho)}{\rho(m\Pi-w)}F(k)\right.\\
&\left.-\frac{\beta(\rho)}{\rho(m\Pi-w)}\left[\left({}^{CF}{}_{w}I^{\rho}F\right)(k)+\left({}^{CF}I^{\rho}_{m\Pi}F\right)(k)\right]\right|\\
\leq\ &\frac{(m\Pi-w)}{4}\left[\int_0^1\left(\Upsilon+\frac{1}{3}\right)\left|F'\left(\frac{1+\Upsilon}{2}w+\frac{1-\Upsilon}{2}m\Pi\right)\right|d\Upsilon\right.\\
&\left.+\int_0^1\left(\Upsilon+\frac{1}{3}\right)\left|F'\left(\frac{1-\Upsilon}{2}w+\frac{1+\Upsilon}{2}m\Pi\right)\right|d\Upsilon\right]\\
\leq\ &\frac{(m\Pi-w)}{4}\left(\int_0^1\left(\Upsilon+\frac{1}{3}\right)d\Upsilon\right)^{1-\frac{1}{q}}\\
&\times\left[\left(\int_0^1\left(\Upsilon+\frac{1}{3}\right)\left|F'\left(\frac{1+\Upsilon}{2}w+\frac{1-\Upsilon}{2}m\Pi\right)\right|^q d\Upsilon\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1\left(\Upsilon+\frac{1}{3}\right)\left|F'\left(\frac{1-\Upsilon}{2}w+\frac{1+\Upsilon}{2}m\Pi\right)\right|^q d\Upsilon\right)^{\frac{1}{q}}\right]\\
\leq\ &\frac{(m\Pi-w)}{4}\left(\int_0^1\left(\Upsilon+\frac{1}{3}\right)d\Upsilon\right)^{1-\frac{1}{q}}\\
&\times\left[\left(\int_0^1\left(\Upsilon+\frac{1}{3}\right)\left(\left(\frac{1+\Upsilon}{2}\right)^s|F'(w)|^q+m\left(\frac{1-\Upsilon}{2}\right)^s|F'(\Pi)|^q\right)d\Upsilon\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1\left(\Upsilon+\frac{1}{3}\right)\left(\left(\frac{1-\Upsilon}{2}\right)^s|F'(w)|^q+m\left(\frac{1+\Upsilon}{2}\right)^s|F'(\Pi)|^q\right)d\Upsilon\right)^{\frac{1}{q}}\right]\\
=\ &\frac{(m\Pi-w)}{4}\left(\frac{5}{6}\right)^{1-\frac{1}{q}}\\
&\times\left[\left(\left(\frac{2^{-s}\left(1+2^{2+s}-s+2^{3+s}s\right)}{6+9s+3s^2}\right)|F'(w)|^q+m\left(\frac{2^{-s}(5+s)}{6+9s+3s^2}\right)|F'(\Pi)|^q\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\left(\frac{2^{-s}(5+s)}{6+9s+3s^2}\right)|F'(w)|^q+m\left(\frac{2^{-s}\left(1+2^{2+s}-s+2^{3+s}s\right)}{6+9s+3s^2}\right)|F'(\Pi)|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Hence, the proof is concluded. □

Corollary 2.15. *If we consider $m=1$ in Theorem 2.14, then it follows that*

$$\left|\frac{1}{3}\left[2F(w)-F\left(\frac{w+\Pi}{2}\right)+2F(\Pi)\right]+\frac{2(1-\rho)}{\rho(\Pi-w)}F(k)\right.$$

$$-\frac{\beta(\rho)}{\rho(\Pi-w)}\left[\left({}^{CF}{}_{w}I^{\rho}F\right)(k)+\left({}^{CF}I^{\rho}_{\Pi}F\right)(k)\right]\Bigg|$$

$$\leq \frac{(\Pi-w)}{4}\left(\frac{5}{6}\right)^{1-\frac{1}{q}}\left[\left(\left(\frac{2^{-s}\left(1+2^{2+s}-s+2^{3+s}s\right)}{6+9s+3s^2}\right)|F'(w)|^q+\left(\frac{2^{-s}(5+s)}{6+9s+3s^2}\right)|F'(\Pi)|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\left(\frac{2^{-s}(5+s)}{6+9s+3s^2}\right)|F'(w)|^q+\left(\frac{2^{-s}\left(1+2^{2+s}-s+2^{3+s}s\right)}{6+9s+3s^2}\right)|F'(\Pi)|^q\right)^{\frac{1}{q}}\right].$$

Corollary 2.16. *If we consider $s=1$ in Corollary 2.15, then it follows that*

$$\left|\frac{1}{3}\left[2F(w)-F\left(\frac{w+\Pi}{2}\right)+2F(\Pi)\right]+\frac{2(1-\rho)}{\rho(\Pi-w)}F(k)\right.$$

$$\left.-\frac{\beta(\rho)}{\rho(\Pi-w)}\left[\left({}^{CF}{}_{w}I^{\rho}F\right)(k)+\left({}^{CF}I^{\rho}_{\Pi}F\right)(k)\right]\right|$$

$$\leq \frac{5(\Pi-w)}{24}\left[\left(\frac{4|F'(w)|^q+|F'(\Pi)|^q}{5}\right)^{\frac{1}{q}}+\left(\frac{|F'(w)|^q+4|F'(\Pi)|^q}{5}\right)^{\frac{1}{q}}\right].$$

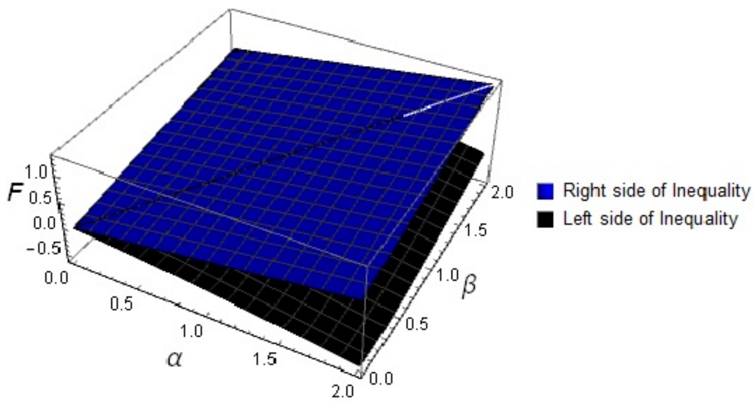

Figure 3: Graphical representation of the error bound for Corollary 2.16, where the left side of the inequality is depicted in black and the right side in blue.

3. Milne-type Inequalities For Bounded Functions Involving Fractional Integrals

In this section, we establish a fractional Milne-type inequality for a bounded function.

Theorem 3.1. *Let F be defined as in Theorem 2.3 such that $F' \in L[w,\Pi]$. If there exist $\underline{m}, M \in \mathbb{R}$ such that $\underline{m} \leq F'(\Upsilon^{\bullet}) \leq M$ for all $\Upsilon^{\bullet} \in [w,\Pi]$, then the following fractional inequality holds:*

$$\left|\frac{1}{3}\left[2F(w)-F\left(\frac{w+m\Pi}{2}\right)+2F(m\Pi)\right]+\frac{2(1-\rho)}{\rho(m\Pi-w)}F(k)\right.$$

$$\left.-\frac{\beta(\rho)}{\rho(m\Pi-w)}\left[\left({}^{CF}{}_{w}I^{\rho}F\right)(k)+\left({}^{CF}I^{\rho}_{m\Pi}F\right)(k)\right]\right|$$

$$\leq \frac{5(m\Pi - w)}{24}\left(M - \underline{m}\right).$$

Proof. Applying Lemma 2.1, we obtain

$$\begin{aligned}
&\frac{1}{3}\left[2F(w) - F\left(\frac{w+m\Pi}{2}\right) + 2F(m\Pi)\right] + \frac{2(1-\rho)}{\rho(m\Pi - w)}F(k) \\
&-\frac{\beta(\rho)}{\rho(m\Pi - w)}\left[\left({}^{CF}{}_{w}I^{\rho} F\right)(k) + \left({}^{CF}I^{\rho}_{m\Pi} F\right)(k)\right] \\
= &\frac{(m\Pi - w)}{4}\int_0^1 \left(\Upsilon + \frac{1}{3}\right)\left[F'\left(\frac{1-\Upsilon}{2}w + \frac{1+\Upsilon}{2}m\Pi\right) - F'\left(\frac{1+\Upsilon}{2}w + \frac{1-\Upsilon}{2}m\Pi\right)\right]d\Upsilon \\
= &\frac{(m\Pi - w)}{4}\left\{\int_0^1 \left(\Upsilon + \frac{1}{3}\right)\left[F'\left(\frac{1-\Upsilon}{2}w + \frac{1+\Upsilon}{2}m\Pi\right) - \frac{\underline{m}+M}{2}\right]d\Upsilon\right. \\
&\left.+\int_0^1 \left(\Upsilon + \frac{1}{3}\right)\left[\frac{\underline{m}+M}{2} - F'\left(\frac{1+\Upsilon}{2}w + \frac{1-\Upsilon}{2}m\Pi\right)\right]d\Upsilon\right\}. \qquad (6)
\end{aligned}$$

By considering the absolute value of (6), we obtain

$$\begin{aligned}
&\left|\frac{1}{3}\left[2F(w) - F\left(\frac{w+m\Pi}{2}\right) + 2F(m\Pi)\right] + \frac{2(1-\rho)}{\rho(m\Pi - w)}F(k)\right. \\
&\left.-\frac{\beta(\rho)}{\rho(m\Pi - w)}\left[\left({}^{CF}{}_{w}I^{\rho} F\right)(k) + \left({}^{CF}I^{\rho}_{m\Pi} F\right)(k)\right]\right| \\
\leq &\frac{(m\Pi - w)}{4}\left[\int_0^1 \left(\Upsilon + \frac{1}{3}\right)\left|F'\left(\frac{1-\Upsilon}{2}w + \frac{1+\Upsilon}{2}m\Pi\right) - \frac{\underline{m}+M}{2}\right|d\Upsilon\right. \\
&\left.+\int_0^1 \left(\Upsilon + \frac{1}{3}\right)\left|\frac{\underline{m}+M}{2} - F'\left(\frac{1+\Upsilon}{2}w + \frac{1-\Upsilon}{2}m\Pi\right)\right|d\Upsilon\right].
\end{aligned}$$

From $\underline{m} \leq F'(\Upsilon^{\bullet}) \leq M$, for all $\Upsilon^{\bullet} \in [w, \Pi]$ and for all $\Upsilon \in [0,1]$, we get

$$\left|F'\left(\frac{1-\Upsilon}{2}w + \frac{1+\Upsilon}{2}m\Pi\right) - \frac{\underline{m}+M}{2}\right| \leq \frac{M-\underline{m}}{2}, \qquad (7)$$

and

$$\left|\frac{\underline{m}+M}{2} - F'\left(\frac{1+\Upsilon}{2}w + \frac{1-\Upsilon}{2}m\Pi\right)\right| \leq \frac{M-\underline{m}}{2}. \qquad (8)$$

Using (7) and (8), we have

$$\begin{aligned}
&\left|\frac{1}{3}\left[2F(w) - F\left(\frac{w+m\Pi}{2}\right) + 2F(m\Pi)\right] + \frac{2(1-\rho)}{\rho(m\Pi - w)}F(k)\right. \\
&\left.-\frac{\beta(\rho)}{\rho(m\Pi - w)}\left[\left({}^{CF}{}_{w}I^{\rho} F\right)(k) + \left({}^{CF}I^{\rho}_{m\Pi} F\right)(k)\right]\right| \\
\leq &\frac{(m\Pi - w)}{4}\left(M - \underline{m}\right)\int_0^1 \left(\Upsilon + \frac{1}{3}\right)d\Upsilon \\
= &\frac{5(m\Pi - w)}{24}\left(M - \underline{m}\right).
\end{aligned}$$

Hence, the proof is concluded. □

Corollary 3.2. *If we consider $m = 1$ in Theorem 3.1, then it follows that*

$$\begin{aligned}&\left|\frac{1}{3}\left[2F(w) - F\left(\frac{w+\Pi}{2}\right) + 2F(\Pi)\right] + \frac{2(1-\rho)}{\rho(\Pi - w)}F(k)\right.\\&\left.-\frac{\beta(\rho)}{\rho(\Pi - w)}\left[\left({}^{CF}{}_{w}I^{\rho}F\right)(k) + \left({}^{CF}I^{\rho}_{\Pi}F\right)(k)\right]\right|\\\leq\;&\frac{5(\Pi - w)}{24}\left(M - \underline{m}\right).\end{aligned}$$

Corollary 3.3. *Assuming the conditions of Corollary 3.2, if there exists $M \in \mathbb{R}^{+}$ satisfying $|F'(\Upsilon)| \leq M$ for all $\Upsilon \in [w, \Pi]$, then it follows that*

$$\begin{aligned}&\left|\frac{1}{3}\left[2F(w) - F\left(\frac{w+\Pi}{2}\right) + 2F(\Pi)\right] + \frac{2(1-\rho)}{\rho(\Pi - w)}F(k)\right.\\&\left.-\frac{\beta(\rho)}{\rho(\Pi - w)}\left[\left({}^{CF}{}_{w}I^{\rho}F\right)(k) + \left({}^{CF}I^{\rho}_{\Pi}F\right)(k)\right]\right|\\\leq\;&\frac{5(\Pi - w)}{12}M.\end{aligned}$$

Remark 3.4. *If we consider $\rho = 1$ and $\beta(0) = \beta(1) = 1$ in Corollary 3.3, then it follows that*

$$\begin{aligned}&\left|\frac{1}{3}\left[2F(w) - F\left(\frac{w+\Pi}{2}\right) + 2F(\Pi)\right] - \frac{1}{\Pi - w}\int_{w}^{\Pi}F(x)\,dx\right|\\\leq\;&\frac{5(\Pi - w)}{12}M,\end{aligned}$$

which was proved by Alomari and Liu [30, Theorem 3.1].

4. Milne-type Inequalities For Lipschitzian Functions Involving Fractional Integrals

In this section, we derive a fractional Milne-type inequality for a Lipschitzian function.

Theorem 4.1. *Let F be defined as in Theorem 2.3 such that $F' \in L[w, \Pi]$. If F' is an L-Lipschitzian function on $[w, \Pi]$, then the following fractional inequality holds:*

$$\begin{aligned}&\left|\frac{1}{3}\left[2F(w) - F\left(\frac{w+m\Pi}{2}\right) + 2F(m\Pi)\right] + \frac{2(1-\rho)}{\rho(m\Pi - w)}F(k)\right.\\&\left.-\frac{\beta(\rho)}{\rho(m\Pi - w)}\left[\left({}^{CF}{}_{w}I^{\rho}F\right)(k) + \left({}^{CF}I^{\rho}_{m\Pi}F\right)(k)\right]\right|\\\leq\;&\frac{(m\Pi - w)^{2}}{8}L.\end{aligned}$$

Proof. By using the Lemma 2.1 and since F' is an L-Lipschitzian function, we get

$$\begin{aligned}&\left|\frac{1}{3}\left[2F(w) - F\left(\frac{w+m\Pi}{2}\right) + 2F(m\Pi)\right] + \frac{2(1-\rho)}{\rho(m\Pi - w)}F(k)\right.\\&\left.-\frac{\beta(\rho)}{\rho(m\Pi - w)}\left[\left({}^{CF}{}_{w}I^{\rho}F\right)(k) + \left({}^{CF}I^{\rho}_{m\Pi}F\right)(k)\right]\right|\end{aligned}$$

$$
\begin{aligned}
&= \left|\frac{(m\Pi-w)}{4}\int_0^1\left(\Upsilon+\frac{1}{3}\right)\left[F'\left(\frac{1-\Upsilon}{2}w+\frac{1+\Upsilon}{2}m\Pi\right)-F'\left(\frac{1+\Upsilon}{2}w+\frac{1-\Upsilon}{2}m\Pi\right)\right]d\Upsilon\right| \\
&\leq \frac{(m\Pi-w)}{4}\int_0^1\left(\Upsilon+\frac{1}{3}\right)\left|F'\left(\frac{1-\Upsilon}{2}w+\frac{1+\Upsilon}{2}m\Pi\right)-F'\left(\frac{1+\Upsilon}{2}w+\frac{1-\Upsilon}{2}m\Pi\right)\right|d\Upsilon \\
&\leq \frac{(m\Pi-w)}{4}\int_0^1\left(\Upsilon+\frac{1}{3}\right)L\Upsilon\left(m\Pi-w\right)d\Upsilon \\
&= \frac{(m\Pi-w)^2}{8}L.
\end{aligned}
$$

Hence, the proof is concluded. □

Remark 4.2. *If we considered $\rho = m = 1$ and $\beta(0) = \beta(1) = 1$ in Theorem 4.1, then it follows that*

$$
\begin{aligned}
&\left|\frac{1}{3}\left[2F(w)-F\left(\frac{w+\Pi}{2}\right)+2F(\Pi)\right]-\frac{1}{\Pi-w}\int_w^\Pi F(x)\,dx\right| \\
\leq\; &\frac{(\Pi-w)^2}{8}L,
\end{aligned}
$$

which is obtained by Budak et al. [29, Corollary 4].

5. Applications

5.1. *Special means:*

The following special means will be utilized:

(a) The Arithmetic Mean:

$$A(w,\Pi) := \frac{w+\Pi}{2},\, w \neq \Pi.$$

(b) The Generalized Logarithmic Mean:

$$L_r(w,\Pi) := \left[\frac{\Pi^{r+1}-w^{r+1}}{(r+1)(\Pi-w)}\right]^{1/r},\, r\in\mathbb{Z}\backslash\{-1,0\},\; w,\Pi>0,\Pi>w.$$

Proposition 5.1. *Let $w,\Pi \in \mathbb{R}$ with $0 < w < \Pi$, then it follows that*

$$\left|4A\left(w^2,\Pi^2\right)-A^2(w,\Pi)-3L_2^2(w,\Pi)\right| \leq \frac{5(\Pi-w)}{24}\left[\left(\frac{4w^q+\Pi^q}{5}\right)^{\frac{1}{q}}+\left(\frac{w^q+4\Pi^q}{5}\right)^{\frac{1}{q}}\right].$$

Proof. The statement directly follows from Corollary 2.16, applying the function $F(x) = \frac{1}{2}x^2$, where $\rho = 1$ and $\beta(0) = \beta(1) = 1$. □

5.2. q-Digamma functions:

Let $0 < q < 1$ and $\ell > 0$, the q-digamma function ψ_q is the q-analogue of the digamma function ψ defined as [31, 32]:

$$\begin{aligned}\psi_q(\ell) &= -\ln(1-q) + (\ln q)\sum_{k=0}^{\infty}\frac{q^{k+\ell}}{1-q^{k+\ell}} \\ &= -\ln(1-q) + (\ln q)\sum_{k=1}^{\infty}\frac{q^{k\ell}}{1-q^{k}}.\end{aligned}$$

For $q > 1$ and $\ell > 0$, the q-digamma function ψ_q defined as

$$\begin{aligned}\psi_q(\ell) &= -\ln(q-1) + (\ln q)\left[\ell - \frac{1}{2} - \sum_{k=0}^{\infty}\frac{q^{-(k+\ell)}}{1-q^{-(k+\ell)}}\right] \\ &= -\ln(q-1) + (\ln q)\left[\ell - \frac{1}{2} - \sum_{k=1}^{\infty}\frac{q^{-k\ell}}{1-q^{-k}}\right].\end{aligned}$$

Proposition 5.2. *Let $w, \Pi \in \mathbb{R}$ with $0 < w < \Pi$, then it follows that*

$$\begin{aligned}&\left|\frac{1}{3}\left[2\psi_q'(w) - \psi_q'\left(\frac{w+\Pi}{2}\right) + 2\psi_q'(\Pi)\right] - \frac{\psi_q(\Pi) - \psi_q(w)}{\Pi - w}\right| \\ \leq\ & \frac{5(\Pi - w)}{24}\left(\left|\psi_q''(w)\right| + \left|\psi_q''(\Pi)\right|\right).\end{aligned}$$

Proof. The statement directly follows from Corollary 2.6, $F(\epsilon) = \psi_q'(\epsilon)$, $\epsilon > 0$ is convex in $(0, \infty)$, $F'(\epsilon) = \psi_q''(\epsilon)$ and $s = \rho = 1$ & $\beta(0) = \beta(1) = 1$. □

Proposition 5.3. *Let $w, \Pi \in \mathbb{R}$ with $0 < w < \Pi$ and $\frac{1}{p} + \frac{1}{q} = 1$, $q > 1$, then it follows that*

$$\begin{aligned}&\left|\frac{1}{3}\left[2\psi_q'(w) - \psi_q'\left(\frac{w+\Pi}{2}\right) + 2\psi_q'(\Pi)\right] - \frac{\psi_q(\Pi) - \psi_q(w)}{\Pi - w}\right| \\ \leq\ & \frac{(\Pi - w)}{12}\left(\frac{4^{p+1}-1}{3(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{3\left|\psi_q''(w)\right|^q + \left|\psi_q''(\Pi)\right|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{\left|\psi_q''(w)\right|^q + 3\left|\varphi_q''(\Pi)\right|^q}{4}\right)^{\frac{1}{q}}\right].\end{aligned}$$

Proof. The statement directly follows from Corollary 2.12, $F(\epsilon) = \psi_q'(\epsilon)$, $\epsilon > 0$ is convex in $(0, \infty)$, $F'(\epsilon) = \psi_q''(\epsilon)$ and $s = \rho = 1$ & $\beta(0) = \beta(1) = 1$. □

6. Conclusion

In this article, we have derived significant contributions to the theory of inequalities by establishing a fractional Milne-type inequality for differentiable mappings. We have extended our analysis to include bounded functions, and Lipschitzian functions, developing Milne-type inequalities for each of these function classes. Furthermore, we have presented generalizations of previous results [28, 29, 30], and obtained several Milne-type inequalities for (s, m)-convex functions. Additionally, we also derived some applications to special means and q-digamma functions. Furthermore, we show graphical behaviour of our attained inequalities. In the future, scholars may explore new results using modified Riemann-Liouville, Atangana-Baleanu (AB) and Atangana-Baleanu-Katugampola (ABK) fractional integral operators. Finally, our results can be extend in quantum calculus.

Competing Interests
The authors declare no competing interests.

Data availability
Data sharing is not relevant to this paper, as there was no generation or analysis of new data during this study.

Authors Contributions
Conceptualization; A.M., Methodology; A.M., Software; A.M., Validation; A.M. and A.K., Writing-original draft; A.M., Writing-review & editing; A.M., S.L., and A.K., Project administration; A.M., S.L., and A.K., Supervision; S.L. All authors have read and agreed to the final version of the manuscript.

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