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On simultaneous characterizations of partner-ruled surfaces with Sasai's interpretation

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Abstract. In this study, the partner-ruled surfaces in Euclidean 3-space are introduced based on the vector fields of a space curve associated with the element of a modified orthogonal frame incorporating curvatures. First, the necessary conditions for each pair of partner-ruled surfaces to be simultaneously developable and minimal are investigated. Furthermore, simultaneous characterizations are provided for the parameter curves of these surfaces to be asymptotic curves, lines of curvature, and geodesics. Finally, to illustrate the theoretical results, an example of partner-ruled surfaces is presented along with their graphical representations.

1. Introduction

The concept of surface has been researched by many mathematicians, philosophers, and scientists over thousands of years throughout history. In this process, the surface theory has been dramatically reinforced by the development of differential geometry. Alongside pioneers in this field, such as Gauss, Riemann, and Poincaré, Monge also made significant contributions to the study of surfaces. Differential geometry of surfaces holds a prominent place in various fields, such as physics, engineering, and computer graphics. One of the most important among these surfaces is the ruled surface, which was introduced by G. Monge and establishes a linear equation satisfied for all ruled surfaces. Monge's approach profoundly influenced the progress of surface theory and its applications in the 19th and 20th centuries and continues to maintain its popularity. Guggenheimer and Hoschek examined ruled surfaces from different perspectives and made significant contributions to differential geometry. Ruled surfaces, referred to by various names, are defined as linear surfaces due to the movement of a line based on a curve or as surfaces composed of an infinite

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number of straight lines. These curves are respectively called the base curve and the directing curve. In Euclidean space, a ruled surface can also be defined as the geometric locus of a family of straight lines dependent on a parameter. Furthermore, ruled surfaces have various applications, especially in fields such as kinematics, architecture, and computer-aided design. In addition, the investigation of certain classes of surfaces, such as developable surfaces and minimal surfaces, is one of the fundamental goals of classical differential geometry. The most well-known ruled surfaces include the surfaces of the plane, cylinder, and cone [1, 2]. Rich information about ruled surfaces can be found in detail in sources [3–7]. These studies are carried out using the Serret-Frenet frame. However, the Serret-Frenet frame is insufficient at points where the curvature of the space curve is zero. Because at points where the curvature is zero, the principal normal and binormal vectors of a space curve become discontinuous. Sasai has defined the modified orthogonal frame (MOF) as an alternative to the Frenet frame to solve this problem [8]. Then, the MOF was defined by Bükçü and Karaca for the non-zero torsion of space curves in Minkowski 3-space [9]. In this study, we examine the partner-ruled surfaces (PRSs), which have not been investigated so far, using the modified orthogonal frame with curvature (MOFC). Detailed information about curves and surfaces created with the MOF can be found in detailed sources [10–21]. Recently, Li et al. investigated partner-ruled surfaces formed by polynomial curves with the Frenet-like curve frame [22, 23], and Soukaina also studied the developability of PRSs using the Darboux frame simultaneously [24].
In this study, we introduce PRSs generated by the vectors of the MOFC of a space curve in Euclidean 3-space. Our research focuses on establishing conditions that determine whether every PRS with the MOFC is simultaneously developable or minimal. In addition, the conditions for the parameter curves to be simultaneously asymptotic, geodesic, and curvature lines are obtained. At the end of the study, an example related to PRSs is given, and the graphics of the PRSs are drawn.

2. Preliminaries

Let α be a moving unit speed curve parametrized by arc-length s in Euclidean 3-space. For the unit speed curve α in Euclidean 3-space, the unit tangent vector $t(s)$ of α at point $\alpha(s)$ is given by $t(s) = \alpha'(s)$, if $\alpha''(s) \neq 0$, the unit normal vector $n(s)$ of α at point $\alpha(s)$ is provided by $n(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}$, and the speed vector $b(s) = n(s) \times t(s)$ is called the binormal vector of α at point $\alpha(s)$. Then, there exists an orthonormal frame moving along $\alpha(s)$ which satisfies the following Frenet-Serret formulae

$$t'(s) = \kappa(s)\, n(s),\ n'(s) = -\kappa(s)\, t(s) + \tau(s)\, b(s),\ b'(s) = -\tau(s)\, n(s),$$

where $\kappa(s)$ and $\tau(s)$ are the curvature and torsion functions of α, respectively. When a space curve's principal normal and binormal vectors become discontinuous at zero curvature points, Sasai proposed a modified orthogonal frame with curvature (MOFC) as an alternative to the Frenet frame [8, 9]. The relations between the MOFC and the Serret-Frenet frame of the curve α exist as follows:

$$T(s) = t(s),\ N(s) = \kappa(s)\, n(s),\ B(s) = \kappa(s)\, b(s),$$

where the curvature $\kappa(s)$ is not zero. So, the MOFC is expressed as follows:

$$T'(s) = N(s),\ N'(s) = -\kappa^2 T(s) + \frac{\kappa'}{\kappa} N(s) + \tau B(s),\ B'(s) = -\tau N(s) + \frac{\kappa'}{\kappa} B(s), \tag{1}$$

where the curve's torsion is

$$\tau(s) = \frac{\det(\alpha'(s), \alpha''(s), \alpha'''(s))}{\kappa(s)^2}.$$

Moreover, the elements of MOFC satisfy

$$\langle T(s), T(s)\rangle = 1,\ \langle N(s), N(s)\rangle = \langle B(s), B(s)\rangle = \kappa(s)^2, \langle T(s), N(s)\rangle = \langle T(s), B(s)\rangle = \langle N(s), B(s)\rangle = 0.$$

Let $\varphi(s,v)$ be a ruled surface generated by the motion of a straight line on a given curve whose parameterized equation is given by

$$\varphi(s,v)=\alpha(s)+vr(s),$$

where the regular curve $\alpha(s)$ is called the base curve (or directrix) and the straight line $r(s)$ is called the ruling (or generator) of the ruled surface. In this case, the various quantities related to the ruled surface are determined as follows:

i. The unit normal vector field of $\varphi(s,v)$ is defined by $U(s,v)=\frac{\varphi_s\times\varphi_v}{\|\varphi_s\times\varphi_v\|}$, where $\varphi_s=\frac{\partial\varphi}{\partial s}$ and $\varphi_v=\frac{\partial\varphi}{\partial v}$ are linearly independent tangent vectors of $\varphi(s,v)$.

ii. The first fundamental form I of $\varphi(s,v)$ is calculated by $I=Eds^2+2Fdsdv+Gdv^2$, and its coefficients

$$E(s,v)=\langle\varphi_s,\varphi_s\rangle,\ F(s,v)=\langle\varphi_s,\varphi_v\rangle,\ G(s,v)=\langle\varphi_v,\varphi_v\rangle. \tag{2}$$

iii. The second fundamental form II of $\varphi(s,v)$ is given by $II=eds^2+2fdsdv+gdv^2$, and its coefficients

$$e(s,v)=\langle\varphi_{ss},U(s,v)\rangle,\ f(s,v)=\langle\varphi_{sv},U(s,v)\rangle,\ g(s,v)=\langle\varphi_{vv},U(s,v)\rangle. \tag{3}$$

iv. The Gaussian and the mean curvatures of $\varphi(s,v)$ are determined by the following formula:

$$K=\frac{eg-f^2}{EG-F^2}\text{ and }H=\frac{Eg-2Ef+Ge}{2(EG-F^2)}, \tag{4}$$

respectively, where $EG-F^2\neq 0$.

Lemma 2.1. *Let $\varphi(s,v)$ a ruled surface, then there exist the following situations;*

i. If the Gaussian curvature vanishes everywhere, then the ruled surface $\varphi(s,v)$ is developable,

ii. If the mean curvature vanishes everywhere, then the ruled surface $\varphi(s,v)$ is minimal [10].

3. Simultaneous Characterizations of Partner-Ruled Surfaces with the Modified Orthogonal Frame with Curvature

In this section, we express the PRSs that are generated by the motion of a straight line on a given curve, where the unit tangent vector T, unit principal normal vector N, and unit binormal vector B of the MOFC are considered as the basic curve and the straight line.

Definition 3.1. *Let the curve $\alpha: I\subset R\to R^3$, for all $s\in I$, be a differentiable unit speed space curve in R^3 with the MOFC elements $\{T,N,B,\kappa,\tau\}$ such that $\kappa(s)\neq 0$. The ruled surfaces that are generated by the motion of the principal normal vector $N(T)$ on the tangent vector $T(N)$ are defined by*

$$\begin{cases}\varphi_N^T(s,v)=T(s)+vN(s),\\ \varphi_T^N(s,v)=N(s)+vT(s),\end{cases}$$

and these surfaces are called $TN-$PRSs concerning the MOFC in Euclidean 3-space.

Theorem 3.2. *Let φ_N^T and φ_T^N be components of $TN-$PRSs with the MOFC in R^3, then the $TN-$PRSs with the MOFC are simultaneously developable and minimal surfaces if and only if the curve α lies in the plane.*

Proof. By differentiating $\varphi_N^T(s,v) = T(s) + vN(s)$ for s and v, using the MOFC given by Eq. (1), it is found as

$$\left(\varphi_N^T\right)_s = -v\kappa^2 T + \left(1 + \frac{v\kappa'}{\kappa}\right)N + v\tau B, \ \left(\varphi_N^T\right)_v = N. \tag{5}$$

From here, the normal vector field of the surface φ_N^T is obtained as follows:

$$U_N^T(s,v) = \frac{\left(\varphi_N^T\right)_s \times \left(\varphi_N^T\right)_v}{\left\|\left(\varphi_N^T\right)_s \times \left(\varphi_N^T\right)_v\right\|} = -\frac{\tau T + \kappa^2 B}{\sqrt{\kappa^6 + \tau^2}} \tag{6}$$

for the condition $\kappa \neq 0$. The first fundamental form I_N^T of the ruled surface φ_N^T , as in Eq. (2), is

$$I_N^T(s,v) = E_N^T(s,v)\,ds^2 + 2F_N^T(s,v)\,dsdv + G_N^T(s,v)\,dv^2$$

and its components are

$$E_N^T(s,v) = \kappa^2\left((1+v\lambda) + v^2\left(\kappa^2+\tau^2\right)\right), \ F_N^T(s,v) = \left(1 + v\frac{\kappa'}{\kappa}\right)\kappa^2, \ G_N^T(s,v) = \kappa^2. \tag{7}$$

By differentiating Eq. (5) in terms of s and v, we have

$$\begin{aligned}
\left(\varphi_N^T\right)_{ss} &= -\kappa\left(\kappa + v\left(\kappa' + 2\kappa'\right)\right)T + \left(\frac{\kappa'}{\kappa} - v\left(\kappa^2 + \tau^2 - \frac{\kappa''}{\kappa}\right)\right)N + \left(\tau + v\left(\frac{2\kappa'\tau}{\kappa} + \left(\frac{\kappa'}{\kappa}\right)'\right)\right)B,\\
\left(\varphi_N^T\right)_{sv} &= -\kappa^2 T + \frac{\kappa'}{\kappa}N + \tau B,\\
\left(\varphi_N^T\right)_{vv} &= 0.
\end{aligned}$$

Moreover, from the scalar product of the last equations and Eq. (6), using Eq. (3), the second fundamental form II_N^T of the ruled surface φ_N^T is

$$II_N^T(s,v) = e_N^T ds^2 + 2f_N^T dsdv + g_N^T dv^2$$

and its components are

$$e_N^T(s,v) = \frac{v\kappa\left(2\tau\kappa' - (\kappa\tau)'\right)}{\sqrt{\kappa^6+\tau^2}}, \ f_N^T(s,v) = 0, \ g_N^T(s,v) = 0. \tag{8}$$

Substituting the above-mentioned Eqs (7) and (8) into Eq (4), the Gauss and mean curvatures of φ_N^T are provided below the formulas, respectively,

$$K_N^T(s,v) = 0 \text{ and } H_N^T(s,v) = \frac{2\tau\kappa' - (\kappa\tau)'}{2\kappa\sqrt{\kappa^6+\tau^2}\left(v\left(\kappa^2+\tau^2\right) - \frac{\kappa'}{\kappa}\left(1 + v\frac{\kappa'}{\kappa}\right)\right)}. \tag{9}$$

On the other hand, by differentiating $\varphi_T^N(s,v) = N(s) + vT(s)$ according to s and v, and also using the MOFC, it is found as

$$\left(\varphi_T^N\right)_s = -\kappa^2 T + \left(v + \frac{\kappa'}{\kappa}\right)N + \tau B, \ \left(\varphi_T^N\right)_v = T. \tag{10}$$

So, from the last equations, the normal vector field of the surface φ_T^N is determined as follows:

$$U_T^N(s,v) = \frac{\left(\varphi_T^N\right)_s \times \left(\varphi_T^N\right)_v}{\left\|\left(\varphi_T^N\right)_s \times \left(\varphi_T^N\right)_v\right\|} = \frac{\kappa\tau N - (\kappa v + \kappa')B}{\kappa\sqrt{(\kappa v + \kappa')^2 + \kappa^2\tau^2}}. \tag{11}$$

Here, the condition $\kappa \neq 0$ requires $(\kappa\upsilon + \kappa')^2 + \kappa^2\tau^2 \neq 0$. Using Eq. (10), the first fundamental form I_T^N of the ruled surface φ_T^N, as in Eq. (2), is

$$I_T^N(s,\upsilon) = E_T^N(s,\upsilon)\,ds^2 + 2F_T^N(s,\upsilon)\,dsd\upsilon + G_T^N(s,\upsilon)\,d\upsilon^2$$

and its components are

$$E_T^N(s,\upsilon) = \kappa^4 + (\upsilon\kappa + \kappa')^2 + \tau^2\kappa^2,\ F_T^N(s,\upsilon) = -\kappa^2,\ G_T^N(s,\upsilon) = 1. \tag{12}$$

By differentiating Eq. (10) for s and υ, we have

$$\begin{aligned}
\left(\varphi_T^N\right)_{ss} &= \left(-\upsilon\kappa^2 - 3\kappa\kappa'\right)T + \left(\frac{\upsilon\kappa' + \kappa''}{\kappa} - \kappa^2 - \tau^2\right)N + \left(\upsilon\tau + \frac{2\kappa'\tau}{\kappa} + \tau'\right)B,\\
\left(\varphi_T^N\right)_{s\upsilon} &= N,\ \left(\varphi_T^N\right)_{\upsilon\upsilon} = 0.
\end{aligned}$$

Moreover, from the scalar product of the last equations and Eq. (11) using Eq. (3), the second fundamental form II_T^N of the ruled surface φ_T^N is

$$II_T^N(s,\upsilon) = e_T^N(s,\upsilon)\,ds^2 + 2f_T^N(s,\upsilon)\,dsd\upsilon + g_T^N(s,\upsilon)\,d\upsilon^2$$

and its components are

$$\begin{aligned}
e_T^N(s,\upsilon) &= -\frac{\tau^3 + \tau\left(\kappa^2 + \left(\upsilon + \frac{\kappa'}{\kappa}\right)^2 - \left(\frac{\kappa'}{\kappa}\right)'\right) + \left(\upsilon + \frac{\kappa'}{\kappa}\right)\tau'}{\sqrt{(\kappa\upsilon + \kappa')^2 + \kappa^2\tau^2}},\\
f_T^N(s,\upsilon) &= \frac{\tau}{\sqrt{(\kappa\upsilon + \kappa')^2 + \kappa^2\tau^2}},\\
g_T^N(s,\upsilon) &= 0.
\end{aligned} \tag{13}$$

Thus, by substituting the first and second fundamental forms given by Eqs. (12) and (3.12) into Eq. (4), the Gaussian and mean curvatures of φ_T^N are provided by

$$\begin{aligned}
K_T^N(s,\upsilon) &= -\frac{\tau^2}{\left((\kappa\upsilon + \kappa')^2 + \kappa^2\tau^2\right)^2},\\
H_T^N(s,\upsilon) &= \frac{\tau\left(\kappa^2 - \left(\upsilon + \frac{\kappa'}{\kappa}\right)^2 + \left(\frac{\kappa'}{\kappa}\right)'\right) - \left(\upsilon + \frac{\kappa'}{\kappa}\right)\tau' - \tau^3}{2\kappa^3\left(\left(\upsilon + \frac{\kappa'}{\kappa}\right)^2 + \tau^2\right)^{3/2}},
\end{aligned} \tag{14}$$

respectively, under the conditions stated in Lemma 2.1, based on Eqs. (9) and (14), one can say that the $TN-$PRSs are simultaneously developable and minimal surfaces. □

Theorem 3.3. *Let φ_N^T and φ_T^N be components of the TN−PRSs with the MOFC in R^3, then the s−parameter curves on the TN−PRSs with the MOFC are simultaneously*

i. not geodesic,

ii. asymptotic if and only if $\tau = 0$ and the curvature of the curve α is different from zero ($\kappa \neq 0$).

Proof. Let φ_N^T and φ_T^N be components of the $TN-$PRSs with the MOFC.

i. Considering the second derivative of the φ_N^T and φ_T^N for s and the normal vectors of the $TN-$PRSs, the following vector product can be calculated:

$$\left(\varphi_N^T\right)_{ss} \times U_N^T = \frac{\kappa^2\left(-\frac{\kappa'}{\kappa} + \upsilon\left(\kappa^2 - \left(\frac{\kappa'}{\kappa}\right)^2 + \tau^2 - \left(\frac{\kappa'}{\kappa}\right)'\right)\right)}{\sqrt{\kappa^6+\tau^2}}T - \frac{\kappa^4\left(1+\upsilon\frac{\kappa'}{\kappa}\right) + 2\upsilon\kappa^3\kappa' + \tau\left(\tau + 2\upsilon\frac{\kappa'\tau}{\kappa} + \upsilon\tau'\right)}{\sqrt{\kappa^6+\tau^2}}N + \frac{\tau\left(\frac{\kappa'}{\kappa} + \upsilon\left(-\kappa^2 + \left(\frac{\kappa'}{\kappa}\right)^2 - \tau^2 + \left(\frac{\kappa'}{\kappa}\right)'\right)\right)}{\sqrt{\kappa^6+\tau^2}}B$$

and

$$\left(\varphi_T^N\right)_{ss} \times U_T^N = \frac{\kappa^4\left(-\frac{\kappa'}{\kappa} + \upsilon\left(\kappa^2 - \left(\frac{\kappa'}{\kappa}\right)^2 + \tau^2 - \left(\frac{\kappa'}{\kappa}\right)'\right)\right)}{\sqrt{\kappa^6+\tau^2}}T - \frac{\kappa^2\left(\kappa^2\left(1+\upsilon\frac{\kappa'}{\kappa}\right) + 2\upsilon\kappa\kappa' + \tau\left(\tau + 2\upsilon\frac{\kappa'\tau}{\kappa} + \upsilon\tau'\right)\right)}{\sqrt{\kappa^6+\tau^2}}N + \frac{\kappa^2\tau\left(\frac{\kappa'}{\kappa} + \upsilon\left(-\kappa^2 + \left(\frac{\kappa'}{\kappa}\right)^2 - \tau^2 + \left(\frac{\kappa'}{\kappa}\right)'\right)\right)}{\sqrt{\kappa^6+\tau^2}}B.$$

So one can say that simultaneously $\left(\varphi_N^T\right)_{ss} \times U_N^T \neq 0$ and $\left(\varphi_T^N\right)_{ss} \times U_T^N \neq 0$ require the $s-$parameter curves on the $TN-$PRSs to be non-geodesic.

ii. Considering the second derivative of the φ_N^T and φ_T^N for s and the normal vectors of the $TN-$PRSs, the following scalar product can be calculated:

$$\left\langle\left(\varphi_N^T\right)_{ss}, U_N^T\right\rangle = \frac{\kappa\left(\kappa\left(1+\upsilon\frac{\kappa'}{\kappa}\right)\tau + 2\upsilon\tau\kappa' - \kappa^3\left(\tau + 2\upsilon\frac{\kappa'}{\kappa}\tau + \upsilon\tau'\right)\right)}{\sqrt{\kappa^6+\tau^2}}$$

and

$$\left\langle\left(\varphi_T^N\right)_{ss}, U_T^N\right\rangle = \frac{\kappa\left(\kappa\left(\upsilon + \frac{\kappa'}{\kappa}\right)\tau + 2\tau\kappa' - \kappa^3\left(\left(\upsilon + 2\frac{\kappa'}{\kappa}\right)\tau + \tau'\right)\right)}{\sqrt{\kappa^6+\tau^2}}.$$

From here, if $\tau = 0$ and $\kappa \neq 0$, then $\left\langle\left(\varphi_N^T\right)_{ss}, U_N^T\right\rangle = 0$ and $\left\langle\left(\varphi_T^N\right)_{ss}, U_T^N\right\rangle = 0$. So, we can say that $s-$parameter curves on the $TN-$PRSs are simultaneously asymptotic curves if and only if $\tau = 0$ and $\kappa \neq 0$.

□

Theorem 3.4. *Let φ_N^T and φ_T^N be components of the TN−PRSs with MOFC in R^3, then the $\upsilon-$parameter curves on the TN−PRSs with MOFC are simultaneously*

i. geodesics,

ii. asymptotic.

Proof. Let φ_N^T and φ_T^N be components of the $TN-$PRSs with MOFC in R^3.

i. Since $\left(\varphi_N^T\right)_{\upsilon\upsilon} \times U_N^T = 0$ and $\left(\varphi_T^N\right)_{\upsilon\upsilon} \times U_T^N = 0$, the $\upsilon-$parameter curves of the $TN-$PRSs simultaneously are geodesics.

ii. Since $\left\langle\left(\varphi_N^T\right)_{\upsilon\upsilon}, U_N^T\right\rangle = 0$ and $\left\langle\left(\varphi_T^N\right)_{\upsilon\upsilon}, U_T^N\right\rangle = 0$, the $\upsilon-$parameter curves of the $TN-$PRSs with MOFC simultaneously are asymptotic.

□

Theorem 3.5. *Let φ_N^T and φ_T^N be components of the TN−PRSs with MOFC, and then the parameter curves of TN−PRSs simultaneously cannot be the line of curvatures.*

Proof. Let φ_N^T and φ_T^N be components of the $TN-$PRSs with MOFC. From Eqs. (7), (8), (12), and (3.12), since $\kappa \neq 0$, we can say that the parameter curves of $TN-$PRSs simultaneously cannot be the line of curvatures. □

Definition 3.6. *Let $\alpha : I \subset R \to R^3$, for all $s \in I$, be a differentiable unit speed space curve with the MOFC elements $\{T, N, B, \kappa, \tau\}$ such that $\kappa(s) \neq 0$, $(\kappa')^2 + \kappa^2(v-\tau) \neq 0$,, $v\tau \neq 1$. The ruled surfaces that are generated by the motion of the binormal vector $B(T)$ on the tangent vector $T(B)$ are defined by*

$$\begin{cases} \varphi_B^T(s,v) = T(s) + vB(s), \\ \varphi_T^B(s,v) = B(s) + vT(s), \end{cases}$$

and these surfaces are called TB−PRSs with respect to the MOFC of the curve α in R^3.

Theorem 3.7. *Let φ_B^T and φ_T^B be components of the TB−PRSs in R^3, then the TB−PRSs are simultaneously*

i. developable if and only if the curvature κ of α is a non-zero constant,

ii. not minimal.

Proof. By differentiating $\varphi_B^T(s,v) = T(s) + vB(s)$ concerning s and v using MOFC, one can find as:

$$\left(\varphi_B^T\right)_s = (1 - v\tau)N + \frac{v\kappa'}{\kappa}B, \ \left(\varphi_B^T\right)_v = B. \tag{15}$$

Through the last equations, the normal vector field of the surface φ_N^T is obtained as follows:

$$U_B^T = \frac{\left(\varphi_B^T\right)_s \times \left(\varphi_B^T\right)_v}{\left\|\left(\varphi_B^T\right)_s \times \left(\varphi_B^T\right)_v\right\|} = \frac{1 - v\tau}{|-1 + v\tau|}T = \pm T. \tag{16}$$

Here $v\tau \neq 1$ satisfies $-1 + v\tau \neq 0$. Using Eq. (15), the first fundamental form I_B^T of the ruled surface φ_N^T, as in Eq. (2), is

$$I_B^T(s,v) = E_B^T(s,v)\,ds^2 + 2F_B^T(s,v)\,dsdv + G_B^T(s,v)\,dv^2$$

and its components are

$$E_B^T(s,v) = (v\kappa')^2 + (-1 + v\tau)^2\kappa^2, \ F_B^T(s,v) = v\kappa'\kappa, \ G_B^T(s,v) = \kappa^2. \tag{17}$$

By differentiating Eq. (15) for s and v, we get

$$\begin{aligned} \left(\varphi_B^T\right)_{ss} &= \kappa^2(-1 + v\tau)T + \left(\frac{\kappa' - 2v\kappa'\tau}{\kappa} - v\tau'\right)N + \left(\frac{v\kappa''}{\kappa} + \tau - v\tau^2\right)B, \\ \left(\varphi_B^T\right)_{sv} &= -\tau N + \frac{\kappa'}{\kappa}B, \\ \left(\varphi_B^T\right)_{vv} &= 0, \end{aligned}$$

By substituting the last equations and (16) into Eq. (**??**2.3), the second fundamental form II_B^T of the ruled surface is presented as

$$II_B^T(s,v) = e_B^T(s,v)\,ds^2 + 2f_B^T(s,v)\,dsdv + g_B^T(s,v)\,dv^2$$

with its components

$$e_B^T(s,v) = -\kappa^2|-1 + v\tau|, \ f_B^T(s,v) = 0, \ g_B^T(s,v) = 0. \tag{18}$$

From here, by substituting Eqs. (17) and (18) into Eq. (4), the Gaussian curvature K_B^T and the mean curvature H_B^T of φ_B^T are determined as

$$K_B^T = 0,\ H_B^T = -\frac{1}{2\left|-1+v\tau\right|}. \tag{19}$$

On the other hand, by differentiating $\varphi_T^B(s,v) = B(s) + vT(s)$ according to parameters s and v, it is obtained as

$$\left(\varphi_T^B\right)_s = (v-\tau)N + \frac{\kappa'}{\kappa}B,\ \left(\varphi_T^B\right)_v = T. \tag{20}$$

So, the normal vector field of the surface φ_T^B using Eq. (20) is found as follows:

$$U_T^B = \frac{\left(\varphi_T^B\right)_s \times \left(\varphi_T^B\right)_v}{\left\|\left(\varphi_T^B\right)_s \times \left(\varphi_T^B\right)_v\right\|} = \frac{\kappa' N + (-v+\tau)\kappa B}{\sqrt{\kappa'^2 + \kappa^2(v-\tau)^2}}, \tag{21}$$

where $\kappa \neq 0$ and $\kappa'^2 + \kappa^2(v-\tau)^2 \neq 0$. Considering Eqs. (2) and (20) together, the first fundamental form I_T^B of φ_T^B is

$$I_T^B(s,v) = E_T^B(s,v)\,ds^2 + 2F_T^B(s,v)\,dsdv + G_T^B(s,v)\,dv^2$$

with components

$$E_T^B(s,v) = \kappa'^2 + (v-\tau)^2\kappa^2, F_T^B(s,v) = 0,\ G_T^B(s,v) = 1. \tag{22}$$

By differentiating Eq. (20) for s and v, we have

$$\begin{aligned}
\left(\varphi_T^B\right)_{ss} &= \kappa^2(-v+\tau)T + \left(\frac{v\kappa' - 2\kappa'\tau}{\kappa} - \tau'\right)N + \left(\frac{\kappa''}{\kappa} + v\tau - \tau^2\right)B,\\
\left(\varphi_T^B\right)_{sv} &= N,\\
\left(\varphi_T^B\right)_{vv} &= 0.
\end{aligned}$$

From the scalar product of the last equations with Eq. (21), the second fundamental form II_T^B of φ_T^B is found as follows:

$$II_T^B(s,v) = e_T^B(s,v)\,ds^2 + 2f_T^B(s,v)\,dsdv + g_T^B(s,v)\,dv^2$$

with components

$$\begin{aligned}
e_T^B(s,v) &= -\frac{\kappa\left(-2v\tau^2 + \tau^3 + \tau\left(v^2 + \left(\frac{\kappa'}{\kappa}\right)^2 - \left(\frac{\kappa'}{\kappa}\right)'\right) + v - \left(\frac{\kappa'}{\kappa}\right)' + \frac{\kappa'\tau'}{\kappa}\right)}{\sqrt{\left(\frac{\kappa'}{\kappa}\right)^2 + (v-\tau)^2}},\\
f_T^B(s,v) &= \frac{\kappa'}{\sqrt{\left(\frac{\kappa'}{\kappa}\right)^2 + (v-\tau)^2}},\ g_T^B(s,v) = 0.
\end{aligned} \tag{23}$$

So, by substituting Eqs. (22) and (23) into Eq. (4), the Gaussian curvature K_T^B and the mean curvature H_T^B of

φ_T^B are determined as

$$K_T^B = -\frac{\left(\frac{\kappa'}{\kappa}\right)^2}{\left(\left(\frac{\kappa'}{\kappa}\right)^2 + (\upsilon - \tau)^2\right)^2},$$
$$H_T^B = \frac{2\upsilon\tau^2 - \tau^3 - \tau\left(\upsilon^2 + \left(\frac{\kappa'}{\kappa}\right)^2 - \left(\frac{\kappa'}{\kappa}\right)'\right) - \upsilon\left(\frac{\kappa'}{\kappa}\right)' - \frac{\kappa'\tau'}{\kappa}}{2\kappa\left(\left(\frac{\kappa'}{\kappa}\right)^2 + (\upsilon - \tau)^2\right)^{3/2}}. \tag{24}$$

Consequently, from Eqs. (19) and (24), it can easily be implied that $TB-$PRSs are simultaneously developable if the curvature κ is a non-zero constant, but not minimal surface since the curvature κ is non-zero. □

Theorem 3.8. *Let φ_B^T and φ_T^B be components of the $TB-$PRSs with the MOFC in R^3, then the $s-$parameter curves of $TB-$PRSs cannot be simultaneously geodesics and asymptotic.*

Proof. The proof is carried out similarly to the proof of the theorem provided for $TN-$PRSs with the MOFC. □

Theorem 3.9. *Let φ_B^T and φ_T^B be a pair of the $TB-$PRSs with the MOFC in R^3, then the $\upsilon-$parameter curves of $TB-$PRSs are simultaneously geodesics and asymptotic curves.*

Proof. The proof is completed similarly to the proof of the theorem provided for $TN-$PRSs with MOFC. □

Theorem 3.10. *Let φ_B^T and φ_T^B be components of the $TB-$PRSs with the MOFC in R^3, then the s and $\upsilon-$parameter curves of $TB-$PRSs are simultaneously lines of curvature if and only if the curvature κ of α is a non-zero constant.*

Proof. The proof is similar to the proof of the theorem for $TN-$PRSs. □

Definition 3.11. *Let $\alpha : I \subset R \to R^3$, for all $s \in I$, be a differentiable unit speed space curve with the MOFC elements $\{T, N, B, \kappa, \tau\}$ such that $\kappa(s) \neq 0$. The ruled surfaces that are generated by the motion of the principal normal vector $N(B)$ on the binormal vector $B(N)$ are defined by*

$$\begin{cases} \varphi_B^N(s,\upsilon) = N(s) + \upsilon B(s), \\ \varphi_N^B(s,\upsilon) = B(s) + \upsilon N(s), \end{cases} \tag{25}$$

and these surfaces are called $NB-$PRSs with respect to the MOFC of the curve α in R^3.

Theorem 3.12. *Let φ_B^N and φ_N^B be components of the $NB-$PRSs with the MOFC in R^3, then $NB-$PRSs with the MOFC are simultaneously*

i. developable if and only if $\tau = 0$ and κ is non-zero constant.

ii. not minimal.

Proof. By differentiating the first equation of Eq. (25) with respect to s and υ, respectively, and using the MOFC derivative formulae, one can obtain

$$\left(\varphi_B^N\right)_s = -\kappa^2 T + \left(\frac{\kappa'}{\kappa} - \upsilon\tau\right)N + \left(\frac{\upsilon\kappa'}{\kappa} + \tau\right)B, \ \left(\varphi_B^N\right)_\upsilon = B. \tag{26}$$

From the cross product of both vectors $\left(\varphi_B^N\right)_s$ and $\left(\varphi_N^B\right)_s$ which are the partial derivatives of φ_B^N given by Eq. (26), the normal vector field of φ_B^N is found as:

$$U_B^N = \frac{\left(\varphi_B^N\right)_s \times \left(\varphi_B^N\right)_\upsilon}{\left\|\left(\varphi_B^N\right)_s \times \left(\varphi_B^N\right)_\upsilon\right\|} = \frac{\left(\frac{\kappa'}{\kappa} - \upsilon\tau\right)T + \kappa^2 N}{\sqrt{\kappa^6 + \left(\frac{\kappa'}{\kappa} - \upsilon\tau\right)^2}}. \tag{27}$$

Here $\kappa \neq 0$ satisfies $\kappa^6 + \left(\frac{\kappa'}{\kappa} - v\tau\right)^2 \neq 0$. By substituting Eq. (26) into Eq. (2), the first fundamental form I_B^N of φ_B^N is

$$I_B^N(s,v) = E_B^N(s,v)\,ds^2 + 2F_B^N(s,v)\,dsdv + G_B^N(s,v)\,dv^2$$

and its components are

$$E_B^N(s,v) = \kappa^4 + (\kappa' - v\tau\kappa)^2 + (v\kappa' + \tau\kappa)^2, F_B^N(s,v) = \kappa(v\kappa' + \tau\kappa),\ G_B^N(s,v) = \kappa^2. \tag{28}$$

By differentiating Eq. (26), we get

$$\left(\varphi_B^N\right)_{ss} = \left(v\kappa^2\tau - 3\kappa\kappa'\right)T + \left(\frac{\kappa'' - 2v\kappa'\tau}{\kappa} - \tau^2 - \kappa^2 - v\tau'\right)N + \left(\frac{v\kappa'' + 2\kappa'\tau}{\kappa} - v\tau^2 + \tau'\right)B,$$
$$\left(\varphi_B^N\right)_{sv} = -N\tau + \frac{\kappa'}{\kappa}B,\ \left(\varphi_B^N\right)_{vv} = 0.$$

By substituting the last equations and (27) into Eq. (3), the second fundamental form II_B^N of φ_B^N is presented as

$$II_B^N(s,v) = e_B^N(s,v)\,ds^2 + 2f_B^N(s,v)\,dsdv + g_B^N(s,v)\,dv^2$$

and its components are

$$\begin{aligned} e_B^N(s,v) &= -\frac{\kappa\left(\kappa^5 + \kappa\left(\frac{\kappa' - v\tau\kappa}{\kappa}\right)^2 + 2\left(\frac{\kappa' - v\tau\kappa}{\kappa}\right)\kappa' + \kappa^3\left(2\frac{v\kappa'\tau}{\kappa} + \tau^2 + v\tau' - \left(\frac{\kappa'}{\kappa}\right)^2 - \left(\frac{\kappa'}{\kappa}\right)'\right)\right)}{\sqrt{\kappa^6 + \left(\frac{\kappa'}{\kappa} - v\tau\right)^2}}, \\ f_B^N(s,v) &= -\frac{\kappa^4\tau}{\sqrt{\kappa^6 + \left(\frac{\kappa'}{\kappa} - v\tau\right)^2}}, \\ g_B^N(s,v) &= 0. \end{aligned} \tag{29}$$

Thus, by substituting Eqs. (28) and (29) into Eq. (4), the Gaussian curvature K_B^N and the mean curvature H_B^N of the ruled surface φ_B^N are calculated as

$$\begin{aligned} K_B^N &= -\frac{\kappa^4\tau^2}{\left(\kappa^2 + \left(\frac{\kappa'}{\kappa} - v\tau\right)^2\right)\left(\kappa^6 + \left(\frac{\kappa'}{\kappa} - v\tau\right)^2\right)}, \\ H_B^N &= \frac{\kappa^3\left(\left(\frac{\kappa'}{\kappa}\right)^2 + \tau^2 + \left(\frac{\kappa'}{\kappa}\right)' - v\tau'\right) - \kappa^5 - \kappa\left(\frac{\kappa'}{\kappa} - v\tau\right)^2 - 2\left(\frac{\kappa'}{\kappa} - v\tau\right)\kappa'}{2\kappa\left(\kappa^2 + \left(\frac{\kappa'}{\kappa} - v\tau\right)^2\right)\sqrt{\kappa^6 + \left(\frac{\kappa'}{\kappa} - v\tau\right)^2}}. \end{aligned} \tag{30}$$

On the other hand, by differentiating the second equation of Eq. (25) with respect to s and v, respectively, and using the MOFC derivative formulae, we find

$$\left(\varphi_N^B\right)_s = -v\kappa^2 T + \left(\frac{v\kappa'}{\kappa} - \tau\right)N + \left(\frac{\kappa'}{\kappa} + v\tau\right)B,\ \left(\varphi_N^B\right)_v = N. \tag{31}$$

From the cross product of both vectors $\left(\varphi_N^B\right)_s$ and $\left(\varphi_N^B\right)_v$ which are the partial derivatives of φ_N^B given by Eq. (31), the normal vector field of the surface φ_N^B is found as:

$$U_N^B = \frac{\left(\varphi_N^B\right)_s \times \left(\varphi_N^B\right)_v}{\left\|\left(\varphi_N^B\right)_s \times \left(\varphi_N^B\right)_v\right\|} = -\frac{\left(\frac{\kappa'}{\kappa} + v\tau\right)T + v\kappa^2 B}{\sqrt{v^2\kappa^6 + \left(\frac{\kappa'}{\kappa} + v\tau\right)^2}}. \tag{32}$$

Here $\kappa \neq 0$ satisfies $\upsilon^2\kappa^6 + \left(\frac{\kappa'}{\kappa} + \upsilon\tau\right)^2 \neq 0$. Substituting Eq. (31) into Eq. (2), the first fundamental form I_N^B of φ_N^B holds:

$$I_N^B(s,\upsilon) = E_N^B(s,\upsilon)\,ds^2 + 2F_N^B(s,\upsilon)\,dsd\upsilon + G_N^B(s,\upsilon)\,d\upsilon^2$$

and its components are

$$E_N^B(s,\upsilon) = \upsilon^2\kappa^4 + (\upsilon\kappa' - \tau\kappa)^2 + (\kappa' + \upsilon\tau\kappa)\,\kappa, \qquad F_N^B(s,\upsilon) = (\upsilon\kappa' - \tau\kappa)\,\kappa, \qquad G_N^B(s,\upsilon) = \kappa^2. \tag{33}$$

By differentiating Eq. (31), it gets

$$\begin{aligned}
\left(\varphi_N^B\right)_{ss} &= \left(\kappa^2\tau - 3\upsilon\kappa\kappa'\right)T + \left(\frac{\upsilon\kappa'' - 2\kappa'\tau}{\kappa} - \upsilon\left(\tau^2+\kappa^2\right) - \tau'\right)N + \left(\frac{\kappa'' + 2\upsilon\kappa'\tau}{\kappa} - \tau^2 + \upsilon\tau'\right)B,\\
\left(\varphi_N^B\right)_{s\upsilon} &= -\kappa^2 T + \frac{\kappa'}{\kappa}N + \tau B,\ \left(\varphi_N^B\right)_{\upsilon\upsilon} = 0.
\end{aligned}$$

By substituting Eq. (3.30) and the last equations into Eq. (2.3), the second fundamental form II_N^B of φ_N^B satisfies

$$II_N^B(s,\upsilon) = e_N^B(s,\upsilon)\,ds^2 + 2f_N^B(s,\upsilon)\,dsd\upsilon + g_N^B(s,\upsilon)\,d\upsilon^2.$$

Its components are

$$\begin{aligned}
e_N^B(s,\upsilon) &= \frac{(\kappa' + \upsilon\tau\kappa)^2 + 2\upsilon\kappa'(\kappa' + \upsilon\tau\kappa) - \upsilon\kappa^4\left(\left(\frac{\kappa'}{\kappa}\right)^2 + 2\frac{\upsilon\kappa'\tau}{\kappa} - \tau^2 + \left(\frac{\kappa'}{\kappa}\right)' + \upsilon\tau'\right)}{\sqrt{\upsilon^2\kappa^6 + \left(\frac{\kappa'}{\kappa} + \upsilon\tau\right)^2}},\\
f_N^B(s,\upsilon) &= \frac{\kappa\left(\kappa' + \upsilon\left(1-\kappa^2\right)\tau\kappa\right)}{\sqrt{\upsilon^2\kappa^6 + \left(\frac{\kappa'}{\kappa} + \upsilon\tau\right)^2}},\\
g_N^B(s,\upsilon) &= 0.
\end{aligned} \tag{34}$$

So, by substituting Eqs. (3.31), (3.32) into Eq. (2.4), the Gaussian curvature K_N^B and the mean curvature H_N^B of φ_N^B are calculated by

$$\begin{aligned}
K_N^B &= -\frac{\left(\frac{\kappa'}{\kappa} - \upsilon\left(-1+\kappa^2\right)\tau\right)^2}{\left(\frac{\kappa'}{\kappa} + \upsilon\left(\upsilon\kappa^2 + \tau\right)\right)\left(\upsilon^2\kappa^6 + \left(\frac{\kappa'}{\kappa} + \upsilon\tau\right)^2\right)},\\
H_N^B &= \frac{\left(\frac{\kappa' + \upsilon\tau\kappa}{\kappa}\right)\left((\tau\kappa - \upsilon\kappa') + 2\upsilon\kappa'\right) - \upsilon\kappa^3\left(\left(\frac{\kappa'}{\kappa}\right)^2 + \tau^2 + \left(\frac{\kappa'}{\kappa}\right)' + \upsilon\tau'\right)}{2\left(\kappa' + \upsilon\left(\upsilon\kappa^2 + \tau\right)\kappa\right)\sqrt{\upsilon^2\kappa^6 + \left(\frac{\kappa'}{\kappa} + \upsilon\tau\right)^2}}.
\end{aligned} \tag{35}$$

In conclusion, it can be said that the $NB-$PRSs are simultaneously developable and minimal. □

Theorem 3.13. *Let φ_B^N and φ_N^B be components of the NB−PRSs with the MOFC, then s−parameter curves of NB−PRSs are simultaneously*

i. non-geodesic,

ii. asymptotic if and only if $\tau = 0$ and κ is a non-zero constant.

Proof. It is proven similarly to the proof of the theorem given for $TN-$PRSs. □

Theorem 3.14. *Let φ_B^N and φ_N^B be components of the NB−PRSs with the MOFC, then the $\upsilon-$parameter curves of TN−PRSs are simultaneously*

i. geodesic,

ii. asymptotic.

Proof. The proof is carried out in a manner similar to that of the theorem provided for $TN-$PRSs. □

Theorem 3.15. *Let φ_B^N and φ_N^B be components of the NB−PRSs with the MOFC, then the s and $\upsilon-$parameter curves of NB−PRSs are simultaneously lines of curvatures if and only if $\tau = 0$ and κ is a non-zero constant.*

Proof. The proof is similar to that in $TN-$PRSs. □

Corollary 3.16. *Let TN, TB, and NB be PRSs, then characterizations of PRSs can be summarized in the table below:*

	TN−PRSs	*TB−PRSs*	*NB−PRSs*
Developability:	*developable if and only if α is planar*	*developable if and only if κ is non-zero constant*	*developable if and only if $\tau = 0$ and κ is non-zero constant*
Minimality:	*minimal if and only if α is planar*	*not minimal*	*not minimal*
s−parameter curves:	*• not geodesic and line of curvature, • asymptotic if and only if $\tau = 0$ and κ is non-zero constant*	*• not geodesic and asymptotic, • line of curvature if and only if κ is non-zero constant*	*• not geodesic, • asymptotic and line of curvature if and only if $\tau = 0$ and κ is non-zero constant*
$\upsilon-$parameter curves:	*• geodesic and asymptotic, • not a line of curvature*	*• geodesic and asymptotic, • line of curvature if and only if κ is non-zero constant*	*• geodesic and asymptotic, • line of curvature if and only if $\tau = 0$ and κ is non-zero constant*

Example 3.17. *Let us consider the Cornu (or Euler) spiral defined by the parametric equation*

$$\alpha(s) = \left(\frac{1}{\sqrt{2}} \int_0^s \cos \frac{\pi t^2}{2} dt, \frac{1}{\sqrt{2}} \int_0^s \sin \frac{\pi t^2}{2} dt, \frac{s}{\sqrt{2}} \right).$$

The Frenet elements of the curve $\alpha(s)$ are presented as

$$\begin{aligned} t &= \left(\frac{1}{\sqrt{2}} \cos \frac{\pi s^2}{2}, \frac{1}{\sqrt{2}} \sin \frac{\pi s^2}{2}, \frac{1}{\sqrt{2}} \right), \\ n &= \left(-\sin \frac{\pi s^2}{2}, \cos \frac{\pi s^2}{2}, 0 \right), \\ b &= \left(-\frac{1}{\sqrt{2}} \cos \frac{\pi s^2}{2}, -\frac{1}{\sqrt{2}} \sin \frac{\pi s^2}{2}, \frac{1}{\sqrt{2}} \right), \\ \kappa &= \frac{\pi s}{\sqrt{2}}, \ \tau = \frac{\pi s}{\sqrt{2}}. \end{aligned}$$

Since $\alpha''(s_0) = 0$ for $s_0 = 0$. At these points, the Frenet frame cannot be defined. Therefore, it is useful to consider the MOFC of the Cornu spiral curve as given below:

$$T = \left(\frac{1}{\sqrt{2}}\cos\frac{\pi s^2}{2}, \frac{1}{\sqrt{2}}\sin\frac{\pi s^2}{2}, \frac{1}{\sqrt{2}}\right),$$
$$N = \left(-\frac{\pi s}{\sqrt{2}}\sin\frac{\pi s^2}{2}, \frac{\pi s}{\sqrt{2}}\cos\frac{\pi s^2}{2}, 0\right),$$
$$B = \left(-\frac{\pi s}{2}\cos\frac{\pi s^2}{2}, -\frac{\pi s}{2}\sin\frac{\pi s^2}{2}, \frac{\pi s}{2}\right).$$

Thus, we get the parametric forms for the $TN-$PRSs based on the MOFC as follows:

$$\begin{cases} \varphi_N^T = \frac{1}{\sqrt{2}}\left(\cos\frac{\pi s^2}{2} - v\sin\frac{\pi^2 s^3}{2},\ \sin\frac{\pi s^2}{2} + v\cos\frac{\pi^2 s^3}{2},\ 1\right) \\ \varphi_T^N = \frac{1}{\sqrt{2}}\left(-\sin\frac{\pi^2 s^3}{2} + v\cos\frac{\pi s^2}{2},\ \cos\frac{\pi^2 s^3}{2} + v\sin\frac{\pi s^2}{2},\ v\right). \end{cases}$$

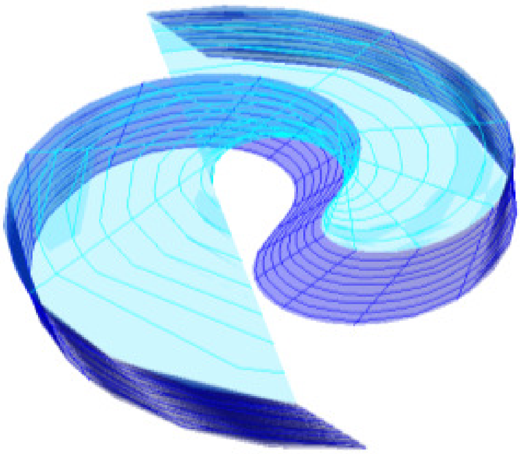

Figure 1: $TN-$PRSs (φ_N^T(cyan) and φ_T^N(blue)) with $s = (-\pi/2,\ \pi/2)$ and $v = (-1, 1)$.

Thus, we get the parametric forms for the $TB-$PRSs based on the MOFC as follows:

$$\begin{cases} \varphi_B^T = \frac{1}{2}\left(\cos\frac{\pi s^2}{2}\left(\sqrt{2} - \pi s\, v\right), \sin\frac{\pi s^2}{2}\left(\sqrt{2} - \pi s\, v\right),\ \sqrt{2} + \pi s\, v\right) \\ \varphi_T^B = \frac{1}{2}\left(\cos\frac{\pi s^2}{2}\left(\sqrt{2}v - \pi s\right), \sin\frac{\pi s^2}{2}\left(\sqrt{2}v - \pi s\right),\ \sqrt{2}v + \pi s\right). \end{cases}$$

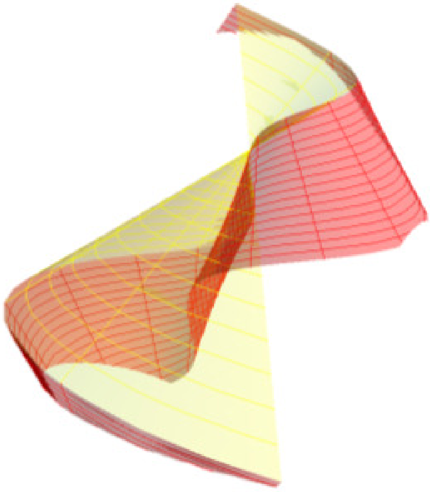

Figure 2: $TB-$PRSs (φ_B^T(yellow) and φ_T^B(red)) with $s = (-\pi/2,\ \pi/2)$ and $\upsilon = (-1, 1)$.

Thus, we get the parametric forms for the NB−PRSs based on the MOFC as follows:

$$\begin{cases} \varphi_B^N = \frac{-\pi s}{2}\left(\sqrt{2}\sin\frac{\pi s^2}{2} + \upsilon\cos\frac{\pi s^2}{2}, -\sqrt{2}\cos\frac{\pi s^2}{2} + \upsilon\sin\frac{\pi s^2}{2}, -\upsilon\right) \\ \varphi_N^B = \frac{-\pi s}{2}\left(\cos\frac{\pi s^2}{2} + \sqrt{2}\upsilon\sin\frac{\pi s^2}{2},\ \sin\frac{\pi s^2}{2} - \sqrt{2}\upsilon\cos\frac{\pi s^2}{2}, -1\right). \end{cases}$$

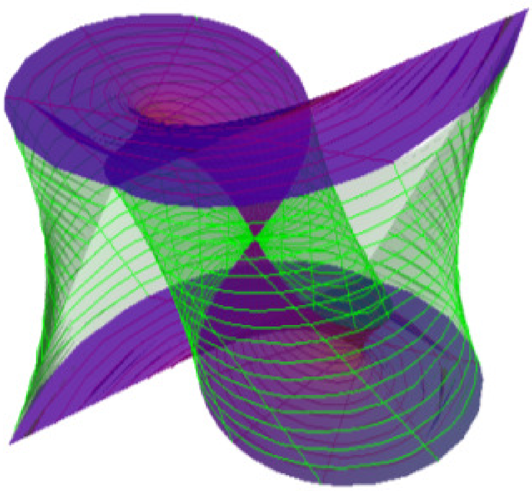

Figure 3: $NB-$PRSs (φ_B^N(green) and φ_N^B(purple)) with $s = (-\pi/2,\ \pi/2)$ and $\upsilon = (-1, 1)$.

4. Conclusion

In this paper, the invariants of the PRSs formed by the modified orthogonal vector fields of a space curve are simultaneously presented in Euclidean 3-space. The necessary conditions for each pair of the PRSs to be simultaneously developable and minimal are given. In addition, several characterizations of the parameter curves of all PRSs are investigated. The parametric equations of the PRSs are derived, and their graphical representations are illustrated.

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