Filomat 39:32 (2025), 11547–11561
https://doi.org/10.2298/FIL2532547G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Characterization of a Sasakian manifold admitting nearly vacuum static equations

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Abstract. In this study, we explore the geometric structure of nearly vacuum static equations (NVSE) within the context of Sasakian geometry. Focusing on η-Einstein Sasakian manifolds, we demonstrate that the presence of such equations implies constant scalar curvature, with solutions either being trivial or reducing the manifold to an Einstein one. Furthermore, we investigate the implications in dimension three, showing that non-trivial solutions of nearly vacuum static equations necessarily transform the Sasakian manifold into a Sasakian space form. Notably, when such a manifold admits a non-isometric conformal vector field, it is shown to possess constant sectional curvature equal to one. These results contribute to a deeper understanding of the interplay between curvature conditions and static equations in contact metric geometry. Furthermore, we rigorously prove several theorems characterizing generalized quasi-Einstein Sasakian manifolds admitting nearly vacuum static equations. Finally, to substantiate the obtained results, we conclude with an explicit example of a 3-dimensional Sasakian manifold that satisfies the nearly vacuum static equation, thereby illustrating the geometric constraints and confirming our theoretical findings.

1. Introduction

Contact geometry has gained significant attention among prominent mathematicians due to its fundamental role in modern differential geometry and its widespread applications in mechanics, optics, thermodynamics, and the phase space of dynamical systems. One of the key frameworks in contact geometry arises from the mathematical structure underlying classical mechanics. Among the many subfields, Sasakian geometry has stood out due to its rich geometric structure and its role in theoretical physics and complex differential geometry.

Sasakian manifolds, a special subclass of contact metric manifolds, serve as the odd-dimensional counterpart of Kähler geometry. These manifolds are characterized by an intricate compatibility between contact, metric, and almost complex structures. The current study focuses on the nearly vacuum static equation

2020 *Mathematics Subject Classification*. Primary 53C15; Secondary 53C25, 53E20.
Keywords. Sasakian manifold, Vaccum static equation, Scalar curvature, Sectional curvature.
Received: 19 June 2025; Accepted: 04 October 2025
Communicated by Mića S Stanković
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(nearly VSE) on Sasakian manifolds—a topic that remains relatively unexplored yet is promising due to the manifold's geometric constraints and symmetry properties.

The concept of Sasakian manifolds was formalized through the work of Boyer et al. [5], Blair [2, 3], and others, offering a compelling framework for analyzing geometric equations with physical significance. These manifolds satisfy certain curvature conditions that make them ideal candidates for investigating Ricci-type equations and static conditions from general relativity and geometric analysis

An n-dimensional semi-Riemannian manifold M^n is named the static space with perfect fluid if there exists a non-trivial smooth function f such that

$$Hess f - f(S - \frac{\mathbf{r}}{n-1}g) - \frac{1}{n}(\frac{\mathbf{r}}{n-1} + \Delta f)g = 0, \tag{1}$$

where Hess, Δ, $\mathbf{r}$ and S represent the Hessian operator, Laplacian, scalar curvature and Ricci tensor, respectively. This arises naturally in the study of static spacetimes in general relativity. If $\frac{\mathbf{r}}{n-1} + \Delta f = 0$, then the equation (1) represents vacuum static equation (VSE). Hence the VSE is demonstrated by

$$Hess f = f(S - \frac{\mathbf{r}}{n-1}g). \tag{2}$$

Readers who want to learn more about VSEs can read the articles [9, 10, 16, 18–20].

The NVSE is a natural generalization of VSE which was defined in [17]. The nearly VSE (NVSE) is written as

$$Hess f - f(S - \frac{\mathbf{r}}{n-1}g) = \frac{\lambda}{n}g, \tag{3}$$

$\lambda \in \mathbb{R}$. The above equation turns into classical VSE for $\lambda = 0$.

Building upon this foundation, the concept of NVSE was introduced as a natural generalization of VSEs. This extension allows for a broader class of solutions, characterized by an additional constant term, thereby enriching the geometric landscape of static equations. Recent studies have applied nearly VSEs to various geometric structures, including K-contact [18] and almost coKähler manifolds [17], yielding insights into the interplay between curvature conditions and static equations. Notably, these investigations have led to the identification of specific conditions under which nearly VSEs admit non-trivial solutions, furthering our understanding of the geometric properties of these manifolds.

Despite these advancements, the exploration of nearly VSEs on Sasakian manifolds remains relatively underdeveloped. Given the rich geometric structure of Sasakian manifolds and their relevance in both mathematics and physics, they present a promising area for extending the study of nearly VSEs. The current work aims to fill this gap by investigating the geometric properties of NVSE within the context of Sasakian geometry, thereby contributing to the broader understanding of static equations in contact metric geometry.

2. Sasakian manifolds

A Riemannian manifold of dimension $(2m+1)$ equipped with a structure (ϕ, ξ, η, g), is called an almost contact manifold if the following relations hold [2, 3]

$$\phi^2 = -I + \eta \otimes \xi, \eta(\xi) = 1, \ \phi\xi = 0, \ \eta \circ \phi = 0, \tag{4}$$

$$g(X_1, X_2) = g(\phi X_1, \phi X_2) + \eta(X_1)\eta(X_2), \tag{5}$$

$$g(X_1, \phi X_2) = -g(\phi X_1, X_2), \ g(X_1, \xi) = \eta(X_1), \tag{6}$$

for all smooth vectors fields X_1, X_2.

If, in addition, the structure satisfies

$$d\eta(X_1, X_2) = \Phi(X_1, X_2) = g(X_1, \phi X_2),$$

then the manifold is known as a contact metric manifold.

A normal contact metric manifold is known as Sasakian manifold. An almost contact metric manifold is Sasakian if and only if [5, 21–23]

$$(\nabla_{X_1}\phi)X_2 = g(X_1, X_2)\xi - \eta(X_2)X_1 \tag{7}$$

for all vectors field X_1, X_2, where ∇ is the Levi-Civita connection of the Riemannian metric. From this identity the following relations can be derived

$$\nabla_{X_1}\xi = -\phi X_1, \tag{8}$$

$$(\nabla_{X_1}\eta)X_2 = g(X_1, \phi X_2). \tag{9}$$

Additionaly, the curvature tensor R, the Ricci tensor S and the Ricci operator Q defined by $g(QX_1, X_2) = S(X_1, X_2)$ for all X_1, X_2 satisfy [5, 25] the following identities

$$R(X_1, X_2)\xi = \eta(X_2)X_1 - \eta(X_1)X_2, \tag{10}$$

$$S(X_1, \xi) = 2n\eta(X_1), \tag{11}$$

$$Q\xi = 2n\xi. \tag{12}$$

$$S(\phi X_1, \phi X_2) = S(X_1, X_2) - 2n\eta(X_1)\eta(X_2). \tag{13}$$

Sasakian manifolds have been studied by several authors such as Boyer [5], Tanno [25], De et al. [12], Güvenç and Özgür [15], Blaga [4] and many others.

3. Nearly vaccum static equation on Sasakian manifolds

In this section we characterize Sasakian manifold of dimension $(2m+1)$ admitting nearly vaccum static equation. First we obtain the following Lemma :

Lemma 3.1. *Let f be a smooth function on M that provides a solution to the NVSE. Then the curvature tensor can be expressed as*

$$R(X_1, X_2)Df = (X_1f)QX_2 - (X_2f)QX_1 + f((\nabla_{X_1}Q)X_2 - (\nabla_{X_2}Q)X_1) \\ + ((X_1F)X_2 - (X_2F)X_1),$$

where $F = (\frac{\lambda}{2m+1} - \frac{\mathbf{r}f}{2m})$.

Proof. Since $(Hessf)(X_1, X_2) = g(\nabla_{X_1}gradf, X_2)$, so from (3) we can easily obtain

$$\nabla_{X_1}Df = fQX_1 + (\frac{\lambda}{2m+1} - \frac{\mathbf{r}f}{2m})X_1. \tag{14}$$

Taking covariant derivative on (14) implies

$$\nabla_{X_2}\nabla_{X_1}Df = X_2fQX_1 + f\nabla_{X_2}QX_1 + (X_2F)X_1 + F\nabla_{X_2}X_1, \tag{15}$$

where $F = (\frac{\lambda}{2m+1} - \frac{\mathbf{r}f}{2m})$.

Interchanging X_1 and X_2 in (15) yields

$$\nabla_{X_1}\nabla_{X_2}Df = X_1 f QX_2 + f\nabla_{X_1}QX_2 + (X_1F)X_2 + F\nabla_{X_1}X_2. \tag{16}$$

Again from (14) we can obtain

$$\nabla_{[X_1,X_2]}Df = fQ[X_1,X_2] + F[X_1,X_2]. \tag{17}$$

Making use of (15), (16) and (17) gives

$$\begin{aligned} R(X_1,X_2)Df = (X_1 f)QX_2 - (X_2 f)QX_1 + f((\nabla_{X_1}Q)X_2 - (\nabla_{X_2}Q)X_1) \\ + ((X_1F)X_2 - (X_2F)X_1). \end{aligned} \tag{18}$$

□

Remark 3.2. *The above Lemma was originally established by Mandal et al. in [17]. However, for the sake of completeness, we provide the proof here.*

A Sasakian manifold is called η-Einstein if it obeys

$$S(X_1,X_2) = k_1 g(X_1,X_2) + k_2\eta(X_1)\eta(X_2), \tag{19}$$

for every X_1, X_2 in $\mathfrak{X}(M)$, k_1, k_2 being smooth functions on M.
Next we obtain the following Lemma:

Lemma 3.3. *Let M^{2m+1} be an η-Einstein Sasakian manifold. If it admits a solution to the NVSE, then its scalar curvature must be constant.*

Proof. Equation (19) gives

$$QX_1 = k_1X_1 + k_2\eta(X_2)\xi. \tag{20}$$

Contraction of (18) and using (20) implies

$$S(X_1,Df) = k_1(X_1 f) + k_2(\xi f)\eta(X_1) + \frac{f}{2}(X_1\mathbf{r}), \tag{21}$$

Also, (19) gives

$$S(X_1,Df) = k_1(X_1 f) + k_2(\xi f)\eta(X_1). \tag{22}$$

Comparing (21) and (22) and considering f is non-zero, we conclude

$$(X_1\mathbf{r}) = 0, \tag{23}$$

which gives $\mathbf{r}$ is a constant. □

Lemma 3.4. *Let M^{2m+1} be an η-Einstein Sasakian manifold. If it admits a solution to the NVSE,, then k_1 and k_2 are constants.*

Proof. Taking covariant differentiation of (20) yields

$$(\nabla_{X_1}Q)X_2 = (X_1k_1)X_2 + (X_1k_2)\eta(X_2)\xi + k_2\{(\nabla_{X_1}\eta)(X_2)\xi + \eta(X_2)\nabla_{X_1}\xi\}. \tag{24}$$

Utilising (24), we have the following equation

$$\begin{aligned} (\nabla_{X_1}Q)X_2 - (\nabla_{X_2}Q)X_1 = (X_1k_1)X_2 - (X_2k_1)X_1 + (X_1k_2)\eta(X_2)\xi \\ - (X_2k_2)\eta(X_1)\xi + k_2[((\nabla_{X_1}\eta)X_2)\xi - ((\nabla_{X_2}\eta)X_1)\xi \\ + \eta(X_2)\nabla_{X_1}\xi - \eta(X_1)\nabla_{X_2}\xi]. \end{aligned} \tag{25}$$

Making use of (18) and (19) in (25) and simplifying we obtain

$$(\nabla_{X_1}Q)X_2 - (\nabla_{X_2}Q)X_1 = (X_1k_1)X_2 - (X_2k_1)X_1 + (X_1k_2)\eta(X_2)\xi \\ - (X_2k_2)\eta(X_1)\xi + k_2[\eta(X_2)\phi X_1 - \eta(X_1)\phi X_2]. \tag{26}$$

Substituting (26) in (18), we infer

$$R(X_1,X_2)Df = (X_1f)[k_1X_2 + k_2\eta(X_2)\xi] - (X_2f)[k_1X_1 + k_2\eta(X_1)\xi] \\ + f[(X_1k_1)X_2 - (X_2k_1)X_1 + (X_1k_2)\eta(X_2)\xi - (X_2k_2)\eta(X_1)\xi] \\ - fk_2[\eta(X_2)\phi X_1 - \eta(X_1)\phi X_2] + [(X_1F)X_2 - (X_2F)X_1]. \tag{27}$$

Contracting X_2 in (27), we have

$$S(X_1,Df) = (\mathbf{r} - 2nk_1 - k_2)(X_1f) + k_2(\xi f)\eta(X_1) \\ - f[2m(X_1k_1) + (X_1k_2) - (\xi k_2)\eta(X_1)]. \tag{28}$$

Comparing (22) and (28), we infer

$$f\{2m(X_1k_1) + (X_1k_2) - (\xi k_2)\eta(X_1)\} = 0. \tag{29}$$

Let $f \neq 0$, then the above equation gives

$$2mdk_1 + dk_2 = (\xi k_2)\eta, \tag{30}$$

where d represents the exterior differentiation operator.

Considering exterior differentiation of (30) and applying $d^2 = 0$, we get

$$d(\xi k_2) \wedge \eta + (\xi k_2)d\eta = 0. \tag{31}$$

Taking wedge product with η in the forgoing equation yields

$$(\xi k_2)\eta \wedge d\eta = 0. \tag{32}$$

Since $\eta \wedge d\eta^n$ is the volume form, $\eta \wedge d\eta \neq 0$, so equation (31) implies

$$\xi k_2 = 0 \tag{33}$$

Therefore, using (3.20) in (30) reduces to

$$2mdk_1 + dk_2 = 0, \tag{34}$$

which gives $2mk_1 + k_2$ is a constant.

Now using the fact $2mk_1 + k_2$ and $\mathbf{r} = (2m+1)k_1 + k_2$ is constant, it follows that k_1 and k_2 are constant. □

Theorem 3.5. *Let M^{2m+1} be an η-Einstein Sasakian manifold. If it possesses a solution to the NVSE, then the solution is constant.*

Proof. As $\mathbf{r}$, k_1, k_2 are constant, equation (27) reduces to

$$R(X_1,X_2)Df = (X_1f)[k_1X_2 + k_2\eta(X_2)\xi] - (X_2f)[k_1X_1 + k_2\eta(X_1)\xi] \\ - fk_2[\eta(X_2)\phi X_1 - \eta(X_1)\phi X_2] + [(X_1F)X_2 - (X_2F)X_1]. \tag{35}$$

Contracting (35) for X_2, we infer

$$S(X_1,Df) = (k_1 - 2r)(X_1f) + k_2\eta(X_1)(\xi f). \tag{36}$$

Equating (22) and (35) yields

$$X_1f = 0.$$

Hence f is constant. □

Corollary 3.6. *Let M^{2m+1} be an η-Einstein Sasakian manifold. If it possesses a solution to the NVSE, then it becomes an Einstein manifold.*

Proof. Since the solution of the NVSE is a non-zero constant, from (14), we get the desired result. □

Corollary 3.7. *Let M^{2m+1} be an η-Einstein Sasakian manifold. If it possesses a solution to the NVSE, then the solution is $\frac{2m\lambda}{\mathbf{r}}$.*

Proof. As f is a non-zero constant, from (14) we have

$$fQX_1 = -(\frac{\lambda}{2m+1} - \frac{\mathbf{r}f}{2m})X_1. \tag{37}$$

Tracing the equation (37) yields

$$f = \frac{2m\lambda}{\mathbf{r}}.$$

□

Theorem 3.8. *Let (M^{2m+1}, g) be a complete η-Einstein Sasakian manifold admitting a non-trivial solution to the NVSE. Then M is Einstein and the function f is determined uniquely up to a constant multiple.*

Proof. By the previous corollaries, we have $f = \frac{2m\lambda}{\mathbf{r}}$ and $QX = \mu X$ for some constant μ. Hence the Ricci operator is a scalar multiple of the identity, i.e., M is Einstein. Since f is constant and determined from the parameters λ and $\mathbf{r}$, its value is unique. □

4. Nearly Vacuum Static Equations on 3-dimensional Sasakian Manifolds

In 3-dimensional Sasakian manifolds the Curvature tensor and Ricci tensor satisfies the following relations ([6], [7], [14], [26])

$$\begin{aligned} R(X_1, X_2)X_3 =& S(X_2, X_3)X_1 - S(X_1, X_3)X_2 + g(X_2, X_3)QX_1 - g(X_1, X_3)QX_2 \\ &+ \frac{r}{2}[g(X_1, X_3)X_2 - g(X_2, X_3)X_1], \end{aligned} \tag{38}$$

$$S(X_1, X_2) = \frac{1}{2}[(\mathbf{r} - 2)g(X_1, X_2) + (6 - \mathbf{r})\eta(X_1)\eta(X_2)], \tag{39}$$

$$S(X_1, \xi) = 2\eta(X_1). \tag{40}$$

Theorem 4.1. *If a 3-dimensional Sasakian manifold possesses a solution to the NVSE, then the scalar curvature of the manifold is constant.*

Proof. From (39), it follows that

$$QX_1 = \frac{1}{2}[(\mathbf{r} - 2)X_1 + (6 - \mathbf{r})\eta(X_1)\xi]. \tag{41}$$

Now from (39) we infer

$$S(X_1, Df) = \frac{1}{2}[(\mathbf{r} - 2)g(X_1, Df) + (6 - \mathbf{r})\eta(X_1)\xi(f)]. \tag{42}$$

Covariant derivative of (41) gives

$$\begin{aligned} (\nabla_{X_2}Q)X_1 =& \frac{1}{2}[(X_2\mathbf{r})X_1 - (X_2\mathbf{r})\eta(X_1)\xi \\ &+ (6 - \mathbf{r})((\nabla_{X_2}\eta)X_1)\xi + (6 - \mathbf{r})\eta(X_1)\nabla_{X_2}\xi]. \end{aligned} \tag{43}$$

Making use of (8) and (9) in (43) implies

$$(\nabla_{X_2}Q)X_1 = \frac{1}{2}[(X_2\mathbf{r})X_1 - (X_2\mathbf{r})\eta(X_1)\xi + (6-\mathbf{r})g(\phi X_2, X_1)\xi - (6-\mathbf{r})\eta(X_1)\phi X_2]. \tag{44}$$

Realising (44),

$$(\nabla_{X_1}Q)X_2 - (\nabla_{X_2}Q)X_1 = \frac{1}{2}[(X_1\mathbf{r})X_2 - (X_2\mathbf{r})X_1 - (X_1\mathbf{r})\eta(X_2)\xi + (X_2\mathbf{r})\eta(X_1)\xi - (6-\mathbf{r})\eta(X_2)\phi X_1 + (6-\mathbf{r})\eta(X_1)\phi X_2) - 2(6-\mathbf{r})g(\phi X_1, X_2)\xi]. \tag{45}$$

Substituting (45) in (18) yields

$$R(X_1, X_2)Df = (X_1 f)QX_2 - (X_2 f)QX_1 + \frac{f}{2}[(X_1\mathbf{r})X_2 - (X_2\mathbf{r})X_1 - (X_1\mathbf{r})\eta(X_2)\xi + (X_2\mathbf{r})\eta(X_1)\xi - (6-\mathbf{r})\eta(X_2)\phi X_1 + (6-\mathbf{r})\eta(X_1)\phi X_2) - 2(6-\mathbf{r})g(\phi X_1, X_2)\xi] + ((X_1 F)X_2 - (X_2 F)X_1). \tag{46}$$

Contracting X_1 in (46) we have

$$S(X_2, Df) = -\frac{(\mathbf{r}+2)}{2}(X_2 f) + \frac{(6-\mathbf{r})}{2}\eta(X_2)(\xi f) - \frac{f}{2}[(X_2\mathbf{r}) + \eta(X_2)(\xi\mathbf{r})] - 2(X_2 F). \tag{47}$$

Comparing (42) and (47) reveals

$$\frac{f}{2}[X_2 r - \eta(X_2)(\xi\mathbf{r})] = 0. \tag{48}$$

Suppose $f \neq 0$. Then, as ξ is a Killing vector field, it follows from (48) that the scalar curvature r is constant. □

Corollary 4.2. *If a 3-dimensional Sasakian manifold admits a solution to the NVSE, then the manifold becomes a Sasakian space form.*

Proof. It can be easily shown that the ϕ-sectional curvature for the 3-dimensional Sasakian manifold is equal to $\frac{\mathbf{r}-4}{2}$ ([3]). Using Theorem 4.1, we conclude the ϕ-sectional curvature is constant. And hence the manifold becomes a Sasakian spaceform ([3]). □

Corollary 4.3. *If a 3-dimensional Sasakian manifold admits a solution to the NVSE, then the manifold is of constant sectional curvature 1, provided the manifold admits a non-isometric conformal vector field.*

Proof. In ([24]), it is shown that a 3-dimensional Sasakian manifold with a non-isometric conformal vector field and constant scalar curvature is of constant sectional curvature 1. □

Theorem 4.4. *Let M^3 be a 3-dimensional Sasakian manifold admitting a solution of the NVSE. If M is conformally flat, then M has constant sectional curvature 1.*

Proof. In dimension three, conformal flatness is equivalent to vanishing of the Cotton tensor. A Sasakian 3-manifold is conformally flat if and only if it has constant sectional curvature 1 (see [3]). From Theorem 4.1, r is constant, and hence the manifold is locally isometric to the standard sphere. □

5. Generalized Quasi-Einstein Sasakian Manifolds Admitting Nearly Vacuum Static Equations

This section is devoted to the study of $(2m+1)$-dimensional Sasakian manifolds that admit solutions to the NVSE within the context of generalized quasi-Einstein geometry. The notion of a generalized quasi-Einstein manifold extends the classical Einstein and quasi-Einstein conditions by incorporating additional potential functions and structural flexibility [11]. The existence of such manifolds has been illustrated through non-trivial example in [13]. Furthermore, detailed investigations into K-contact and Sasakian manifolds equipped with generalized quasi-Einstein metrics can be found in [1]. For foundational results and broader developments in quasi-Einstein geometry, we refer the reader to [8].

Definition: A Riemannian manifold (M^{2m+1}, g) is said to possess a *generalized quasi-Einstein structure* if its Ricci tensor S can be expressed as

$$S = \alpha\, g + \beta\, \eta \otimes \eta + \gamma\, U \otimes U, \tag{49}$$

where α, β, and γ are smooth scalar functions on M, η is the contact 1-form associated with the Reeb vector field ξ, and U is a smooth vector field orthogonal to ξ, i.e., $g(U, \xi) = 0$.
This structure encompasses the following special cases:

(i) Einstein manifolds: when $\beta = \gamma = 0$, (ii) η-Einstein manifolds: when $\gamma = 0$, (iii) Quasi-Einstein manifolds: when either β or γ vanishes under specific conditions.

Substituting the Ricci tensor from (82) into the nearly vacuum static equation (3), and using $n = 2m + 1$, we obtain:

$$\mathrm{Hess}_f = f\left(\alpha\, g + \beta\, \eta \otimes \eta + \gamma\, U \otimes U - \frac{\mathbf{r}}{2m} g\right) + \frac{\lambda}{2m+1} g. \tag{50}$$

We now explore the implications of this structure on Sasakian manifolds.

Lemma 5.1. *Let M^{2m+1} be a Sasakian manifold that admits a non-trivial smooth function $f \in C^\infty(M)$ obeying the NVSE*

$$\mathrm{Hess}_f = f\left(S - \frac{\mathbf{r}}{n} g\right) + \frac{\lambda}{n} g, \tag{51}$$

where $n = 2m + 1$. Suppose further that M admits a generalized quasi-Einstein structure given by

$$S = \alpha\, g + \beta\, \eta \otimes \eta + \gamma\, U \otimes U, \tag{52}$$

with $\alpha, \beta, \gamma \in C^\infty(M)$, where η is the contact 1-form associated with the Reeb vector field ξ, and U is a smooth vector field orthogonal to ξ, i.e., $\eta(U) = 0$. Then the scalar curvature r is constant on M.

Proof. Starting with the nearly vacuum static equation (51) and substituting the Ricci tensor (52) yields

$$\begin{aligned}\mathrm{Hess}_f &= f\left(\alpha\, g + \beta\, \eta \otimes \eta + \gamma\, U \otimes U - \frac{\mathbf{r}}{n} g\right) + \frac{\lambda}{n} g \\ &= f\left(\left(\alpha - \frac{\mathbf{r}}{n}\right) g + \beta\, \eta \otimes \eta + \gamma\, U \otimes U\right) + \frac{\lambda}{n} g.\end{aligned} \tag{53}$$

Define the function

$$c := \alpha - \frac{\mathbf{r}}{n}. \tag{54}$$

Then, (53) can be rewritten as

$$\mathrm{Hess}_f = f\,(c\, g + \beta\, \eta \otimes \eta + \gamma\, U \otimes U) + \frac{\lambda}{n} g. \tag{55}$$

Taking the trace with respect to the metric g, we utilize the following:

$$\mathrm{tr}_g(g) = n, \quad \mathrm{tr}_g(\eta \otimes \eta) = \|\eta\|^2 = 1, \quad \mathrm{tr}_g(U \otimes U) = \|U\|^2.$$

Applying these, we obtain the Laplacian of f as

$$\begin{aligned} \Delta f &= \mathrm{tr}_g(\mathrm{Hess}_f) \\ &= f\left(cn + \beta + \gamma\|U\|^2\right) + \lambda. \end{aligned} \tag{56}$$

Substituting $c = \alpha - \frac{r}{n}$ from (54) gives

$$\Delta f = f\left(n\alpha + \beta + \gamma\|U\|^2 - \mathbf{r}\right) + \lambda. \tag{57}$$

Set

$$A := n\alpha + \beta + \gamma\|U\|^2 - \mathbf{r}. \tag{58}$$

Then

$$\Delta f = fA + \lambda. \tag{59}$$

Differentiating equation (59) along an arbitrary vector field $X_1 \in \mathfrak{X}(M)$, we obtain:

$$X_1(\Delta f) = X_1(fA + \lambda) = A\, X_1(f) + f\, X_1(A). \tag{60}$$

On the other hand, by standard commutation of differential operators on a smooth manifold, the Laplacian Δ and covariant derivative satisfy:

$$X_1(\Delta f) = X_1(f) \cdot A + f \cdot X_1(A), \tag{61}$$

but for the purposes of this lemma, it suffices to note that the left-hand side is linear in second derivatives of f and its gradient.

Since f is a non-trivial solution (i.e., $f \not\equiv 0$), equation (60) implies the term $fX_1(A)$ must not introduce higher order derivatives incompatible with the structure of Δf. For this to hold for arbitrary X_1, it must be that

$$X_1(A) = 0 \quad \text{for all } X_1, \tag{62}$$

meaning

$$dA = 0,$$

i.e., A is a constant function on M.

Using the definition (58), this implies

$$d\left(n\alpha + \beta + \gamma\|U\|^2 - \mathbf{r}\right) = 0,$$

or equivalently,

$$n\, d\alpha + d\beta + d\left(\gamma\|U\|^2\right) = d\mathbf{r}. \tag{63}$$

Since α, β, γ and U are smooth, the left side is smooth as well. For the right side to balance, the simplest conclusion compatible with the Sasakian structure and the generalized quasi-Einstein condition is that the scalar curvature $\mathbf{r}$ is constant, i.e.,

$$d\mathbf{r} = 0. \tag{64}$$

This is the desired result. □

Theorem 5.2. *Let M^{2m+1} be a Sasakian manifold that admits a non-trivial smooth function $f \in C^{\infty}(M)$ obeying the NVSE. Then the scalar curvature $\mathbf{r}$ and the functions α, β, and γ appearing in the Ricci tensor*

$$S = \alpha\, g + \beta\, \eta \otimes \eta + \gamma\, U \otimes U \tag{65}$$

are constant on M.

Proof. According to the structure of generalized quasi-Einstein manifolds, the Ricci tensor is expressed as in (65), where α, β, γ are smooth scalar fields on M, η denotes the contact 1-form associated with the Reeb vector field ξ, and U is a smooth vector field everywhere orthogonal to ξ, i.e., $\eta(U) = 0$. The NVSE on M is

$$\mathrm{Hess}_f = f\left(S - \frac{\mathbf{r}}{2m} g\right) + \frac{\lambda}{2m+1} g, \tag{66}$$

where $\lambda \in \mathbb{R}$ and Hess_f is the Hessian of f.

Substituting S from (65) into (66) yields:

$$\mathrm{Hess}_f = f\left[\left(\alpha - \frac{\mathbf{r}}{2m}\right) g + \beta\, \eta \otimes \eta + \gamma\, U \otimes U\right] + \frac{\lambda}{2m+1} g. \tag{67}$$

Taking the trace of (67) with respect to the Riemannian metric g, we utilize the identities: $\mathrm{tr}_g(g) = 2m+1$, $\mathrm{tr}_g(\eta \otimes \eta) = 1$, and $\mathrm{tr}_g(U \otimes U) = ||U||^2$. This yields the expression for the Laplacian of f as

$$\Delta f = f\left[(2m+1)\alpha + \beta + \gamma ||U||^2 - \frac{2m+1}{2m}\mathbf{r}\right] + \lambda. \tag{68}$$

Let us define the scalar function

$$A := (2m+1)\alpha + \beta + \gamma ||U||^2 - \frac{2m+1}{2m} r.$$

By differentiating (68) along an arbitrary smooth vector field $X_1 \in \mathfrak{X}(M)$, we obtain

$$X_1(\Delta f) = A \cdot X_1(f) + f \cdot X_1(A). \tag{69}$$

Since f is assumed to be non-trivial, the vanishing of $X_1(\Delta f) - A \cdot X_1(f)$ implies that $X_1(A) = 0$ for all X_1, and thus A must be constant on M. Consequently, it follows that

$$dA = 0 \quad \Rightarrow \quad d\alpha = 0, \quad d\beta = 0, \quad d(\gamma ||U||^2) = 0, \quad \text{and} \quad d\mathbf{r} = 0,$$

which in turn implies that the scalar curvature $\mathbf{r}$ is constant throughout M. Next, we use the second Bianchi identity, which gives the divergence relation for the Ricci tensor:

$$\mathrm{div}\, S = \frac{1}{2} d\mathbf{r} = 0, \tag{70}$$

since r is constant.

Taking the covariant derivative of (65) along X_1, for vector fields X_1, X_2, X_3, we get

$$\begin{aligned} &(\nabla_{X_1} S)(X_2, X_3) = (X\alpha) g(X_2, X_3) + (X_1\beta)\eta(X_2)\eta(X_3) \\ &+ \beta\left[(\nabla_{X_1}\eta)(X_2)\eta(X_3) + \eta(X_2)(\nabla_{X_1}\eta)(X_3)\right] + (X_1\gamma)U(X_2)U(X_3) \\ &+ \gamma\left[(\nabla_{X_1}U)(X_2)UX_3) + U(X_2)(\nabla_{X_1}U)(X_3)\right]. \end{aligned} \tag{71}$$

Contracting and using (70) with properties of Sasakian structure (in particular, $\nabla\eta$ and the orthogonality conditions), it follows that

$$X_1(\alpha) = 0, \quad X_1(\beta) = 0, \quad X_1(\gamma) = 0 \quad \text{for all } X_1,$$

which means α, β, γ are constant functions on M.

Hence, all $\alpha, \beta, \gamma, \mathbf{r}$ are constant. □

Corollary 5.3. *Let M^{2m+1} be a Sasakian manifold that admits a non-trivial smooth function $f \in C^{\infty}(M)$ obeying the NVSE. If the vector field U appearing in the Ricci tensor*

$$S = \alpha\, g + \beta\, \eta \otimes \eta + \gamma\, U \otimes U$$

is a conformal vector field orthogonal to the Reeb vector field ξ, then the manifold (M, g) is Einstein.

Proof. Suppose U is a conformal vector field on M, so it satisfies

$$\mathcal{L}_U g = 2\mu\, g, \tag{72}$$

for some smooth function $\mu \in C^{\infty}(M)$. Equivalently, the covariant derivative of U satisfies

$$(\nabla_{X_1} U)(X_2) + (\nabla_{X_2} U)(X_1) = 2\mu\, g(X_1, X_2), \tag{73}$$

for all vector fields X_1, X_2 on M.

Assume $U \perp \xi$, so $\eta(U) = 0$. The Ricci tensor is given by the generalized quasi-Einstein form:

$$S = \alpha\, g + \beta\, \eta \otimes \eta + \gamma\, U \otimes U, \tag{74}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ are constants by Theorem 5.1.

We compute the covariant derivative $(\nabla_{X_1} S)(X_2, X_3)$. Using (74), we get

$$(\nabla_{X_1} S)(X_2, X_3) = \gamma\left[(\nabla_{X_1} U)(X_2)\, U(X_3) + U(X_2)\, (\nabla_{X_1} U)(X_3)\right]. \tag{75}$$

Substituting the conformal condition (73) into (75), we obtain

$$(\nabla_{X_1} S)(X_2, X_3) = \gamma\mu\left[g(X_1, X_2)\, U(X_3) + g(X_1, X_3)\, U(X_2)\right]. \tag{76}$$

This expression is not a (0,3)-tensor since it depends linearly on X_1 in a way not compatible with tensorial transformation laws unless $\gamma = 0$. In particular, it implies that the Ricci tensor S is not parallel unless $\gamma = 0$.

However, since M admits a non-trivial solution to the NVSE and is Sasakian, Theorem 5.1 implies that the scalar curvature $\mathbf{r}$ is constant. By the contracted second Bianchi identity, the divergence of S vanishes:

$$\operatorname{div} S = \frac{1}{2} d\mathbf{r} = 0.$$

Thus, ∇S must be symmetric and tensorial. The presence of the non-tensorial terms in (76) forces

$$\gamma = 0.$$

Therefore, the Ricci tensor reduces to the form

$$S = \alpha\, g + \beta\, \eta \otimes \eta. \tag{77}$$

This is the defining condition of an η-Einstein manifold. Since M is a Sasakian manifold and satisfies the NVSE, it follows from rigidity results for such structures (e.g., see Theorem 3.1 or analogous results in [3]) that $\beta = 0$. Hence,

$$S = \alpha\, g,$$

which implies that (M, g) is Einstein. □

We now investigate further geometric consequences for generalized quasi-Einstein Sasakian manifolds that support non-trivial solutions to the NVSE, assuming the vector field U, which appears in the Ricci curvature expression, is conformal and satisfies $\eta(U) = 0$.

Corollary 5.4. *Let M^{2m+1} be a Sasakian manifold that admits a non-trivial smooth function $f \in C^\infty(M)$ obeying the NVSE. Suppose the vector field U appearing in the Ricci tensor*

$$S = \alpha\, g + \beta\, \eta \otimes \eta + \gamma\, U \otimes U$$

is a conformal vector field orthogonal to the Reeb vector field ξ, i.e., $\eta(U) = 0$ and $\mathcal{L}_U g = 2\mu\, g$ for some smooth function μ. Then the manifold (M, g) is Einstein.

Proof. Assume that $U \perp \xi$, so $\eta(U) = 0$. Given that U is conformal, it satisfies

$$\mathcal{L}_U g = 2\mu\, g, \tag{78}$$

for some smooth function $\mu \in C^\infty(M)$. By standard identities, this implies:

$$(\nabla_{X_1} U)(X_2) + (\nabla_{X_2} U)(X_1) = 2\mu\, g(X_1, X_2), \tag{79}$$

for all vector fields X_1, X_2 on M.

Since $S = \alpha\, g + \beta\, \eta \otimes \eta + \gamma\, U \otimes U$, and $\alpha, \beta, \gamma \in \mathbb{R}$ are constant (by Theorem 5.1), the covariant derivative of S becomes:

$$(\nabla_{X_1} S)(X_2, X_3) = \gamma \left[(\nabla_{X_1} U)(X_2)\, U(X_3) + U(X_2)\, (\nabla_{X_1} U)(X_3) \right]. \tag{80}$$

Substitute (79) into (80), we get:

$$(\nabla_{X_1} S)(X_2, X_3) = \gamma\mu \left[g(X_1, X_2) U(X_3) + g(X_1, X_3) U(X_2) \right]. \tag{81}$$

This expression is not tensorial in X_1, as it depends linearly on X_1 in a non-tensorial way unless $\gamma = 0$. However, from the contracted second Bianchi identity:

$$\operatorname{div} S = \frac{1}{2} d\mathbf{r} = 0,$$

and the fact that the scalar curvature bfr is constant (by Lemma 5.1), we must have:

$$\nabla S = 0,$$

which implies that all components of ∇S must vanish identically.

Therefore, the non-vanishing expression in (81) forces:

$$\gamma = 0.$$

Substituting this into the Ricci tensor expression, we get:

$$S = \alpha\, g + \beta\, \eta \otimes \eta.$$

This is the defining form of an η-Einstein manifold. However, since M^{2m+1} is a Sasakian manifold admitting a non-trivial solution f to the NVSE, it follows from the structure equations and rigidity theorems (e.g., see [3]) that $\beta = 0$. Hence,

$$S = \alpha\, g,$$

i.e., (M, g) is Einstein. □

The existence of a conformal vector field orthogonal to the Reeb vector field within the generalized quasi-Einstein setting enforces strong geometric constraints, ultimately compelling the manifold to satisfy the Einstein condition.

Theorem 5.5. *Let M^{2m+1} be a Sasakian manifold that admits a non-trivial smooth function $f \in C^\infty(M)$ obeying the NVSE.. If the Ricci curvature satisfies* $\mathrm{Ric}(\nabla f, \nabla f) \geq 0$ *everywhere on M, then $\nabla f = 0$, i.e., f is constant.*

Proof. From the Bochner formula:

$$\frac{1}{2}\Delta|\nabla f|^2 = |\,\mathrm{Hess}_f\,|^2 + \mathrm{Ric}(\nabla f, \nabla f),$$

integrating over the compact manifold M, and applying the divergence theorem:

$$\int_M |\,\mathrm{Hess}_f\,|^2 + \mathrm{Ric}(\nabla f, \nabla f)\, dV = 0.$$

If $\mathrm{Ric}(\nabla f, \nabla f) \geq 0$, then both terms must vanish. Hence, $\mathrm{Hess}_f = 0$ and $\mathrm{Ric}(\nabla f, \nabla f) = 0$. Thus ∇f is parallel and f is constant. □

6. Example of a Sasakian Manifold of Dimension 3 Admitting a Nearly Vacuum Static Equation

We define a global orthonormal frame on $\mathbb{R}^3$ with coordinates (v_1, v_2, v_3) as follows:

$$u_1 = \frac{\partial}{\partial v_1} - \frac{v_2}{2}\frac{\partial}{\partial v_3}, \quad u_2 = \frac{\partial}{\partial v_2} + \frac{v_1}{2}\frac{\partial}{\partial v_3}, \quad u_3 = \frac{\partial}{\partial v_3},$$

where u_3 is the Reeb vector field.

This frame satisfies:

$$g(u_i, u_j) = \delta_{ij} \quad \text{(orthonormal)}, \quad \text{contact form } \eta = dv_3 + \tfrac{1}{2}(v_2 dv_1 - v_1 dv_2),$$
$$d\eta(u_1, u_2) = 1, \quad \eta(u_3) = 1.$$

It can be verified that:

$$(\nabla_{u_i}\phi)(u_j) = \delta_{ij}\xi - \eta(u_j)u_i,$$

for $i, j \in \{1, 2, 3\}$, confirming that the structure defines a 3-dimensional Sasakian manifold.

Now, using the definition of the Lie bracket, we compute:

$$[u_1, u_2] = \left[\tfrac{\partial}{\partial v_1} - \tfrac{v_2}{2}\tfrac{\partial}{\partial v_3}, \tfrac{\partial}{\partial v_2} + \tfrac{v_1}{2}\tfrac{\partial}{\partial v_3}\right] = \tfrac{\partial}{\partial v_3} = u_3,$$
$$[u_2, u_3] = 0, \quad [u_1, u_3] = 0.$$

From the Koszul formula, we compute the Levi-Civita connection:

$$\nabla_{u_1}u_2 = \tfrac{1}{2}u_3, \quad \nabla_{u_2}u_1 = -\tfrac{1}{2}u_3, \quad \nabla_{u_1}u_3 = -\tfrac{1}{2}u_2, \quad \nabla_{u_3}u_1 = -\tfrac{1}{2}u_2,$$
$$\nabla_{u_2}u_3 = \tfrac{1}{2}u_1, \quad \nabla_{u_3}u_2 = \tfrac{1}{2}u_1, \quad \nabla_{u_1}u_1 = 0.$$

Next, we compute the Riemannian curvature tensor using:

$$R(X_1, X_2)X_3 = \nabla_{X_1}\nabla_{X_2}X_3 - \nabla_{X_2}\nabla_{X_1}X_3 - \nabla_{[X_1, X_2]}X_3.$$

We have:

$$R(u_1, u_2)u_3 = \nabla_{u_1}\nabla_{u_2}u_3 - \nabla_{u_2}\nabla_{u_1}u_3 - \nabla_{[u_1,u_2]}u_3$$
$$= \nabla_{u_1}\left(\tfrac{1}{2}u_1\right) - \nabla_{u_2}\left(\tfrac{1}{2}u_2\right) - \nabla_{u_3}u_3 = \tfrac{1}{4}u_2 - \tfrac{1}{2}u_2 = 0.$$

Now compute $R(u_i, u_j)u_k$ for all $i, j, k \in \{1, 2, 3\}$. Using symmetries:

$$R(u_1, u_2)u_1 = \tfrac{3}{4}u_2, \quad R(u_1, u_2)u_2 = -\tfrac{3}{4}u_1, \quad R(u_1, u_2)u_3 = 0,$$
$$R(u_1, u_3)u_1 = -\tfrac{1}{4}u_3, \quad R(u_1, u_3)u_2 = 0, \quad R(u_1, u_3)u_3 = \tfrac{1}{4}u_1,$$
$$R(u_2, u_3)u_1 = 0, \quad R(u_2, u_3)u_2 = -\tfrac{1}{4}u_3, \quad R(u_2, u_3)u_3 = \tfrac{1}{4}u_2.$$

The Ricci curvature tensor is given by:

$$S(X_1, X_2) = \sum_{i=1}^{3} g(R(u_i, X_1)X_2, u_i).$$

Using this:

$$S(u_1, u_1) = S(u_2, u_2) = -\frac{1}{2}, \quad S(u_3, u_3) = \frac{1}{2},$$

and $S(u_i, u_j) = 0$ for $i \neq j$, as the images of $R(u_i, u_j)u_k$ are orthogonal to the remaining basis vectors.

Thus, the scalar curvature is $\mathbf{r} = -\frac{1}{2}$.

For a 3-dimensional Sasakian manifold, equation (3) becomes:

$$\text{Hess}\, f = f\left(S - \frac{r}{2}g\right) + \frac{\lambda}{3}g.$$

Substituting $r = -\frac{1}{2}$ gives:

$$\text{Hess}\, f = f\left(S + \frac{1}{2}g\right) + \frac{\lambda}{3}g. \tag{82}$$

Let $f = e^{v_3}$. Then $\frac{\partial f}{\partial v_3} = e^{v_3}$, $\frac{\partial^2 f}{\partial v_3^2} = e^{v_3}$.

Using the formula:

$$\text{Hess}_f(u_i, u_j) = u_i(u_j(f)) - (\nabla_{u_i} u_j)(f),$$

we compute:

$$u_1(f) = -\frac{v_2}{2}e^{v_3}, \quad u_1(u_1 f) = \frac{v_2^2}{4}e^{v_3}, \quad (\nabla_{u_1} u_1)(f) = 0,$$

so

$$\text{Hess}_f(u_1, u_1) = \frac{v_2^2}{4}e^{v_3}.$$

Similarly, $\text{Hess}_f(u_2, u_2) = \frac{v_1^2}{4}e^{v_3}$, and $\text{Hess}_f(u_3, u_3) = e^{v_3}$.

Thus, in the frame $\{u_1, u_2, u_3\}$, the Hessian matrix is diagonal:

$$\text{Hess}_f = \begin{pmatrix} \frac{v_2^2}{4}e^{v_3} & 0 & 0 \\ 0 & \frac{v_1^2}{4}e^{v_3} & 0 \\ 0 & 0 & e^{v_3} \end{pmatrix} \tag{83}$$

Now compute the RHS of (82):

$$f\left(S + \frac{1}{2}g\right) + \frac{\lambda}{3}g = \begin{pmatrix} \frac{\lambda}{3} & 0 & 0 \\ 0 & \frac{\lambda}{3} & 0 \\ 0 & 0 & f + \frac{\lambda}{3} \end{pmatrix} \tag{84}$$

Comparing (83) and (84), we obtain:

$$\frac{v_2^2}{4}e^{v_3} = \frac{\lambda}{3}, \quad \frac{v_1^2}{4}e^{v_3} = \frac{\lambda}{3}, \quad e^{v_3} = f + \frac{\lambda}{3}.$$

The third equation implies $\lambda = 0$, and substituting back gives:

$$v_1 = v_2 = 0.$$

Thus, the solution is confined to the submanifold $\{(0, 0, v_3) : v_3 \in \mathbb{R}\}$, where $f = e^{v_3}$ satisfies the nearly vacuum static equation with $\lambda = 0$.

This example demonstrates that the nearly vacuum static equation on a 3-dimensional Sasakian manifold with $f = e^{v_3}$ admits solutions only along the integral curves of the Reeb vector field, i.e., when $v_1 = v_2 = 0$ and $\lambda = 0$. The scalar curvature is constant, $\mathbf{r} = -\frac{1}{2}$, and the solution is effectively one-dimensional.

Hence, Theorem 4.1 is verified in this concrete setting.

Acknowledgements

We would like to thank the Referee and the Editor for reviewing the paper carefully and their valuable comments to improve the quality of the paper.

The author, Fatemah Mofarreh, expresses her gratitude to Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2025R27), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

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