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Optimality extension criteria to the 3D double-diffusive magneto convection system

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Abstract. We establish optimal extension criteria for the three-dimensional double-diffusive magneto convection system in terms of the planar components of the solution or their horizontal derivatives. More precisely, we prove that a unique local strong solution does not blow up at time T provided that $(\nabla_h\tilde{u}, \nabla_h\tilde{b}) \in L^2(0, T; \dot{V}^{-1}_{\infty,\infty,2})$ or $(\tilde{u}, \tilde{b}) \in L^2(0, T; \dot{V}^{0}_{\infty,\infty,2})$, where the Vishik space $\dot{V}^{s}_{p,q,\theta}$ strictly contains the homogeneous Besov space $\dot{B}^{s}_{p,q}$. Our approach relies on a logarithmic interpolation inequality combined with the well-known Kato-Ponce commutator estimates.

1. Introduction

Consider the three-dimensional Cauchy problem for the double-diffusive magneto convection system

$$\begin{cases} u_t - \Delta u + (u\cdot\nabla)u + \nabla p = (b\cdot\nabla)b + (\theta - s)\mathbf{k}, \\ b_t - \Delta b + (u\cdot\nabla)b = (b\cdot\nabla)u \\ \theta_t - \Delta\theta + u\cdot\nabla\theta = u\cdot\mathbf{k}, \\ s_t - \Delta s + u\cdot\nabla s = u\cdot\mathbf{k}, \\ \nabla\cdot u = \nabla\cdot b = 0, \\ u(0,x) = u_0(x), b(0,x) = b_0(x), \theta(0,x) = \theta_0(x), s(0,x) = s_0(x), \end{cases} \tag{1}$$

where $u = u(t,x)$, $b = b(t,x)$ and $p = p(t,x)$ are unknown velocity, magnetic field and dynamic pressure of the fluid, respectively. $\theta = \theta(t,x)$, $s = s(t,x)$ and $\mathbf{k}$ are scalar quantities affecting the density of the fluid and vertical unit vector, respectively.

Double-diffusive convection behavior is driven by the interaction of two fluids components diffusing at different rates plays an important role in oceanography and many other fields. (see [8, 9, 17] for details). We now analyze the structures of the system (1). If $b \equiv \theta \equiv s \equiv 0$, the system (1) reduces to incompressible Navier-Stokes equations. If $\theta \equiv s \equiv 0$, it reduces to incompressible magnetohydrodynamics (MHD) equations and the system (1) at $b \equiv 0$ degenerates to the double-diffusive convection system. All of those

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systems have been studied intensively, particularly regarding whether the given initial data is sufficiently smooth for the solution to remain smooth or experience a finite time singularity. Very recently, the second author of the present paper [25] established the existence and uniqueness of local strong solution to (1) for given a initial data $(u_0, b_0, \theta_0, s_0) \in H^1$ and proved the global existence of a strong solution when the L^2 norm of the initial data is small.

It is an open problem whether a local strong solution to the 3D Navier-Stokes equations can be smoothly extended beyond time T up to infinity. There is a vast literature providing sufficient conditions to ensure the extension of the local strong solution (see [14, 15] and references therein).

One of the major contributions is the Beale-Kato-Majda's regularity criterion [3], which holds for Euler as well as for Navier-Stokes, and states that if a smooth solution to either the Euler or Navier-Stokes equations develop singularities in finite time, then

$$\int_0^{T_{\max}} \|\omega(\cdot,t)\|_{L^\infty} \mathrm{d}t = +\infty. \tag{2}$$

This result was also extended to the strain tensor [10]. Recently, Guo-Kučera-Skalák [7] using a different approach based on Bony decomposition proved that a weak solution of Navier-Stokes equations is regular provided

$$\begin{aligned} &\tilde{\omega} \in L^p(0,T;\dot{B}^{-\frac{3}{q}}_{\infty,\infty}), \text{ for } q \in (3,\infty) \text{ and } \frac{2}{p}+\frac{3}{q}=2,\\ &\tilde{\omega} \in L^p(0,T;\dot{B}^{-3(\frac{1}{q}-\frac{1}{\theta})}_{\theta,\infty}), \text{ for } q \in (\frac{3}{2},3],\ \theta \in [q,\frac{3q}{3-q}) \text{ and } \frac{2}{p}+\frac{3}{q}=2, \end{aligned} \tag{3}$$

where $\tilde{\omega} = (\omega_1, \omega_2, 0)$. Later, O in [14] proved that the local strong solution can be smoothly extended after T provided

$$\tilde{\omega} \in L^2(0,T;BMO^{-1}). \tag{4}$$

And then, this result was then extended in the aforementioned paper [15] to the Besov space.They proved that the local strong solution can be extended beyond T if

$$\widetilde{\omega} \in L^2(0,T;\dot{B}^{-1}_{\infty,\infty}). \tag{5}$$

In addition to strengthening regularity criteria to larger spaces, there have also been results not involving all the components of u, for instance regularity criteria on the planar components or its partial derivatives. Denote by $\tilde{u} = (u_1, u_2, 0)$, $\tilde{b} = (b_1, b_2, 0)$ and $\nabla_h = (\partial_1, \partial_2, 0)$ the planar components vector of u, b and the partial derivatives, respectively. Dong and Zhang [5] proved the BKM type criterion to the 3D Navier-Stokes equations via the partial derivatives of planar velocity $\nabla_h \widetilde{u}$

$$\nabla_h \widetilde{u} \in L^1(0,T;\dot{B}^0_{\infty,\infty}). \tag{6}$$

In [2], Benbernou-Gala-Ragusa showed that if $(\tilde{u}, \tilde{b})$ belongs to the space $L^2(0,T;BMO)$, then the Leray-Hopf solution (u,b) of the 3D incompressible magnetohydrodynamic equations is regular. Later, Zhang-Yang [27] established the following MHD equations regularity criterion:

$$\tilde{u} \in L^2(0,T;\dot{B}^0_{\infty,\infty}). \tag{7}$$

Recently, Kanamaru in [11] proved a logarithmic interpolation inequality using the Vishik space $\dot{V}^s_{p,\sigma,\theta}$ (see definition 2.4). As an application of this inequality, they obtained a regularity criteria of Beale-Kato-Majda type. Specifically, if a weak solution u satisfies

$$\int_0^T \|\operatorname{rot} u(\tau)\|^{\theta}_{\dot{V}^0_{p,\infty,\theta}} \mathrm{d}\tau < \infty, \quad \frac{2}{\theta}+\frac{3}{p}=2, p \in (3,\infty]. \tag{8}$$

Then the solution u is smooth on $(0,T]$. Later, the author used the Littlewood-Paley decomposition method to obtain Navier-Stokes regularity criteria in Vishik spaces [22]. This was done by considering two components of the vorticity vector or the scalar pressure. For more recent progress in Vishik spaces, see [12, 26]. Furthermore, Farwig and Kanamaru in [6] proved that a strong solution u to the Navier-Stokes equations on $(0,T)$ can be extended if either $u \in L^{\theta}\left(0,T;\dot{U}^{-\alpha}_{\infty,\frac{1}{\theta},\infty}\right)$ for $\frac{2}{\theta}+\alpha = 1, 0 < \alpha < 1$ or $u \in L^2\left(0,T;\dot{V}^0_{\infty,\infty,2}\right)$, where $\dot{U}^s_{p,\beta,\sigma}$ and $\dot{V}^s_{p,q,\theta}$ are Banach spaces(See Definition 2.4, 2.5) that may be larger than the homogeneous Besov space $\dot{B}^s_{p,q}$. Very recently, Ma, Ragusa, and Wu [13] extended the criteria of Farwing and Kanamaru by employing horizontal components of the velocity in Vishik-type spaces. They showed that the solution remains regular provided either

$$\tilde{u} \in L^2\left(0,T;\dot{V}^0_{\infty,\infty,2}\right) \tag{9}$$

or

$$\nabla_h\tilde{u} \in L^1\Big(0,T;\dot{U}^{\frac{3}{p}}_{p,\frac{1}{\theta},\infty}(\mathbb{R}^3)\Big) \quad \text{with} \quad 1 \le p \le \infty,\ 1 \le \theta < \infty. \tag{10}$$

In papers [18, 23, 24], the author established some global regularity/stability results to the 3D double-diffusive convection system. In particular, the author in [23] proved that if partial derivatives of the planar components of the velocity field (i.e. $\nabla_h\tilde{u}$) belong to the Besov space:

$$\nabla_h\tilde{u} \in L^{\frac{2}{2-r}}\Big(0,T;\dot{B}^{-r}_{\infty,\infty}(\mathbb{R}^3)\Big) \quad \text{with} \quad 0 \le r < 1, \tag{11}$$

then the local solution (u,θ,s) can be extended smoothly beyond $t = T$. It is an open question for the case of end-point $r = 1$ in (11) at that time. In the case of the 3D double-diffusive magneto convection system, O-Wu [16] proved the BKM type blow-up criterion involving the partial derivatives of planar components. Namely, it was proved that the local strong solution can be extended beyond T if

$$(\nabla_h\tilde{u},\nabla_h\tilde{b}) \in L^2(0,T;\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)). \tag{12}$$

Motivated by the references cited above, it is interesting to extend the criteria (7) and (12) to the new Banach spaces (i.e., Vishik spaces), which larger than the homogeneous Besov spaces. Moreover, we will consider the complex fluid of double-diffusive magneto convection system and show the following.

Theorem 1.1. Let $(u_0,b_0,\theta_0,s_0) \in H^2$ and (u,b,θ,s) be a unique local strong solution to (1). If $(\tilde{u},\tilde{b})$ satisfies

$$\int_0^T \left(\|\nabla_h\tilde{u}\|^2_{\dot{V}^{-1}_{\infty,\infty,2}} + \|\nabla_h\tilde{b}\|^2_{\dot{V}^{-1}_{\infty,\infty,2}}\right)dt < \infty, \tag{13}$$

or

$$\int_0^T \left(\|\tilde{u}\|^2_{\dot{V}^{0}_{\infty,\infty,2}} + \|\tilde{b}\|^2_{\dot{V}^{0}_{\infty,\infty,2}}\right)dt < \infty, \tag{14}$$

then the solution (u,b,θ,s) can be smoothly extended beyond time T.

Remark 1.2. It follows by Definition 2.4,2.5 that

$$\dot{B}^s_{\infty,\infty} \hookrightarrow \dot{V}^s_{\infty,\infty,\theta} \hookrightarrow \dot{U}^s_{\infty,\frac{1}{\theta},\infty} \tag{15}$$

for $s \in \mathbb{R}$ and $1 \le \theta < \infty$. In view of (15), regularity criteria (13) and (14) can be viewed as a generalization of [6, 11, 14–16, 23, 25, 27].

The proofs of Theorem 1.1 is based on a bilinear estimate, a logarithmic interpolation inequality and the well-known commutator estimate due to Kato and Ponce [10].

2. Preliminaries

Let $\mathcal{S}(\mathbb{R}^3)$ be the Schwartz class of rapidly decreasing functions. Given $f \in \mathcal{S}(\mathbb{R}^3)$, its Fourier transform $\mathcal{F}f = \hat{f}$ is defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}^3} f(x)e^{-ix\cdot\xi}dx.$$

Let (χ, φ) be a couple of smooth functions valued in $[0,1]$ such that χ is supported in $B = \{\xi \in \mathbb{R}^3 : |\xi| \le \frac{4}{3}\}$, φ is supported in $C = \{\xi \in \mathbb{R}^3 : \frac{3}{4} \le |\xi| \le \frac{8}{3}\}$ such that

$$\chi(\xi) + \sum_{j\ge 0} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^3,$$

$$\sum_{j\in Z} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^3 \setminus \{0\}.$$

Denoting $\varphi_j = \varphi(2^{-j}\xi)$, $h = \mathcal{F}^{-1}\varphi$ and $\widetilde{h} = \mathcal{F}^{-1}\chi$, the dyadic blocks are defined as follows, respectively.

$$\dot{\Delta}_j f = \varphi(2^{-j}D)f = 2^{3j}\int_{\mathbb{R}^3} h(2^j y)f(x-y)dy, \ j \in \mathbb{Z},$$

$$\dot{S}_j f = \sum_{k\le j-1} \Delta_j f = \chi(2^{-j}D)f = 2^{3j}\int_{\mathbb{R}^3} \widetilde{h}(2^j y)f(x-y)dy, \ j \in \mathbb{Z}.$$

Furthermore, the above dyadic decomposition has nice properties of quasi-orthogonality, namely,

$$\dot{\Delta}_j\dot{\Delta}_q f \equiv 0 \quad \text{with } |j-q| \ge 2.$$

We have the following formal decomposition:

$$f = \sum_{-\infty}^{\infty} \dot{\Delta}_j f, \quad f \in \mathcal{S}'(\mathbb{R}^3) \setminus \mathcal{P}(\mathbb{R}^3),$$

where $\mathcal{P}(\mathbb{R}^3)$ is the set of polynomials, which can be found in [1] about the details of the Littlewood-Paley decomposition theory.

Definition 2.1. *Let $s \in \mathbb{R}$, $(p,\sigma) \in [1,\infty]^2$, the homogeneous Besov space $\dot{B}^s_{p,\sigma}$ is defined by*

$$\dot{B}^s_{p,\sigma} = \{f \in \mathcal{Z}'(\mathbb{R}^3); \|f\|_{\dot{B}^s_{p,\sigma}} < \infty\},$$

where

$$\|f\|_{\dot{B}^s_{p,\sigma}} = \begin{cases} (\sum_{j\in Z} 2^{js\sigma}\|\dot{\Delta}_j f\|_p^\sigma)^{\frac{1}{\sigma}} & \sigma < \infty, \\ \sup_{j\in Z} 2^{js}\|\dot{\Delta}_j f\|_p & \sigma = \infty. \end{cases}$$

Here $\mathcal{Z}'(\mathbb{R}^3)$ is the dual space of

$$\mathcal{Z}(\mathbb{R}^3) = \{f \in \mathcal{S}(\mathbb{R}^3); D^\alpha \widehat{f}(0) = 0, \forall \alpha \in \mathbb{N}^3\}.$$

The following Bernstein's inequalities [1] will be frequently used throughout the paper.

Lemma 2.2. *Let $k \in N$, then for all $1 \le p \le q \le \infty$ there holds the inequality*

$$\sup_{|\alpha|=k} \|\partial^\alpha \dot{\Delta}_j f\|_{L^q} \le C2^{jk+3j(\frac{1}{p}-\frac{1}{q})}\|\dot{\Delta}_j f\|_{L^p}.$$

where C is an absolute constant independent of f, j.

Remark 2.3. *From the above Bernstein's estimate, we easily conclude that*

$$\left\|\dot{\Delta}_j f\right\|_{L^q} \leq C2^{3j\left(\frac{1}{p}-\frac{1}{q}\right)} \left\|\dot{\Delta}_j f\right\|_{L^p}.$$

We now introduce Banach spaces $\dot{V}^s_{p,q,\theta}$ and $\dot{U}^s_{p,\beta,\sigma}$ which are larger than the homogeneous Besov spaces $\dot{B}^s_{p,q}$. These spaces may be regarded as modified versions of spaces defined by Nakao-Taniuchi [20] and Vishik [21].

Definition 2.4. *Let $s \in \mathbb{R}, 1 \leq p, q, \theta \leq \infty$. Then, $\dot{V}^s_{p,q,\theta}(\mathbb{R}^n) := \left\{f \in Z'; \|f\|_{\dot{V}^s_{p,q,\theta}} < \infty\right\}$ is introduced by the norm*

$$\|f\|_{\dot{V}^s_{p,q,\theta}} := \begin{cases} \sup_{N=1,2,\cdots} \frac{\left(\sum_{|j|\leq N} 2^{js\theta}\|\dot{\Delta}_j f\|^\theta_{L^p}\right)^{\frac{1}{\theta}}}{N^{\frac{1}{\theta}-\frac{1}{q}}}, & \theta \neq \infty, \\ \sup_{N=1,2,\cdots} N^{\frac{1}{q}} \max_{|j|\leq N} 2^{js} \left\|\dot{\Delta}_j f\right\|_{L^p}, & \theta = \infty. \end{cases}$$

Definition 2.5. *Let $s, \beta \in \mathbb{R}, 1 \leq p, \sigma \leq \infty$. Then, $\dot{U}^s_{p,\beta,\sigma}(\mathbb{R}^n) := \left\{f \in Z'; \|f\|_{\dot{U}^s_{p,\beta,\sigma}} < \infty\right\}$ is equipped with the norm*

$$\|f\|_{\dot{U}^s_{p,\beta,\sigma}} := \begin{cases} \sup_{N=1,2,\cdots} \frac{\left(\sum_{|j|\leq N} 2^{js\sigma}\|\dot{\Delta}_j f\|^\sigma_{L^p}\right)^{\frac{1}{\sigma}}}{N^\beta}, & \sigma \neq \infty, \\ \sup_{N=1,2,\cdots} \frac{\max_{|j|\leq N} 2^{js}\|\dot{\Delta}_j f\|_{L^p}}{N^\beta}, & \sigma = \infty. \end{cases}$$

We see from the following lemma that $\dot{V}^s_{p,q,\theta}$ and $\dot{U}^s_{p,\beta,\sigma}$ are extensions of $\dot{B}^s_{p,q}$ and $\dot{V}^s_{p,q,\theta}$, respectively.

Lemma 2.6. *[6] (i) Let $s \in \mathbb{R}, 1 \leq p, q \leq \infty$ and $1 \leq \theta_1 \leq \theta_2 \leq q < \theta_3$. Then, it holds that*

$$\{0\} = \dot{V}^s_{p,q,\theta_3} \subset \dot{B}^s_{p,q} = \dot{V}^s_{p,q,q} \subset \dot{V}^s_{p,q,\theta_2} \subset \dot{V}^s_{p,q,\theta_1}.$$

(ii) Let $s \in \mathbb{R}, 1 \leq p, \sigma \leq \infty$ and $\beta_1 < 0 \leq \beta_2 \leq \beta_3$. Then, it holds that

$$\{0\} = \dot{U}^s_{p,\beta_1,\sigma} \subset \dot{B}^s_{p,\sigma} = \dot{U}^s_{p,0,\sigma} \subset \dot{U}^s_{p,\beta_2,\sigma} \subset \dot{U}^s_{p,\beta_3,\sigma}.$$

(iii) Let $s, \beta \in \mathbb{R}, 1 \leq p, q, \theta \leq \infty, \tilde{\beta} = \frac{1}{\theta} - \frac{1}{q}$ and $1 \leq \sigma_1 \leq \sigma_2 \leq \infty$. Then, it holds that

$$\dot{V}^s_{p,q,\theta} = \dot{U}^s_{p,\tilde{\beta},\theta} \quad \text{and} \quad \dot{U}^s_{p,\beta,\sigma_1} \subset \dot{U}^s_{p,\beta,\sigma_2}.$$

The space BMO consists of locally integrable functions f such that

$$\|f\|_{BMO} = \sup_B \frac{1}{|B|}\int_B |f - f_B| dx < \infty \text{ with } f_B = \frac{1}{|B|}\int_B f dx,$$

where the the supremum is taken over all balls B in $\mathbb{R}^3$. The space BMO^{-1} is defined by

$$BMO^{-1} = \{ f \in \mathcal{S}' \,|\, \text{There exists } g = (g_1, g_2, g_3) \in BMO \text{ such that } f = \sum_{i=1}^3 \partial_i g_i\}$$

with the norm

$$\|f\|_{BMO^{-1}} = \inf_{g\in BMO} \sum_{i=1}^3 \|g_i\|_{BMO}.$$

We recall some lemmas that will be used in the proof of our result.

Lemma 2.7 ([14]). *Let $1 < q, r < \infty$ be such that $1/q + 1/r = 1$. Then there exists an absolute constant $C > 0$ such that*

$$\int_{\mathbb{R}^3} fgh\,dx \le C\|f\|_{BMO^{-1}}(\|g\|_q\|\nabla h\|_r + \|\nabla g\|_q\|h\|_r) \tag{16}$$

for any $f \in BMO^{-1}, g \in W^1_q$ and $h \in W^1_r$.

Lemma 2.8 ([6]). *Let $s_0, s_1, s_2 \in \mathbb{R}$ satisfy $s_1 < s_0 < s_2$, let $0 \le \beta < \infty$ and $1 \le p, \sigma \le \infty$. Then there exists a positive constant C depending only on s_0, s_1, s_2, but not on p, β, σ such that*

$$\|f\|_{\dot{B}^{s_0}_{p,\sigma}} \le C\Big(1 + \|f\|_{\dot{U}^{s_0}_{p,\beta,\sigma}} \log^{\beta}\Big(e + \|f\|_{\dot{B}^{s_1}_{p,\infty}\cap\dot{B}^{s_2}_{p,\infty}}\Big)\Big)$$

for all $f \in \dot{B}^{s_1}_{p,\infty} \cap \dot{B}^{s_2}_{p,\infty}$. Indeed, by setting $\beta = \frac{1}{\theta} - \frac{1}{q}, \sigma = \theta(1 \le q \le \infty, 1 \le \theta \le q)$, it holds that

$$\|f\|_{\dot{B}^{s_0}_{p,\theta}} \le C\Big(1 + \|f\|_{\dot{V}^{s_0}_{p,q,\theta}} \log^{\frac{1}{\theta}-\frac{1}{q}}\Big(e + \|f\|_{\dot{B}^{s_1}_{p,\infty}\cap\dot{B}^{s_2}_{p,\infty}}\Big)\Big).$$

We will use the following commutator estimate due to Kato and Ponce.

Lemma 2.9 ([10]). *Let $1 < p < \infty$ and $s > 0$. Then there exists an constant C such that*

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \le C(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1}g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{p_3}}\|g\|_{L^{p_4}}), \tag{17}$$

for $f \in \dot{W}^{1,p_1} \cap \dot{W}^{s,p_3}$, $g \in L^{p_4} \cap \dot{W}^{s-1,p_2}$ and $1 < p_2, p_3 < \infty$ satisfying $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$, where $\Lambda^s = (-\Delta)^{\frac{s}{2}}$.

3. Proof of the main result

This section is devoted to the proof of the Theorem 1.1, which is based on the establishment of a priori estimates under condition (13) or (14). Throughout the paper, C stands for some real positive constants which may be different in each occurrence.

Proof of Theorem 1.1. Case I: If (13) holds, for any small constant $\epsilon > 0$, there exists $T_0 = T_0(\epsilon) \in (0, T)$ such that

$$\int_{T_0}^{T}\Big(\|\nabla_h\tilde{u}\|^2_{\dot{V}^{-1}_{\infty,\infty,2}} + \|\nabla_h\tilde{b}\|^2_{\dot{V}^{-1}_{\infty,\infty,2}}\Big)d\tau \le \epsilon. \tag{18}$$

For any $t \in (T_0, T)$, we denote

$$X(t) = \max_{\tau\in[T_0,t]}(\|u(\tau)\|^2_{H^2} + \|b(\tau)\|^2_{H^2})$$

Note that $X(t)$ is nondecreasing. The proof is divided into two steps.

First, we show the basic energy estimate. Taking the L^2 inner product to systems $(1)_1$, $(1)_2$, $(1)_3$ and $(1)_4$ with u, b, θ and s, respectively, we obtain

$$\|(u, b, \theta, s)(t)\|^2_{L^2} + 2\int_0^T \|\nabla(u, b, \theta, s)(\tau)\|^2_{L^2}\,d\tau \le C_0. \tag{19}$$

Taking the gradient operator to $(1)_1$ and $(1)_2$, and multiply the resulting equations by ∇u and ∇b, respectively. Integrating over whole space and summing up the resulting identities gives that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&(\|\nabla u\|^2_{L^2} + \|\nabla b\|^2_{L^2}) + (\|\Delta u\|^2_{L^2} + \|\Delta b\|^2_{L^2})\\ &= -\int_{\mathbb{R}^3}\nabla(u\cdot\nabla u)\cdot\nabla u dx + \int_{\mathbb{R}^3}\nabla(b\cdot\nabla b)\cdot\nabla u dx + \int_{\mathbb{R}^3}\nabla(\theta - s)\mathbf{k}\cdot\nabla u dx\\ &\quad - \int_{\mathbb{R}^3}\nabla(u\cdot\nabla b)\cdot\nabla b dx + \int_{\mathbb{R}^3}\nabla(b\cdot\nabla u)\cdot\nabla b dx\\ &= \sum_{i=1}^{5} I_i.\end{aligned} \tag{20}$$

As argued in [14, 25], it follows from Lemma 2.7 and Young's inequality that

$$\begin{aligned} I_1 + I_2 + I_4 + I_5 \le\, & C\left(\|\nabla_h\widetilde{u}\|^2_{BMO^{-1}} + \|\nabla_h\widetilde{b}\|^2_{BMO^{-1}}\right)\left(\|\nabla u\|^2_{L^2} + \|\nabla b\|^2_{L^2}\right) \\ & + \frac{1}{4}\left(\|\Delta u\|^2_{L^2} + \|\Delta b\|^2_{L^2}\right). \end{aligned} \tag{21}$$

For I_3, we have

$$I_3 = \int_{\mathbb{R}^3} \nabla(\theta - s)\mathbf{k}\cdot\nabla u dx \le \frac{1}{2}\left(\|\nabla u\|^2_{L^2} + \|\nabla\theta\|^2_{L^2} + \|\nabla s\|^2_{L^2}\right). \tag{22}$$

Summing up (20), (21) and (22), and applying Gronwall's lemma yields that

$$\begin{aligned} (\|\nabla u\|^2_{L^2} + \|\nabla b\|^2_{L^2}) & + \int_{T_0}^t (\|\Delta u\|^2_{L^2} + \|\Delta b\|^2_{L^2})\,d\tau \\ & \le \left(\|\nabla u(T_0)\|^2_{L^2} + \|\nabla b(T_0)\|^2_{L^2} + \int_{T_0}^t (\|\nabla\theta\|^2_{L^2} + \|\nabla s\|^2_{L^2})\,d\tau\right) \\ & \quad \times \exp\left(C\int_{T_0}^t \left(1 + \|\nabla_h\widetilde{u}\|^2_{BMO^{-1}} + \|\nabla_h\widetilde{b}\|^2_{BMO^{-1}}\right)d\tau\right) \\ & \le C(T_0)\exp\left(C\int_{T_0}^t \left(\|\nabla_h\widetilde{u}\|^2_{BMO^{-1}} + \|\nabla_h\widetilde{b}\|^2_{BMO^{-1}}\right)d\tau\right), \end{aligned} \tag{23}$$

where we have used the basic energy inequality (19). By the embeddings $\dot{B}^{-1}_{\infty,2} \hookrightarrow BMO^{-1}$, $\dot{B}^0_{2,\infty} \subset \dot{B}^{-\frac{3}{2}}_{\infty,\infty}$, $\dot{B}^1_{2,\infty} \subset \dot{B}^{-\frac{1}{2}}_{\infty,\infty}$, $H^1 \subset B^1_{2,\infty} = L^2 \cap \dot{B}^1_{2,\infty} \subset \dot{B}^0_{2,\infty} \cap \dot{B}^1_{2,\infty}$, and Lemma 2.8, we obtain from (23) that

$$\begin{aligned} & \|\nabla u\|^2_{L^2} + \|\nabla b\|^2_{L^2} + \int_{T_0}^t (\|\Delta u\|^2_{L^2} + \|\Delta b\|^2_{L^2})\,d\tau \\ & \le C(T_0)\exp\left(C\int_{T_0}^t \left(1 + \|\nabla_h\tilde{u}\|^2_{\dot{V}^{-1}_{\infty,\infty,2}} \log\left(e + \|\nabla_h\tilde{u}\|_{\dot{B}^{-\frac{3}{2}}_{\infty,\infty}\cap\dot{B}^{-\frac{1}{2}}_{\infty,\infty}}\right)\right)d\tau\right) \\ & \qquad \times \exp\left(C\int_{T_0}^t \left(1 + \|\nabla_h\tilde{b}\|^2_{\dot{V}^{-1}_{\infty,\infty,2}} \log\left(e + \|\nabla_h\tilde{b}\|_{\dot{B}^{-\frac{3}{2}}_{\infty,\infty}\cap\dot{B}^{-\frac{1}{2}}_{\infty,\infty}}\right)\right)d\tau\right) \\ & \le C(T_0)\exp\left(C\int_{T_0}^t \left(1 + \|\nabla_h\tilde{u}\|^2_{\dot{V}^{-1}_{\infty,\infty,2}} \log\left(e + \|\nabla_h\tilde{u}\|_{H^1}\right)\right)d\tau\right) \\ & \qquad \times \exp\left(C\int_{T_0}^t \left(1 + \|\nabla_h\tilde{b}\|^2_{\dot{V}^{-1}_{\infty,\infty,2}} \log\left(e + \|\nabla_h\tilde{b}\|_{H^1}\right)\right)d\tau\right) \\ & \le C(T_0)\exp\left(C\int_{T_0}^T \left(\|\nabla_h\tilde{u}\|^2_{\dot{V}^{-1}_{\infty,\infty,2}} + \|\nabla_h\tilde{b}\|^2_{\dot{V}^{-1}_{\infty,\infty,2}}\right)\log\left(e + \|\nabla_h\tilde{u}\|_{H^1} + \|\nabla_h\tilde{b}\|_{H^1}\right)d\tau\right) \\ & \le C(T_0)\exp\left(C\int_{T_0}^T \left(\|\nabla_h\tilde{u}\|^2_{\dot{V}^{-1}_{\infty,\infty,2}} + \|\nabla_h\tilde{b}\|^2_{\dot{V}^{-1}_{\infty,\infty,2}}\right)\log(e + \|u\|_{H^2} + \|b\|_{H^2})d\tau\right) \\ & \le C(T_0)\,(e + X(t))^{C\epsilon}, \end{aligned} \tag{24}$$

where we have used that

$$\|f(\tau)\|_{\dot{B}^{-\frac{3}{2}}_{\infty,\infty}\cap\dot{B}^{s-\frac{3}{2}}_{\infty,\infty}} \le C\|f(\tau)\|_{\dot{B}^0_{2,\infty}\cap\dot{B}^s_{2,\infty}} \le C\|f(\tau)\|_{B^s_{2,\infty}} \le C\|f(\tau)\|_{H^s}.$$

Next, we show the H^2 estimate. Taking Λ^2 to $(1)_1$ and $(1)_2$, and multiply the resulting equations by $\Lambda^2 u$ and

$\Lambda^2 b$, respectively. Integrating over whole space and summing up the resulting identity gives that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\Lambda^2 u(t)\|_{L^2}^2 + \|\Lambda^2 b(t)\|_{L^2}^2) + (\|\Lambda^3 u(t)\|_{L^2}^2 + \|\Lambda^3 b(t)\|_{L^2}^2)\\
&\quad = -\int_{\mathbb{R}^3} \Lambda^2(u\cdot\nabla u)\cdot\Lambda^2 u dx + \int_{\mathbb{R}^3} \Lambda^2(b\cdot\nabla b)\cdot\Lambda^2 u dx + \int_{\mathbb{R}^3} \Lambda^2(\theta - s)\mathbf{k}\cdot\Lambda^2 u dx\\
&\qquad - \int_{\mathbb{R}^3} \Lambda^2(u\cdot\nabla b)\cdot\Lambda^2 b dx + \int_{\mathbb{R}^3} \Lambda^2(b\cdot\nabla u)\cdot\Lambda^2 b dx\\
&\quad = \sum_{i=1}^{5} J_i.
\end{aligned} \tag{25}$$

We now estimate each terms on the right hand side of (25) in view of commutator estimate. Taking into account $\nabla\cdot u = 0$, and applying Lemma 2.9, Gagliardo-Nirenberg's inequality and basic energy inequlity (19), we get

$$\begin{aligned}
J_1 = \int_{\mathbb{R}^3} \Lambda^2(u\cdot\nabla u)\cdot\Lambda^2 u dx &= \int_{\mathbb{R}^3} \left(\Lambda^2(u\cdot\nabla u) - (u\cdot\Lambda^2\nabla u)\right)\cdot\Lambda^2 u dx\\
&\le \left\|\Lambda^2(u\cdot\nabla u) - (u\cdot\Lambda^2\nabla u)\right\|_{L^{\frac{4}{3}}} \|\Lambda^2 u\|_{L^4} \le C\|\nabla u\|_{L^2}\|\Lambda^2 u\|_{L^4}^2\\
&\le C\|\nabla u\|_{L^2}\|u\|_{L^2}^{\frac{2}{12}}\|\Lambda^3 u\|_{L^2}^{\frac{22}{12}} \le C\|\nabla u\|_{L^2}^{12} + \frac{1}{8}\|\Lambda^3 u\|_{L^2}^2.
\end{aligned} \tag{26}$$

In the same way as in the estimate of J_1 we have for J_4

$$\begin{aligned}
J_4 = \int_{\mathbb{R}^3} \Lambda^2(u\cdot\nabla b)\cdot\Lambda^2 b dx &= \int_{\mathbb{R}^3} \left(\Lambda^2(u\cdot\nabla b) - (u\cdot\Lambda^2\nabla b)\right)\cdot\Lambda^2 b dx\\
&\le \left\|\Lambda^2(u\cdot\nabla b) - (u\cdot\Lambda^2\nabla b)\right\|_{L^{\frac{4}{3}}} \|\Lambda^2 b\|_{L^4}\\
&\le C\left(\|\nabla u\|_{L^2}\|\Lambda^2 b\|_{L^4} + \|\Lambda^2 u\|_{L^4}\|\nabla b\|_{L^2}\right)\|\Lambda^2 b\|_{L^4}\\
&\le C(\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2})(\|\Lambda^2 u\|_{L^4}^2 + \|\Lambda^2 b\|_{L^4}^2)\\
&\le C(\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2})\left(\|u\|_{L^2}^{\frac{2}{12}}\|\Lambda^3 u\|_{L^2}^{\frac{22}{12}} + \|b\|_{L^2}^{\frac{2}{12}}\|\Lambda^3 b\|_{L^2}^{\frac{22}{12}}\right)\\
&\le C(\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2})^{12} + \frac{1}{8}\left(\|\Lambda^3 u\|_{L^2}^2 + \|\Lambda^3 b\|_{L^2}^2\right).
\end{aligned} \tag{27}$$

Similarily we have for J_2 and J_5 as follows:

$$\begin{aligned}
J_2 + J_5 &= \int_{\mathbb{R}^3} \Lambda^2(b\cdot\nabla b)\cdot\Lambda^2 u dx + \int_{\mathbb{R}^3} \Lambda^2(b\cdot\nabla u)\cdot\Lambda^2 b dx\\
&= \int_{\mathbb{R}^3} (\Lambda^2(b\cdot\nabla b) - (b\cdot\Lambda^2\nabla b))\cdot\Lambda^2 u dx + \int_{\mathbb{R}^3} (\Lambda^2(b\cdot\nabla u) - (b\cdot\Lambda^2\nabla u))\cdot\Lambda^2 b dx\\
&\le \left\|\Lambda^2(b\cdot\nabla b) - (b\cdot\Lambda^2\nabla b)\right\|_{L^{\frac{4}{3}}} \|\Lambda^2 u\|_{L^4} + \left\|\Lambda^2(b\cdot\nabla u) - (b\cdot\Lambda^2\nabla u)\right\|_{L^{\frac{4}{3}}} \|\Lambda^2 b\|_{L^4}\\
&\le C(\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2})(\|\Lambda^2 u\|_{L^4}^2 + \|\Lambda^2 b\|_{L^4}^2)\\
&\le C(\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2})\left(\|u\|_{L^2}^{\frac{2}{12}}\|\Lambda^3 u\|_{L^2}^{\frac{22}{12}} + \|b\|_{L^2}^{\frac{2}{12}}\|\Lambda^3 b\|_{L^2}^{\frac{22}{12}}\right)\\
&\le C(\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2})^{12} + \frac{1}{8}\left(\|\Lambda^3 u\|_{L^2}^2 + \|\Lambda^3 b\|_{L^2}^2\right),
\end{aligned} \tag{28}$$

where we have used the fact that

$$\int_{\mathbb{R}^3} (b\cdot\Lambda^2\nabla b)\cdot\Lambda^2 u dx + \int_{\mathbb{R}^3} (b\cdot\Lambda^2\nabla u)\cdot\Lambda^2 b dx = 0. \tag{29}$$

It remains to estimate the third term J_3. By virtue of Leibniz rule, we have that

$$J_3 = \int_{\mathbb{R}^3} \Lambda^2(\theta - s)\mathbf{k} \cdot \Lambda^2 u dx = -\int_{\mathbb{R}^3} \Lambda(\theta - s)\mathbf{k} \cdot \Lambda^3 u dx \leq C(\|\nabla\theta\|_{L^2}^2 + \|\nabla s\|_{L^2}^2) + \frac{1}{4}\|\Lambda^3 u\|_{L^2}^2. \tag{30}$$

Substituting above estimates into (25), we obtain

$$\begin{aligned}\frac{d}{dt}(\|\Lambda^2 u(t)\|_{L^2}^2 + \|\Lambda^2 b(t)\|_{L^2}^2) &+ (\|\Lambda^3 u(t)\|_{L^2}^2 + \|\Lambda^3 b(t)\|_{L^2}^2) \\ &\leq C\left(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2\right)^6 + C(\|\nabla\theta\|_{L^2}^2 + \|\nabla s\|_{L^2}^2)\end{aligned} \tag{31}$$

Integrating (31) over (T_0, t) in view of basic energy inequality (19) and (24), we have

$$\begin{aligned}\|u(t)\|_{\dot{H}^2}^2 + \|b(t)\|_{\dot{H}^2}^2 &\leq \|u(T_0)\|_{\dot{H}^2}^2 + \|b(T_0)\|_{\dot{H}^2}^2 + C_0 + C\int_{T_0}^t \left(\|\nabla u\|_2^2 + \|\nabla b\|_2^2\right)^6 d\tau \\ &\leq \|u(T_0)\|_{\dot{H}^2}^2 + \|b(T_0)\|_{\dot{H}^2}^2 + C_0 + C(T_0)\int_{T_0}^t (e + X(\tau))^{6C\epsilon} d\tau.\end{aligned} \tag{32}$$

Taking $0 < \epsilon \leq \frac{1}{6C}$, and combining (32) with the basic energy inequality yields that

$$X(t) \leq X(T_0) + C_0 + C(T_0)\int_{T_0}^t (e + X(\tau)) d\tau. \tag{33}$$

Applying Gronwall's inequality, one concludes that

$$X(t) \leq C(T_0) + C_0 < \infty, \text{ for all } t \in [T_0, T],$$

which together with (24) implies that

$$u, b \in L^\infty(0, T; H^2(\mathbb{R}^3)) \cap L^2(0, T; H^3(\mathbb{R}^3)).$$

Naturally, it yields

$$\theta, s \in L^\infty(0, T; H^2(\mathbb{R}^3)) \cap L^2(0, T; H^3(\mathbb{R}^3)).$$

Case II: If (14) holds, for any small constant $\epsilon > 0$, there exists $T_0 = T_0(\epsilon) \in (0, T)$ such that

$$\int_{T_0}^T \left(\|\tilde{u}\|_{\dot{V}^0_{\infty,\infty,2}}^2 + \|\tilde{b}\|_{\dot{B}^0_{\infty,\infty,2}}^2\right) d\tau \leq \epsilon. \tag{34}$$

For any $t \in (T_0, T)$, we denote

$$X(t) = \max_{\tau \in [T_0, t]} (\|u(\tau)\|_{\dot{H}^2}^2 + \|b(\tau)\|_{\dot{H}^2}^2)$$

Note that $X(t)$ is nondecreasing.

Multiplying the first equation of (1) by $-\Delta u$, the second one by $-\Delta b$, after integration by and taking the divergence-free property into account, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2\right) + \|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2 \\ =& -\sum_{i=1}^3 \int_{\mathbb{R}^3} (\partial_i u \cdot \nabla) u \partial_i u dx + \sum_{i=1}^3 \int_{\mathbb{R}^3} (\partial_i b \cdot \nabla) b \partial_i u dx + \int_{\mathbb{R}^3} \nabla(\theta - s)\mathbf{k} \cdot \nabla u dx \\ &- \sum_{i=1}^3 \int_{\mathbb{R}^3} (\partial_i u \cdot \nabla) b \partial_i b dx + \sum_{i=1}^3 \int_{\mathbb{R}^3} (\partial_i b \cdot \nabla) u \partial_i b dx \\ =& K_1 + K_2 + K_3 + K_4 + K_5.\end{aligned} \tag{35}$$

First, we decompose K_4 into three parts.

$$\begin{aligned} K_4 &= -\sum_{i=1}^{3}\int_{\mathbb{R}^3}(\partial_i\tilde{u}\cdot\nabla_h)\,b\partial_i b dx-\sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i u_3\partial_3\tilde{b}\partial_i\tilde{b}dx-\sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i u_3\partial_3 b_3\partial_i b_3 dx\\ &= K_{41}+K_{42}+K_{43}. \end{aligned}$$

Integrating by parts and using the Hölder inequality, we have

$$\begin{aligned} K_{41} &= \sum_{i=1}^{3}\int_{\mathbb{R}^3}\left[(\tilde{u}\cdot\nabla_h)\partial_i b\partial_i b+(\tilde{u}\cdot\nabla_h)b\partial_i^2 b\right]dx\\ &\le C\int_{\mathbb{R}^3}|\tilde{u}||\nabla b||\Delta b|dx. \end{aligned} \tag{36}$$

Since $\nabla\cdot b=0=\nabla\cdot\Delta b$ and curl $\nabla b=0$, it follows from the div-curl lemma that

$$\begin{aligned} K_{41} &\le C\|\tilde{u}\|_{BMO}\|\nabla b\cdot\Delta b\|_{\mathcal{H}^1}\\ &\le C\|\tilde{u}\|_{BMO}\|\nabla b\|_{L^2}\|\Delta b\|_{L^2}\\ &\le C\|\tilde{u}\|_{BMO}^2\|\nabla b\|_{L^2}^2+\frac{1}{12}\|\Delta b\|_{L^2}^2. \end{aligned} \tag{37}$$

Similarly to the estimates for K_{41}, we apply integration by parts to have

$$\begin{aligned} K_{42} &= \sum_{i=1}^{3}\int_{\mathbb{R}^3}\tilde{b}\partial_3\left(\partial_i u_3\partial_i\tilde{b}\right)dx\\ &\le C\|\tilde{b}\|_{BMO}\left(\|\nabla u\|_{L^2}\|\Delta b\|_{L^2}+\|\nabla b\|_{L^2}\|\Delta u\|_{L^2}\right)\\ &\le C\|\tilde{b}\|_{BMO}^2\left(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2\right)+\frac{1}{12}\left(\|\Delta u\|_{L^2}^2+\|\Delta b\|_{L^2}^2\right). \end{aligned}$$

For K_{43}, the divergence-free condition $\partial_3 b_3=-\partial_1 b_1-\partial_2 b_2$ and integration by parts imply

$$\begin{aligned} K_{43} &= \sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i u_3(\nabla_h\cdot\tilde{b})\partial_i b_3 dx\\ &= \sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i u_3\left(\partial_1 b_1+\partial_2 b_2\right)\partial_i b_3 dx\\ &= -\sum_{i=1}^{3}\int_{\mathbb{R}^3}\left(b_1\partial_1\left(\partial_i u_3\partial_i b_3\right)+b_2\partial_2\left(\partial_i u_3\partial_i b_3\right)\right)dx\\ &\le \int_{\mathbb{R}^3}\left|\nabla^2 u\right||\tilde{b}||\nabla b|dx+\int_{\mathbb{R}^3}|\nabla u||\tilde{b}|\left|\nabla^2 b\right|dx\\ &\le \|\tilde{b}\|_{BMO}\|\nabla b\cdot\Delta u\|_{\mathcal{H}^1}+\|\tilde{b}\|_{BMO}\|\Delta b\cdot\nabla u\|_{\mathcal{H}^1}\\ &\le C\|\tilde{b}\|_{BMO}^2\left(\left\|\nabla u\right\|_{L^2}^2+\|\nabla b\|_{L^2}^2\right)+\frac{1}{12}\left(\|\Delta u\|_{L^2}^2+\|\Delta b\|_{L^2}^2\right). \end{aligned}$$

Combining all the estimates of K_{41}, K_{42} and K_{43}, we have

$$K_4\le C\left(\|\tilde{u}\|_{BMO}^2+\|\tilde{b}\|_{BMO}^2\right)\left(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2\right)+\frac{1}{3}\left(\|\Delta u\|_{L^2}^2+\|\Delta b\|_{L^2}^2\right). \tag{38}$$

Similarly, we split K_5 as

$$\begin{aligned}K_5 &= \sum_{i=1}^{3}\int_{\mathbb{R}^3}\left(\partial_i\tilde{b}\cdot\nabla_h\right)u\partial_i b dx + \sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i b_3\partial_3\tilde{u}\partial_i\tilde{b}dx + \sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i u_3\partial_3 u_3\partial_i b_3 dx\\ &= K_{51}+K_{52}+K_{53}.\end{aligned}$$

K_{51}, K_{52} and K_{53} can be estimated the same as K_{41}, K_{42} and K_{43}, respectively. One has

$$K_5 \le C\left(\|\tilde{u}\|_{BMO}^2+\|\tilde{b}\|_{BMO}^2\right)\left(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2\right)+\frac{1}{3}\left(\|\Delta u\|_{L^2}^2+\|\Delta b\|_{L^2}^2\right). \tag{39}$$

Therefore, obviously, K_1 and K_2 can be estimated as follows

$$K_1 \le C\|\tilde{u}\|_{BMO}^2\|\nabla u\|_{L^2}^2+\frac{1}{3}\|\Delta u\|_{L^2}^2 \tag{40}$$

and

$$K_2 \le C\left(\|\tilde{u}\|_{BMO}^2+\|\tilde{b}\|_{BMO}^2\right)\left(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2\right)+\frac{1}{3}\left(\|\Delta u\|_{L^2}^2+\|\Delta b\|_{L^2}^2\right). \tag{41}$$

$$K_3 = \int_{\mathbb{R}^3}\nabla(\theta-s)\mathbf{k}\cdot\nabla u dx \le \frac{1}{2}\left(\|\nabla u\|_2^2+\|\nabla\theta\|_2^2+\|\nabla s\|_2^2\right). \tag{42}$$

Combining (38)-(42) with (35), and applying Gronwall's lemma to obtain

$$\begin{aligned}&(\|\nabla u\|_2^2+\|\nabla b\|_2^2)+\int_{T_0}^{t}(\|\Delta u\|_2^2+\|\Delta b\|_2^2)\,d\tau\\
&\quad\le\left(\|\nabla u(T_0)\|_2^2+\|\nabla b(T_0)\|_2^2+\int_{T_0}^{t}(\|\nabla\theta\|_2^2+\|\nabla s\|_2^2)\,d\tau\right)\\
&\qquad\times\exp\left(C\int_{T_0}^{t}\left(1+\|\tilde{u}\|_{BMO}^2+\|\tilde{b}\|_{BMO}^2\right)d\tau\right)\\
&\quad\le C(T_0)\exp\left(C\int_{T_0}^{t}\left(\|\tilde{u}\|_{BMO}^2+\|\tilde{b}\|_{BMO}^2\right)d\tau\right).\end{aligned}\tag{43}$$

By the embeddings $\dot{B}^0_{\infty,2}\hookrightarrow BMO$, we get

$$(\|\nabla u\|_2^2+\|\nabla b\|_2^2)+\int_{T_0}^{t}(\|\Delta u\|_2^2+\|\Delta b\|_2^2)\,d\tau \le C(T_0)\exp\left(C\int_{T_0}^{t}\left(\|\tilde{u}\|_{\dot{B}^0_{\infty,2}}^2+\|\tilde{b}\|_{\dot{B}^0_{\infty,2}}^2\right)d\tau\right). \tag{44}$$

Now, by applying the logarithmic interpolation inequality, i.e., Lemma 2.8, it follows that

$$\|\tilde{u}\|_{\dot{B}^0_{\infty,2}} \le C\left(1+\|\tilde{u}\|_{\dot{V}^0_{\infty,\infty,2}}\log^{\frac{1}{2}}\left(e+\|\tilde{u}\|_{\dot{B}^{-\frac{3}{2}}_{\infty,\infty}\cap\dot{B}^{-\frac{1}{2}}_{\infty,\infty}}\right)\right)$$

and

$$\|\tilde{b}\|_{\dot{B}^0_{\infty,2}} \le C\left(1+\|\tilde{u}\|_{\dot{V}^0_{\infty,\infty,2}}\log^{\frac{1}{2}}\left(e+\|\tilde{u}\|_{\dot{B}^{-\frac{3}{2}}_{\infty,\infty}\cap\dot{B}^{-\frac{1}{2}}_{\infty,\infty}}\right)\right).$$

Finally, we find that

$$\begin{aligned}&\|\nabla u\|_2^2+\|\nabla b\|_2^2+\int_{T_0}^{t}(\|\Delta u\|_2^2+\|\Delta b\|_2^2)\,d\tau\\
\le& C(T_0)\exp\left(C\int_{T_0}^{T}\left(\|\tilde{u}\|_{\dot{V}^0_{\infty,\infty,2}}^2+\|\tilde{b}\|_{\dot{V}^0_{\infty,\infty,2}}^2\right)\log(e+\|u\|_{H^2}+\|b\|_{H^2})d\tau\right)\\
\le& C(T_0)\left(e+X(t)\right)^{C\epsilon},\end{aligned}\tag{45}$$

then, we repeat the H^2 estimates of u and b. So the bound on H^2 norm of (u, b, θ, s) is enough to guarantee that the solution can be extended beyond time T. Thus, we obtain the desired result.

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