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C^*-basic construction on field algebras of G-spin models

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Abstract. Let G be a finite group. Starting from the field algebra $\mathcal{F}$ of G-spin models, we show that the C^*-basic construction for the field algebra $\mathcal{F}$ and the $D(G)$-invariant subalgebra of $\mathcal{F}$ can be represented as the crossed product C^*-algebra $\mathcal{F} \rtimes D(G)$. Moreover, under the natural $\widehat{D(G)}$-module action on $\mathcal{F} \rtimes D(G)$, the iterated crossed product C^*-algebra can be obtained, which is C^*-isomorphic to the C^*-basic construction for $\mathcal{F} \rtimes D(G)$ and the field algebra $\mathcal{F}$. In addition, it is proved that the iterated crossed product C^*-algebra is a new field algebra, and the concrete structures with the order and disorder operators are given.

1. Introduction

Jones index theory, a pivotal component of operator algebra theory, was introduced by V. F. R. Jones in 1983 [7]. This groundbreaking work reveals the relationship between operator algebras and other fields of mathematics as well as physics such as topology, quantum physics, dynamical systems, and noncommutative geometry. Consequently, the index theory has garnered substantial attention in the scholarly community [3, 11, 17, 18, 20, 21]. In [11], M. Pimsner and S. Popa introduced the probabilistic index for conditional expectations, which is the optimal constant in the renowned Pimsner-Popa inequality. Alternatively, H. Kosaki explored in [9] a spatial approach to defining the index for normal semifinite faithful conditional expectations onto subfactors, leveraging Connes' spatial theory and the framework of operator-valued weights. While the probabilistic index excels in analytical contexts, even within C^*-algebras (refer to, e.g., [12]), it may not align seamlessly with algebraic operations, such as the basic construction within the C^*-algebra framework. Motivated by the Pimsner-Popa basis [11], Kosaki's index formula [9], and Casimir elements in semi-simple Lie algebras, Y. Watatani [18] proposed to assume the existence of a quasi-basis for conditional expectations, a generalization of the Pimsner-Popa basis in von Neumann algebras, which is intimately tied to the K-theory of C^*-algebras [4, 18]. Moreover, the index theory has been further explored on Hilbert C^*-bimodules by T. Kajiwara, C. Pinzari, and Y. Watatani in [8]. S. Popa [13]

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delved into the W^*-representation of subfactors, specifically focusing on its implications for Jones index. More recently, A. Jaffe et al., in [5], embarked on a comprehensive study of quantum Fourier analysis, heavily leveraging the foundational principles of index theory, etc.

It is noteworthy that the basic construction plays a pivotal role in establishing that the index of a subfactor resides within the set $\{4\cos^2\frac{\pi}{n}, n = 3, 4, \cdots\} \cup [4, +\infty]$. Furthermore, the Jones projections derived during this process adhere to the Temperley-Lieb relation, which has profound implications across various mathematical areas, including Hecke algebras, braid groups, and more.

In the realm of algebraic quantum field theory, quantum spin models occupy a prominent position, which are kind of models with the profound background in theoretical physics. These models, particularly viewed as quantum chains in 1+1-dimensional quantum field theory, exhibit intriguing characteristics such as braid group statistics and quantum symmetry. The classical quantum spin model includes Ising models, XY models, Heisenberg models and G-spin models. Notably, G-spin models serve as the simplest instances of lattice field theories that manifest quantum symmetry, which cannot be described by a group and is also the starting point for exploring quantum spin models and superselection principle. In 1993, K. Szlachányi and P. Vecsernyés [15] conducted a comprehensive study of quantum symmetry and braid group statistics within the framework of the G-spin model on a one-dimensional lattice, defined the C^*-algebra (i.e. quantum field algebra) generated by ordered and disordered operators, and the action of quantum double $D(G)$ of a finite group G on the quantum field algebra. Finally, they proved that these models have an order-disorder type of quantum symmetry given by $D(G)$ which generalizes $Z(2)\times Z(2)$ symmetry of the lattice Ising model. Subsequently, scholars have made further research on G-spin models [6, 10, 19, 22, 23]. For example, Xin and Jiang [22, 23] investigated the quantum symmetry and index theory in the G-spin model determined by the normal subgroup in the 1-dimensional lattice.

As a continuation of [22, 23], the paper studies the C^*-basic construction on field algebras of G-spin models, and is organized as follows. In Section 2, we review the necessary definitions and facts about G-spin models. In Section 3, under the natural $D(G)$-module action on $\mathcal{F}$, the crossed product C^*-algebra $\mathcal{F} \rtimes D(G)$ is obtained, which is C^*-isomorphic to the C^*-basic construction $\langle \mathcal{F}, e_{\mathcal{A}}\rangle_{C^*}$ constructed from the C^*-basic construction for the inclusion $\mathcal{A} \subseteq \mathcal{F}$. In Section 4, we construct the iterated crossed product C^*-algebra $\mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)}$, which is canonically isomorphic to $M_{|G|^2}(\mathcal{F})$ by Takai duality [16], and also prove that the C^*-algebra $\mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)}$ is C^*-isomorphic to $\langle \mathcal{F} \rtimes D(G), e_2\rangle_{C^*}$ constructed from the C^*-basic construction for the inclusion $\mathcal{F} \subseteq \mathcal{F} \rtimes D(G)$. Finally, we give the concrete description of the new field algebra $\mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)}$ by means of the order and disorder operators.

All the algebras in this paper will be unital associative algebras over the complex field $\mathbb{C}$. The unadorned tensor product $\otimes$ will stand for the usual tensor product over $\mathbb{C}$. For general results on Hopf algebras please refer to the books of Abe [1] and Sweedler [14]. We shall follow their notations, such as S, $\triangle$, ε for the antipode, the comultiplication and the counit, respectively. Also we shall use the so-called "Sweedler-type notation" for the image of $\triangle$. That is

$$\triangle(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}.$$

2. Preliminaries

We first recall the main features of G-spin models, considered in the C^*-algebraic framework for quantum lattice systems, and then give the C^*-basic construction for C^*-algebras.

2.1. G-spin models

Assume that G is a finite group with a unit u. The G-valued spin configuration on the two-dimensional square lattice is the map $\sigma\colon \mathbb{Z}^2 \to G$ with Euclidean action functional:

$$S(\sigma) = \sum_{(x,y)} f(\sigma_x^{-1}\sigma_y),$$

in which the summation runs over the nearest neighbor pairs in $\mathbb{Z}^2$ and $f\colon G \to \mathbb{R}$ is a function of the positive type. These kind of classical statistical systems are called G-spin models. In general, G-spin models with an Abelian group G are known to have a symmetry group $G \times \widetilde{G}$, where $\widetilde{G}$ is the group of characters of G. If G is non-Abelian, the models have a symmetry of a quantum double $D(G)$ [15], which is defined as follows.

Definition 2.1. *Let $C(G)$ be the algebra of complex-valued functions on G and consider the adjoint action of G on $C(G)$ according to $\alpha_g\colon f \mapsto f \circ Ad(g^{-1})$. The quantum double $D(G)$ is defined as the crossed product $D(G) = C(G) \rtimes_\alpha G$ of $C(G)$ by this action. In terms of generators, $D(G)$ is the algebra generated by elements U_g and V_h $(g, h \in G)$, with the relations*

$$\begin{aligned} U_g U_h &= \delta_{g,h} U_g, \\ V_g V_h &= V_{gh}, \\ V_h U_g &= U_{hgh^{-1}} V_h, \end{aligned}$$

and the identification $\sum_{g \in G} U_g = V_u = 1$, *where* $\delta_{g,h} = \begin{cases} 1, & \text{if } g = h, \\ 0, & \text{if } g \neq h. \end{cases}$

It is easy to see that $D(G)$ is of finite dimension, where as a convenient basis one may choose $U_g V_h$, $g, h \in G$, multiplying according to $U_{g_1} V_{h_1} U_{g_2} V_{h_2} = \delta_{g_1 h_1, h_1 g_2} U_{g_1} V_{h_1 h_2}$.

Here and from now on, by (g, h) we always denote the element $U_g V_h$ for notational convenience.

Also, the structure maps are given by

$$\begin{aligned} \triangle(g, h) &= \sum_{t \in G} (t, h) \otimes (t^{-1} g, h), && \text{(coproduct)} \\ \varepsilon(g, h) &= \delta_{g,u}, && \text{(counit)} \\ S(g, h) &= (h^{-1} g^{-1} h, h^{-1}), && \text{(antipode)} \end{aligned}$$

on the linear basis $\{(g, h), g, h \in G\}$ and are extended in $D(G)$ by linearity. One can prove that $D(G)$ is a Hopf algebra, with a unique element $E = \frac{1}{|G|} \sum_{g \in G} (u, g)$, called an integral element, satisfying for any $a \in D(G)$,

$$aE = Ea = \varepsilon(a) E.$$

Moreover, with the definition

$$(g, h)^* = (h^{-1} g h, h^{-1}),$$

and the appropriate extension, $D(G)$ is a semisimple $*$-algebra of finite dimension [14], which implies that $D(G)$ becomes a Hopf C^*-algebra.

As in the traditional case, one can define the local quantum field algebra as follows.

Definition 2.2. *The local field algebra of a G-spin model $\mathcal{F}_{loc}$ is an associative algebra with a unit I generated by $\{\delta_g(x), \rho_h(l) \colon g, h \in G, x \in \mathbb{Z}, l \in \mathbb{Z} + \frac{1}{2}\}$ subject to*

$$\begin{aligned} \sum_{g \in G} \delta_g(x) &= I = \rho_u(l), \\ \delta_{g_1}(x) \delta_{g_2}(x) &= \delta_{g_1, g_2} \delta_{g_1}(x), \\ \rho_{h_1}(l) \rho_{h_2}(l) &= \rho_{h_1 h_2}(l), \\ \delta_{g_1}(x) \delta_{g_2}(x') &= \delta_{g_2}(x') \delta_{g_1}(x), \\ \rho_h(l) \delta_g(x) &= \begin{cases} \delta_{hg}(x) \rho_h(l), & \text{if } l < x, \\ \delta_g(x) \rho_h(l), & \text{if } l > x, \end{cases} \\ \rho_{h_1}(l) \rho_{h_2}(l') &= \begin{cases} \rho_{h_2}(l') \rho_{h_2^{-1} h_1 h_2}(l), & \text{if } l > l', \\ \rho_{h_1 h_2 h_1^{-1}}(l') \rho_{h_1}(l), & \text{if } l < l', \end{cases} \end{aligned}$$

for $x, x' \in \mathbb{Z}, l, l' \in \mathbb{Z} + \frac{1}{2}$ *and* $h_1, h_2, g_1, g_2 \in G$.

The $*$-operation is defined on the generators as $\delta_g^*(x) = \delta_g(x)$, $\rho_h^*(l) = \rho_{h^{-1}}(l)$ and can be extended to an involution on $\mathcal{F}_{\text{loc}}$. In this way, $\mathcal{F}_{\text{loc}}$ becomes a unital $*$-algebra. Using the C^*-inductive limit [2], $\mathcal{F}_{\text{loc}}$ can be extended to a C^*-algebra $\mathcal{F}$, called the field algebra of G-spin models.

There is an action γ of $D(G)$ on $\mathcal{F}$ in the following. For $x \in \mathbb{Z}$, $l \in \mathbb{Z} + \frac{1}{2}$ and $g, h \in G$, set

$$\begin{array}{rcll} (g,h)\delta_f(x) &=& \delta_{g,u}\delta_{hf}(x), & \forall f \in G, \\ (g,h)\rho_t(l) &=& \delta_{g,hth^{-1}}\rho_g(l), & \forall t \in G. \end{array}$$

The map γ can be extended for products of generators inductively in the number of generators by the rule

$$(g,h)(fT) = \sum_{(g,h)} (g,h)_{(1)}(f)(g,h)_{(2)}(T),$$

where f is one of the generators in $\mathcal{F}_{\text{loc}}$ and T is a finite product of generators. Finally, it is linearly extended both in $D(G)$ and $\mathcal{F}_{\text{loc}}$.

Proposition 2.3. *([15]) The field algebra $\mathcal{F}$ is a $D(G)$-module algebra with respect to the map γ. Namely, the map γ satisfies the following relations:*

$$\begin{array}{rcl} (ab)(T) &=& a(b(T)), \\ a(T_1T_2) &=& \sum_{(a)} a_{(1)}(T_1)a_{(2)}(T_2), \\ a(T^*) &=& (S(a^*)(T))^*, \end{array}$$

for $a, b \in D(G)$, $T_1, T_2, T \in \mathcal{F}$.

Set

$$\mathcal{A} = \{F \in \mathcal{F} : a(F) = \varepsilon(a)(F), \ \forall a \in D(G)\}.$$

We call it an observable algebra in the field algebra $\mathcal{F}$ of G-spin models. Furthermore, one can show that $\mathcal{A}$ is a nonzero C*-subalgebra of $\mathcal{F}$, and

$$\mathcal{A} = \{F \in \mathcal{F} : E(F) = F\} \equiv E(\mathcal{F}).$$

Indeed, from the following proposition, one can see that $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{F}$.

Proposition 2.4. *([15]) The map $E : \mathcal{F} \to \mathcal{A}$ satisfies the following conditions:*

(1) *$E(I) = I$ where I is the unit of $\mathcal{F}$;*

(2) *(bimodular property) $\forall\, F_1, F_2 \in \mathcal{A}$, $F \in \mathcal{F}$,*

$$E(F_1FF_2) = F_1E(F)F_2;$$

(3) *E is positive.*

In the following a linear map Γ from a unital C^*-algebra B onto its unital C^*-subalgebra A with properties (1)-(3) in Proposition 2.4 is called a conditional expectation. If Γ is a conditional expectation from B onto A, then Γ is a projection of norm one [2]. In addition, if $E(Bb) = 0$ implies $b = 0$, for $b \in B$, then we say E is faithful.

We next review some basic facts about the index for C^*-algebras in [18].

Definition 2.5. *Let Γ be a conditional expectation from a unital C^*-algebra B onto its unital C^*-subalgebra A. A finite family $\{(u_1, v_1), (u_2, v_2), \cdots, (u_n, v_n)\} \subseteq B \times B$ is called a quasi-basis for Γ if for all $b \in B$,*

$$\sum_{i=1}^n u_i\Gamma(v_ib) = b = \sum_{i=1}^n \Gamma(bu_i)v_i.$$

Furthermore, if there exists a quasi-basis for Γ, we call Γ of index-finite type. In this case we define the index of Γ by

$$\text{Index}\,\Gamma = \sum_{i=1}^n u_iv_i.$$

Remark 2.6. *(1) If Γ is a conditional expectation of index-finite type, then the C^*-index Index Γ is a central element of B and does not depend on the choice of quasi-basis. In particular, if $A \subseteq B$ are simple unital C^*-algebras, then we can choose one of the form $\{(w_i, w_i^*) : i = 1, 2, \cdots, n\}$, which shows that Index Γ is a positive element [18].*

(2) Let $N \subseteq M$ be factors of type II_1 and $\Gamma : M \to N$ the canonical conditional expectation determined by the unique normalized trace on M, then Index Γ is exactly Jones index $[M : N]$ based on the coupling constant [11]. More generally, let M be a (σ-finite) factor with a subfactor N and Γ a normal conditional expectation from M onto N, then Γ is of index-finite if and only if Index Γ is finite in the sense of Ref. [9], and the values of Index Γ are equal.

Proposition 2.7. *([22]) The map $E : \mathcal{F} \to \mathcal{A}$ is a conditional expectation of index-finite type, and Index $E = |G|^2 I$, where $|G|$ denotes the order of the group G.*

2.2. The C^*-basic construction

This section will give a brief description of the C^*-basic construction for C^*-algebras.

Let $\Gamma : B \to A$ be a faithful conditional expectation. Then B_A(viewing B as a right A-module) is a pre-Hilbert module over A with an A-valued inner product $\langle x, y \rangle = \Gamma(x^* y)$ for $x, y \in B_A$. Let $\overline{B_A}$ be the completion of B_A with respect to the norm on B_A defined by

$$\|x\|_{B_A} = \|\Gamma(x^* x)\|_A^{\frac{1}{2}}, \ x \in B_A.$$

Then $\overline{B_A}$ is a Hilbert C^*-module over A. Since Γ is faithful, the canonical map $B \to \overline{B_A}$ is injective. Let $L_A(\overline{B_A})$ be the set of all (right) A-module homomorphisms $T : \overline{B_A} \to \overline{B_A}$ with an adjoint A-module homomorphism $T^* : \overline{B_A} \to \overline{B_A}$ such that

$$\langle T\xi, \eta \rangle = \langle \xi, T^* \eta \rangle.$$

Then $L_A(\overline{B_A})$ is a C^*-algebra with the operator norm

$$\|T\| = \sup\{\|T\xi\| : \|\xi\| = 1\}.$$

There is an injective $*$-homomorphism $\lambda : B \to L_A(\overline{B_A})$ defined by $\lambda(b)x = bx$ for $x \in B_A$ and $b \in B$, so that B can be viewed as a C^*-subalgebra of $L_A(\overline{B_A})$. Note that the map $\gamma_A : B_A \to B_A$ defined by $\gamma_A(x) = \Gamma(x)$ for $x \in B_A$ is bounded and thus it can be extended to a bounded linear operator on $\overline{B_A}$, denoted by γ_A again. Then $\gamma_A \in L_A(\overline{B_A})$ and $\gamma_A = \gamma_A^2 = \gamma_A^*$; that is, γ_A is a projection in $L_A(\overline{B_A})$. From now on we call γ_A the Jones projection of Γ. The (reduced) C^*-basic construction is a C^*-subalgebra of $L_A(\overline{B_A})$ defined to be

$$\langle B, \gamma_A \rangle_{C^*} = \overline{\operatorname{span}\{\lambda(x)\gamma_A\lambda(y) \in L_A(\overline{B_A}) : x, y \in B\}}^{\|\cdot\|}.$$

3. The C^*-basic construction for the inclusion $\mathcal{A} \subseteq \mathcal{F}$

In this section, we will construct the crossed product C^*-algebra $\mathcal{F} \rtimes D(G)$ extending $\mathcal{F} \equiv \mathcal{F} \rtimes I_{D(G)}$ by means of a Hopf module left action of $D(G)$ on $\mathcal{F}$, such that $\mathcal{F} \rtimes D(G)$ coincides with the C^*-algebra $\langle \mathcal{F}, e_{\mathcal{A}} \rangle_{C^*}$ constructed from the C^*-basic construction for the inclusion $\mathcal{A} \subseteq \mathcal{F}$.

For the conditional expectation $E : \mathcal{F} \to \mathcal{A}$ given in Proposition 2.7, we shall consider the C^*-basic construction $\langle \mathcal{F}, e_{\mathcal{A}} \rangle_{C^*}$, which is a C^*-subalgebra of $L_{\mathcal{A}}(\overline{\mathcal{F}})$ linearly generated by $\{\lambda(x)e_{\mathcal{A}}\lambda(y) : x, y \in \mathcal{F}\}$, where $\overline{\mathcal{F}}$ is the completion of $\mathcal{F}_{\mathcal{A}}$ with respect to the norm $\|x\|_{\mathcal{F}_{\mathcal{A}}} = \|E(x^* x)\|_{\mathcal{A}}^{\frac{1}{2}}$ and $e_{\mathcal{A}}$ is the Jones projection of E.

We will give some properties about the elements in $L_{\mathcal{A}}(\overline{\mathcal{F}})$ as follows.

Proposition 3.1. *(1) As operators on $\overline{\mathcal{F}}$, we have $e_{\mathcal{A}} T e_{\mathcal{A}} = E(T) e_{\mathcal{A}}$.*

(2) Let $T \in \mathcal{F}$, then $T \in \mathcal{A}$ if and only if $e_{\mathcal{A}} T = T e_{\mathcal{A}}$.

As we have known, the field algebra $\mathcal{F}$ is a $D(G)$-module algebra in G-spin models, and one can construct the crossed product $*$-algebra $\mathcal{F}_{\text{loc}} \rtimes D(G)$, as a vector space $\mathcal{F}_{\text{loc}} \otimes D(G)$ with the $*$-algebra structure

$$\begin{aligned}\Big(T \otimes (g,h)\Big)\Big(F \otimes (s,t)\Big) &= \sum_{(g,h)} T(g,h)_{(1)}(F) \otimes (g,h)_{(2)}(s,t),\\ \Big(T \otimes (g,h)\Big)^* &= \Big(I_{\mathcal{F}} \otimes (g,h)^*\Big)\Big(T^* \otimes I_{D(G)}\Big).\end{aligned}$$

Using the C^*-inductive limit [2], the crossed product $\mathcal{F}_{\text{loc}} \rtimes D(G)$ can be extended naturally to a C^*-algebra $\mathcal{F} \rtimes D(G)$.

Now, we consider a special element $I_{\mathcal{F}} \rtimes \frac{1}{|G|} \sum\limits_{g\in G} (u,g)$ in $\mathcal{F} \rtimes D(G)$.

Proposition 3.2. *The element $I_{\mathcal{F}} \rtimes \frac{1}{|G|} \sum\limits_{g\in G} (u,g)$ is a self-adjoint idempotent element. That is*

$$\Big(I_{\mathcal{F}} \rtimes \tfrac{1}{|G|} \sum_{g\in G}(u,g)\Big)^2 = I_{\mathcal{F}} \rtimes \tfrac{1}{|G|} \sum_{g\in G}(u,g) = \Big(I_{\mathcal{F}} \rtimes \tfrac{1}{|G|} \sum_{g\in G}(u,g)\Big)^*.$$

Proof. We can compute that

$$\begin{aligned}&\Big(I_{\mathcal{F}} \rtimes \tfrac{1}{|G|} \sum_{g\in G}(u,g)\Big)^2\\ =\ &\Big(\sum_{g_1,g_2\in G} \delta_{g_1}(1)\delta_{g_2}(2) \rtimes \tfrac{1}{|G|}\sum_{h\in G}(u,h)\Big)\Big(\sum_{s_1,s_2\in G} \delta_{s_1}(1)\delta_{s_2}(2) \rtimes \tfrac{1}{|G|}\sum_{t\in G}(u,t)\Big)\\ =\ &\tfrac{1}{|G|^2}\delta_{g_1,hs_1}\delta_{g_2,hs_2} \sum_{g_1,g_2,s_1,s_2,h,t\in G} \delta_{g_1}(1)\delta_{g_2}(2) \rtimes (u,ht)\\ =\ &\tfrac{1}{|G|^2} \sum_{g_1,g_2,h,t\in G} \delta_{g_1}(1)\delta_{g_2}(2) \rtimes (u,ht)\\ =\ &\tfrac{1}{|G|}\Big(I_{\mathcal{F}} \rtimes \sum_{h\in G}(u,h)\Big),\end{aligned}$$

and

$$\begin{aligned}\Big(I_{\mathcal{F}} \rtimes \tfrac{1}{|G|} \sum_{g\in G}(u,g)\Big)^* &= \tfrac{1}{|G|}\Big(I_{\mathcal{F}} \rtimes \sum_{g\in G}(u,g)^*\Big)\Big(I_{\mathcal{F}} \rtimes I_{D(G)}\Big)\\ &= \tfrac{1}{|G|}\Big(I_{\mathcal{F}} \rtimes \sum_{g\in G}(u,g^{-1})\Big)\Big(I_{\mathcal{F}} \rtimes I_{D(G)}\Big)\\ &= \tfrac{1}{|G|}\Big(I_{\mathcal{F}} \rtimes \sum_{g\in G}(u,g)\Big).\end{aligned}$$

Hence, $I_{\mathcal{F}} \rtimes \frac{1}{|G|} \sum\limits_{g\in G}(u,g)$ is a self-adjoint idempotent element. □

Proposition 3.3. *The element $T \rtimes I_{D(G)}$ in $\mathcal{F} \rtimes D(G)$ satisfies the following covariant relation*

$$\Big(I_{\mathcal{F}} \rtimes \frac{1}{|G|}\sum_{g\in G}(u,g)\Big)\Big(T \rtimes I_{D(G)}\Big)\Big(I_{\mathcal{F}} \rtimes \frac{1}{|G|}\sum_{g\in G}(u,g)\Big) = \Big(E(T) \rtimes I_{D(G)}\Big)\Big(I_{\mathcal{F}} \rtimes \frac{1}{|G|}\sum_{g\in G}(u,g)\Big).$$

Proof. Suppose that $T = \delta_{s_1}(1)\delta_{s_2}(2)\rho_{t_1}(\frac{1}{2})\rho_{t_2}(\frac{3}{2})$, we can obtain

$$\begin{aligned}&\Big(I_{\mathcal{F}} \rtimes \tfrac{1}{|G|} \sum_{g\in G}(u,g)\Big)\Big(T \rtimes I_{D(G)}\Big)\\ =\ &\tfrac{1}{|G|}\Big(\sum_{g_1,g_2\in G} \delta_{g_1}(1)\delta_{g_2}(2) \rtimes \sum_{g\in G}(u,g)\Big)\Big(\delta_{s_1}(1)\delta_{s_2}(2)\rho_{t_1}(\tfrac{1}{2})\rho_{t_2}(\tfrac{3}{2}) \rtimes \sum_{s\in G}(s,u)\Big)\\ =\ &\tfrac{1}{|G|} \sum_{g_1,g_2,g,s\in G} \delta_{u,t_1t_2s}\delta_{g_1,gs_1}\delta_{g_2,gs_2}\delta_{g_1}(1)\delta_{g_2}(2)\rho_{gt_1g^{-1}}(\tfrac{1}{2})\rho_{gt_2g^{-1}}(\tfrac{3}{2}) \rtimes (gsg^{-1},g)\\ =\ &\tfrac{1}{|G|} \sum_{g\in G} \delta_{gs_1}(1)\delta_{gs_2}(2)\rho_{gt_1g^{-1}}(\tfrac{1}{2})\rho_{gt_2g^{-1}}(\tfrac{3}{2}) \rtimes (gt_2^{-1}t_1^{-1}g^{-1},g),\end{aligned}$$

and then

$$
\begin{aligned}
&\Big(I_{\mathcal{F}} \rtimes \tfrac{1}{|G|}\sum_{g\in G}(u,g)\Big)\Big(T \rtimes I_{D(G)}\Big)\Big(I_{\mathcal{F}} \rtimes \tfrac{1}{|G|}\sum_{g\in G}(u,g)\Big)\\
= &\ \tfrac{1}{|G|}\sum_{g\in G}\delta_{gs_1}(1)\delta_{gs_2}(2)\rho_{gt_1g^{-1}}(\tfrac{1}{2})\rho_{gt_2g^{-1}}(\tfrac{3}{2}) \rtimes (gt_2^{-1}t_1^{-1}g^{-1},g)\Big(\sum_{g_1,g_2\in G}\delta_{g_1}(1)\delta_{g_2}(2) \rtimes \tfrac{1}{|G|}\sum_{g\in G}(u,g)\Big)\\
= &\ \tfrac{1}{|G|^2}\sum_{g,g_1,g_2,f\in G}\delta_{t_1t_2,u}\delta_{s_1,t_1g_1}\delta_{s_2,g_2}\delta_{gs_1}(1)\delta_{gs_2}(2)\rho_{gt_1g^{-1}}(\tfrac{1}{2})\rho_{gt_2g^{-1}}(\tfrac{3}{2}) \rtimes (u,gf)\\
= &\ \tfrac{1}{|G|^2}\delta_{t_1t_2,u}\sum_{g,f\in G}\delta_{gs_1}(1)\delta_{gs_2}(2)\rho_{gt_1g^{-1}}(\tfrac{1}{2})\rho_{gt_2g^{-1}}(\tfrac{3}{2}) \rtimes (u,gf)\\
= &\ \tfrac{1}{|G|^2}\delta_{t_1t_2,u}\sum_{g,f\in G}\delta_{gs_1}(1)\delta_{gs_2}(2)\rho_{gt_1g^{-1}}(\tfrac{1}{2})\rho_{gt_2g^{-1}}(\tfrac{3}{2}) \rtimes (u,f).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
E(T) &= E\Big(\delta_{s_1}(1)\delta_{s_2}(2)\rho_{t_1}(\tfrac{1}{2})\rho_{t_2}(\tfrac{3}{2})\Big)\\
&= \tfrac{1}{|G|}\delta_{t_1t_2,u}\sum_{f\in G}\delta_{fs_1}(1)\delta_{fs_2}(2)\rho_{ft_1f^{-1}}(\tfrac{1}{2})\rho_{ft_2f^{-1}}(\tfrac{3}{2}),
\end{aligned}
$$

and then

$$
\begin{aligned}
&\Big(E(T) \rtimes I_{D(G)}\Big)\Big(I_{\mathcal{F}} \rtimes \tfrac{1}{|G|}\sum_{g\in G}(u,g)\Big)\\
= &\ \Big(\tfrac{1}{|G|}\delta_{t_1t_2,u}\sum_{f\in G}\delta_{fs_1}(1)\delta_{fs_2}(2)\rho_{ft_1f^{-1}}(\tfrac{1}{2})\rho_{ft_2f^{-1}}(\tfrac{3}{2}) \rtimes \sum_{s\in G}(s,u)\Big)\Big(\sum_{g_1,g_2\in G}\delta_{g_1}(1)\delta_{g_2}(2) \rtimes \tfrac{1}{|G|}\sum_{t\in G}(u,t)\Big)\\
= &\ \tfrac{1}{|G|^2}\delta_{t_1t_2,u}\sum_{f,t,g_1,g_2,s\in G}\delta_{s,u}\delta_{s_1,t_1f^{-1}g_1}\delta_{fs_2,g_2}\delta_{fs_1}(1)\delta_{fs_2}(2)\rho_{ft_1f^{-1}}(\tfrac{1}{2})\rho_{ft_2f^{-1}}(\tfrac{3}{2}) \rtimes (u,t)\\
= &\ \tfrac{1}{|G|^2}\delta_{t_1t_2,u}\sum_{f,t\in G}\delta_{fs_1}(1)\delta_{fs_2}(2)\rho_{ft_1f^{-1}}(\tfrac{1}{2})\rho_{ft_2f^{-1}}(\tfrac{3}{2}) \rtimes (u,t).
\end{aligned}
$$

From the above, we can obtain the desired result. □

The following theorem is one of main results of this paper, which gives a characterization of the C^*-algebra $\langle \mathcal{F}, e_{\mathcal{A}}\rangle_{C^*}$ constructed from the C^*-basic construction for the inclusion $\mathcal{A} \subseteq \mathcal{F}$.

Theorem 3.4. *There exists a C^*-isomorphism between the crossed product C^*-algebra $\mathcal{F} \rtimes D(G)$ and the C^*-algebra $\langle \mathcal{F}, e_{\mathcal{A}}\rangle_{C^*}$. That is,*

$$\mathcal{F} \rtimes D(G) \cong \langle \mathcal{F}, e_{\mathcal{A}}\rangle_{C^*}.$$

Proof. We first show that the action of $D(G)$ on $\mathcal{F}$ is faithful, that is, $(g,h)F = (s,t)F$ for any $F \in \mathcal{F}$ implies $(g,h) = (s,t)$. To this end let $F = \sum_{f\in G}\delta_f(x)$, then $(g,h)(\sum_{f\in G}\delta_f(x)) = (s,t)(\sum_{f\in G}\delta_f(x))$, and therefore

$$\sum_{f\in G}\delta_{g,u}\delta_{hf}(x) = \sum_{f\in G}\delta_{s,u}\delta_{tf}(x).$$

Using $\sum_{f\in G}\delta_{hf}(x) = I$, we have $\delta_{g,u} = \delta_{s,u}$ and then $g = s$. Let now $F = \delta_u(x)\rho_{h^{-1}gh}(l)$, then we conclude $(g,h)(\delta_u(x)\rho_{h^{-1}gh}(l)) = (s,t)(\delta_u(x)\rho_{h^{-1}gh}(l))$, which proves

$$
\begin{aligned}
\sum_{a\in G}(a,h)\delta_u(x)(a^{-1}g,h)\rho_{h^{-1}gh}(l) &= \sum_{b\in G}(b,t)\delta_u(x)(b^{-1}g,t)\rho_{h^{-1}gh}(l),\\
\sum_{a\in G}\delta_{a,u}\delta_h(x)\delta_{a^{-1}g,g}\rho_{a^{-1}g}(l) &= \sum_{b\in G}\delta_{b,u}\delta_t(x)\delta_{b^{-1}g,th^{-1}ght^{-1}}\rho_{b^{-1}g}(l).
\end{aligned}
$$

Putting $a = u = b$, we get

$$\delta_h(x)\rho_g(l) = \delta_{g,th^{-1}ght^{-1}}\delta_t(x)\rho_g(l),$$

and therefore $h = t$. Hence we get the action of $\mathcal{F} \rtimes D(G)$ on $\mathcal{F}$ is also faithful.

Secondly, Propositon 2.7 tells us that the conditional expectation $E\colon \mathcal{F} \to \mathcal{A}$ is of index-finite type in the sense of Watatani. It then follows from Propositon 1.3.3 in [18] that $\langle \mathcal{F}, e_{\mathcal{A}} \rangle_{C^*}$ is the same as $B(\mathcal{F})$, the algebra of bounded and adjointable operators in $\mathcal{F}_{\mathcal{A}}$ (with the standard $\mathcal{A}$-valued inner product).

Finally, the element $e_{\mathcal{A}}$ is represented by an element in $D(G)$, so combined with the previous bullet point, this yields C^*-algebra $\langle \mathcal{F}, e_{\mathcal{A}} \rangle_{C^*}$ coincides with the crossed product C^*-algebra $\mathcal{F} \rtimes D(G)$. □

Remark 3.5. *From Theorem 3.4, we know that the C^*-basic constructions do not depend on the choice of conditional expectations, which can also be seen in Proposition 2.10.11 [18].*

4. The C^*-basic construction for the inclusion $\mathcal{F} \subseteq \mathcal{F} \rtimes D(G)$

In this section, we continue to investigate the crossed product C^*-algebra $\mathcal{F} \rtimes D(G)$, and the natural $\widehat{D(G)}$-module algebra action on $\mathcal{F} \rtimes D(G)$, which gives rise to the iterated crossed product C^*-algebra $\mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)}$. The fixed point algebra under this action is given by $\mathcal{F} \equiv \mathcal{F} \rtimes I_D(G)$, which is consistent with the range of the conditional expectation E_2. We then prove that the C^*-algebra $\langle \mathcal{F} \rtimes D(G), e_2 \rangle_{C^*}$ constructed from the C^*-basic construction for the inclusion $\mathcal{F} \subseteq \mathcal{F} \rtimes D(G)$ is precisely C^*-isomorphic to the iterated crossed product C^*-algebra $\mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)}$.

Since $D(G)$ is of finite dimension and $\widehat{C(G) \otimes \mathbb{C}G} \cong \mathbb{C}G \otimes C(G)$ as algebras, $\{(y, \delta_x)\colon y, x \in G\}$ can be viewed as a linear basis of $\widehat{D(G)}$. As the above states, the structure maps on $\widehat{D(G)}$ are the following

$$\begin{array}{rcll}
\widetilde{\triangle}(y, \delta_x) & = & \sum\limits_{t \in G} (y, \delta_{t^{-1}}) \otimes (tyt^{-1}, \delta_{tx}), & \text{(coproduct)} \\
(y, \delta_x)(w, \delta_z) & = & \delta_{x,z}(yw, \delta_x), & \text{(multiplication)} \\
(y, \delta_x)^* & = & (y^{-1}, \delta_x), & \text{(}*\text{-operation)} \\
\widetilde{\varepsilon}(y, \delta_x) & = & \delta_{x,u}, & \text{(counit)} \\
\widetilde{S}(y, \delta_x) & = & (x^{-1}y^{-1}x, \delta_{x^{-1}}). & \text{(antipode)}
\end{array}$$

It is easy to see that $I_{\widehat{D(G)}} = \sum\limits_{x \in G} (u, \delta_x)$, and there is a unique element $E_2 = \frac{1}{|G|} \sum\limits_{y \in G} (y, \delta_u)$, called an integral element, satisfying for any $b \in \widehat{D(G)}$,

$$bE_2 = E_2 b = \widetilde{\varepsilon}(b) E_2.$$

The map $\sigma\colon \widehat{D(G)} \times (\mathcal{F} \rtimes D(G)) \to \mathcal{F} \rtimes D(G)$ given on the generating elements of $\mathcal{F} \rtimes D(G)$ as

$$\sigma\Big((y, \delta_x) \times (F \otimes (g, h))\Big) = \delta_{x,h}\Big(F \otimes (gy^{-1}, h)\Big)$$

for $(g, h) \in D(G)$, can be linearly extended both in $\widehat{D(G)}$ and $\mathcal{F} \rtimes D(G)$. Here and from now on, by $(y, \delta_x)(F \otimes (g, h))$ we always denote $\sigma((y, \delta_x) \times (F \otimes (g, h)))$ for notational convenience.

In particular, considering the action of E_2 on $\mathcal{F} \rtimes D(G)$, we can obtain that

$$E_2(F \otimes (g, h)) = \frac{1}{|G|} \sum_{y \in G} (y, \delta_u)(F \otimes (g, h)) = \frac{1}{|G|} \sum_{y \in G} \delta_{h,u}(F \otimes (gy^{-1}, h)) = \frac{1}{|G|} \delta_{h,u}(F \otimes I_{D(G)}),$$

which means that the range of E_2 on $\mathcal{F} \rtimes D(G)$ is contained in $\mathcal{F}$. Moreover, we can show that E_2 is a positive map preserving the unit and possessing the bimodular property. Namely,

Proposition 4.1. *The map $E_2\colon \mathcal{F} \rtimes D(G) \to \mathcal{F}$ is a conditional expectation.*

Proof. (1) $E_2(I_{\mathcal{F} \rtimes D(G)}) = E_2(I_{\mathcal{F}} \otimes \sum\limits_{g \in G} (g, u)) = \frac{1}{|G|} \sum\limits_{g \in G} (I_{\mathcal{F}} \otimes I_{D(G)}) = I_{\mathcal{F}}$.

(2) $\forall T_1, T_2 \in \mathcal{F}, \widetilde{T} \in \mathcal{F} \rtimes D(G)$, we have

$$\begin{aligned} E_2(T_1\widetilde{T}T_2) &= \tfrac{1}{|G|}\sum_{y\in G}(y,\delta_u)(T_1\widetilde{T}T_2) \\ &= \tfrac{1}{|G|}\sum_{y,t_1,t_2\in G}(y,\delta_{t_1^{-1}})(T_1)(t_1yt_1^{-1},\delta_{t_1t_2^{-1}})(\widetilde{T})(t_2yt_2^{-1},\delta_{t_2})(T_2) \\ &= \tfrac{1}{|G|}\sum_{y,t_1,t_2\in G}\widetilde{\varepsilon}(y,\delta_{t_1^{-1}})(T_1)(t_1yt_1^{-1},\delta_{t_1t_2^{-1}})(\widetilde{T})\widetilde{\varepsilon}(t_2yt_2^{-1},\delta_{t_2})(T_2) \\ &= \tfrac{1}{|G|}\sum_{y,t_1,t_2\in G}\delta_{t_1^{-1},u}(T_1)(t_1yt_1^{-1},\delta_{t_1t_2^{-1}})(\widetilde{T})\delta_{t_2,u}(T_2) \\ &= \tfrac{1}{|G|}\sum_{y\in G}T_1(y,\delta_u)(\widetilde{T})T_2 \\ &= T_1E_2(\widetilde{T})T_2. \end{aligned}$$

(3) We note the relation for any $\widetilde{T} \in \mathcal{F} \rtimes D(G)$,

$$\begin{aligned} E_2(\widetilde{T^*}\widetilde{T}) &= \tfrac{1}{|G|}\sum_{y\in G}(y,\delta_u)(\widetilde{T^*}\widetilde{T}) \\ &= \tfrac{1}{|G|}\sum_{y,t\in G}(y,\delta_{t^{-1}})(\widetilde{T^*})(tyt^{-1},\delta_t)(\widetilde{T}) \\ &= \tfrac{1}{|G|}\sum_{y,t\in G}(\widetilde{S}(y,\delta_{t^{-1}})^*(\widetilde{T}))^*(tyt^{-1},\delta_t)(\widetilde{T}) \\ &= \tfrac{1}{|G|}\sum_{y,t\in G}((tyt^{-1},\delta_t)(F))^*(tyt^{-1},\delta_t)(\widetilde{T}), \end{aligned}$$

which means E_2 is a positive map on $\mathcal{F} \rtimes D(G)$. □

Proposition 4.2. *The map σ defines a Hopf module left action of $\widehat{D(G)}$ on $\mathcal{F} \rtimes D(G)$. That is $\mathcal{F} \rtimes D(G)$ is a left $\widehat{D(G)}$-module algebra.*

Proof. It suffices to check that the map $\sigma\colon \widehat{D(G)} \times (\mathcal{F} \rtimes D(G)) \to \mathcal{F} \rtimes D(G)$ satisfies the following relations:

$$\begin{aligned} \big((y,\delta_x)(w,\delta_z)\big)\big(F\otimes(g,h)\big) &= (y,\delta_x)\big((w,\delta_z)(F\otimes(g,h))\big), \\ (y,\delta_x)\big((F\otimes(g_1,h_1))(T\otimes(g_2,h_2))\big) &= \sum_{(y,\delta_x)}\big((y,\delta_x)_{(1)}(F\otimes(g_1,h_1))\big)\big((y,\delta_x)_{(2)}(T\otimes(g_2,h_2))\big), \\ (y,\delta_x)\big(F\otimes(g,h)\big)^* &= \big(\widetilde{S}(y,\delta_x)^*(F\otimes(g,h))\big)^*, \end{aligned}$$

for $(y,\delta_x),(w,\delta_z) \in \widehat{D(G)}$, $T, F \in \mathcal{F}$ and $(g_i,h_i),(g,h) \in D(G)$ for $i = 1,2$.

As to the first equality, we compute

$$\begin{aligned} \big((y,\delta_x)(w,\delta_z)\big)(F\otimes(g,h)) &= \delta_{x,z}(yw,\delta_x)(F\otimes(g,h)) \\ &= \delta_{x,z}\delta_{x,h}(F\otimes(gw^{-1}y^{-1},h)) \\ &= \delta_{z,h}\delta_{x,h}(F\otimes(gw^{-1}y^{-1},h)) \\ &= \delta_{z,h}(y,\delta_x)(F\otimes(gw^{-1},h)) \\ &= (y,\delta_x)\big((w,\delta_z)(F\otimes(g,h))\big). \end{aligned}$$

Next,

$$\begin{aligned} & (y,\delta_x)\Big((F\otimes(g_1,h_1))(T\otimes(g_2,h_2))\Big) \\ = & \sum_{(g_1,h_1)}(y,\delta_x)\Big(F(g_1,h_1)_{(1)}T\otimes(g_1,h_1)_{(2)}(g_2,h_2)\Big) \\ = & \sum_{f\in G}(y,\delta_x)\Big(F(f,h_1)T\otimes(f^{-1}g_1,h_1)(g_2,h_2)\Big) \\ = & \sum_{f\in G}(y,\delta_x)\Big(F(f,h_1)T\otimes\delta_{f^{-1}g_1h_1,h_1g_2}(f^{-1}g_1,h_1h_2)\Big) \\ = & (y,\delta_x)\Big(F(g_1h_1g_2^{-1}h_1^{-1},h_1)T\otimes(h_1g_2h_1^{-1},h_1h_2)\Big) \\ = & \delta_{x,h_1h_2}F(g_1h_1g_2^{-1}h_1^{-1},h_1)T\otimes(h_1g_2h_1^{-1}y^{-1},h_1h_2), \end{aligned}$$

and

$$\begin{aligned} & \sum_{(y,\delta_x)}\Big((y,\delta_x)_{(1)}(F\otimes(g_1,h_1))\Big)\Big((y,\delta_x)_{(2)}(T\otimes(g_2,h_2))\Big) \\ = & \sum_{t\in G}\Big((y,\delta_{t^{-1}})(F\otimes(g_1,h_1))\Big)\Big((tyt^{-1},\delta_{tx})(T\otimes(g_2,h_2))\Big) \\ = & \sum_{t\in G}\delta_{t^{-1},h_1}\delta_{tx,h_2}\Big(F\otimes(g_1y^{-1},h_1)\Big)\Big(T\otimes(g_2tyt^{-1}t^{-1},h_2)\Big) \\ = & \delta_{x,h_1h_2}\Big(F\otimes(g_1y^{-1},h_1)\Big)\Big(T\otimes(g_2h_1^{-1}y^{-1}h_1,h_2)\Big) \\ = & \sum_{f\in G}\delta_{x,h_1h_2}F(f,h_1)T\otimes(f^{-1}g_1y^{-1},h_1)(g_2h_1^{-1}y^{-1}h_1,h_2) \\ = & \delta_{x,h_1h_2}\delta_{f^{-1}g_1,h_1g_2h_1^{-1}}F(f,h_1)T\otimes(f^{-1}g_1y^{-1},h_1h_2) \\ = & \delta_{x,h_1h_2}F(g_1h_1g_2^{-1}h_1^{-1},h_1)T\otimes(h_1g_2h_1^{-1}y^{-1},h_1h_2). \end{aligned}$$

Thus, we obtain that

$$(y,\delta_x)\Big((F\otimes(g_1,h_1))(T\otimes(g_2,h_2))\Big)=\sum_{(y,\delta_x)}\Big((y,\delta_x)_{(1)}(F\otimes(g_1,h_1))\Big)\Big((y,\delta_x)_{(2)}(T\otimes(g_2,h_2))\Big).$$

To prove the third equation, we can calculate

$$\begin{aligned} (y,\delta_x)\Big(F\otimes(g,h)\Big)^* &= (y,\delta_x)\Big((I_{\mathcal{F}}\otimes(g,h)^*)(F^*\otimes I_{D(G)})\Big) \\ &= (y,\delta_x)\Big((I_{\mathcal{F}}\otimes(h^{-1}gh,h^{-1}))(F^*\otimes I_{D(G)})\Big) \\ &= \sum_{f\in G}(y,\delta_x)\Big((f,h^{-1})F^*\otimes(f^{-1}h^{-1}gh,h^{-1})\Big) \\ &= \sum_{f\in G}\delta_{x,h^{-1}}(f,h^{-1})F^*\otimes(f^{-1}h^{-1}ghy^{-1},h^{-1}), \end{aligned}$$

and

$$\begin{aligned} \Big(\widetilde{S}(y,\delta_x)(F\otimes(g,h))\Big)^* &= \Big((x^{-1}yx,\delta_{x^{-1}})(F\otimes(g,h))\Big)^* \\ &= \delta_{x^{-1},h}\Big(F\otimes(gx^{-1}y^{-1}x,h)\Big)^* \\ &= \sum_{f\in G}\delta_{x,h^{-1}}(f,h^{-1})F^*\otimes(f^{-1}h^{-1}ghy^{-1},h^{-1}). \end{aligned}$$

□

From Proposition 4.2, we can construct the crossed product $(\mathcal{F}\rtimes D(G))\rtimes\widehat{D(G)}$, which is called the iterated crossed product C^*-algebra.

In the following, we will consider the $\widehat{D(G)}$-invariant subalgebra of $\mathcal{F} \rtimes D(G)$. To do this, set

$$(\mathcal{F} \rtimes D(G))^{\widehat{D(G)}} = \{\widetilde{T} \in \mathcal{F} \rtimes D(G) \colon b(\widetilde{T}) = \widetilde{\varepsilon}(b)(\widetilde{T}), \ \forall b \in \widehat{D(G)}\}.$$

One can show that $(\mathcal{F} \rtimes D(G))^{\widehat{D(G)}}$ is a C^*-subalgebra of $\mathcal{F} \rtimes D(G)$. Furthermore,

$$(\mathcal{F} \rtimes D(G))^{\widehat{D(G)}} = \{\widetilde{T} \in \mathcal{F} \rtimes D(G) \colon E_2(\widetilde{T}) = \widetilde{T}\}.$$

In fact, for $\widetilde{T} \in (\mathcal{F} \rtimes D(G))^{\widehat{D(G)}} \subseteq \mathcal{F} \rtimes D(G)$, we can compute that

$$E_2(\widetilde{T}) = \tfrac{1}{|G|} \sum_{y \in G} (y, \delta_u)(\widetilde{T}) = \tfrac{1}{|G|} \sum_{y \in G} \widetilde{\varepsilon}(y, \delta_u)(\widetilde{T}) = \tfrac{1}{|G|} \sum_{y \in G} \widetilde{T} = \widetilde{T}.$$

For the converse, suppose that $\widetilde{T} \in \mathcal{F} \rtimes D(G)$ with $E_2(\widetilde{T}) = \widetilde{T}$. Then for any $(w, \delta_z) \in \widehat{D(G)}$, we have

$$\begin{aligned}
(w, \delta_z)(\widetilde{T}) &= (w, \delta_z) E_2(\widetilde{T}) \\
&= (w, \delta_z) \tfrac{1}{|G|} \sum_{y \in G} (y, \delta_u)(\widetilde{T}) \\
&= \tfrac{1}{|G|} \sum_{y \in G} \delta_{z,u}(wy, \delta_u)(\widetilde{T}) \\
&= \widetilde{\varepsilon}(w, \delta_z) E_2(\widetilde{T}) \\
&= \widetilde{\varepsilon}(w, \delta_z)(\widetilde{T}).
\end{aligned}$$

This can be linearly extended in $\widehat{D(G)}$. Hence, $\widetilde{T} \in (\mathcal{F} \rtimes D(G))^{\widehat{D(G)}}$.

Remark 4.3. *$(\mathcal{F} \rtimes D(G))^{\widehat{D(G)}}$ is the subalgebra of $\mathcal{F} \rtimes D(G)$ corresponding to the trivial representation $\widetilde{\varepsilon}$ of $\widehat{D(G)}$.*

Naturally, we consider the C^*-algebra $\langle \mathcal{F} \rtimes D(G), e_2 \rangle_{C^*}$ constructed from the C^*-basic construction for the inclusion $\mathcal{F} \subseteq \mathcal{F} \rtimes D(G)$ in the following, where e_2 is the Jones projection of E_2.

Theorem 4.4. *There exists a C^*-isomorphism of C^*-algebras between $\mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)}$ and $\langle \mathcal{F} \rtimes D(G), e_2 \rangle_{C^*}$. That is,*

$$\mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)} \cong \langle \mathcal{F} \rtimes D(G), e_2 \rangle_{C^*}.$$

Proof. The proof is similar to that of Theorem 3.4. □

Moreover, by Takai duality [16], the iterated crossed product C^*-algebra $\mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)}$ is canonically isomorphic to $M_{|G|^2}(\mathcal{F})$.

Remark 4.5. *In the following we will give the concrete construction for $M_{|G|^2}(\mathcal{F})$.*

The local field $M_{|G|^2}(\mathcal{F}_{loc})$ of a $M_{|G|^2}(G)$-spin model is a $*$-algebra with a unit $I_{M_{|G|^2}(\mathcal{F})}$ generated by $\Big\{\delta_g(x) \otimes M, \rho_h(l) \otimes$

$N\colon g,h \in G,\ x \in \mathbb{Z},\ l \in \mathbb{Z} + \frac{1}{2},\ M, N \in LB(M_{|G|^2})\big\}$ *satisfying the following relations*

$$\begin{aligned}
O^g_M(x)O^h_N(x) &= \delta_{g,h}O^g_{MN}(x),\\
D^g_M(l)D^h_N(l) &= D^{gh}_{MN}(l),\\
\sum_{g\in G} O^g_I(x) &= I_{M_{|G|^2}(\mathcal{F})} = D^u_I(l),\\
O^g_M(x)O^h_N(x') &= O^h_M(x')O^g_N(x),\\
D^g_M(l)O^h_N(x) &= \begin{cases} O^{gh}_M(x)D^g_N(l), & \text{if } l < x,\\ O^h_M(x)D^g_N(l), & \text{if } l > x,\end{cases}\\
D^g_M(l)D^h_N(l') &= \begin{cases} D^h_M(l')D^{h^{-1}gh}_N(l), & \text{if } l > l',\\ D^{ghg^{-1}}_M(l')D^g_N(l), & \text{if } l < l',\end{cases}\\
(O^g_M(x))^* &= O^g_{M^*}(x),\\
(D^h_N(l))^* &= D^{h^{-1}}_{N^*}(l),
\end{aligned}$$

for $x, x' \in \mathbb{Z}$, $l, l' \in \mathbb{Z} + \frac{1}{2}$ *and* $g, h \in G$, *where by* $O^g_M(x)$, $D^h_N(l)$ *and* $LB(M_{|G|^2})$ *we denote* $\delta_g(x) \otimes M$, $\rho_h(l) \otimes N$ *and the linear basis of* $M_{|G|^2}(\mathbb{C})$ *for convenience, respectively.*

Similar to the case of the field algebra $\mathcal{F}$ of G-spin models, one can show that $M_{|G|^2}(\mathcal{F})$ is the C^*-algebra. From now on, we call $M_{|G|^2}(\mathcal{F})$ the field algebra of $M_{|G|^2}(G)$-spin models, and we call $O^g_M(x)$ and $D^h_N(l)$ the order and disorder operators, respectively.

Now one can show that the field algebra $\mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)}$ is $D(G)$-module algebra. Indeed, the map

$$\tau\colon D(G) \times (\mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)}) \to \mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)}$$

given on the generating elements of $\mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)}$ as

$$\tau((g,h) \times (\widetilde{F} \otimes (y, \delta_x))) = \delta_{h^{-1}gh, x^{-1}yx}(\widetilde{F} \otimes (y, \delta_{xh^{-1}}))$$

for any $\widetilde{F} \in \mathcal{F} \rtimes D(G)$, can be linearly extended both in $D(G)$ and $\mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)}$.

The observable algebra of $M_{|G|^2}(G)$-spin models is defined as $(\mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)})^{D(G)}$. So it is clear that $(\mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)})^{D(G)} = E_2(\mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)}) = \mathcal{F} \rtimes D(G)$.

Remark 4.6. *Let* $\mathcal{A} \subseteq \mathcal{F}$ *be an inclusion of unital* C^**-algebras with a conditional expectation* $E\colon \mathcal{F} \to \mathcal{A}$ *of index-finite type [22]. Set* $\mathcal{F}_{-1} = \mathcal{A}$, $\mathcal{F}_0 = \mathcal{F}$, *and* $E_1 = E$, *and recall the* C^**-basic construction (the* C^**-algebra version of the basic construction). We inductively define* $e_{k+1} = e_{\mathcal{F}_{k-1}}$ *and* $\mathcal{F}_{k+1} = \langle \mathcal{F}_k, e_{k+1}\rangle_{C^*}$, *the Jones projection and* C^**-basic construction applied to* $E_{k+1}\colon \mathcal{F}_k \to \mathcal{F}_{k-1}$, *and take* $E_{k+2}\colon \mathcal{F}_{k+1} \to \mathcal{F}_k$ *to be the dual conditional expectation* $E_{\mathcal{F}_k}$ *of Definition 2.3.3 in [18]. Then this gives the inclusion tower of iterated basic constructions*

$$\mathcal{A} \subseteq \mathcal{F} \subseteq \mathcal{F} \rtimes D(G) \subseteq \mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)} \subseteq \mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)} \rtimes D(G) \subseteq \cdots.$$

It follows from Proposition 2.10.11 in [18] that this tower does not depend on the choice of E.

Notice that $\mathcal{F}_2 = \mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)}$ *is* C^**-isomorphic to* $M_{|G|^2}(\mathcal{F})$, *the field algebra of a* $M_{|G|^2}(G)$*-spin model, and* $\mathcal{F}_4 = \mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)} \rtimes D(G) \rtimes \widehat{D(G)}$ *is* C^**-isomorphic to* $M_{|G|^4}(\mathcal{F})$, *called the field algebra of a* $M_{|G|^4}(G)$*-spin model, where the order and disorder operators can be defined similar to those in Remark 4.5.*

References

[1] E. Abe, *Hopf Algebras*, Cambridge University Press, Cambridge, 2004.
[2] O. Bratteli, D. W. Robinson, *Operator algebras and quantum statistical mechanics 1*, Springer, Berlin, Heidelberg, 1987.
[3] S. Carpi, Y. Kawahigashi, R. Longo, *On the Jones index values for conformal subnets*, Lett. Math. Phys. **92** (2010), 99–108.
[4] M. Izumi, *Inclusions of simple C^*-algebras*, J. Reine Angew. Math. **547** (2002) 97–138.
[5] A. Jaffe, C. L. Jiang, Z. W. Liu, Y. X. Ren, J. S. Wu, *Quantum fourier analysis*, Proc. Natl. Acad. Sci. USA **117** (2020), 10715–10720.
[6] L. N. Jiang, *C^*-index of observable algebras in G-spin models*, Sci. China Math. **48** (2005), 57–66.
[7] V. F. R. Jones, *Index for subfactors*, Invent. Math. **72** (1983), 1–25.
[8] T. Kajiwara, C. Pinzari, Y.Watatani, *Jones index theory for Hilbert C^*-bimodules and its equivalence with conjugation theory*, J. Funct. Anal. **215** (2004), 1–49.
[9] H. Kosaki, *Extension of Jones' theory on index to arbitrary factors*, J. Funct. Anal. **66** (1986), 123–140.
[10] F. Nill, K. Szlachányi,*Quantum chains of Hopf algebras with quantum double cosymmetry*, Comm. Math. Phys. **187** (1997), 159–200.
[11] M. Pimsner, S. Popa, *Entropy and index for subfactors*, Ann. Sci. Éc. Norm. Supér. **19** (1986), 57–106.
[12] S. Popa, *On the relative Dixmier property for inclusions of C^*-algebras*, J. Funct. Anal. **171** (2000), 139–154.
[13] S. Popa, *W^*-representations of subfactors and restrictions on the Jones index*, Enseign. Math. **69** (2023), 149–215.
[14] M. E. Sweedler, *Hopf algebras*, W.A. Benjamin, New York, 1969.
[15] K. Szlachányi, P. Vecsernyés, *Quantum symmetry and braided group statistics in G-spin models*, Comm. Math. Phys. **156** (1993), 127–168.
[16] H. Takai, *On a duality for crossed products of C^*-algebras*, J. Funct. Anal. **19** (1957), 25–39.
[17] S. Vaes, V. Stefan, *Property (T) discrete quantum groups and subfactors with triangle presentations*, Adv. Math. **345** (2019), 382–428.
[18] Y. Watatani, *Index for C^*-subalgebras*, Mem. Amer. Math. Soc. **83** (1990), 1–117.
[19] X. M. Wei, L. N. Jiang, Q. L. Xin, *The field algebra in Hopf spin models determined by a Hopf $*$-subalgebra and its symmetric structure*, Acta Math. Sci. Ser. B (Engl. Ed.) **41** (2021), 907–924.
[20] E. Witten, *Quantum theory and the Jones polynomials*, Comm. Math. Phys. **121** (1989), 351–399.
[21] Q. L. Xin, T. Q. Cao, *C^*-basic construction between non-balanced quantum doubles*, Czechoslovak Math. J. **74** (2024), 611–621.
[22] Q. L. Xin, T. Q. Cao, L. N. Jiang, *C^*-index of observable algebra in the field algebra determined by a normal group*, Math. Methods Appl. Sci. **45** (2022), 3689–3697.
[23] Q. L. Xin, L. N. Jiang, *Symmetric structure of field algebra of G-spin models determined by a normal subgroup*, J. Math. Phys. **55** (2014), 091703.