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Spectral radius and spanning trees of graphs with leaf distance at least four

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Abstract. Let G be a connected graph of order n. The leaf distance of a tree is defined as the minimum distance between any two leaves. Let $\mathcal{D}(G)$ and $\mathcal{Q}(G)$ be the distance matrix and signless Laplacian distance matrix of G, respectively. Motivated by the structure condition of Kaneko, Kano, and Suzuki (2007), we obtain the lower bound for the size of G and the upper bound for the spectral radius of $\mathcal{D}(G)$ (resp. $\mathcal{Q}(G)$) to guarantee that G contains a spanning tree with leaf distance at least four. Furthermore, we improve the adjacency spectral and signless Laplacian spectral results of Chen, Lv, Li, and Xu (2025) to determine whether G contains a spanning tree with leaf distance at least four, and propose two questions for further research.

1. Introduction

Throughout this paper, we only consider simple, undirected and connected graphs. Let $G = (V, E)$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The *order* and *size* of G are the number of its vertices and edges, respectively. For $v \in V(G)$, the degree of v in G, denoted by $d_G(v)$(or $d(v)$ for short), is the number of vertices adjacent to v in G. The minimum degree of G is denoted by $\delta(G)$. A *pendant vertex* of G is a vertex of degree one and a pendant vertex is a *leaf* if G is a tree. The distance d_{ij} between two vertices v_i and v_j is the length of the shortest (v_i, v_j)-path in G, and the *transmission* $tr(v)$ of vertex v in G is the sum of the distances from v to all other vertices of G. For a subset $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S, and by $G - S$ the subgraph obtained from G by removing the vertices in S and their incident edges.

Let G_1 and G_2 be vertex disjoint graphs. The *union* $G_1 \cup G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. For any positive integer t, let tG denote the disjoint union of t copies of G. The *join* $G_1 \vee G_2$ is derived from $G_1 \cup G_2$ by joining every vertex of G_1 with every vertex of G_2 by an edge. Other undefined notations can be found in [2].

Let G be a graph of order n, the *adjacency matrix* $A(G) = (a_{ij})$ of G be the $n \times n$ symmetric matrix, where $a_{ij} = 1$ if v_i and v_j are adjacent in G, zero otherwise. Let $D(G)$ be the diagonal degree matrix of G. Then $Q(G) = D(G) + A(G)$ is called the *signless Laplacian matrix* of G. The largest eigenvalue of $A(G)$ (resp. $Q(G)$),

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denoted by $\lambda(G)$ (resp. $q(G)$), is called the adjacency spectral radius (resp. signless Laplacian spectral radius) of G.

The *distance matrix* of G, denoted by $\mathcal{D}(G)$, is the $n \times n$ symmetric matrix whose (i, j)-entry is d_{ij}. The *distance signless Laplacian matrix* of G is defined by $\mathcal{Q}(G) = Tr(G) + \mathcal{D}(G)$, where $Tr(G) = diag(tr(v_1), tr(v_2), \ldots, tr(v_n))$ is the diagonal matrix of the vertex transmissions in G. The largest eigenvalue of $\mathcal{D}(G)$ (resp. $\mathcal{Q}(G)$), denoted by $\mu(G)$ (resp. $\eta(G)$), is called the distance spectral radius ($\mathcal{D}$-index for short) (resp. distance signless Laplacian spectral radius ($\mathcal{Q}$-index for short)) of G.

A *spanning tree* T of a connected graph G is a spanning subgraph of G that is a tree, including all vertices of G. For an integer $k \geq 2$, a *k-tree* is a tree with the maximum degree at most k, and a *k-ended tree* is a tree with at most k leaves. In particular, a spanning 2-ended tree is called a Hamilton path of G.

Let T be a tree. For a vertex $v \in V(T)$, the *leaf degree* of v is defined as the number of leaves adjacent to v, and the leaf degree of T is the maximum leaf degree among all vertices of T. The *leaf distance* of T is defined as the minimum distance between any two leaves in T.

In graph theory, deciding whether a graph has spanning trees of particular types is a fundamental problem. For example, the Hamilton path problem, that is to find a spanning 2-ended tree, is a famous problem.

In mathematical literature, the study on the existence of spanning trees attracted much attention. For example, Win [16] made a connection between the existence of spanning k-trees in a graph and its toughness, and provided a Chvátal-Erdős type condition to ensure that a t-connected graph contains a spanning k-ended tree [15]; Gu and Liu [8] characterized a connected graph with a spanning k-tree by applying Laplacian eigenvalues; Fan, Goryainov, Huang and Lin [6] studied the existence of a spanning k-tree in a connected graph via the adjacency spectral radius and the signless Laplacian spectral radius, respectively; Kaneko[9] introduced the concept of leaf degree of a spanning tree, and gave the following criterion for a connected graph to possess a spanning tree with leaf degree at most k; Ao, Liu and Yuan [1] established the tight spectral conditions for the existence of a spanning tree with leaf degree at most k in a connected graph; Zhou, Sun and Liu [18] provided the $\mathcal{D}$-index and $\mathcal{Q}$-index for spanning trees with leaf degree at most k in connected graphs; Lin, Liu and You [11] studied some tight conditions for the existence of spanning trees with leaf degree at most k in graphs with a given minimum degree.; and so on.

Theorem 1.1. ([9]) *Let G be a connected simple graph and $k \geq 1$ be an integer. Then G has a spanning tree with leaf degree at most k if and only if $i(G - S) < (k + 1)|S|$ for any nonempty subset $S \subseteq V(G)$, where $i(G - S)$ is the number of isolated vertices in $G - S$.*

Note that when the leaf degree of a tree is one, then its leaf distance is at least three. So with $k = 1$, both the result of Ao, Liu, and Yuan [1, Theorem 1.5] and Theorem 1.1 can be used to guarantee the existence of a spanning tree with leaf distance at least three. The following conjecture, posed in [9], claims that decreasing the coefficient of $|S|$ in the above condition guarantees the existence of a spanning tree with larger leaf distance.

Conjecture 1.2. ([9]) *Let $d \geq 3$ be an integer and G be a connected graph of order $n \geq d + 1$. If $i(G - S) < \frac{2|S|}{d-2}$ for any nonempty subset $S \subseteq V(G)$, then G has a spanning tree with leaf distance at least d.*

By the previous discussion, Theorem 1.1 proves Conjecture 1.2 for $d = 3$. The following case $d = 4$ was later proven by Kaneko, Kano, and Suzuki in [10].

Theorem 1.3. ([10]) *Let G be a connected graph of order $n \geq 5$. Then G has a spanning tree with leaf distance at least four if $i(G - S) < |S|$ for any nonempty subset $S \subseteq V(G)$.*

Actually, it is not easy that one determines a graph having a spanning tree with leaf distance at least four by Theorem 1.3, because we need justify $i(G - S) < |S|$ for all nonempty $S \subseteq V(G)$. Motivated by [10, 18], in this paper, we study the sufficient conditions to ensure that a graph G with the minimum degree $\delta(G)$ has a spanning tree with leaf distance at least four in terms of its lower bound of size and spectral radius of G, and obtain Theorems 1.4, 1.5, 1.6 as follows.

Theorem 1.4. *Let G be a connected graph of order $n \geq 5$ and $t \leq \delta(G)$. If*

$$|E(G)| > \begin{cases} \binom{n-t}{2} + \frac{3}{2}t^2 - t + \frac{1}{2}, & \text{if } n = 5 \text{ or } n \geq 6t+1, \quad (1a) \\ \binom{n-t}{2} + \frac{3}{2}t^2 + \frac{t}{2}, & \text{if } 6 \leq n \leq 6t, \quad (1b) \end{cases}$$

then G has a spanning tree with leaf distance at least four.

When $t = 1$, we can derive the result [4, Theorem 4] from Theorem 1.4.

Theorem 1.5. *Let G be a connected graph of order $n \geq 7t + 2$ with $t \leq \delta(G)$. If $\mu(G) \leq \mu(K_t \vee (K_{n-2t} \cup tK_1))$, then G has a spanning tree with leaf distance at least four.*

Theorem 1.6. *Let G be a connected graph of order $n \geq 9t + 3$ with $t \leq \delta(G)$. If $\eta(G) \leq \eta(K_t \vee (K_{n-2t} \cup tK_1))$, then G has a spanning tree with leaf distance at least four.*

Using Theorem 1.3, Chen, Lv, Li, and Xu [4] establish the following spectral conditions for the existence of a spanning tree with leaf distance at least four in a graph G with the minimum degree $\delta = \delta(G)$.

Theorem 1.7. ([4]) *Let G be a graph of order $n \geq 6\delta + 2$. If $\lambda(G) \geq \lambda(K_\delta \vee (K_{n-2\delta} \cup \delta K_1))$, then G has a spanning tree with leaf distance at least four.*

Theorem 1.8. ([4]) *Let G be a graph of order $n \geq \frac{1}{2}(11\delta^2 + 9\delta + 4)$. If $q(G) \geq q(K_\delta \vee (K_{n-2\delta} \cup \delta K_1))$, then G has a spanning tree with leaf distance at least four.*

In this paper, we use different methods to improve Theorems 1.6 - 1.7 as follows.

Theorem 1.9. *Let G be a connected graph of order $n \geq 5t + 2$ with $t \leq \delta(G)$. If $\lambda(G) \geq \lambda(K_t \vee (K_{n-2t} \cup tK_1))$, then G has a spanning tree with leaf distance at least four.*

Theorem 1.10. *Let G be a connected graph of order $n \geq 7t + 1$ with $t \leq \delta(G)$. If $q(G) \geq q(K_t \vee (K_{n-2t} \cup tK_1))$, then G has a spanning tree with leaf distance at least four.*

The present paper is organized in the following way. In Section 2, we recall some well known results and show several lemmas. In Section 3, we show Theorems 1.4, 1.5, 1.6, 1.9 and 1.10 in order. In Section 4, we propose two questions for further research.

2. Preliminaries

In this section, we introduce some necessary preliminary lemmas, which are very useful in the following proofs.

Definition 2.1. ([3]) *Let M be a complex matrix of order n described in the following block form*

$$M = \begin{pmatrix} M_{11} & \cdots & M_{1l} \\ \vdots & \ddots & \vdots \\ M_{l1} & \cdots & M_{ll} \end{pmatrix}$$

where the blocks M_{ij} are $n_i \times n_j$ matrices for any $1 \leq i, j \leq l$ and $n = n_1 + \cdots + n_l$. For $1 \leq i, j \leq l$, let b_{ij} denote the average row sum of M_{ij}, i.e. b_{ij} is the sum of all entries in M_{ij} divided by the number of rows. Then $B(M) = (b_{ij})$ (or simply B) is called the quotient matrix of M. If, in addition, for each pair i, j, M_{ij} has a constant row sum, i.e., $M_{ij}\vec{e}_{n_j} = b_{ij}\vec{e}_{n_i}$, then B is called the equitable quotient matrix of M, where $\vec{e}_k = (1, 1, \ldots, 1)^T \in C^k$, and C denotes the field of complex numbers.

Lemma 2.2. ([17]) *Let M be a nonnegative matrix, B be the equitable quotient matrix of M as defined in Definition 2.1. Then $\rho(B) = \rho(M)$, where $\rho(B)$ and $\rho(M)$ are the spectral radius of B and M, respectively.*

Lemma 2.3. ([7, 12]) *Let e be an edge of G such that $G - e$ is still connected. Then*

$$\mu(G) < \mu(G - e),\ \eta(G) < \eta(G - e).$$

Lemma 2.4. ([17]) *Let G be a graph and F be a spanning subgraph of G. Then $\lambda(F) \leq \lambda(G)$ and $q(F) \leq q(G)$. In particular, if G is connected, and F is a proper subgraph of G, then $\lambda(F) < \lambda(G)$ and $q(F) < q(G)$.*

In order to establish better results subsequently, we introduce and prove the following lemma.

Lemma 2.5. *Let n, t be positive integers with $n > 2t$, $H = K_t \vee (K_{n-2t} \cup tK_1)$,*

$$g(x) = x^3 - (n + t - 4)x^2 - (2nt + 3n - 5t^2 + t - 5)x + nt^2 - 2nt - 2n + 5t^2 - 2t^3 + 2,$$

$$h(x) = x^3 - (5n + t - 8)x^2 + (8n^2 - nt - 26n + 8t^2 - 4t + 20)x - 4n^3 + 2n^2t + 20n^2 - 8nt^2 - 2nt - 32n - 2t^3 + 18t^2 - 4t + 16,$$

$$p(x) = x^3 + (-n + t + 2)x^2 + (t + 1 - n - t^2)x + nt^2 - 2t^3 - t^2,$$

$$r(x) = x^3 + (t - 3n + 4)x^2 + (2n^2 + nt - 6n - 4t^2 + 4)x - 2n^2t + 4nt^2 + 6nt - 2t^3 - 6t^2 - 4t.$$

Then $\mu(H)$, $\eta(H)$, $\lambda(H)$, $q(H)$ are the largest roots of $g(x) = 0$, $h(x) = 0$, $p(x) = 0$, $r(x) = 0$, respectively. In particular, we have
(i) *if $n \geq 5t + 1$, then $\mu(H) > n + \frac{3}{2}t - 1$.*
(ii) *if $n \geq 9t + 3$, then $2n + 5t - 2 < \eta(H) < 3n - 3t - 5$.*
(iii) *$\lambda(H) > n - t - 1$.*
(iv) *if $n \geq t + 2$, then $q(H) > 2n - 2t - 2$.*

Proof. By considering the partition $V(H) = V(K_t) \cup V(K_{n-2t}) \cup V(tK_1)$, the corresponding equitable quotient matrices of $\mathcal{D}(H)$, $\mathcal{Q}(H)$, $A(H)$, $Q(H)$ can be written as follows:

$$B_{\mathcal{D}} = \begin{pmatrix} t-1 & n-2t & t \\ t & n-2t-1 & 2t \\ t & 2(n-2t) & 2(t-1) \end{pmatrix},\ B_{\mathcal{Q}} = \begin{pmatrix} n+t-2 & n-2t & t \\ t & 2n-t-2 & 2t \\ t & 2(n-2t) & 2n+t-4 \end{pmatrix},$$

$$B_A = \begin{pmatrix} t-1 & n-2t & t \\ t & n-2t-1 & 0 \\ t & 0 & 0 \end{pmatrix},\ B_Q = \begin{pmatrix} n+t-2 & n-2t & t \\ t & 2n-3t-2 & 0 \\ t & 0 & t \end{pmatrix}.$$

By direct computation, we obtain the characteristic polynomials of the matrices $B_{\mathcal{D}}$, $B_{\mathcal{Q}}$, B_A, B_Q as $g(x)$, $h(x)$, $p(x)$, $r(x)$, respectively. By Lemma 2.2, $\mu(H)$, $\eta(H)$, $\lambda(H)$, $q(H)$ are the largest roots of $g(x) = 0$, $h(x) = 0$, $p(x) = 0$, $r(x) = 0$, respectively.

Let $\widetilde{g}(x) = -\frac{3t}{2}x^2 + (\frac{5}{2}t + \frac{9}{2}t^2)x + \frac{15}{4}t^2 + \frac{53}{8}t^3$. Then $\widetilde{g}'(x) = 0$ yields $x = \frac{5t+9t^2}{6t}$. Since $\frac{5t+9t^2}{6t} < 5t + 1 \leq n$, we have $g(n + \frac{3}{2}t - 1) = \widetilde{g}(n) \leq \widetilde{g}(5t + 1) = -\frac{67}{8}t^3 + \frac{23}{4}t^2 + t < 0$, which implies $\mu(H) > n + \frac{3}{2}t - 1$ since $\mu(H)$ is the largest root of $g(x) = 0$.

Let $\widetilde{h}(x) = -4tx^2 + (10t + 8t^2)x + 52t^2 + 138t^3$. Then $\widetilde{h}'(x) = 0$ yields $x = \frac{10t+8t^2}{8t}$. By $\frac{10t+8t^2}{8t} < 9t + 3 \leq n$, we have $h(2n + 5t - 2) = \widetilde{h}(n) \leq \widetilde{h}(9t + 3) = -114t^3 - 50t^2 - 6t < 0$, which implies $\eta(H) > 2n + 5t - 2$ since $\eta(H)$ is the largest root of $h(x) = 0$.

Note that $h'(x) = 3x^2 - 2(5n + t - 8)x + 8n^2 - nt - 26n + 8t^2 - 4t + 20$ and $\frac{5n+t-8}{3} < 3n - 3t - 5$. Then we can easily see that $h'(x)$ is increasing in the interval $[3n - 3t - 5, +\infty)$, and thus $h'(x) \geq h'(3n - 3t - 5) = 5n^2 - (18 + 31t)n + 15 + 48t + 41t^2 \geq 5(9t + 3)n - (18 + 31t)n + 15 + 48t + 41t^2 = (14t - 3)n + 15 + 48t + 41t^2 > 0$ when $x \geq 3n - 3t - 5$ and $n \geq 9t + 3$, which yields that $h(x)$ is increasing in the interval $[3n - 3t - 5, +\infty)$. Therefore, when $x \geq 3n-3t-5$, we have $h(x) \geq h(3n-3t-5) = (n - 2t - 1)\left(2n^2 - (21t + 9)n + 31t^2 + 36t + 9\right) \geq$

$(n-2t-1)\left(2(9t+3)^2-(21t+9)(9t+3)+31t^2+36t+9\right)=(n-2t-1)4t^2>0$, which implies $\eta(H)<3n-3t-5$ since $\eta(H)$ is the largest root of $h(x)=0$.

By direct computation, $p(n-t-1)=-t^3<0$ and $r(2n-2t-2)=2t^3+2t^2-2nt^2=2t^2(t+1-n)<0$ for $n\geq t+2$, then $\lambda(H)>n-t-1$ and $q(H)>2n-2t-2$. □

Lemma 2.6. *Let s be a positive integer, $H=K_s\vee sK_1$,*

$$g^*(x)=x^2-3(s-1)x+s^2-4s+2,\ h^*(x)=x^2-(8s-6)x+14s^2-22s+8,$$

$$p^*(x)=x^2-(s-1)x-s^2,\ r^*(x)=x^2-(4s-2)x+2s^2-2s.$$

Then $\mu(H)$, $\eta(H)$, $\lambda(H)$, $q(H)$ are the largest roots of $g^*(x)=0$, $h^*(x)=0$, $p^*(x)=0$, $r^*(x)=0$, respectively.

Proof. By considering the partition $V(H)=V(K_s)\cup V(sK_1)$, the corresponding equitable quotient matrices of $\mathcal{D}(H)$, $\mathcal{Q}(H)$, $A(H)$, $Q(H)$ can be written as follows:

$$B^*_{\mathcal{D}}=\begin{pmatrix}s-1 & s\\ s & 2(s-1)\end{pmatrix}, B^*_{\mathcal{Q}}=\begin{pmatrix}3s-2 & s\\ s & 5s-4\end{pmatrix}, B^*_{A}=\begin{pmatrix}s-1 & s\\ s & 0\end{pmatrix}, B^*_{Q}=\begin{pmatrix}3s-2 & s\\ s & s\end{pmatrix}.$$

By direct computation, we obtain the characteristic polynomials of the matrices $B^*_{\mathcal{D}}$, $B^*_{\mathcal{Q}}$, B^*_{A}, B^*_{Q} as $g^*(x)$, $h^*(x)$, $p^*(x)$, $r^*(x)$, respectively. By Lemma 2.2, $\mu(H)$, $\eta(H)$, $\lambda(H)$, $q(H)$ are the largest roots of $g^*(x)=0$, $h^*(x)=0$, $p^*(x)=0$, $r^*(x)=0$, respectively. □

3. Proofs of Main Results

In this section, we give the proofs of Theorems 1.4, 1.5, 1.6, 1.9 and 1.10. Firstly, it is easy to see that both $K_t\vee(K_{n-2t}\cup tK_1)$ and $K_t\vee tK_1$ contain a Hamilton path. Then we have the following result.

Proposition 3.1. *Let n, t be positive integers with $n\geq\max\{5,2t\}$,*

$$H=\begin{cases}K_t\vee(K_{n-2t}\cup tK_1), & \text{if } n\geq 2t+1;\\ K_t\vee tK_1, & \text{if } n=2t.\end{cases}$$

Then H has a spanning tree with leaf distance at least four.

3.1. Proof of Theorem 1.4

Suppose that G has no spanning trees with leaf distance at least four, then there exists a nonempty subset $S\subseteq V(G)$ such that $i(G-S)\geq|S|$ by Theorem 1.3.

We choose the connected graph R satisfying $V(R)=V(G)$, $\delta(R)\geq t$ and $i(R-S)\geq|S|$ so that its size is as large as possible. Then we have $|E(G)|\leq|E(R)|$. According to the choice of R, we see that the induced subgraph $R[S]$ and each connected component of $R-S$ are complete graphs, and $R=R[S]\vee(R-S)$.

Firstly, we claim that there is at most one non-trivial connected component in $R-S$. Otherwise, we can obtain a new graph R' by adding edges among all non-trivial connected components to derive a bigger non-trivial connected component. Obviously, R is a proper spanning subgraph of R', which is a contradiction to the choice of R. Let $i(R-S)=i$ and $|S|=s$ for short. Then $i\geq s$ by $i(R-S)\geq|S|$.

We complete the proof by considering the following two cases.

Case 1. $R-S$ has exactly one non-trivial connected component.

In this case, let R_1 be the unique non-trivial connected component of $R-S$ and $V(R_1)=n_1\geq 2$. Now, we show $i=s$.

If $i\geq s+1$, then we take a new graph R'' obtained from R by joining each vertex of R_1 with one vertex in $V(R-S)\backslash V(R_1)$. Then we have

$$|E(R'')|=|E(R)|+n_1>|E(R)|,\ \delta(R'')\geq\delta(R)\geq t,\ i(R''-S)=i-1\geq|S|=s,$$

which is a contradiction with the choice of R. Thus $i = s$ by $i \geq s$, $R = K_s \vee (K_{n-2s} \cup sK_1)$ and $\delta(R) = s$.

Clearly, we have $n = 2s + n_1 \geq 2s + 2 \geq 2t + 2$ since $s = \delta(R) \geq t \geq 1$, and $|E(R)| = s^2 + \binom{n-s}{2}$. Then

$$\begin{aligned}\binom{n-t}{2} + \frac{3}{2}t^2 - t + \frac{1}{2} - |E(R)| &= \binom{n-t}{2} + \frac{3}{2}t^2 - t + \frac{1}{2} - s^2 - \binom{n-s}{2} \\ &= \frac{3}{2}t^2 - t + \frac{1}{2} - s^2 + \frac{(s-t)(2n-s-t-1)}{2} \\ &= \frac{1}{2}t^2 - t + \frac{1}{2} + \frac{(s-t)(2n-3s-3t-1)}{2} \\ &\geq \frac{(t-1)^2}{2} + \frac{(s-t)(s-3t+3)}{2}.\end{aligned}$$

Let $f_1(x) = (x-t)(x-3t+3) + (t-1)^2 = (x - \frac{4t-3}{2})^2 + t - \frac{5}{4}$. Then

$$f_1(s) \geq \begin{cases} f_1(1) = 0, & \text{if } t = 1, \\ f_1(\frac{4t-3}{2}) > 0, & \text{if } t \geq 2. \end{cases}$$

Therefore, $|E(G)| \leq |E(R)| \leq \binom{n-t}{2} + \frac{3}{2}t^2 - t + \frac{1}{2} < \binom{n-t}{2} + \frac{3}{2}t^2 + \frac{1}{2}t$, a contradiction with (1a) and (1b).

Case 2. $R - S$ has no non-trivial connected component.

In this case, we show $i \leq s + 1$. If $i \geq s + 2$, then we take a new graph R''' obtained from R by adding an edge between two vertices in $V(R - S)$. Thus $i(R''' - S) = i - 2 \geq s = |S|$, $\delta(R''') \geq \delta(R) \geq t$, and $|E(R''')| = |E(R)| + 1$. This contradicts with the choice of R, which implies $i = s$ or $i = s + 1$.

Subcase 2.1. $i = s$.

In this subcase, $n = 2s$, $R = K_s \vee sK_1$ and $|E(R)| = s^2 + \binom{s}{2}$.

For $n = 2s \geq 6t + 2$, we deduce

$$\binom{n-t}{2} + \frac{3}{2}t^2 - t + \frac{1}{2} - |E(R)| = \frac{(s-t)(s-3t-1)}{2} + \frac{(t-1)^2}{2} \geq 0.$$

Therefore, $|E(G)| \leq |E(R)| \leq \binom{n-t}{2} + \frac{3}{2}t^2 - t + \frac{1}{2}$, which is a contradiction with (1a).

For $6 \leq n \leq 6t$ (i.e. $3 \leq s \leq 3t$),

$$\binom{n-t}{2} + \frac{3}{2}t^2 + \frac{t}{2} - |E(R)| = \frac{1}{2}(s-2t)(s-2t-1) \geq 0,$$

which implies $|E(G)| \leq |E(R)| \leq \binom{n-t}{2} + \frac{3}{2}t^2 + \frac{t}{2}$, a contradiction with (1b).

Subcase 2.2. $i = s + 1$.

In this subcase, $n = 2s + 1 \geq 2t + 1$, $R = K_s \vee (s+1)K_1$ and $|E(R)| = s(s+1) + \binom{s}{2}$.

For $n = 2s + 1 \geq 6t + 1$, we have

$$f_2(s) = \binom{n-t}{2} + \frac{3}{2}t^2 - t + \frac{1}{2} - |E(R)| = \frac{1}{2}\left[\left(s - \frac{4t-1}{2}\right)^2 - t + \frac{3}{4}\right] \geq f_2(3t) = \frac{t^2+1}{2} > 0.$$

For $n = 5$, we admit $s = 2$ and $1 \leq t \leq 2$ by $n = 2s + 1 \geq 2t + 1$. Then $f_2(2) = \frac{1}{2}(4t^2 - 11t + 7) \geq 0$. Hence $|E(G)| \leq |E(R)| \leq \binom{n-t}{2} + \frac{3}{2}t^2 - t + \frac{1}{2}$, which is a contradiction with (1a).

For $6 \leq n \leq 6t$, we obtain

$$\binom{n-t}{2} + \frac{3}{2}t^2 + \frac{t}{2} - |E(R)| = \frac{(s-2t+1)(s-2t)}{2} + t > 0.$$

Then $|E(G)| \leq |E(R)| < \binom{n-t}{2} + \frac{3}{2}t^2 + \frac{t}{2}$, which is a contradiction with (1b).

In view of Case 1 and Case 2, the proof of Theorem 1.4 is complete. ■

3.2. Proof of Theorem 1.5

Let $H = K_t \vee (K_{n-2t} \cup tK_1)$, G be a connected graph of order $n \geq 7t+2$ with $\delta(G) \geq t$ and $\mu(G) \leq \mu(H)$. We suppose to the contrary that G contains no spanning trees with leaf distance at least four. By applying Theorem 1.3, there exists some nonempty subset S of $V(G)$ satisfying $i(G-S) \geq |S|$. For convenience, we take $|S| = s$. Then $G^* = \begin{cases} K_s \vee (K_{n-2s} \cup sK_1), & \text{if } n \geq 2s+1 \\ K_s \vee sK_1, & \text{if } n = 2s \end{cases}$ is the graph with the maximum size such that $V(G^*) = V(G)$ and $i(G^*-S) \geq |S|$. Clearly, G^* has a spanning tree with leaf distance at least four by Proposition 3.1, then $G^* \ncong G$, and thus G is a proper spanning subgraph of G^* with $s = \delta(G^*) \geq \delta(G) \geq t$. Therefore, $\mu(G^*) < \mu(H)$ by Lemma 2.3 and $\mu(G) \leq \mu(H)$.

In what follows, we proceed to prove Theorem 1.5 by considering the following two cases.

Case 1. $s = t$.

In this case, $G^* = H$, then $\mu(G^*) = \mu(H)$, a contradiction with $\mu(G^*) < \mu(H)$.

Case 2. $s \geq t+1$.

Now we show $\mu(G^*) > \mu(H)$ by considering the following two subcases since $n \geq 2s$.

Subcase 2.1. $n \geq 2s+1$.

By Lemma 2.5, $\mu_1 = \mu(G^*)$, $\mu = \mu(H)$ are the largest roots of $g_1(x) = 0$, $g(x) = 0$, respectively, where $g_1(x) = x^3 - (n+s-4)x^2 - (2ns+3n-5s^2+s-5)x + ns^2 - 2ns - 2n + 5s^2 - 2s^3 + 2$, and $g(x)$ defined in Lemma 2.5.

Now we compare the $\mathcal{D}$-index of G^* and H. Using Lemma 2.5 and $n \geq 7t+2 > 5t+1$, we have $\mu = \mu(H) > n + \frac{3}{2}t - 1$. By direct computation, we derive

$$g_1(\mu) = g_1(\mu) - g(\mu) = (s-t)g_2(\mu), \tag{2}$$

where $g_2(\mu) = -\mu^2 - (2n - 5(s+t) + 1)\mu + (5+n)(s+t) - 2(s^2 + st + t^2 + n)$, then

$$g_2(\mu) < g_2(n + \frac{3}{2}t - 1) = -2s^2 + (\frac{11}{2}t + 6n)s + n - 3n^2 + \frac{3}{2}t + \frac{13}{4}t^2 \tag{3}$$

by $n \geq \max\{2s+1, 7t+2\}$ and $-\dfrac{2n-5(s+t)+1}{2} < n + \dfrac{3}{2}t - 1 < \mu$.

Let $g_3(x) = -2x^2 + (\frac{11}{2}t + 6n)x + n - 3n^2 + \frac{3}{2}t + \frac{13}{4}t^2$. Then $g_3(x)$ is increasing in the interval $(-\infty, \frac{n-1}{2}]$ by $\frac{\frac{11}{2}t+6n}{4} > \frac{n-1}{2}$. Since $n \geq 2s+1$, we have

$$g_3(s) \leq g_3(\frac{n-1}{2}) = -\frac{n^2}{2} + (\frac{11}{4}t - 1)n - \frac{1}{2} - \frac{5}{4}t + \frac{13}{4}t^2. \tag{4}$$

Let $g_4(x) = -\frac{x^2}{2} + (\frac{11}{4}t - 1)x - \frac{1}{2} - \frac{5}{4}t + \frac{13}{4}t^2$. Then $g_4(x)$ is decreasing in the interval $[7t+2, +\infty)$ by $\frac{11}{4}t - 1 \leq 7t+2 \leq n$ and thus

$$g_4(n) \leq g_4(7t+2) = -2t^2 - \frac{67}{4}t - \frac{9}{2} < 0. \tag{5}$$

By (2), (3), (4), (5) and $s \geq t+1$, we have

$$\begin{aligned} g_1(\mu) =& (s-t)g_2(\mu) < (s-t)g_2(n + \frac{3}{2}t - 1) \\ =& (s-t)g_3(s) \leq (s-t)g_3(\frac{n-1}{2}) \\ =& (s-t)g_4(n) \leq (s-t)g_4(7t+2) \\ <& 0, \end{aligned}$$

then $\mu(G^*) = \mu_1 > \mu = \mu(H)$, a contradiction.

Subcase 2.2. $n = 2s$.

In this subcase, $G^* = K_s \vee sK_1$. By Lemma 2.6, $\mu^* = \mu(G^*)$ is the largest root of $g^*(x) = 0$. Recall that $\mu = \mu(H)$ is the largest root of $g(x) = 0$. Then we obtain

$$\begin{aligned}\mu g^*(\mu) =& \mu g^*(\mu) - g(\mu)\\ =& -(s-t+1)\mu^2 + (s^2+2s+4st-5t^2+t-3)\mu - 2st^2 + 4st + 4s - 5t^2 + 2t^3 - 2.\end{aligned} \tag{6}$$

Let $g_5(x) = -(s-t+1)x^2 + (s^2+2s+4st-5t^2+t-3)x - 2st^2 + 4st + 4s - 5t^2 + 2t^3 - 2$. Clearly, $\frac{s^2+2s+4st+t-5t^2-3}{2(s-t+1)} < 2s + \frac{3}{2}t - 1$ by $s = \frac{n}{2} \geq 3t+1$, and then $g_5(x)$ is decreasing in the interval $[2s + \frac{3}{2}t - 1, +\infty)$. Using Lemma 2.5 and $n = 2s \geq 7t + 2 \geq 5t + 1$, we have $\mu = \mu(H) > n + \frac{3}{2}t - 1$. Then we deduce

$$g_5(\mu) < g_5(n + \frac{3}{2}t - 1) = -2s^3 + \frac{6+15t}{2}s^2 - \frac{4+8t+9t^2}{4}s - \frac{6t+15t^2+13t^3}{4}. \tag{7}$$

Let $g_6(x) = -2x^3 + \frac{6+15t}{2}x^2 - \frac{4+8t+9t^2}{4}x - \frac{6t+15t^2+13t^3}{4}$. Then we possess $g_6'(x) = -6x^2 + (6+15t)x - (1 + 2t + \frac{9}{4}t^2)$. Since $s \geq \frac{7t+2}{2} > \frac{6+15t}{12}$, $g_6'(s) \leq g_6'(\frac{7t+2}{2}) = -\frac{93}{4}t^2 - 8t - 1 < 0$, which implies

$$g_6(s) \leq g_6(\frac{7t+2}{2}) = -5t^3 + \frac{11}{4}t^2 + \frac{t}{2} < 0. \tag{8}$$

By (6), (7) and (8), we admit $\mu g^*(\mu) = g_5(\mu) < g_5(n + \frac{3}{2}t - 1) = g_6(s) \leq g_6(\frac{7t+2}{2}) < 0$, which yields $g^*(\mu) < 0$, and thus $\mu(G^*) = \mu^* > \mu = \mu(H)$, a contradiction.

Combining Case 1, Case 2, G has a spanning tree with leaf distance at least four. ■

3.3. Proof of Theorem 1.6

Let $H = K_t \vee (K_{n-2t} \cup tK_1)$, G be a connected graph of order $n \geq 9t + 3$ with $\delta(G) \geq t$ and $\eta(G) \leq \eta(H)$. We assume the contrary that G contains no spanning trees with leaf distance at least four. By applying Theorem 1.3, there exists some nonempty subset S of $V(G)$ satisfying $i(G - S) \geq |S|$. For convenience, we take $|S| = s$. Then $G^* = \begin{cases} K_s \vee (K_{n-2s} \cup sK_1), & \text{if } n \geq 2s+1 \\ K_s \vee sK_1, & \text{if } n = 2s \end{cases}$ is the graph with the maximum size such that $V(G^*) = V(G)$ and $i(G^* - S) \geq |S|$. Clearly, G^* has a spanning tree with leaf distance at least four by Proposition 3.1, then $G^* \not\cong G$, and thus G is a proper spanning subgraph of G^* with $s = \delta(G^*) \geq \delta(G) \geq t$. Therefore, $\eta(G^*) < \eta(H)$ by Lemma 2.3 and $\eta(G) \leq \eta(H)$.

In what follows, we now consider the following two cases to prove Theorem 1.6.

Case 1. $s = t$.

In this case, $G^* = H$, then $\eta(G^*) = \eta(H)$, a contradiction with $\eta(G^*) < \eta(H)$.

Case 2. $s \geq t + 1$.

Now we verify $\eta(G^*) > \eta(H)$. Combining $n \geq 2s$, it suffices to deal with the following two subcases.

Subcase 2.1. $n \geq 2s + 1$.

By Lemma 2.5, $\eta = \eta(H)$, $\eta_1 = \eta(G^*)$ are the largest roots of $h(x) = 0$, $h_1(x) = 0$, respectively, where $h_1(x) = x^3 - (5n+s-8)x^2 + (8n^2 - ns - 26n + 8s^2 - 4s + 20)x - 4n^3 + 2n^2s + 20n^2 - 8ns^2 - 2ns - 32n - 2s^3 + 18s^2 - 4s + 16$, and $h(x)$ defined in Lemma 2.5.

Next, we compare the Q-index of G^* and H. Together with Lemma 2.5 and $n \geq 9t + 3$, we obtain $\eta = \eta(H) > 2n + 5t - 2$. By direct calculation, we deduce

$$h_1(\eta) = h_1(\eta) - h(\eta) = (s - t)h_2(\eta), \tag{9}$$

where $h_2(x) = -x^2 + (-n + 8(s+t) - 4)x + 2n^2 - 8n(s+t) - 2n - 2(s^2 + st + t^2) + 18(s+t) - 4$.

Clearly, $h_2(x)$ is decreasing in the interval $[2n + 5t - 2, +\infty)$ by $n \geq 2s + 1$ and $\frac{-n+8(s+t)-4}{2} < 2n + 5t - 2 < \eta$, which implies

$$h_2(\eta) < h_2(2n + 5t - 2) = -2s^2 + (2 + 38t + 8n)s - 4n^2 + 2t + 13t^2 - 17nt. \tag{10}$$

Let $h_3(x) = -2x^2 + (2 + 38t + 8n)x - 4n^2 + 2t + 13t^2 - 17nt$. Then $h_3(x)$ is increasing in the interval $(-\infty, \frac{n-1}{2}]$ by $\frac{2+38t+8n}{4} > \frac{n-1}{2}$. By $n \geq 2s + 1$, we have

$$h_3(s) \leq h_3(\frac{n-1}{2}) = -\frac{n^2}{2} + (2t-2)n - \frac{3}{2} - 17t + 13t^2. \tag{11}$$

Let $h_4(x) = -\frac{x^2}{2} + (2t-2)x - \frac{3}{2} - 17t + 13t^2$. Therefore, $h_4(x)$ is decreasing in the interval $[9t + 3, +\infty)$ by $2t - 2 < 9t + 3 \leq n$ and so

$$h_4(n) \leq h_4(9t+3) = -\frac{19}{2}t^2 - 56t - 12 < 0. \tag{12}$$

By (9), (10), (11), (12) and $s \geq t + 1$, we obtain

$$\begin{aligned} h_1(\eta) =&(s-t)h_2(\eta) < (s-t)h_2(2n+5t-2) \\ =&(s-t)h_3(s) \leq (s-t)h_3(\frac{n-1}{2}) \\ =&(s-t)h_4(n) \leq (s-t)h_4(9t+3) \\ <&0, \end{aligned}$$

then $\eta(G^*) = \eta_1 > \eta = \eta(H)$, a contradiction.

Subcase 2.2. $n = 2s$.

In this subcase, $G^* = K_s \vee sK_1$. By Lemma 2.6, $\eta^* = \eta(G^*)$ is the largest root of $h^*(x) = 0$. Recall that $\eta = \eta(H)$ is the largest root of $h(x) = 0$. Then we have

$$\eta h^*(\eta) = \eta h^*(\eta) - h(\eta) = h_5(\eta), \tag{13}$$

where $h_5(x) = (2s+t-2)x^2 + (-18s^2+30s+2st-8t^2+4t-12)x + 32s^3 - 8s^2t - 80s^2 + 16st^2 + 4st + 64s + 2t^3 - 18t^2 + 4t - 16$. By $n \geq 9t+3$ and Lemma 2.5, we have $4s + 5t - 2 < \eta < 6s - 3t - 5$. Since $h_5(x)$ is a quadratic convex function, then

$$h_5(\eta) \leq \max\{h_5(4s+5t-2), h_5(6s-3t-5)\}. \tag{14}$$

By direct calculation, we obtain

$$h_5(4s+5t-2) = -8s^3 + (12+6t)s^2 + (-4+30t+84t^2)s - 20t - 52t^2 - 13t^3.$$

Let $h_6(x) = -8x^3 + (12+6t)x^2 + (-4+30t+84t^2)x - 20t - 52t^2 - 13t^3$. Then $h_6'(x) = -24x^2 + 2(12+6t)x - 4 + 30t + 84t^2$ and $h_6'(x)$ is decreasing in the interval $[\frac{9t+3}{2}, +\infty)$ by $\frac{12+6t}{24} = \frac{1}{2} + \frac{t}{4} < \frac{9t+3}{2} \leq s$. Hence when $x \geq \frac{9t+3}{2}$, we have $h_6'(x) \leq h_6'(\frac{9t+3}{2}) = -348t^2 - 168t - 22 < 0$, which implies $h_6(x)$ is decreasing in the interval $[\frac{9t+3}{2}, +\infty)$. By $s \geq \frac{9t+3}{2}$,

$$h_6(s) \leq h_6(\frac{9t+3}{2}) = -\frac{485}{2}t^3 - 196t^2 - \frac{121}{2}t - 6 < 0,$$

then $h_5(4s+5t-2) = h_6(s) < 0$.

By direct computation, we have

$$h_5(6s-3t-5) = -4s^3 + (22t-2)s^2 + (12-56t^2)s - 6 - 15t + 22t^2 + 35t^3.$$

Let $h_7(x) = -4x^3 + (22t-2)x^2 + (12-56t^2)x - 6 - 15t + 22t^2 + 35t^3$. Then $h_7'(x) = -12x^2 + 2(22t-2)x + 12 - 56t^2$ and $\frac{22t-2}{12} < \frac{9t+3}{2} \leq s$, thus $h_7'(x)$ is decreasing in the interval $[\frac{9t+3}{2}, +\infty)$. By $x \geq \frac{9t+3}{2}$, we have $h_7'(x) \leq h_7'(\frac{9t+3}{2}) = -101t^2 - 114t - 21 < 0$. Then $h_7(x)$ is decreasing in the interval $[\frac{9t+3}{2}, +\infty)$. By $s \geq \frac{9t+3}{2}$,

$$h_7(s) \leq h_7(\frac{9t+3}{2}) = -136t^3 - 170t^2 - 60t - 6 < 0,$$

which yields $h_5(6s-3t-5) = h_7(s) < 0$.

It follows from (14) that $h_5(\eta) < 0$ for $n \geq 9t + 3$, which implies $h^*(\eta) < 0$ by (13), that is, $\eta(G^*) = \eta^* > \eta = \eta(H)$, a contradiction.

Combining Case 1, Case 2, G has a spanning tree with leaf distance at least four. ∎

3.4. Proof of Theorem 1.9

Let $H = K_t \vee (K_{n-2t} \cup tK_1)$, G be a connected graph of order $n \geq 5t + 2$ with $\delta(G) \geq t$ and $\lambda(G) \geq \lambda(H)$. Suppose to the contrary that G has no spanning trees with leaf distance at least four. In light of Theorem 1.3, there is a nonempty subset $S \subseteq V(G)$ such that $i(G - S) \geq |S|$. For convenience, we take $|S| = s$. Then $G^* = \begin{cases} K_s \vee (K_{n-2s} \cup sK_1), & \text{if } n \geq 2s + 1 \\ K_s \vee sK_1, & \text{if } n = 2s \end{cases}$ is the graph with the maximum size satisfying $V(G^*) = V(G)$ and $i(G^* - S) \geq |S|$. Clearly, G^* contains a spanning tree with leaf distance at least four by Proposition 3.1, then $G^* \ncong G$, and thus G is a proper spanning subgraph of G^* with $s = \delta(G^*) \geq \delta(G) \geq t$. Hence $\lambda(H) < \lambda(G^*)$ by Lemma 2.4 and $\lambda(G) \geq \lambda(H)$.

Next, we proceed to prove Theorem 1.9 by considering the following two cases.

Case 1. $s = t$.

In this case, $G^* = H$, then $\lambda(G^*) = \lambda(H)$, a contradiction with $\lambda(H) < \lambda(G^*)$.

Case 2. $s \geq t + 1$.

Now we show $\lambda(G^*) < \lambda(H)$ by considering the following two subcases.

Subcase 2.1. $n \geq 2s + 1$.

By Lemma 2.5, $\lambda_1 = \lambda(G^*)$, $\lambda = \lambda(H)$ are the largest roots of $p_1(x) = 0$, $p(x) = 0$, respectively, where $p_1(x) = x^3 + (-n + s + 2)x^2 + (s + 1 - n - s^2)x + ns^2 - 2s^3 - s^2$, $p(x)$ defined in Lemma 2.5, and $\lambda = \lambda(H) > n - t - 1$. By direct calculation, we derive

$$p_1(\lambda) = p_1(\lambda) - p(\lambda) = (s - t)p_2(\lambda), \tag{15}$$

where $p_2(x) = x^2 + (1 - s - t)x + (n - 1)(s + t) - 2(s^2 + st + t^2)$, then

$$p_2(\lambda) > p_2(n - t - 1) = -2s^2 - st - n + n^2 + t - 2nt \tag{16}$$

by $n \geq 2s + 1$ and $\frac{s+t-1}{2} < n - t - 1 < \lambda$.

Let $p_3(x) = -2x^2 - tx - n + n^2 + t - 2nt$. Then $p_3(x)$ is decreasing in the interval $[t + 1, +\infty)$ immediately. By $t + 1 \leq s \leq \frac{n-1}{2}$, we have

$$p_3(s) \geq p_3(\frac{n-1}{2}) = \frac{n^2}{2} - \frac{5}{2}tn - \frac{1}{2} + \frac{3}{2}t. \tag{17}$$

Let $p_4(x) = \frac{x^2}{2} - \frac{5}{2}tx - \frac{1}{2} + \frac{3}{2}t$. Then $p_4(x)$ is increasing in the interval $[5t, +\infty)$. By $5t + 2 \leq n$, we have

$$p_4(n) \geq p_4(5t + 2) = \frac{13t}{2} + \frac{3}{2} > 0. \tag{18}$$

By (15), (16), (17), (18) and $s \geq t + 1$, we have

$$\begin{aligned} p_1(\lambda) =& (s - t)p_2(\lambda) > (s - t)p_2(n - t - 1) = (s - t)p_3(s) \\ \geq& (s - t)p_3(\frac{n-1}{2}) = (s - t)p_4(n) \geq (s - t)p_4(5t + 2) > 0. \end{aligned}$$

Now, we proceed to prove $p_1'(x) > 0$ in the interval $[\lambda, +\infty)$. By simple calculation, we know $p_1'(x) = 3x^2 + 2(-n + s + 2)x + s + 1 - n - s^2$. Since $\frac{n-s-2}{3} < n - t - 1 < \lambda$, $p_1'(x)$ is increasing in the interval $[n - t - 1, +\infty)$. When $x \geq \lambda$, $p_1'(x) \geq p_1'(\lambda) > p_1'(n - t - 1) = -s^2 + (2n - 2t - 1)s - n + n^2 + 2t + 3t^2 - 4nt$.

Let $p_5(x) = -x^2 + (2n - 2t - 1)x - n + n^2 + 2t + 3t^2 - 4nt$. Then $p_5(x)$ is increasing in the interval $(-\infty, \frac{n-1}{2}]$ by $n \geq 5t + 2$ and $\frac{2n-2t-1}{2} > \frac{n-1}{2} \geq s \geq t + 1$ and so we admit $p_5(s) \geq p_5(t + 1) = n^2 + (1 - 2t)n - 2 - 3t \geq (5t+2)n + (1-2t)n - 2 - 3t = (3t+3)n - 2 - 3t > 0$. Therefore, when $x \geq \lambda$, we have $p_1'(x) \geq p_1'(\lambda) > p_1'(n-t-1) = p_5(s) \geq p_5(t + 1) > 0$, and $p_1(x)$ is increasing in the interval $[\lambda, +\infty)$, which implies $p_1(x) \geq p_1(\lambda) > 0$ when $x \geq \lambda$. Then $\lambda(G^*) = \lambda_1 < \lambda = \lambda(H)$, a contradiction.

Subcase 2.2. $n = 2s$.

In this subcase, $G^* = K_s \vee sK_1$. In view of Lemma 2.6, $\lambda^* = \lambda(G^*)$ is the largest root of $p^*(x) = 0$. Then we have

$$\begin{aligned}\lambda p^*(\lambda) =&\lambda p^*(\lambda) - p(\lambda)\\ =&(s-t-1)\lambda^2 + (2s + t^2 - s^2 - t - 1)\lambda - 2st^2 + 2t^3 + t^2.\end{aligned} \tag{19}$$

Let $p_6(x) = (s-t-1)x^2+(2s+t^2-s^2-t-1)x-2st^2+2t^3+t^2$. Clearly, $\frac{2s+t^2-s^2-t-1}{-2(s-t-1)} < 2s-t-1$ by $n = 2s \geq 5t+2$, and then $p_6(x)$ is increasing in the interval $[2s - t - 1, +\infty)$ by $s - t - 1 > 0$. Using Lemma 2.5, we have $\lambda = \lambda(H) > 2s - t - 1$. Then we have

$$p_6(\lambda) > p_6(2s - t - 1) = 2s^3 - (3 + 7t)s^2 + (1 + 6t + 5t^2)s - t - 2t^2. \tag{20}$$

Let $p_7(x) = 2x^3 - (3 + 7t)x^2 + (1 + 6t + 5t^2)x - t - 2t^2$. Then $p_7'(x) = 6x^2 - (6 + 14t)x + 1 + 6t + 5t^2$. When $x \geq \frac{5t+2}{2}$, $p_7'(x) \geq 6 \cdot \frac{5t+2}{2}x - (6 + 14t)x + 1 + 6t + 5t^2 = tx + 1 + 6t + 5t^2 > 0$. Then $p_7(x)$ is increasing in the interval $[\frac{5t+2}{2}, +\infty)$. Thus we obtain

$$p_7(s) \geq p_7(\frac{5t + 2}{2}) = \frac{7}{4}t^2 + \frac{t}{2} > 0. \tag{21}$$

It can be inferred from (19), (20) and (21) that

$$\lambda p^*(\lambda) = p_6(\lambda) > p_6(2s - t - 1) = p_7(s) \geq p_7(\frac{5t + 2}{2}) > 0.$$

So $p^*(\lambda) > 0$. Since $\frac{s-1}{2} < 2s - t - 1 < \lambda$, $p^*(x)$ is increasing in the interval $[\lambda, +\infty)$, which implies $p^*(x) \geq p^*(\lambda) > 0$ when $x \geq \lambda$. Hence $\lambda(G^*) = \lambda^* < \lambda = \lambda(H)$, a contradiction.

Combining Case 1, Case 2, G has a spanning tree with leaf distance at least four. ■

3.5. *Proof of Theorem 1.10*

Let $H = K_t \vee (K_{n-2t} \cup tK_1)$, G be a connected graph of order $n \geq 7t + 1$ with $\delta(G) \geq t$ and $\eta(G) \leq \eta(H)$. We assume the contrary that G contains no spanning trees with leaf distance at least four. On account of Theorem 1.3, there exists some nonempty subset $S \subseteq V(G)$ satisfying $i(G - S) \geq |S|$. For convenience, we take $|S| = s$. Then $G^* = \begin{cases} K_s \vee (K_{n-2s} \cup sK_1), & \text{if } n \geq 2s + 1 \\ K_s \vee sK_1, & \text{if } n = 2s \end{cases}$ is the graph with the maximum size such that $V(G^*) = V(G)$ and $i(G^* - S) \geq |S|$. Clearly, G^* has a spanning tree with leaf distance at least four by Proposition 3.1, then $G^* \ncong G$, and thus G is a proper spanning subgraph of G^* with $s = \delta(G^*) \geq \delta(G) \geq t$. Therefore, $q(H) < q(G^*)$ by Lemma 2.4 and $q(G) \geq q(H)$.

Next, we proceed to prove Theorem 1.10 by considering two cases.

Case 1. $s = t$.

In this case, $G^* = H$, then $q(G^*) = q(H)$, a contradiction with $q(H) < q(G^*)$.

Case 2. $s \geq t + 1$.

Now we verify $q(G^*) < q(H)$. Combining $n \geq 2s$, it suffices to deal with $n \geq 2s + 1$ and $n = 2s$.

Subcase 2.1. $n \geq 2s + 1$.

By Lemma 2.5, $q_1 = q(G^*)$, $q = q(H)$ are the largest roots of $r_1(x)$, $r(x)$, respectively, where $r_1(x) = x^3 + (s - 3n + 4)x^2 + (2n^2 + ns - 6n - 4s^2 + 4)x - 2n^2s + 4ns^2 + 6ns - 2s^3 - 6s^2 - 4s$, $r(x)$ defined in Lemma 2.5, and $q = q(H) > 2n - 2t - 2$. By direct computation, we derive

$$r_1(q) = r_1(q) - r(q) = (s - t)r_2(q), \tag{22}$$

where $r_2(x) = x^2 + (n - 4(s + t))x - 2n^2 + 4n(s + t) + 6n - 2(s^2 + st + t^2) - 6(s + t) - 4$. Then $r_2(x)$ is increasing in the interval $[2n - 2t - 2, +\infty)$ by $n \geq \max\{7t + 1, 2s + 1\}$ and $\frac{4(s+t)-n}{2} < 2n - 2t - 2 < q$, which implies

$$r_2(q) > r_2(2n - 2t - 2) = -2s^2 + (2 - 4n + 6t)s - 4n + 4n^2 + 10t + 10t^2 - 14nt. \tag{23}$$

Let $r_3(x) = -2x^2 + (2 - 4n + 6t)x - 4n + 4n^2 + 10t + 10t^2 - 14nt$. It is easy to check that $r_3(x)$ is decreasing on the interval $[s, \frac{n-1}{2}]$ by $\frac{2-4n+6t}{4} < 0 < s \le \frac{n-1}{2}$. Thus

$$r_3(s) \ge r_3(\frac{n-1}{2}) = \frac{3n^2}{2} - 11nt - \frac{3}{2} + 7t + 10t^2. \tag{24}$$

Let $r_4(x) = \frac{3}{2}x^2 - 11tx - \frac{3}{2} + 7t + 10t^2$. Since $\frac{11t}{3} < 7t + 1 \le n$, then $r_4(x)$ is increasing in the interval $[7t + 1, +\infty)$ and

$$r_4(n) \ge r_4(7t + 1) = \frac{13}{2}t^2 + 17t > 0. \tag{25}$$

Together with (22), (23), (24), (25) and $s \ge t + 1$, we obtain

$$\begin{aligned} r_1(q) =&(s - t)r_2(q) > (s - t)r_2(2n - 2t - 2) = (s - t)r_3(s) \\ \ge&(s - t)r_3(\frac{n-1}{2}) = (s - t)r_4(n) \ge (s - t)r_4(7t + 1) > 0. \end{aligned}$$

Now, we proceed to prove $r_1'(x) > 0$ in the interval $[q, +\infty)$. By simple calculation, we have $r_1'(x) = 3x^2 + 2(s - 3n + 4)x + 2n^2 + ns - 6n - 4s^2 + 4$. Since $\frac{3n-s-4}{3} < 2n - 2t - 2 < q$, $r_1'(x)$ is increasing in the interval $[2n-2t-2, +\infty)$. Then $r_1'(x) \ge r_1(q) > r_1'(2n-2t-2) = -4s^2+(5n-4t-4)s-2n+2n^2+8t+12t^2-12nt = r_5(s)$ when $x \ge q$ and $r_5(s) \ge r_5(t+1) = 2n^2+(3-7t)n+4t^2-8t-8 \ge 2(7t+1)^2+(3-7t)(7t+1)+4t^2-8t-8 = 53t^2+34t-3 > 0$ by $\frac{5n-4t-4}{8} > \frac{n-1}{2} \ge s \ge t + 1$ and $\frac{7t-3}{4} < 7t + 1 < n$. Hence when $x \ge q$, $r_1'(x) \ge r_1'(q) > r_1'(2n - 2t - 2) = r_5(s) \ge r_5(t + 1) > 0$, which implies $r_1(x)$ is increasing in the interval $[q, +\infty)$. So $r_1(x) \ge r_1(q) > 0$ when $x \ge q$. Then $q(G^*) = q_1 < q = q(H)$, a contradiction.

Subcase 2.2. $n = 2s$.

In this subcase, $G^* = K_s \vee sK_1$. By Lemma 2.6, $q^* = q(G^*)$ is the largest root of $r^*(x) = 0$, where $r^*(x)$ defined in Lemma 2.6. Then we obtain

$$qr^*(q) = qr^*(q) - r(q) = r_6(q), \tag{26}$$

where $r_6(x) = (2s - t - 2)x^2 + (-6s^2 + 10s - 2st + 4t^2 - 4)x + 8s^2t - 8st^2 - 12st + 2t^3 + 6t^2 + 4t$. Clearly, $\frac{-6s^2+10s-2st+4t^2-4}{-2(2s-t-2)} < 4s - 2t - 2$ by $s \ge \frac{7t+1}{2}$ and then $r_6(x)$ is increasing in the interval $[4s - 2t - 2, +\infty)$. By Lemma 2.5, Thus we have

$$r_6(q) > r_6(4s - 2t - 2) = 2\,(2s - t - 1)\,r_7(s), \tag{27}$$

where $r_7(x) = 2s^2 - 8st + 5t^2 - 2s + 4t$. Then $r_7(x)$ is increasing in the interval $[\frac{7t+1}{2}, +\infty)$ by $2t < \frac{7t+1}{2} \le s$ and thus

$$r_7(s) \ge r_7(\frac{7t+1}{2}) = \frac{3t^2}{2} - \frac{1}{2} > 0. \tag{28}$$

It can be inferred from (26), (27) and (28) that

$$qr^*(q) = r_6(q) > r_6(4s - 2t - 2) = 2\,(2s - t - 1)\,r_7(s) \ge 2\,(2s - t - 1)\,r_7(\frac{7t+1}{2}) > 0.$$

Hence $r^*(q) > 0$. Since $\frac{4s-2}{2} < 4s - 2t - 2 < q$, $r^*(x)$ is increasing in the interval $[q, +\infty)$. When $x \ge q$, $r^*(x) \ge r^*(q) > 0$. Thus, $q(G^*) = q^* < q = q(H)$, a contradiction.

Combining Case 1, Case 2, G has a spanning tree with leaf distance at least four. ■

4. Two Questions for Further Research

For any $\alpha \in [0,1]$, Nikiforov [13] defined the A_α-matrix of G as $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$. Obviously, $A_\alpha(G) = A(G)$ if $\alpha = 0$, and $A_\alpha(G) = \frac{1}{2}Q(G)$ if $\alpha = \frac{1}{2}$. The largest eigenvalue of $A_\alpha(G)$, denoted by $\rho_\alpha(G)$, is called the A_α-spectral radius of G.

Problem 4.1. *For a connected graph G of order n with the minimum degree $\delta(G)$ and $\alpha \in [0,1]$, find an A_α-spectral radius condition of G to determine whether G has a spanning tree with leaf distance at least four.*

Theorems 1.7, 1.8, 1.9 and 1.10 study Problem 4.1 when $\alpha \in \{0, \frac{1}{2}\}$.

In [5], Erbes, Molla, Mousley, and Santana proved the Conjecture 1.2 is true when $\alpha(G) \leq 5$ or $d \geq \frac{n}{3}$, where $\alpha(G)$ is the independence number of G. Based on these results, it is natural that the following question can be proposed for further research.

Problem 4.2. *For a connected graph G of order n and $5 \leq d \leq n-1$, find some spectral radius conditions of G to determine whether G has a spanning tree with leaf distance at least d.*

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