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Geometric structures of sequential warped products and their relations with the gradient Ricci-Yamabe solitons

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Abstract. In this paper, we investigate the geometric properties of sequential warped product manifolds which is the most general concept of the warped product manifolds. We then establish some necessary and sufficient conditions for sequential warped product manifolds admitting a gradient Ricci-Yamabe soliton. We focus particularly on the triviality of the sequential warped product manifold admitting the gradient Ricci-Yamabe soliton. For the physical applications, we investigate the behavior of these solitons in sequential generalized Robertson-Walker spacetimes and sequential standard static spacetimes.

1. Introduction

The concept of the warped product manifolds [3] was introduced by Bishop and O'Neill and has been studied extensively in differential geometry. They are helpful tools to derive more interesting geometric properties of Riemannian manifolds with new concrete examples [3], [19]. It was shown that the standard spacetime models used in cosmology, as well as the solutions to Einstein's field equations, can be expressed as warped products. As a result, warped product manifolds have become important tools in both differential geometry and the general theory of relativity. Among the well-known examples are the Schwarzschild and de Sitter spacetimes. Motivated by this, De, Shenawy and Ünal have introduced a more general class of warped product metrics, where the base and/or the fiber manifold themselves may carry a warped product structure. These are referred to as sequential warped products [6, 22]. Formally, this notion is defined as follows:

Definition 1.1. *Let* (M_i, g_i) *be* m_i*-dimensional semi-Riemannian manifolds, respectively for any* $i = 1, 2, 3$ *and* $f\colon M_1 \to (0,\infty)$, $h\colon M_1 \times M_2 \to (0,\infty)$ *be two smooth functions. Then the sequential warped product (briefly SWP) is defined as the product manifold* $\bar{M} = (M_1 \times_f M_2) \times_h M_3$ *furnished with the metric* $\bar{g} = (g_1 \oplus f^2 g_2) \oplus h^2 g_3$ *[6].*

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In 2021, Güler [10] investigated the geometry of sequential warped products, examining the properties of the Riemann and Ricci curvatures and exploring applications in Lorentzian settings. In [21], more general formulas were derived for a specific format of sequential warped product semi-Riemannian manifolds to be Einstein. For recent work on sequential warped products manifolds; we refer [10], [15], [16] and [20]. In physics, warped products serve as fundamental structures for modeling spacetime. Two significant examples of such models are the generalized Robertson-Walker (GRW) spacetime and the standard static (SS) spacetime, which have been widely explored in the context of general relativity. These two models extend classical spacetime structures: the GRW spacetime generalizes the well-known Robertson-Walker spacetime, while the SS spacetime generalizes the Einstein static universe. Further extending these ideas, De, Shenawy, and Ünal introduced the concepts of sequential generalized Robertson-Walker spacetime and sequential standard static spacetime in their work [6]. Formally, these notions are defined as below:

Definition 1.2. *Let (M_i, g_i) be m_i-dimensional semi-Riemannian manifolds, respectively for any $i = 2, 3$ and $f\colon I \to (0, \infty)$, $h\colon I \times M_2 \to (0, \infty)$ be two smooth functions. Then the sequential generalized Robertson-Walker (briefly SGRW) spacetime is defined as the product manifold $\bar{M} = (I \times_f M_2) \times_h M_3$ furnished with the metric $\bar{g} = ((-dt^2) \oplus f^2 g_2) \oplus h^2 g_3$ where dt^2 denotes the Euclidean metric tensor on a connected and open interval $I \subset \mathbb{R}$, [6].*

Definition 1.3. *Let (M_i, g_i) be m_i-dimensional semi-Riemannian manifolds, respectively for any $i = 1, 2$ and $f\colon M_1 \to (0, \infty)$, $h\colon M_1 \times M_2 \to (0, \infty)$ be two smooth functions. Then the sequential standard static (briefly SSS) spacetime is defined as the product manifold $\bar{M} = (M_1 \times_f M_2) \times_h I$ furnished with the metric $\bar{g} = (g_1 \oplus f^2 g_2) \oplus h^2(-dt^2)$ where $I \subset \mathbb{R}$ is an open and connected interval and dt^2 denotes the Euclidean metric tensor on I [6].*

Since the early 1980s, various flow equations have gained significant attention in mathematics. Among them, the Ricci flow and the Yamabe flow, introduced by Hamilton, are the most well-known [11], [12]. Over the past two decades, geometric flow theory has become a major area of interest for many mathematicians, with a particular focus on the Yamabe and Ricci flows. In 2019, a new geometric flow was introduced by the first author and Crasmareanu, which combines both the Ricci and Yamabe flows [8]. This new flow is called the Ricci-Yamabe flow and has important applications in differential geometry, physics and general relativity [8]. Furthermore, in 2021, it was shown that the limit of the solutions for the Ricci-Yamabe flow leads to a special structure known as the Ricci-Yamabe soliton [7]. Formally, this notion is defined as below:

Definition 1.4. *On a (semi)-Riemannian manifold (M^n, g), $\left(M, g, \phi, \lambda, \alpha, \beta\right)$ is called a gradient Ricci-Yamabe soliton (shortly GRYS) if*

$$Hess\phi + \alpha Ric = \left(\lambda - \frac{1}{2}\beta\tau\right) g, \tag{1}$$

is satisfied for a a differentiable function $\phi : M \to \mathbb{R}$ where Ric is the Ricci tensor, τ is the scalar curvature of (M, g), $Hess\phi$ denotes the Hessian of ϕ and $\lambda, \alpha, \beta \in \mathbb{R}$ [7], [25]. GRYS is called expanding, if $\lambda < 0$; steady, if $\lambda = 0$ and shrinking, if $\lambda > 0$. GRYS said to be proper if $\alpha \neq 0, 1$.

In [18], twisted product manifolds admitting GRYS were investigated. In [13], the second author focused on GRYS with warped product metrics. In [14], authors studied multiply warped products admitting gradient Ricci-Yamabe solitons. Moreover, Ricci-Yamabe solitons also have some applications in physics, we refer [1], [4], [5], [9], [23], [24], [26]. In this paper, we find some geometric properties for sequential warped product manifolds. Next, we give the main relations of the gradient Ricci-Yamabe soliton with sequential warped product. We also obtain some physical applications by investigating the sequential generalized Robertson-Walker spacetime and the sequential standard static spacetime for these solitons.

2. Geometric Properties on Sequential Warped Products

Notation 2.1. *(i) In this paper, we study smooth objects and connected manifolds.*

(ii) Objects belonging to the sequential warped product manifold are represented by "bar" symbol and objects of the base or fibers (M_i, g_i), for $i = 1, 2, 3$, are represented by indices or powers i.

(iii) Every vector field $X \in \chi(M)$ can be decomposed as $X = X_1 + X_2 + X_3$, where $X_i \in \chi(M_i)$, $i = 1, 2, 3$ and $\bar{M} = (M_1 \times_f M_2) \times_h M_3$.

In this section, we investigate some geometric properties of sequential warped product manifolds. Firstly, we deal with harmonicity on sequential warped product manifolds. Recall that any function ϕ on the (semi-)Riemannian manifold $(\bar{M}, \bar{g})$ is harmonic if its Laplacian $\Delta\phi$ also zeroes. If $\{E_j\}_j$ is an orthonormal frame on $\bar{M}$, then

$$\Delta\phi = trace(\nabla d\phi) = E_j E_j(\phi) - (\nabla_{E_j} E_j)\phi, \tag{2}$$

where $Hess\phi(X, Y) = \bar{g}(\nabla_X grad\phi, Y)$ is defined as the Hessian of ϕ, for all $X, Y \in \chi(\bar{M})$. That is, $\Delta\phi = div(grad\phi)$. The divergence of the vector field $X \in \chi(\bar{M})$ is defined by $divX = \bar{g}(\bar{\nabla}_{E_j} X, E_j)$ and the gradient of a function ϕ is $grad\phi = \bar{\nabla}\phi = (d\phi)^\sharp$, where $\sharp$ denotes the musical isomorphism with respect to $\bar{g}$.

We use Notations 2.1 below. Simply performing calculations, we obtain the following:

Lemma 2.2. *[10] On $\bar{M} = (M_1 \times_f M_2) \times_h M_3$, for all $X_i, Y_i \in \chi(M_i)$, $(i = 1, 2, 3)$ the components of the Hessian tensor $Hess\phi$ of ϕ are given by*

(1) $Hess\phi(X_1, Y_1) = Hess_1\phi(X_1, Y_1)$,
(2) $Hess\phi(X_1, Y_2) = -X_1(\ln f)Y_2(\phi)$,
(3) $Hess\phi(X_1, Y_3) = -X_1(\ln h)Y_3(\phi)$,
(4) $Hess\phi(X_2, Y_2) = f grad\phi(f) g_2(X_2, Y_2) + Hess_2\phi(X_2, Y_2)$,
(5) $Hess\phi(X_2, Y_3) = -X_2(\ln h)Y_3(\phi)$,
(6) $Hess\phi(X_3, Y_3) = h grad\phi(h) g_3(X_3, Y_3) + Hess_3\phi(X_3, Y_3)$.

It is possible to verify that for $\phi : \bar{M} \to \mathbb{R}$,

$$\frac{m}{\phi} Hess\phi = Hess\left(m \ln \phi\right) + \frac{1}{m} \mathrm{d}(m \ln \phi) \otimes \mathrm{d}(m \ln \phi), \quad m \in \mathbb{R}. \tag{3}$$

For the next results, we need the consider following construction:

Remark 2.3. *If $\{e_i\}_{i=1}^{m_1}$, $\{u_l\}_{l_1}^{m_2}$ and $\{\xi_\alpha\}_{\alpha=1}^{m_3}$ denote an orthonormal frame on the manifold M_i with respect to g_i $(i = 1, 2, 3)$ respectively, then each of $\{e_i\}_{i=1}^{m_1}$, $\{\frac{u_l}{f}\}_{l_1}^{m_2}$ and $\{\frac{\xi_\alpha}{h}\}_{\alpha=1}^{m_3}$ is an orthonormal frame on M_i $(i = 1, 2, 3)$ all with respect to g, respectively. Therefore, $\{E_j\}_{j_1}^{n} = \{e_i\}_{i=1}^{m_1} \bigcup \{\frac{u_l}{f}\}_{l_1}^{m_2} \bigcup \{\frac{\xi_\alpha}{h}\}_{\alpha=1}^{m_3}$ is an orthonormal frame on the sequential warped product manifold $\bar{M}$ with respect to $\bar{g}$.*

Now, we will examine the results on Laplacian, one of the most used operators in both partial differential equations and differential geometry, on $(\bar{M} = (M_1 \times_f M_2) \times_h M_3, \bar{g})$:

Proposition 2.4. *For any smooth function $\phi : \bar{M} \to \mathbb{R}$, we have:*

$$\Delta\phi = \Delta_1\phi + \frac{1}{f^2}\Delta_2\phi + \frac{1}{h^2}\Delta_3\phi + \frac{m_2}{f} grad_1 f(\phi) + \frac{m_3}{h} grad h(\phi). \tag{4}$$

Proof. Remark 2.3 and equation (2) yield

$$\begin{aligned} \Delta\phi =& E_j E_j(\phi) - (\bar{\nabla}_{E_j} E_j)\phi \\ =& [e_i e_i(\phi) - (\bar{\nabla}_{e_i} e_i)\phi] + \frac{1}{f^2}[u_l u_l(\phi) - (\bar{\nabla}_{u_l} u_l)\phi] + \frac{1}{h^2}[\xi_\alpha \xi_\alpha(\phi) - (\bar{\nabla}_{\xi_\alpha} \xi_\alpha)\phi]. \end{aligned} \tag{5}$$

From Proposition 2.1 in [6], we obtain

$$\Delta\phi = \Delta_1\phi + \frac{1}{f^2}[u_l u_l(\phi) - (\nabla^2_{u_l} u_l)\phi + f g_2(u_l, u_l) grad_1(f)\phi] + \frac{1}{h^2}[\xi_\alpha \xi_\alpha(\phi) - (\nabla^3_{\xi_\alpha}\xi_\alpha)\phi + h g_3(\xi_\alpha, \xi_\alpha) gradh(\phi)], \quad (6)$$

which gives the equation (4) by straightforward computation. □

As consequences of Proposition 2.4, we obtain the following:

Proposition 2.5. *For any smooth function ϕ_1 on M_1, we have:*

(i) ϕ_1 is harmonic on the sequential warped product $(\bar{M}, \bar{g})$ if and only if $\Delta_1\phi_1 = -grad\phi_1(m_2 \ln f + m_3 \ln h)$;

(ii) Any two of the following statements refer to the third:
(a) ϕ_1 is harmonic on the sequential warped product $(\bar{M}, \bar{g})$;
(b) ϕ_1 is harmonic on (M_1, g_1);
(c) $grad\phi_1$ is orthogonal to $grad(m_2 \ln f + m_3 \ln h)$.

Proposition 2.6. *Any smooth function ϕ_2 on M_2, one has:*

(i) ϕ_2 is harmonic on the sequential warped product $(\bar{M}, \bar{g})$ if and only if $\Delta_2\phi_2 = \frac{-m_3 f^2}{h} gradh(\phi_2)$.

(ii) Any two of the following statements refer to the third:
(a) ϕ_2 is harmonic on the sequential warped product $(\bar{M}, \bar{g})$;
(b) ϕ_2 is harmonic on (M_2, g_2);
(c) $gradh$ is orthogonal to $grad\phi_2$.

Proposition 2.7. *If $(\bar{M} = (M_1 \times_f M_2) \times_h M_3, \bar{g})$ is a SWP, then for any smooth function ϕ_3 on M_3, ϕ_3 is harmonic on $(\bar{M}, \bar{g})$ if and only if ϕ_3 is harmonic (M_3, g_3).*

Now we work with harmonic forms on the sequential warped product $(\bar{M} = (M_1 \times_f M_2) \times_h M_3, \bar{g})$ for which we recall the following:

Definition 2.8. *A p-form $\omega \in \mathcal{A}_p(\bar{M})$ is said to be co-closed if its co-differential $\delta^{\bar{g}}\omega$ given by*

$$\delta^{\bar{g}}\omega(X_1, \dots, X_{p-1}) = (\bar{\nabla}.\omega)(\cdot, X_1, \dots, X_{p-1}), \quad \forall X_1, \dots, X_{p-1} \in \chi(\bar{M}) \quad (7)$$

zeroes. Besides, both closed and co-closed p-form on $(\bar{M}, \bar{g})$ is said to be harmonic.

Proposition 2.9. *The co-differential operator $\delta^{\bar{g}}$ of $(\bar{M} = (M_1 \times_f M_2) \times_h M_3, \bar{g})$ is related to the co-differential operators δ^{g_j} of (M_j, g_j), $j = 1, 2, 3$, by:*

$$\delta^{\bar{g}}\omega_1 = \delta^{g_1}\omega_1 + \frac{m_2}{f} w_1(grad_1 f) + \frac{m_3}{h} w_1(gradh); \quad \forall \omega_1 \in \mathcal{A}_1(M_1), \quad (8)$$

$$\delta^{\bar{g}}\omega_2 = \frac{1}{f^2}\delta^{g_2}\omega_2 + \frac{m_3}{h} w_2(gradh); \quad \forall \omega_2 \in \mathcal{A}_1(M_2), \quad (9)$$

$$\delta^{\bar{g}}\omega_3 = \frac{1}{h^2}\delta^{g_3}\omega_3; \quad \forall \omega_3 \in \mathcal{A}_1(M_3). \quad (10)$$

Proof. From Remark 2.3, we can write $\delta^{\bar{g}}$ by

$$\begin{aligned}\delta^{g}\omega =&(\nabla_{E_j}\omega)(E_j) = E_j(\omega(E_j)) - \omega(\bar{\nabla}_{E_j}E_j) \\ =&e_i(\omega(e_i)) - \omega(\bar{\nabla}_{e_i}e_i) + \frac{u_l}{f}(\omega(\frac{u_l}{f})) - \omega(\bar{\nabla}_{\frac{u_l}{f}}\frac{u_l}{f}) \\ &+\frac{\xi_\alpha}{h}(\omega(\frac{\xi_\alpha}{h})) - \omega(\bar{\nabla}_{\frac{\xi_\alpha}{h}}\frac{\xi_\alpha}{h}).\end{aligned} \tag{11}$$

By using Proposition 2.1 in [6] in (11), we complete the proof. □

As a consequence, we obtain the following results for $(\bar{M} = (M_1 \times_f M_2) \times_h M_3, \bar{g})$:

Theorem 2.10. *For each $\omega \in \mathcal{A}_1(\bar{M})$, any four of the following statements imply the fifth:*

(i) *ω is co-closed with respect to $\bar{g}$;*

(ii) *ω restricted to M_1 is co-closed with respect to g_1;*

(iii) *ω restricted to M_2 is co-closed with respect to g_2;*

(iv) *ω restricted to M_3 is co-closed with respect to g_3;*

(v) *ω satisfies the relation*

$$\frac{m_2}{f}w(grad_1 f) + \frac{m_3}{h}w(gradh) = 0. \tag{12}$$

Corollary 2.11. *For each $\omega \in \mathcal{A}_1(\bar{M})$, any four of the following statements imply the fifth:*

(i) *ω is harmonic with respect to $\bar{g}$;*

(ii) *ω restricted to M_1 is harmonic with respect to g_1;*

(iii) *ω restricted to M_2 is harmonic with respect to g_2;*

(iv) *ω restricted to M_3 is harmonic with respect to g_3;*

(v) *ω is closed and satisfies the relation* (12).

Note that the Betti numbers give the dimension of the spaces of harmonic forms. Then from Corollary 2.11, we have the following:

Corollary 2.12. *Let $(\bar{M} = (M_1 \times_f M_2) \times_h M_3, \bar{g})$ be a SWP with (M_i, g_i) are all closed manifolds and the warping functions f and h are constants. If ω_j is harmonic on (M_j, g_j), $j = 1, 2, 3$ and not all three are zero at the same time, then the first Betti number of $\bar{M}$ is non-zero.*

Remark 2.13. *Proposition 2.9, Theorem* 2.10*, Corollary* 2.11 *and Corollary 2.12 can be generalized to arbitrary p-forms.*

Now our purpose is to express the sectional curvature on $(\bar{M} = (M_1 \times_f M_2) \times_h M_3, \bar{g})$:
By using Remark 2.3, we have the sectional curvature

$$K\Big(\sum\Big) = K(E, F) = \bar{g}(\bar{R}(E, F)F, E),$$

where $\sum$ is any plane spanned by an orthonormal (with respect to $\bar{g}$) basis $\{E, F\}$. Then we can prove the following:

Theorem 2.14. *For any plane $\sum$ tangent to $(\bar{M} = (M_1 \times_f M_2) \times_h M_3, \bar{g})$, the sectional curvature K of the metric $\bar{g}$ is obtained by the following six cases, where each K_j denotes the sectional curvatures of g_j, for $j = 1, 2, 3$:*

(i) If $\sum$ is tangent to M_1, then

$$K\Big(\sum\Big) = K_1\Big(\sum\Big); \tag{13}$$

(ii) If $\sum$ is tangent to M_2, then

$$K\Big(\sum\Big) = \frac{1}{f^2}[K_2\Big(\sum\Big) - \|grad_1 f\|^2]; \tag{14}$$

(iii) If $\sum$ is tangent to M_3, then

$$K\Big(\sum\Big) = \frac{1}{h^2}[K_3\Big(\sum\Big) - \|gradh\|^2]; \tag{15}$$

(iv) If $\sum$ is spanned by arbitrary unit vector fields $e_1 \in \chi(M_1)$ and $\frac{u_2}{f} \in \chi(M_2)$ with respect to $\bar{g}$, then

$$K\Big(\sum\Big) = -\frac{1}{f} Hess_1 f(e_1, e_1); \tag{16}$$

(v) If $\sum$ is spanned by arbitrary unit vector fields $e_1 \in \chi(M_1)$ and $\frac{\xi_3}{h} \in \chi(M_3)$ with respect to $\bar{g}$, then

$$K\Big(\sum\Big) = -\frac{1}{h} Hess_1 h(e_1, e_1). \tag{17}$$

(vi) If $\sum$ is spanned by arbitrary unit vector fields $\frac{u_2}{f} \in \chi(M_2)$ and $\frac{\xi_3}{h} \in \chi(M_3)$ with respect to $\bar{g}$, then

$$K\Big(\sum\Big) = -\frac{1}{fh} gradh(f) - \frac{1}{h} Hess_2 h(u_2, u_2). \tag{18}$$

Proof.

- *Case (i):* Let $\sum$ be a plane tangent to M_1. Then by Lemma 2.2-(1) of [10] for any orthonormal vector fields $\{e_1, e_2\}$ on M_1 with respect to $\bar{g}$, we have

$$K\Big(\sum\Big) = K(e_1, e_2) = g(R(e_1, e_2)e_2, e_1) = g_1(R_1(e_1, e_2)e_2, e_1) = K_1(e_1, e_2).$$

- *Case (ii):* Let $\sum$ be a plane tangent to M_2. Then by Lemma 2.2-(2) of [10] for any orthonormal vector fields $\{\frac{u_1}{f}, \frac{u_2}{f}\}$ on M_2 with respect to $\bar{g}$, we have

$$K\Big(\sum\Big) = K(\frac{u_1}{f}, \frac{u_2}{f}) = \frac{1}{f^2} g_2(R_2(u_1, u_2)u_2, u_1) - \frac{1}{f^2}\|grad_1 f\|^2.$$

- *Case (iii):* Let $\sum$ be a plane tangent to M_3. Then by Lemma 2.2-(7) of [10] for any orthonormal vector fields $\{\frac{\xi_1}{h}, \frac{\xi_2}{h}\}$ on M_3 with respect to $\bar{g}$, we have

$$K\Big(\sum\Big) = K(\frac{\xi_1}{h}, \frac{\xi_2}{h}) = \frac{1}{h^2} g_3(R_3(\xi_1, \xi_2)\xi_2, \xi_1) - \frac{1}{h^2}\|gradh\|^2.$$

- *Case (iv):* If $\sum$ is spanned by arbitrary unit vector fields $e_1 \in \chi(M_1)$ and $\frac{u_2}{f} \in \chi(M_2)$ with respect to $\bar{g}$, then by Lemma 2.2-(4) of [10], we have

$$K\Big(\sum\Big) = K(e_1, \frac{u_2}{f}) = -\frac{1}{f} g_1(\nabla^1_{e_1} grad_1 f, e_1) = -\frac{1}{f} Hess_1 f(e_1, e_1).$$

- *Case (v) and (vi)* are obtained analogously by Lemma 2.2-(6) of [10].

  Therefore, the proof is completed.

  □

Theorem 2.14 naturally allows us to obtain the following result:

Corollary 2.15. *If the sectional curvature $K(E,F)$ of $(\bar{M} = (M_1 \times_f M_2) \times_h M_3, \bar{g})$ has constant sign, then:*

(i) K_1 *has constant sign on* M_1*;*

(ii) $Hess_1 f(e_1, e_1)$ *and* $Hess_1 h(e_1, e_1)$ *have constant sign, for all unit vector field* $e_1 \in \chi(M_1)$*;*

(iii) K*,* $Hess_1 f(e_1, e_1)$ *and* $Hess_1 h(e_1, e_1)$ *have opposite sign for all unit vector field* $e_1 \in \chi(M_1)$*.*

In [2], the classical warped product manifolds of constant sectional curvature were studied. If the sectional curvature is constant, then the manifold is a space form. Thereby, it is natural to extend this study in our context.

Corollary 2.16. *If $(\bar{M} = (M_1 \times_f M_2) \times_h M_3, \bar{g})$ is a space form, with positive sectional curvature $K = c > 0$, then:*

(i) (M_1, g_1) *is a space form with positive sectional curvature* $K_1 = c > 0$*;*

(ii) (M_2, g_2) *is a space form with positive sectional curvature* $K_2 = cf^2 + \|grad_1 f\|^2$*;*

(iii) (M_3, g_3) *is a space form with positive sectional curvature* $K_2 = ch^2 + \|gradh\|^2$*;*

(iv) If $(\bar{M}, \bar{g})$ *is flat, then* (a) (M_1, g_1) *is flat.* (b) *Both components* (M_j, g_j)*,* $j = 2, 3$*, are flat if and only if the warping functions* f *and* h *are constants.*

3. Sequential Warped Products admitting Gradient Ricci-Yamabe Solitons

In recent years, there are many important studies that examine Ricci solitons, Ricci-harmonic solitons, gradient Shouten solitons etc., investigating inheritable properties and providing conditions under which these solitons can reduce Einstein manifolds, (for instance we refer [20]). Also, in [17], the authors study such conditions by considering the potential function of a Ricci soliton as a Killing and conformal vector fields and getting some cases where the potential field is concurrent.

Our aim in this section is to find the necessary and sufficient conditions for SWP admitting a GRYS in both Riemannian and Lorentzian settings. For this purpose, a characterization for the potential function of the soliton is obtained and the conditions that must be satisfied for each component of the SWP are obtained. We also investigate a SGRW spacetime and a SSS spacetime for this kind of solitons. Now, we have the following characterization for $(\bar{M} = (M_1 \times_f M_2) \times_h M_3, \bar{g})$:

Proposition 3.1. *If $(\bar{M}, \bar{g}, \phi, \lambda, \alpha, \beta)$ is a GRYS with $\phi : \bar{M} \to \mathbb{R}$, then ϕ depends only on M_1.*

Proof. Assume that $(\bar{M}, \bar{g}, \phi, \lambda, \alpha, \beta)$ is a gradient Ricci-Yamabe soliton with $\phi : \bar{M} \to \mathbb{R}$. Using Proposition 2.3 in [6] and lemma 2.2 into (1), we have

$$-X_1(\ln f)Y_2(\phi) = 0, \tag{19}$$

for $X_1 \in \chi(M_1)$, $Y_2 \in \chi(M_2)$. Using (19), we obtain that the potential function ϕ may depend on M_1 and/or M_3.Similarly, using Proposition 2.3 in [6] and lemma 2.2 into (1), we have

$$-X_1(\ln h)Y_3(\phi) = 0, \tag{20}$$

for $X_1 \in \chi(M_1)$, $Y_3 \in \chi(M_3)$. Using (20), we obtain that the potential function ϕ may depend on M_1 and/or M_2.From (19) and (20), we can conclude that ϕ depends only on M_1. □

Using Proposition 3.1, we prove the following:

Theorem 3.2. *$(\bar{M},\bar{g},\phi,\lambda,\alpha,\beta)$ is a GRYS and $m_i>1$ if and only if f, h, ϕ, λ, α, β verify:*
(i) (M_1,g_1) satisfies the following equation

$$\alpha Ric_1+Hess_1\left(\phi-\alpha m_2\ln f-\alpha m_3\ln h\right)-\frac{1}{\alpha m_2}d\left(m_2\ln f\right)\otimes d\left(m_2\ln f\right)$$

$$-\frac{1}{\alpha m_3}d\left(m_3\ln h\right)\otimes d\left(m_3\ln h\right)=\left(\lambda-\frac{1}{2}\beta\tau\right)g_1. \tag{21}$$

(ii) (M_2,g_2) is gradient almost η-Ricci soliton with $\alpha Ric_2+Hess_2\left(-\alpha m_3\ln h\right)-\frac{1}{\alpha m_3}d\left(m_3\ln h\right)\otimes d\left(m_3\ln h\right)=\Phi g_2$, where

$$\Phi=f^2\left(\lambda-\frac{1}{2}\beta\tau\right)-fgrad\phi\left(f\right)$$

$$+\alpha\left[f\Delta_1 f+(m_2-1)\|grad_1 f\|^2-\alpha f m_3\left(grad\ln h\right)(f)\right]. \tag{22}$$

(iii) (M_3,g_3) is an Einstein manifold with $Ric_3=\mu g_3$, where

$$\mu=\left(\lambda-\frac{1}{2}\beta\tau\right)\frac{1}{\alpha}h^2-\alpha h\Delta h-(m_3-1)\left\|gradh\right\|^2-\frac{1}{\alpha}h\left(grad\phi\right)(h). \tag{23}$$

Proof. Let $(\bar{M},\bar{g},\phi,\lambda,\alpha,\beta)$ be a GRYS and $m_i>1$. Using the equation (3), Proposition 2.3 in [6] and Proposition 3.1 in (1) for $X_1,Y_1\in\chi\left(M_1\right)$, we obtain the equation (21). Similarly, using Proposition 2.3 in [6], Lemma 2.2 and Proposition 3.1 in (1), we find

$$\alpha Ric_2\left(X_2,Y_2\right)+\frac{\alpha m_3}{h}Hess\left(h\right)\left(X_2,Y_2\right)$$

$$=\left[f^2\left(\lambda-\frac{1}{2}\beta\tau\right)-fgrad\phi\left(f\right)+\alpha f^{\sharp}\right]g_2\left(X_2,Y_2\right), \tag{24}$$

for $X_2,Y_2\in\chi\left(M_2\right)$. From the equation (3), we have

$$\frac{\alpha m_3}{h}Hess\left(h\right)=Hess_2\left(\alpha m_3\ln h\right)+\frac{1}{\alpha m_3}d\left(m_3\ln h\right)\otimes d\left(m_3\ln h\right). \tag{25}$$

Therefore, Substituting the equation (25) in (24), we have the desired result. Finally, using Proposition 2.3 in [6], Lemma 2.2 and Proposition 3.1 in (1), we have

$$\alpha Ric_3\left(X_3,Y_3\right)=\left[\left(\lambda-\frac{1}{2}\beta\tau\right)h^2-\alpha h^{\sharp}-h\left(grad\phi\right)(h)\right]g_3\left(X_3,Y_3\right),$$

for $X_3,Y_3\in\chi\left(M_3\right)$. Thus, M_3 is an Einstein manifold. This proves the theorem. □

Theorem 3.3. *$(\bar{M},\bar{g},\phi,\lambda,\alpha,\beta)$ is a GRYS and $m_i>1$ satisfying the following condition*

$$d\left[\|grad_1\phi\|^2-\left(\frac{2}{2\alpha+\beta m_1}+\frac{2}{\alpha}\right)\Delta_1\phi-\frac{2}{\alpha^2}\left(\lambda-\frac{1}{2}\beta\bar{\tau}\right)\phi\right](X)$$

$$=\frac{2m_2}{f}d\left(\Delta f\right)(X)-\frac{2m_2m_3}{fh}Hessh(grad_1 f,X)-\frac{2m_3}{h}div(Hessh)(X)$$

$$-\frac{2m_3}{\alpha h}Hessh(grad_1\phi,X)-\frac{2m_2}{\alpha f^2}dfgrad_1\phi(f)(X)-\frac{\beta}{\alpha^2}d\left(\tau\right)\phi\left(X\right), \tag{26}$$

then:
(i) If $\left(\lambda-\frac{1}{2}\beta\bar{\tau}\right)\left(f^2\right)+\frac{\alpha}{2}(m_2-1)\|grad_1 f\|^2+\frac{\alpha^2}{2\alpha+\beta n_1}(f\Delta_1 f)-f\left(grad_1\phi(f)\right)=c_1$ (c_1 is constant) and h reaches both maximum and minimum with $\alpha,\beta>0$ then h is constant on $M_1\times M_2$.
(ii) If $\frac{2\alpha}{2\alpha+\beta n_1}(h\Delta h)-\left\|gradh\right\|^2=c_2$ (c_2 is constant) and f reaches both maximum and minimum with $\alpha,\beta>0$ and $\lambda<\frac{1}{2}\beta\bar{\tau}$ then f is constant on M_1.

Proof. If we take the trace the case (1) on Proposition 2.3 in [6], then we obtain

$$\bar{\tau} = \tau_1 - m_2 \frac{\Delta_1 f}{f} - m_3 \frac{\Delta h}{h}. \tag{27}$$

Then taking the trace of the equation (1) and using (27), we find

$$\alpha\tau_1 = \left(\lambda - \frac{1}{2}\beta\bar{\tau}\right) m_1 + \alpha m_2 \frac{\Delta_1 f}{f} + \alpha m_3 \frac{\Delta h}{h} - \Delta_1 \phi. \tag{28}$$

From the equation (28), we have

$$\alpha d\tau_1 = -\frac{1}{2}\beta m_1 d\bar{\tau} - \alpha m_2 \frac{1}{f^2} df \Delta_1 f + \alpha m_2 \frac{1}{f} d\left(\Delta_1 f\right)$$

$$-\alpha m_3 \frac{1}{h^2} dh \Delta h + \alpha m_3 \frac{1}{h} d\left(\Delta h\right) - d\left(\Delta_1 \phi\right). \tag{29}$$

From the second Bianchi identity, we have

$$d\tau_1 = 2div(Ric_1). \tag{30}$$

From Proposition 2.3 in [6], the equations (28) and (1), we get

$$div(\alpha Ric_1) = m_2 \alpha div\left(\frac{Hess_1 f}{f}\right) + m_3 \alpha div\left(\frac{Hessh}{h}\right) - div\left(Hess_1 \phi\right). \tag{31}$$

By the definition of divergence, we have

$$div\left(\frac{Hess_1 f}{f}\right)(X) = -\frac{1}{f^2} Hess_1 f(grad_1 f, X) + \frac{1}{f} div Hess_1 f(X) \tag{32}$$

for any vector field X on M_1. Since $Hess_1 f(grad_1 f, X) = \frac{1}{2} d(\left\|grad_1 f\right\|^2)(X)$, the equation (32) turns into

$$div(\frac{Hess_1 f}{f})(X) = -\frac{1}{2f^2} d(\left\|grad_1 f\right\|^2)(X) + \frac{1}{f} div Hess_1 f(X). \tag{33}$$

Similarly, we obtain

$$div(\frac{Hessh}{h})(X) = -\frac{1}{2h^2} d(\left\|gradh\right\|^2)(X) + \frac{1}{h} div Hessh(X). \tag{34}$$

It is well known that

$$div(Hess\phi)(X) = Ric\left(grad\phi, X\right) - \Delta\left(d\phi\right)(X). \tag{35}$$

Using the equation (35), we get

$$div\left(Hess_1 \phi\right) = Ric_1\left(grad_1 \phi, X\right) - \Delta_1\left(d\phi\right)(X). \tag{36}$$

Using equations (35) in (33), we find

$$\begin{aligned} div(\frac{Hess_1 f}{f})(X) = & \frac{1}{2f^2}\left[(m_2 - 1) d(\left\|grad_1 f\right\|^2)(X) + (\frac{2\lambda - \beta\bar{\tau}}{\alpha}) f df(X) - (2f) d\left(\Delta_1 f\right)(X)\right] \\ & - \frac{1}{\alpha f} Hess_1 \phi\left(grad_1 f, X\right) + \frac{m_3}{fh} Hessh(grad_1 f, X). \end{aligned} \tag{37}$$

Similarly, we obtain

$$div\left(Hess_1\phi\right)(X) = \frac{1}{2\alpha}\left[-d(\left\|grad_1\phi\right\|^2)(X) + (2\lambda-\beta\bar{\tau})d\phi(X) + (2\alpha)d\left(\Delta_1\phi\right)(X)\right] + \frac{m_2}{f}Hess_1 f\left(grad_1\phi, X\right) + \frac{m_3}{h}Hessh(grad_1\phi, X). \tag{38}$$

Substituting the equations (34), (37) and (38) into (31), we find

$$\begin{aligned} div(Ric_1)(X) = &\frac{\alpha m_2}{2f^2}\left[(m_2-1)\,d(\left\|grad_1 f\right\|^2)(X) + (\frac{2\lambda-\beta\bar{\tau}}{\alpha})f df(X) - (2f)d\left(\Delta_1 f\right)(X)\right] \\ &-\frac{m_2}{f}Hess_1\phi\left(grad_1 f, X\right) + \frac{\alpha m_2 m_3}{fh}Hessh(grad_1 f, X) \\ &-\frac{\alpha m_3}{2h^2}d(\left\|gradh\right\|^2)(X) + \frac{\alpha m_3}{h}divHessh(X) \\ &-\frac{1}{2\alpha}\left[-d(\left\|grad_1\phi\right\|^2)(X) + (2\lambda-\beta\bar{\tau})d\phi(X) + (2\alpha)d\left(\Delta_1\phi\right)(X)\right] \\ &-\frac{m_2}{f}Hess_1 f\left(grad_1\phi, X\right) - \frac{m_3}{h}Hessh(grad_1\phi, X). \end{aligned} \tag{39}$$

By the use of the equations (29) and (39) into (30), we have

$$\begin{aligned} &\frac{\alpha m_2}{f^2}\left[(m_2-1)\,d(\left\|grad_1 f\right\|^2)(X) + (\frac{2\lambda-\beta\bar{\tau}}{\alpha})f df(X) - (2f)d\left(\Delta_1 f\right)(X)\right] \\ &-\frac{2m_2}{f}Hess_1\phi\left(grad_1 f, X\right) + \frac{2\alpha m_2 m_3}{fh}Hessh(grad_1 f, X) \\ &-\frac{\alpha m_3}{h^2}d(\left\|gradh\right\|^2)(X) + \frac{2\alpha m_3}{h}divHessh(X) \\ &-\frac{1}{\alpha}\left[-d(\left\|grad_1\phi\right\|^2)(X) + (2\lambda-\beta\bar{\tau})d\phi(X) + (2\alpha)d\left(\Delta_1\phi\right)(X)\right] \\ &-\frac{2m_2}{f}Hess_1 f\left(grad_1\phi, X\right) - \frac{2n_3}{h}Hessh(grad_1\phi, X) \\ &-\frac{1}{2}\beta m_1 d\bar{\tau} - \alpha m_2\frac{1}{f^2}df\Delta_1 f + \alpha m_2\frac{1}{f}d\left(\Delta_1 f\right) \\ &-\alpha m_3\frac{1}{h^2}dh\Delta h + \alpha m_3\frac{1}{h}d\left(\Delta h\right) - d\left(\Delta_1\phi\right) = 0. \end{aligned} \tag{40}$$

Using the condition (26), we obtain

$$\begin{aligned} &\frac{1}{f^2}d\left[\frac{2m_2}{\alpha}\left(\lambda-\frac{1}{2}\beta\bar{\tau}\right)\left(f^2\right) + m_2(m_2-1)\|grad_1 f\|^2 + \frac{2m_2\alpha}{2\alpha+\beta n_1}(f\Delta_1 f) - \frac{2m_2}{\alpha}f\left(grad_1\phi(f)\right)\right](X) \\ &+\frac{1}{h^2}d\left[\frac{2m_3\alpha}{2\alpha+\beta m_1}(h\Delta h) - m_3\|gradh\|^2\right](X) = 0. \end{aligned} \tag{41}$$

Then it is necessary to examine the following two cases:

Case I: If $\left(\lambda-\frac{1}{2}\beta\bar{\tau}\right)\left(f^2\right) + \frac{\alpha}{2}(m_2-1)\|grad_1 f\|^2 + \frac{\alpha^2}{2\alpha+\beta m_1}(f\Delta_1 f) - f\left(grad_1\phi(f)\right) = c_1$ =constant, then we have $\frac{2m_3\alpha}{2\alpha+\beta m_1}(h\Delta h) - m_3\left\|gradh\right\|^2 = \nu$, where ν is constant. Let (p_1,q_1), $(p_2,q_2) \in M_1\times M_2$ be the points where h attains its maximum and minimum in $M_1\times M_2$. Hence we have $gradh\,(p_1,q_1) = 0 = gradh\,(p_2,q_2)$ and $\Delta h\,(p_1,q_1) \le 0 \le \Delta h\,(p_2,q_2)$. Since $h>0$ and $\alpha,\beta>0$, we obtain

$$0 \ge \frac{2\alpha}{2\alpha+\beta m_1}h\,(p_1,q_1)\,\Delta h\,(p_1,q_1) = \nu = \frac{2\alpha}{2\alpha+\beta m_1}h\,(p_2,q_2)\,\Delta h\,(p_2,q_2) \ge 0.$$

By the strong maximum principle, this implies h is constant on $M_1 \times M_2$ when $\alpha, \beta > 0$.

Case II: If $\frac{2m_3\alpha}{2\alpha+\beta m_1}(h\Delta h) - m_3 \left\|gradh\right\|^2 = c_2 =$ constant, then we have $\left(\lambda - \frac{1}{2}\beta\overline{\tau}\right)\left(f^2\right) + \frac{\alpha}{2}m_2(m_2-1)\|grad_1 f\|^2 + \frac{\alpha^2}{2\alpha+\beta m_1}(f\Delta_1 f) - f\left(grad_1\phi(f)\right) = \omega$, where ω is constant. Let p, $q \in M_1$ be the points where f attains its maximum and minimum in M_1. Hence we have $gradf(p) = 0 = gradf(q)$ and $\Delta_1 f(p) \leq 0 \leq \Delta_1 f(q)$. Since $f > 0$, $\alpha, \beta > 0$ and $\lambda < \frac{1}{2}\beta\overline{\tau}$, we find

$$0 \geq \frac{\alpha^2}{2\alpha + \beta m_1} f(p)\,\Delta_1 f(p) = \omega(p) + f(p)\left(grad_1\phi(f)(p)\right) - \left(\lambda - \frac{1}{2}\beta\overline{\tau}\right)\left(f^2\right)(p)$$

$$\geq \omega(q) + f(q)\left(grad_1\phi(f)(q)\right) - \left(\lambda - \frac{1}{2}\beta\overline{\tau}\right)\left(f^2\right)(q) \geq 0.$$

By the strong maximum principle, this implies f is constant on M_1 when $\alpha, \beta > 0$ and $\lambda < \frac{1}{2}\beta\overline{\tau}$. This completes the proof. □

As physical applications, we obtain the behavior of these solitons in SGRW spacetimes and SSS spacetimes.

From Theorem 3.2, we have following characterization for a SGRW spacetime ($\bar{M} = (I \times_f M_2) \times_h M_3, \overline{g} = (-dt^2) \oplus f^2 g_2 \oplus h^2 g_3$):

Corollary 3.4. *$(\bar{M}, \bar{g}, \phi, \lambda, \alpha, \beta)$ is a GRYS and $m_i > 1$, $i = 2, 3$ if and only if f, h, ϕ, λ, α, β verify:*
(i) (M_1, g_1) satisfies the following equation

$$\phi'' + Hess_1\left(-\alpha m_2 \ln f - \alpha m_3 \ln h\right) - \frac{1}{\alpha m_2} d\left(m_2 \ln f\right) \otimes d\left(m_2 \ln f\right)$$

$$-\frac{1}{\alpha m_3} d\left(m_3 \ln h\right) \otimes d\left(m_3 \ln h\right) = \left(\frac{1}{2}\beta\tau - \lambda\right),$$

(ii) (M_2, g_2) is gradient almost η-Ricci soliton with $\alpha Ric_2 + Hess_2\left(-\alpha m_3 \ln h\right) - \frac{1}{\alpha m_3} d\left(m_3 \ln h\right) \otimes d\left(m_3 \ln h\right) = \Phi g_2$, where

$$\Phi = f^2\left(\lambda - \frac{1}{2}\beta\tau\right) - ff'\phi' - \alpha\left(ff'' + (m_2 - 1)\left(f'\right)^2\right) - \alpha f m_3\left(grad \ln h\right)(f),$$

(iii) (M_3, g_3) is an Einstein manifold with $Ric_3 = \mu g_3$, where

$$\mu = \left(\lambda - \frac{1}{2}\beta\tau\right)\frac{1}{\alpha}h^2 - \alpha h\Delta h - (m_3 - 1)\left\|gradh\right\|^2 - \frac{1}{\alpha}h\left(grad\phi\right)(h).$$

Proof. By substituting $grad_1 f = -f'$, $Hess_1 f(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = f''$, $Hess_1\phi = \phi''$, $\Delta_1 f = -f''$, $g_1(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = -1$, $\left\|grad_1 f\right\|^2 = -\left(f'\right)^2$ in Theorem 3.2, we complete the proof. □

From Theorem 3.2, we have following characterization for a SSS spacetime ($\bar{M} = (M_1 \times_f M_2) \times_h I, \overline{g} = g_3 \oplus f^2 g_2 \oplus h^2(-dt^2)$):

Corollary 3.5. *$(\bar{M}, \bar{g}, \phi, \lambda, \alpha, \beta)$ is a GRYS and $m_i > 1$, $i = 1, 2$ if and only if f, h, ϕ, λ, α, β verify:*
(i) (M_1, g_1) satisfies the following equation

$$\alpha Ric_1 + Hess_1\left(\phi - \alpha m_2 \ln f - \alpha m_3 \ln h\right) - \frac{1}{\alpha m_2} d\left(m_2 \ln f\right) \otimes d\left(m_2 \ln f\right)$$

$$-\frac{1}{\alpha m_3} d\left(m_3 \ln h\right) \otimes d\left(m_3 \ln h\right) = \left(\lambda - \frac{1}{2}\beta\tau\right) g_1,$$

(ii) (M_2, g_2) *is gradient almost* η*-Ricci soliton with* $\alpha Ric_2 + Hess_2(-\alpha m_3 \ln h) - \frac{1}{\alpha m_3} d(m_3 \ln h) \otimes d(m_3 \ln h) = \Phi g_2$, *where*

$$\Phi = f^2\left(\lambda - \frac{1}{2}\beta\tau\right) - f grad\phi(f)$$

$$+\alpha\left(f\Delta_1 f + (m_2 - 1)\|grad_1 f\|^2 - \alpha f m_3 (grad \ln h)(f)\right),$$

(iii) (M_3, g_3) *satisfies the following equation*

$$\left(\lambda - \frac{1}{2}\beta\tau\right)\frac{1}{\alpha}h^2 - \alpha h\Delta h - (m_3 - 1)\left\|gradh\right\|^2 - \frac{1}{\alpha}h\left(grad\phi\right)(h) = 0.$$

Proof. By substituting $g_3(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = -1$, $Ric_3(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = 0$ in Theorem 3.2, we complete the proof. □

4. Conslusions

Sequential warped product manifolds and gradient Ricci-Yamabe solitons play an important role in both differential geometry and relativity theory. Sequential warped products are useful for modeling layered spacetime structures, while gradient Ricci-Yamabe solitons contribute to self-similar solutions of geometric flows with physical relevance in cosmology and gravitational theory. So, we firstly explored the geometric structure of sequential warped product manifolds.Then we derived necessary and sufficient conditions under which these manifolds admitting gradient Ricci-Yamabe solitons and examined their implications. Furthermore, we applied our general findings to important physical models by considering sequential generalized Robertson-Walker spacetimes and sequential standard static spacetimes. These applications demonstrate how the geometry of sequential warped products influences the behavior of gradient Ricci-Yamabe solitons in a relativistic setting.

Declarations

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