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Binding number and degree conditions for path-factors in graphs

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Abstract. For a graph G and an integer $k \geq 2$, a $P_{\geq k}$-factor of G is a spanning subgraph of G with each component isomorphic to some path of order at least k. A graph G is $P_{\geq k}$-factor uniform if for any distinct edges e_1 and e_2, G admits a $P_{\geq k}$-factor including e_1 and excluding e_2. For a non-negative integer n, G is $(P_{\geq k}, n)$-critical uniform if for any $V' \subseteq V(G)$ with $|V'| = n$, $G - V'$ is $P_{\geq k}$-factor uniform. G is $(P_{\geq k}, n)$-critical deleted if for any $V' \subseteq V(G)$ with $|V'| = n$ and $e \in E(G - V')$, $G - V' - e$ contains a $P_{\geq k}$-factor. In this note, we give some binding number and degree conditions for a graph to be $(P_{\geq 2}, n)$-critical uniform and $(P_{\geq 3}, n)$-critical deleted, which improve some known results.

1. Introduction

In real life, many type of relations can be modeled as graphs. Path factor problem is a classical topic in graph theory and path factor can be seen as a generalization of perfect matching. Research on the existence of path factors can present theoretical guidance for data transmission and help to design and construct networks with high data transmission rates [20]. In this paper, we mainly focus on the conditions of graphs for the existence of special path factors.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $u \in V(G)$, we use $d_G(u)$ and $N_G(u)$ to denote the degree of u and the set of neighbors of u in G, respectively. Let $\delta(G) = \min\{d_G(u) | u \in V(G)\}$. For $S \subseteq V(G)$, we use $G - S$ to denote the graph obtained from G by deleting the vertices of S and edges with at least one endpoint in S. For $e \in E(G)$, $G - e$ is the graph obtained from G by deleting e. For two disjoint graphs G and H, $G \cup H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$, which is called the *union* of G and H. For a positive integer k and a graph H, we also use kH to denote the union of k graphs isomorphic to H. The *join* of G and H, denoted by $G \vee H$, is the graph obtained from $G \cup H$ by adding the edges $\{uv | u \in V(G), v \in V(H)\}$. We use $I(G)$ to denote the set of isolated vertices of G and write $i(G) = |I(G)|$. $\alpha(G)$ and $\omega(G)$ are used to denote the independence number and number of components of G, respectively. The connectivity (resp. edge connectivity) of G is denoted by $\kappa(G)$ (resp. $\kappa'(G)$). For a positive integer k, G is called *k-connected* (resp. *k-edge-connected*) if $\kappa(G) \geq k$ (resp. $\kappa'(G) \geq k$).

The *binding number* of G was introduced by Woodall [22] and defined as

$$bind(G) = \min\left\{\frac{|N_G(S)|}{|S|} : \emptyset \neq S \subseteq V(G), N_G(S) \neq V(G)\right\}.$$

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A subgraph H of G is called a *spanning subgraph* of G if $V(H) = V(G)$. For a graph G and a set $\mathcal{H}$ of connected graphs, an *$\mathcal{H}$-factor* of G is a spanning subgraph H of G with each component isomorphic to some member in $\mathcal{H}$. If each component of H is isomorphic to a path, then we call the $\mathcal{H}$-factor a path-factor. Let $t \geq 2$ be an integer and P_t be the path of order t. A $\{P_t, P_{t+1}, ...\}$-factor is also written as a $P_{\geq t}$-factor.

Tutte [19] in 1947 gave a necessary and sufficient condition, which is called Tutte's condition, for graphs containing P_2-factors. After that the path factor problems have received a lot of attention, see for example [2, 7, 9, 12–15, 20, 21, 25–27, 30, 33]. There are also many interesting results on other factors, we refer the readers to [1, 18, 23, 28, 29, 32].

A graph G is called *factor-critical* if $G - \{u\}$ contains a P_2-factor for any $u \in V(G)$. A graph H is called the *corona* of graph R if H is obtained from R by adding a new vertex $w = w(v)$ together with a new edge vw for every $v \in V(R)$. The concept of sun was introduced by Kaneko [12]. A graph H is called a *sun* if $H \cong K_1$, $H \cong K_2$, or H is the corona of a factor-critical graph R with $|V(R)| \geq 3$. A sun with order one, order two and order at least six are called a *trivial sun*, *a K_2 sun* and a *big sun*, respectively. A component of a graph which is isomorphic to a sun is called a *sun component*. We denote by $sun(G)$ the number of sun components of G.

For $P_{\geq 3}$-factors, Kaneko [12] first gave the following necessary and sufficient condition. Kano, Katona and Kiraly [14] proved the same result, independently.

Theorem 1.1 (Kaneko [12], Kano, Katona and Király [14]). *Let G be a graph. Then G contains a $P_{\geq 3}$-factor if and only if $sun(G - S) \leq 2|S|$ for any $S \subseteq V(G)$.*

Zhang and Zhou [24] introduced the definition of $P_{\geq k}$-factor covered graphs. A graph G is called *$P_{\geq k}$-factor covered* if G has a $P_{\geq k}$-factor containing e for any $e \in E(G)$. They [24] also gave the following characterization for graphs to be $P_{\geq 2}$-factor covered and $P_{\geq 3}$-factor covered.

Theorem 1.2 (Zhang and Zhou [24]). *Let G be a connected graph. Then G is $P_{\geq 2}$-factor covered if and only if*

$$i(G - S) \leq 2|S| - \epsilon_1(S)$$

for any $S \subseteq V(G)$, where $\epsilon_1(S)$ is defined as follows:

$$\epsilon_1(S) = \begin{cases} 2, & if\ S \neq \emptyset \text{ and } S \text{ is not an independent set,} \\ 1, & if\ S \text{ is a nonempty independent set and } G - S \text{ admits a non-trivial component,} \\ 0, & otherwise. \end{cases}$$

Theorem 1.3 (Zhang and Zhou [24]). *Let G be a connected graph. Then G is $P_{\geq 3}$-factor covered if and only if*

$$sun(G - S) \leq 2|S| - \epsilon_2(S)$$

for any $S \subseteq V(G)$, where $\epsilon_2(S)$ is defined as follows:

$$\epsilon_2(S) = \begin{cases} 2, & if\ S \neq \emptyset \text{ and } S \text{ is not an independent set,} \\ 1, & if\ S \text{ is a nonempty independent set and } G - S \text{ admits a non-sun component,} \\ 0, & otherwise. \end{cases}$$

These necessary and sufficient conditions play important roles for further research. Many other classical parameter conditions have been studied for path-factor covered graphs, see for example [5, 8].

Zhou and Sun [31] generalized the definition of $P_{\geq k}$-factor covered graphs to $P_{\geq k}$-factor uniform graphs. A graph G is called *$P_{\geq k}$-factor uniform* if $G - f$ is $P_{\geq k}$-factor covered for any $f \in E(G)$. They [31] also gave the following binding number condition for graphs to be $P_{\geq 3}$-factor uniform.

Theorem 1.4 (Zhou and Sun [31]). *Let G be a 2-edge-connected graph. If $bind(G) > \frac{9}{4}$, then G is $P_{\geq 3}$-factor uniform.*

Gao and Wang [10] improved the above binding number condition to $bind(G) > \frac{5}{3}$, which is tight. Hua [11] presented some toughness and isolated toughness conditions for graphs to be $P_{\geq 3}$-factor uniform. Dai [6] gave two degree sum conditions for graphs to be $P_{\geq 2}$-factor uniform and $P_{\geq 3}$-factor uniform.

Liu [16] first introduced the concept of path-factor critical uniform graphs as follows. For a non-negative integer n, a graph G is $(P_{\geq k}, n)$-*critical uniform* if for any $V' \subseteq V(G)$ with $|V'| = n$, $G - V'$ is $P_{\geq k}$-factor uniform. Note that G is $(P_{\geq k}, 0)$-critical uniform if and only if G is $P_{\geq k}$-factor uniform. For $(P_{\geq 2}, n)$-critical uniform and $(P_{\geq 3}, n)$-critical uniform graphs, Liu [16] first gave two sufficient binding number conditions. After that, Liu and Pan [17] gave the following two independence number and minimum degree conditions.

Theorem 1.5 (Liu and Pan [17]). *Let n be a non-negative integer and G be an $(n+2)$-connected graph. If $\delta(G) > \frac{\alpha(G)+2n+3}{2}$, then G is $(P_{\geq 2}, n)$-critical uniform.*

Theorem 1.6 (Liu and Pan [17]). *Let n be a non-negative integer and G be an $(n+2)$-connected graph. If $\delta(G) > \frac{\alpha(G)+2n+4}{2}$, then G is $(P_{\geq 3}, n)$-critical uniform.*

The condition of Theorem 1.6 is tight. We improve the independence number and minimum degree condition of Theorem 1.5 to $\delta(G) > \frac{\alpha(G)+2n+2}{2}$ and show that it is tight.

A graph G is $(P_{\geq k}, n)$-*critical deleted* if for any $V' \subseteq V(G)$ with $|V'| = n$ and any $e \in E(G - V')$, $G - V' - e$ has a $P_{\geq k}$-factor. Zhou, Bian and Pan [30] gave the following binding number condition for $(n+2)$-connected graphs to be $(P_{\geq 3}, n)$-critical deleted.

Theorem 1.7 (Zhou, Bian and Pan [30]). *Let n be a non-negative integer and G be an $(n+2)$-connected graph. If $bind(G) > \frac{3+n}{2}$, then G is $(P_{\geq 3}, n)$-critical deleted.*

Chen and Dai [3] improved the above binding number condition to $bind(G) > \frac{4+n}{3}$ for $n \geq 1$. Inspired by the known results, we further show that $bind(G) > \frac{5+n}{4}$ is sufficient for $(n+2)$-connected graphs to be $(P_{\geq 3}, n)$-critical deleted, where n is a positive integer.

2. Our Main Results

Theorem 2.1. *Let n be a non-negative integer and G be an $(n+2)$-connected graph. If $\delta(G) > \frac{\alpha(G)+2n+2}{2}$, then G is $(P_{\geq 2}, n)$-critical uniform.*

Theorem 2.2. *Let n be a positive integer and G be an $(n+2)$-connected graph. If $bind(G) > \frac{5+n}{4}$, then G is $(P_{\geq 3}, n)$-critical deleted.*

3. Proof of Theorem 2.1

Now we give the proof of Theorem 2.1. Suppose, to the contrary, that G is an $(n+2)$-connected graph with $\delta(G) > \frac{\alpha(G)+2n+2}{2}$ and G is not $(P_{\geq 2}, n)$-critical uniform. That is, there exist $V' \subseteq V(G)$ with $|V'| = n$ and $e = xy \in E(G - V')$ such that $G - V' - e$ is not $P_{\geq 2}$-factor covered. Let $G' = G - V'$ and $H = G' - e$. Then by Theorem 1.2, there is a subset $S \subseteq V(H)$ such that

$$i(H - S) \geq 2|S| - \epsilon_1(S) + 1. \tag{1}$$

Since G is $(n+2)$-connected and $G' = G - V'$, we have $\kappa(G') \geq \kappa(G) - |V'| = \kappa(G) - n \geq 2$. So G' is 2-connected, $|V(G')| \geq 3$ and $\kappa'(G') \geq \kappa(G') \geq 2$. Hence, $|V(H)| = V(G') \geq 3$ and $H = G' - e$ is connected.

Claim 3.1 $|S| \geq 1$.

Proof. Suppose, to the contrary, that $S = \emptyset$. By the definition of $\epsilon_1(S)$, we have $\epsilon_1(S) = 0$. Then by (1), we have

$$i(H) = i(H - S) \geq 1. \tag{2}$$

Combining with (2) and the connectivity of H, we have $H \cong K_1$, which contradicts $|V(H)| \geq 3$. □

By the definition of $\epsilon_1(S)$, we have $\epsilon_1(S) \leq \max\{|S|, 2\}$. Then combining with (1) and Claim 3.1, we have

$$i(H - S) \geq \max\{|S| + 1, 2|S| - 1\} \geq 2. \quad (3)$$

Note that $H = G' - e$. So $i(G' - S) \geq i(H - S) - 2$. Combining with (3) and Claim 3.1, we have

$$i(G' - S) \geq i(H - S) - 2 \geq |S| - 1 \geq 0. \quad (4)$$

Claim 3.2 $i(G' - S) = 0$.

Proof. Suppose, to the contrary, that $i(G' - S) \geq 1$. Let $z \in I(G' - S)$. Then

$$d_G(z) = d_{V'}(z) + d_S(z) \leq |V'| + |S| = n + |S|. \quad (5)$$

If $|S| = 1$, then $d_G(z) \leq n + 1$, a contradiction to the $(n + 2)$-connectivity of G. So we may assume that $|S| \geq 2$. By (3), we have $i(H - S) \geq 2|S| - 1$, which implies

$$|S| \leq \frac{i(H - S) + 1}{2}. \quad (6)$$

Combining with (5), (6) and the condition $\delta(G) > \frac{\alpha(G)+2n+2}{2}$, we have $\frac{\alpha(G)+2n+2}{2} < \delta(G) \leq d_G(z) \leq n + \frac{i(H-S)+1}{2}$, which means

$$\alpha(G) < i(H - S) - 1. \quad (7)$$

Let $C = \begin{cases} I(H - S), \ if \ \{x, y\} \not\subseteq I(H - S), \\ I(H - S) \setminus \{x\}, \ otherwise. \end{cases}$ Then $|C| \geq i(H - S) - 1$ and C is an independent set of G, which contradicts (7). □

Combining with (4) and Claim 3.2, we have $|S| = 1$ and $i(H - S) = 2$. Since $i(G' - S) = 0$, $i(H - S) = 2$ and $H = G' - xy$, we have that $I(H - S) = \{x, y\}$. Then $d_G(x) \leq d_{V'}(x) + d_S(x) + 1 \leq n + 2$. On the other hand, since G is $(n + 2)$-connected, we have $d_G(x) \geq n + 2$. So $d_G(x) = n + 2$. It follows that

$$\frac{\alpha(G) + 2n + 2}{2} < \delta(G) \leq d_G(x) = n + 2,$$

which implies $\alpha(G) < 2$. So $\alpha(G) = 1$, which implies G is a complete graph. It is obvious that G' is a complete graph of order at least three, H is isomorphic to a graph obtained from G' by deleting an edge. Note that for any edge of a complete graph with order at least three, there is a Hamilton cycle containing this edge. So for any $f \in E(H)$, there is a Hamilton path containing f. Thus H is $P_{\geq 2}$-factor covered, a contradiction.

This completes the proof of Theorem 2.1. □

4. Proof of Theorem 2.2

Now we give the proof of Theorem 2.2. Suppose, to the contrary, that G is an $(n + 2)$-connected graph with $bind(G) > \frac{n+5}{4}$ and G is not $(P_{\geq 3}, n)$-critical deleted. That is, there exist $V' \subseteq V(G)$ with $|V'| = n$ and $e = xy \in E(G - V')$ such that $G - V' - e$ has no $P_{\geq 3}$-factor. Let $G' = G - V'$ and $H = G' - e$. Then by Theorem 1.1, there is some $S \subseteq V(H)$ such that

$$sun(H - S) \geq 2|S| + 1. \quad (8)$$

By (8), we have the following statements.

Claim 4.1 $|S| \geq 1$.

Proof. Suppose, to the contrary, that $S = \emptyset$. Then by (8), we have

$$sun(H) = sun(H - S) \geq 1. \quad (9)$$

Since G is $(n+2)$-connected and $G' = G - V'$, we have $\kappa(G') \geq \kappa(G) - |V'| \geq 2$. So G' is 2-connected, $|V(G')| \geq 3$ and $\kappa'(G') \geq \kappa(G') \geq 2$. Hence, $H = G' - e$ is connected, which means $\omega(H) = 1$. Note that $\omega(H) \geq sun(H)$. Then combining with (9), we have $sun(H) = \omega(H) = 1$, which means H is a sun. Since $|V(H)| = |V(G')| \geq 3$, we have that H is a big sun and $|V(H)| \geq 6$. Let R be the factor-critical graph of H. Then $|V(R)| = \frac{|V(H)|}{2} \geq 3$. Let $u \in V(H) \setminus (V(R) \cup \{x, y\})$. Note that $d_H(u) = d_{G'}(u) = 1$. Let $N_H(u) = \{v\}$. Then $\omega(G' - \{v\}) \geq 2$. So v is a cut-vertex of G', which contradicts the 2-connectivity of G'. □

Claim 4.2 $|S| \geq 2$ and $sun(H - S) \geq 5$.

Proof. By Claim 4.1, we have that $|S| \geq 1$. Suppose, to the contrary, that $|S| = 1$. Then by (8), we have $sun(H - S) \geq 2|S| + 1 = 3$, which implies

$$\omega(H - S) \geq 3. \tag{10}$$

Since G is $(n + 2)$-connected and $G' = G - V'$, we have $\kappa(G') \geq \kappa(G) - |V'| \geq 2$. So $G' - S$ is connected, which means $\omega(G' - S) = 1$. Note that $H = G' - e$. So

$$\omega(H - S) = \omega(G' - e - S) \leq \omega(G' - S) + 1 = 2, \tag{11}$$

which contradicts (10). Hence, $|S| \geq 2$. Furthermore, by (8), we have $sun(H - S) \geq 5$. □

Now we suppose that there exist a trivial sun components, b K_2 sun components and c big sun components in $H - S$, where a, b, c are all non-negative integers. Then combining with Claim 4.2 and (8), we have

$$a + b + c = sun(H - S) \geq 2|S| + 1 \geq 5. \tag{12}$$

Let A, B, C be the vertex set of a trivial sun components, b K_2 sun components and c big sun components, respectively. Then $|A| = a, |B| = 2b, |C| \geq 6c$.

Claim 4.3 $x \notin S$ and $y \notin S$.

Proof. Suppose, to the contrary, that $x \in S$ or $y \in S$. Note that $H = G' - xy$. Then by (12), we have $sun(G' - S) = sun(H - S) = a + b + c \geq 5$. We divide the following proof into two cases.

Case 1. $a \geq 1$.

Let $u \in A$ and $Y = A \cup B \cup C$. Then $Y \neq \emptyset$ and $N_G(Y) \neq V(G)$ since $u \notin N_G(Y)$. So we obtain that

$$\begin{aligned} \frac{n+5}{4} < bind(G) &\leq \frac{|N_G(Y)|}{|Y|} \\ &\leq \frac{|V'| + |S| + |B| + |C|}{|A| + |B| + |C|} \\ &= \frac{n + |S| + 2b + |C|}{a + 2b + |C|}, \end{aligned}$$

which implies $4|S| > 5a + 2b + |C| + (a + 2b + |C| - 4)n$. Note that $a \geq 1, b \geq 0, c \geq 0, a + b + c \geq 2|S| + 1 \geq 5$ and $|C| \geq 6c$. Then we have

$$\begin{aligned} 4|S| &> 5a + 2b + |C| + (a + 2b + |C| - 4)n \\ &\geq 5a + 2b + 6c + (a + 2b + 6c - 4)n \\ &\geq 2(a + b + c) \\ &\geq 2(2|S| + 1) \\ &= 4|S| + 2, \end{aligned}$$

a contradiction.

Case 2. $a = 0$.

By (12), we have $b + c \geq 2|S| + 1 \geq 5$. Let v be a degree one vertex in some sun component of $H - S$ and w be the neighbor vertex of v in $H - S$. Let $Y = (B \cup C) \setminus \{w\}$. Then $Y \neq \emptyset$ and $N_G(Y) \neq V(G)$ since $v \notin N_G(Y)$.

So we obtain that

$$\begin{aligned}\frac{n+5}{4} < bind(G) &\leq \frac{|N_G(Y)|}{|Y|}\\ &\leq \frac{|V'|+|S|+|B|+|C|-1}{|B|+|C|-1}\\ &= \frac{n+|S|+2b+|C|-1}{2b+|C|-1},\end{aligned}$$

which implies $4|S| > 2b+|C|-1+(2b+|C|-5)n$. Note that $n \geq 1, b \geq 0, c \geq 0, b+c \geq 2|S|+1 \geq 5$ and $|C| \geq 6c$. Then we have

$$\begin{aligned}4|S| &> 2b+|C|-1+(2b+|C|-5)n\\ &\geq 2b+6c-1+(2b+6c-5)n\\ &\geq 2(b+c)-1+[2(b+c)-5]n\\ &\geq 2(2|S|+1)\\ &= 4|S|+2,\end{aligned}$$

a contradiction. □

Claim 4.4 $x \notin A$ and $y \notin A$.

Proof. Suppose, to the contrary, that $x \in A$ or $y \in A$. Without loss of generality, we may assume $x \in A$. It implies $a \geq 1$. No matter $y \in A \cup B \cup C$ or not, let $Y = (A \cup B \cup C) \setminus \{y\}$. Then $Y \neq \emptyset$ and $N_G(Y) \neq V(G)$ since $x \notin N_G(Y)$. So we obtain that

$$\begin{aligned}\frac{n+5}{4} < bind(G) &\leq \frac{|N_G(Y)|}{|Y|}\\ &\leq \frac{|V'|+|S|+|B|+|C|+1}{|A|+|B|+|C|-1}\\ &= \frac{n+|S|+2b+|C|+1}{a+2b+|C|-1},\end{aligned}$$

which implies $4|S| > 5a+2b+|C|-9+(a+2b+|C|-5)n$. Note that $n \geq 1, a \geq 1, b \geq 0, c \geq 0, a+b+c \geq 2|S|+1 \geq 5$ and $|C| \geq 6c$. Then combining with $n \geq 1$, we have

$$\begin{aligned}4|S| &> 5a+2b+|C|-9+(a+2b+|C|-5)n\\ &\geq 5a+2b+6c-9+(a+2b+6c-5)n\\ &= 2(a+b+c)+(a+b+c-5)n+(b+5c)n+3a+4c-9\\ &\geq 2(a+b+c)+(a+b+c-5)n+(a+b+c-5)+(2a+8c-4)\\ &\geq 2(a+b+c)-2\\ &\geq 2(2|S|+1)-2\\ &= 4|S|,\end{aligned}$$

a contradiction. □

Now we divide the following proof into three cases. For each case, we construct a vertex set Y such that $Y \neq \emptyset$ and $N_G(Y) \neq V(G)$.

Case 1. $x \in B$ or $y \in B$.

Without loss of generality, we may assume that $x \in B$ and $N_B(x) = \{w\}$. Let $Y = (A \cup B \cup C) \setminus \{x\}$. Then $w \notin N_G(Y)$.

Case 2. $x \in C$ or $y \in C$.

Let D be the set of vertices of all the factor-critical subgraphs of C, $t \in C \setminus (D \cup \{x, y\})$ and $N_C(t) = \{z\}$. We choose $Y = (A \cup B \cup C) \setminus \{z\}$. Then $t \notin N_G(Y)$.

Case 3. $x \in V(G) \setminus (V' \cup S \cup A \cup B \cup C)$ and $y \in V(G) \setminus (V' \cup S \cup A \cup B \cup C)$.

Let $Y = A \cup B \cup C$. Then $x \notin N_G(Y)$ and $y \notin N_G(Y)$.

For all the three cases above, we get $Y \subseteq V(G)$ such that $Y \neq \emptyset$ and $N_G(Y) \neq V(G)$. Then we have

$$\begin{aligned} \frac{n+5}{4} < bind(G) &\leq \frac{|N_G(Y)|}{|Y|} \\ &\leq \frac{|V'| + |S| + |B| + |C|}{|A| + |B| + |C| - 1} \\ &= \frac{n + |S| + 2b + |C|}{a + 2b + |C| - 1}, \end{aligned}$$

which implies $4|S| > 5a + 2b + |C| - 5 + (a + 2b + |C| - 5)n$. Note that $a \geq 0, b \geq 0, c \geq 0, a + b + c \geq 2|S| + 1 \geq 5$ and $|C| \geq 6c$. Then combining with $n \geq 1$, we have

$$\begin{aligned} 4|S| &> 5a + 2b + |C| - 5 + (a + 2b + |C| - 5)n \\ &\geq 5a + 2b + 6c + (a + 2b + 6c - 5)n - 5 \\ &= 2(a + b + c) + (a + b + c - 5)n + 3a + 4c + (b + 5c)n - 5 \\ &\geq 2(a + b + c) + (a + b + c - 5)n + (a + b + c - 5) \\ &\geq 2(2|S| + 1) \\ &= 4|S| + 2, \end{aligned}$$

a contradiction.

This completes the proof of Theorem 2.2. □

5. Concluding Remarks

Remark 5.1 We now show that the minimum degree condition in Theorem 2.1 is tight. Let n and r be non-negative integers with $n \geq 2r + 3$, $G_1 \cong K_n$, $G_2 \cong K_2 \cup (2r + 1)K_1$, $G_3 \cong K_2 \cup rK_1$ and $G \cong G_1 \vee G_2 \vee G_3$. Then G is $(n + r + 2)$-connected, $\delta(G) = n + r + 2$ and $\alpha(G) = 2r + 2$. So $\delta(G) = \frac{\alpha(G)+2n+2}{2}$. Let $V' = V(G_1)$, $e \in E(G_2)$, $G' = G - V'$ and $H = G' - e$. Then for $S = V(G_3)$, we have $|S| = r + 2$, $\epsilon_1(S) = 2$ and $i(H - S) = 2r + 3 > 2r + 2 = 2|S| - \epsilon_1(S)$. By Theorem 1.2, H is not $P_{\geq 2}$-factor covered. So G' is not $P_{\geq 2}$-factor uniform and G is not $(P_{\geq 2}, n)$-critical uniform.

Note that we only need G_3 to be a graph of order $r + 2$ with at least one edge. So as long as G_3 is such a graph, we can get the same conclusion.

Remark 5.2 Maybe the binding number condition in Theorem 2.2 is not best possible. The maximum known binding number of an $(n + 2)$-connected graph that is not $(P_{\geq 3}, n)$-critical deleted is $\frac{n+4}{5}$ for $n \geq 9$, which was given in [3]. We list the graph here. Let $G_1 \cong K_{n+2}$, $G_2 \cong K_2 \cup 3K_1$ and $G \cong G_1 \vee G_2$. Then $bind(G) = \frac{|N_G(V(G_2))|}{|V(G_2)|} = \frac{n+4}{5}$. Let $e \in E(G_2), V' = V(K_n) \subseteq V(G_1), G' = G - V'$ and $H = G' - e$. Then for $S = V(G_1 - V')$, $sun(H - S) = 5 > 2|S| = 4$. By Theorem 1.1, H does not contain a $P_{\geq 3}$-factor. So G' is not $P_{\geq 3}$-factor deleted and G is not $(P_{\geq 3}, n)$-critical deleted.

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Declaration of Competing Interest

We declare that we have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data Availability statement

No data was used for the research described in the article.

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