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On P-equi-statistical relative convergence in sequences of fuzzy-valued functions with applications to Korovkin-type approximation

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Abstract. In this study, we introduce and investigate new forms of convergence, namely, P-statistical relative pointwise, uniform, and P-equi-statistical relative convergence, for sequences of functions whose values lie in the space of fuzzy numbers. These notions, motivated by a synthesis of statistical and relative convergence frameworks, are explored both in terms of their structural characteristics and mutual interrelationships. Furthermore, we examine the behavior of their corresponding r-level sets to provide deeper insight into their convergence dynamics. As a principal application, we apply approximation theorems of Korovkin-type for sequences of functions with fuzzy values under the newly proposed modes of convergence, and we compute the rate of convergence.

1. Introduction and Preliminaries

Due to the generation of intriguing and noteworthy results, the topic has evolved into a dynamic research area, actively discussed across various fields. Researchers have delved into diverse domains, including matrix summation, series theory, Fourier analysis, and Banach spaces. This has led to the formulation of novel generalizations of statistical convergence, comprehensive studies on its properties, and the exploration of its applications. Yet, in a broad context, exact calculations or measurements of either limits or statistical limits are unattainable. In order to capture and represent this inherent imprecision using mathematical frameworks, several methodologies have emerged in mathematics, including fuzzy set theory and fuzzy logic. This is why in this article we present a new version of statistical convergence with respect to power series methods on fuzzy sequences and we give Korovkin-type approximation theorems as applications.

Classic Korovkin theory is mainly based on the problem of approximation of a function by a sequence of positive linear operators [21]. Recent advancements in approximation theory have extended important findings from spaces of real functions to spaces of fuzzy functions. The incorporation of fuzziness into classical approximation theory was initiated by Anastassiou [3], who formulated a fuzzy analogue of classical Korovkin theory (see also [2, 4, 17]). Fast [16] introduced the concept of statistical convergence for

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sequences of real numbers. Building on this framework, various authors have developed statistical fuzzy approximation theorems, enriching the intersection of summability and fuzzy analysis. Notably, in 2008, Anastassiou and Duman [5] demonstrated a Korovkin-type approximation theorem applicable to fuzzy positive linear operators. Their approach involved leveraging the concept of A-statistical convergence, where A represents a non-negative regular summability matrix. Subsequently, Demirci and Orhan [14] expanded on this work, introducing the notion of statistical relative uniform convergence through the utilization of relative uniform convergence principles (see [12, 13, 23]) in 2016 and building upon this convergence method, Orhan et al. [26] conducted a study in 2017, exploring the fuzzy analogue of the Korovkin theorem. More recently, Baxhaku et al. [10] obtained a power series summability-based Korovkin type approximation theorem for any sequence of fuzzy positive linear operators and Yurdakadim and Taş [31] obtained it for P-statistical convergence, i.e., statistical convergence with respect to power series method. Since a sequence can be neither convergent nor statistically convergent and still be P-statistically convergent, it is effective to use this type of convergence.

In addition to these, further contributions have been made regarding Korovkin-type theorems and summability methods in both classical and weighted approximation theory. For example, results concerning statistical and equi-statistical convergence, and their generalizations, have been studied in [20, 25, 30], while the application of Korovkin-type approximation theorems in weighted spaces, as well as operator theory, have been examined in [1, 6, 8, 9]. These works provide a deeper insight into approximation processes in various functional settings and motivate the study of new convergence types.

Inspired by these studies, based on the notions of P-statistical and relative convergence, we introduce and study the notions of P-statistical relative pointwise, uniform and P-equi-statistical relative convergence of sequences of fuzzy-number-valued functions. Besides, we show some of their properties and relations. Also, their r-level cuts are discussed. Conversely, we utilize the convergence methods introduced in this paper to apply Korovkin-type approximation theorems specifically designed for sequences of functions with fuzzy values.

Now, we show some notions that are very useful for the development of this article.

A fuzzy number is formally defined as a function $\widetilde{u}:\mathbb{R}\to[0,1]$ satisfying the properties of normality, convexity, and upper semi-continuity, with the additional requirement that the closure of its support set, $supp(\widetilde{u}):=\{z\in\mathbb{R}:\ \widetilde{u}(z)>0\}$ is compact. The collection of all such fuzzy numbers is denoted by $\mathbb{R}_{\mathcal{F}}$. For any $\widetilde{u}\in\mathbb{R}_{\mathcal{F}}$ and level $r\in(0,1]$, the r-level set of $\widetilde{u}$ is defined as

$$[\widetilde{u}]^r := \{z\in\mathbb{R}:\ \widetilde{u}(z)\geq r\} \text{ and } [\widetilde{u}]^0 := \overline{\{z\in\mathbb{R}:\ \widetilde{u}(z)>0\}}.$$

It is a well-established result that for each $r\in[0,1]$, the level set $[\widetilde{u}]^r$ forms a non-empty, closed, and bounded interval in $\mathbb{R}$ ([18]). Given fuzzy numbers $\widetilde{u},\widetilde{v}\in\mathbb{R}_{\mathcal{F}}$ and a scalar $\lambda\in\mathbb{R}$, one can define their addition and scalar multiplication via their level sets as follows:

$$[\widetilde{u}\oplus\widetilde{v}]^r=[\widetilde{u}]^r+[\widetilde{v}]^r \text{ and } [\lambda\odot\widetilde{u}]^r=\lambda\,[\widetilde{u}]^r,\ \ (0\leq r\leq 1)$$

where + and scalar multiplication are taken in the sense of classical interval arithmetic in $\mathbb{R}$,as detailed in [19].

Each level set $[\widetilde{u}]^r$ can be represented in the form $\left[u_-^{(r)},u_+^{(r)}\right]$, where $u_-^{(r)}\leq u_+^{(r)}$ and both endpoints are real numbers. Then, for $\widetilde{u},\widetilde{v}\in\mathbb{R}_{\mathcal{F}}$, we denote

$$\widetilde{u}\preceq\widetilde{v}\Leftrightarrow u_-^{(r)}\leq v_-^{(r)} \text{ and } u_+^{(r)}\leq v_+^{(r)} \text{ for all } r\in[0,1].$$

A commonly used metric $D:\mathbb{R}_{\mathcal{F}}\times\mathbb{R}_{\mathcal{F}}\to\mathbb{R}_+$ is given by:

$$D(\widetilde{u},\widetilde{v})=\sup_{r\in[0,1]}\ \max\ \left\{\left|u_-^{(r)}-v_-^{(r)}\right|,\left|u_+^{(r)}-v_+^{(r)}\right|\right\}$$

under which $(\mathbb{R}_{\mathcal{F}},D)$ constitutes a complete metric space (cf. [3], [32]). If $\widetilde{f}$ and $\widetilde{g}$ are fuzzy-number-valued functions defined on a closed interval $[a,b]$. The distance between $\widetilde{f}$ and $\widetilde{g}$ is determined by:

$$D^*(\widetilde{f},\widetilde{g})=\sup_{\mathbf{z}\in[a,b]}\sup_{r\in[0,1]}\max\left\{\left|f(z)_-^{(r)}-g(z)_-^{(r)}\right|,\left|f(z)_+^{(r)}-g(z)_+^{(r)}\right|\right\},$$

which can be compactly expressed as $D^{*}(\widetilde{f},\widetilde{g}) = \sup_{\mathbf{z}\in[a,b]} D(\widetilde{f}(z), \widetilde{g}(z))$.

Nuray and Savas [24] introduced the fuzzy analog of statistical convergence by using the metric D. Let $(\widetilde{u}_m)$ be a fuzzy number valued sequence. Then, $(\widetilde{u}_m)$ is *statistically convergent* to $\widetilde{u} \in \mathbb{R}_{\mathcal{F}}$, which is denoted by $st - \lim_m D(\widetilde{u}_m, \widetilde{u}) = 0$ and write $\widetilde{u}_m \xrightarrow{st} \widetilde{u}$ if for every $\varepsilon > 0$,

$$\lim_m \frac{1}{m} \left| \{ m \in \mathbb{N} \ : D(\widetilde{u}_m, \widetilde{u}) \geq \varepsilon \} \right| = 0$$

holds, where $|.|$ denotes the cardinality of the set.

We now turn our attention to the power series method, which plays a foundational role in the formulation of P-statistical convergence. Consider a non-negative real sequence (p_m) with $p_0 > 0$, and define the associated power series by

$$p(t) := \sum_{m=0}^{\infty} p_m t^m$$

which is assumed to have a radius of convergence R_p with $0 < R_p \leq \infty$. A sequence $z = (z_m)$ of real numbers is said to be convergent in the sense of power series method to a value $\kappa \in \mathbb{R}$ if the limit

$$\lim_{0<t\to R_p^-} \frac{1}{p(t)} \sum_{m=0}^{\infty} z_m p_m t^m = \kappa$$

exists. This mode of convergence, introduced in [22, 28]. Importantly, the power series method is regular iff $\lim_{0<t\to R_p^-} \frac{p_m t^m}{p(t)} = 0$ for every $m \in \mathbb{N}_0$ (see, [11]).

Here and in the sequel, power series method is regular.

Definition 1.1. *[29] Let $F \subset \mathbb{N}_0$. If the limit*

$$\delta_P(F) := \lim_{0<t\to R_p^-} \frac{1}{p(t)} \sum_{m=0}^{\infty} p_m t^m \chi_F$$

exists, then $\delta_P(F)$ is called the P-density of F.

Obviously,
i) $\delta_P(\mathbb{N}) = 1$,
ii) if $E \subset F$ then $\delta_P(E) \leq \delta_P(F)$,
iii) if E has P-density then $\delta_P(\mathbb{N}/E) = 1 - \delta_P(E)$.

Definition 1.2. *[31] Let $(\widetilde{u}_m)$ be a sequence of fuzzy numbers in $\mathbb{R}_{\mathcal{F}}$. The sequence is said to be P-statistically convergent to $\widetilde{u} \in \mathbb{R}_{\mathcal{F}}$, denoted by $st_P - \lim D(\widetilde{u}_m, \widetilde{u}) = 0$ or simply $\widetilde{u}_m \xrightarrow{st_P} \widetilde{u}$ if for every $\varepsilon > 0$, the set of indices*

$$\{ m : D(\widetilde{u}_m, \widetilde{u}) \geq \varepsilon \}$$

has P-density zero, i.e.,

$$\lim_{0<t\to R_p^-} \frac{1}{p(t)} \sum_{m=0}^{\infty} p_m t^m \chi_{\{m:D(\widetilde{u}_m,\widetilde{u})\geq\varepsilon\}} = 0.$$

In particular, if the set $\{ m : D(\widetilde{u}_m, \widetilde{u}) \geq \varepsilon \}$ is finite for every $\varepsilon > 0$, then the sequence $(\widetilde{u}_m)$ converges (in the usual metric sense) to $\widetilde{u}$ and write simply $\widetilde{u}_m \to \widetilde{u}$. Moreover, if the fuzzy numbers under consideration are degenerate, i.e., reduce to real numbers, and the metric D coincides with the absolute value on $\mathbb{R}$, then P-statistical convergence for fuzzy numbers coincides with the classical P-statistical convergence for real sequences.

2. Convergence Methods for Fuzzy-Number-Valued Sequences

This section is aimed to give two main results, first, we will define P-statistical type convergence methods for sequences of functions with fuzzy values, and second, we will discuss the results obtained on these types of statistical convergence of sequences of functions with fuzzy values. For these purposes, we now define, study and extend the notions of P-statistical convergence for sequences of functions with fuzzy values.

Let $\widetilde{f}, \widetilde{f_m} : [a,b] \to \mathbb{R}_{\mathcal{F}}$, $m \in \mathbb{N}_0$, be given fuzzy-number-valued functions. Next, we proceed to define, study and extend the notions of P-statistical convergence for sequences of functions with fuzzy values.

Definition 2.1. *We say that a sequence $\left(\widetilde{f_m}\right)$ of fuzzy-number-valued functions defined on a closed interval $[a,b]$ is P-statistically relatively pointwise convergent to a limiting function $\widetilde{f}$, if there exists a scale function $\sigma(z)$ such that for each $z \in [a,b]$ and for every $\varepsilon > 0$,*

$$\lim_{0<t\to R_p^-} \frac{1}{p(t)} \sum_{m=0}^{\infty} p_m t^m \chi_{\left\{m: D_\sigma(\widetilde{f_m}(z), \widetilde{f}(z)) \geq \varepsilon\right\}} = 0,$$

where the relative distance D_σ is defined by

$$D_\sigma(\widetilde{f_m}(z), \widetilde{f}(z)) = \sup_{r\in[0,1]} \max\left\{\left|\frac{f_m(z)_-^{(r)} - f(z)_-^{(r)}}{\sigma(z)}\right|, \left|\frac{f_m(z)_+^{(r)} - f(z)_+^{(r)}}{\sigma(z)}\right|\right\}$$

Equivalently, this means that for every point $z \in [a,b]$ and every $\varepsilon > 0$, there exists a set $S_z \subset \mathbb{N}_0$ with P-density zero such that

$$\forall m \in \mathbb{N}_0 \backslash S_z, D_\sigma(\widetilde{f_m}(z), \widetilde{f}(z)) < \varepsilon.$$

In this case, we write $\widetilde{f_m}(z) \overset{st_P}{\to} \widetilde{f}(z)\,(\sigma)$ for each $z \in [a,b]$.

Theorem 2.1. *The sequence $\left(\widetilde{f_m}\right)$ of fuzzy-number-valued functions is said to be P-statistically relatively convergent to a fuzzy function $\widetilde{f}$ if and only if, for each level $r \in [0,1]$, the corresponding level cuts $\left[\widetilde{f_m}\right]^r$ is uniformly P-statistically relatively converge to $\left[\widetilde{f}\right]^r$, uniformly with respect to the level parameter r.*

Proof. *($\Longrightarrow$) : Let $\widetilde{f_m}(z) \overset{st_P}{\to} \widetilde{f}(z)\,(\sigma)$ for each $z \in [a,b]$. Then, for given $\varepsilon > 0$ and $z_0 \in [a,b]$, there exists $S_{z_0} \subset \mathbb{N}_0$ with $\delta_P(S_{z_0}) = 0$ such that $D_\sigma(\widetilde{f_m}(z_0), \widetilde{f}(z_0)) < \varepsilon$ for any $m \in \mathbb{N}_0 \backslash S_{z_0}$. Then, we can write*

$$\sup_{r\in[0,1]} \max\left\{\left|\frac{f_m(z_0)_-^{(r)} - f(z_0)_-^{(r)}}{\sigma(z_0)}\right|, \left|\frac{f_m(z_0)_+^{(r)} - f(z_0)_+^{(r)}}{\sigma(z_0)}\right|\right\} < \varepsilon.$$

Clearly, for all $r \in [0,1]$, we get

$$\left|\frac{f_m(z_0)_-^{(r)} - f(z_0)_-^{(r)}}{\sigma(z_0)}\right| < \varepsilon \text{ and } \left|\frac{f_m(z_0)_+^{(r)} - f(z_0)_+^{(r)}}{\sigma(z_0)}\right| < \varepsilon.$$

Also, we know that

$$\left[\widetilde{f_m}(z_0)\right]^r = \left[f_m(z_0)_-^{(r)}, f_m(z_0)_+^{(r)}\right], \left[\widetilde{f}(z_0)\right]^r = \left[f(z_0)_-^{(r)}, f(z_0)_+^{(r)}\right].$$

Hence, we get desired result.

$(\Longleftarrow)$: *Now, let* $\left[\widetilde{f_m}(z_0)\right]^r$ *is uniformly P-statistically relatively convergent to* $\left[\widetilde{f}(z_0)\right]^r$ *with respect to r. For any* $\varepsilon > 0$ *and* $z_0 \in [a,b]$, *there exists* $S'_{z_0}, S''_{z_0} \subset \mathbb{N}_0$ *with* $\delta_P\left(S'_{z_0}\right) = 0$ *and* $\delta_P\left(S''_{z_0}\right) = 0$ *such that* $\left|\frac{f_m(z_0)_-^{(r)} - f(z_0)_-^{(r)}}{\sigma(z_0)}\right| < \varepsilon$ *for any* $m \in \mathbb{N}_0 \backslash S'_{z_0}$ *and* $\left|\frac{f_m(z_0)_+^{(r)} - f(z_0)_+^{(r)}}{\sigma(z_0)}\right| < \varepsilon$ *for any* $m \in \mathbb{N}_0 \backslash S''_{z_0}$. *Set* $S_{z_0} = S'_{z_0} \cup S''_{z_0}$. *Then,* $\delta_P\left(S_{z_0}\right) = 0$ *and*

$$\sup_{r\in[0,1]} \max \left\{ \left| \frac{f_m(z_0)_-^{(r)} - f(z_0)_-^{(r)}}{\sigma(z_0)} \right|, \left| \frac{f_m(z_0)_+^{(r)} - f(z_0)_+^{(r)}}{\sigma(z_0)} \right| \right\} < \varepsilon$$

for all $m \in \mathbb{N}_0 \backslash S_{z_0}$. *So, it is clear that* $D_\sigma(\widetilde{f_m}(z_0), \widetilde{f}(z_0)) < \varepsilon$ *whence the result.* □

Theorem 2.2. *Let* $\left(\widetilde{f_m}\right)$ *be a sequence of fuzzy-number-valued functions. Then, the following expressions are equivalent:*

(i) $\widetilde{f_m}(z) \overset{st_P}{\to} \widetilde{f}(z)\,(\sigma)$ *for each* $z \in [a,b]$.

(ii) *There exists a subset* $S_z \subset \mathbb{N}_0$ *with* $\delta_P(S_z) = 1$ *and* $\widetilde{f_{m_k}}(z) \to \widetilde{f}(z)\,(\sigma)$ *for each* $z \in [a,b]$ *where* $m_k \in S_z$.

(iii) *There exist two sequences of fuzzy-number-valued functions* $(\widetilde{g_m})$ *and* $\left(\widetilde{h_m}\right)$ *such that* $\widetilde{f_m}(z) = \widetilde{g_m}(z) + \widetilde{h_m}(z)$ *with* $\widetilde{g_m}(z) \to \widetilde{f}(z)\,(\sigma)$ *and* $\widetilde{h_m}(z) \overset{st_P}{\to} \widetilde{0}\,(\sigma)$ *for all* $z \in [a,b]$.

Proof. $(i) \Longrightarrow (ii)$: *Let* $\widetilde{f_m}(z) \overset{st_P}{\to} \widetilde{f}(z)\,(\sigma)$ *for each* $z \in [a,b]$. *Then, for given* $z_0 \in [a,b]$, *define the following sets:*

$$S_k^1 := \left\{ m : D_\sigma(\widetilde{f_m}(z_0), \widetilde{f}(z_0)) \geq \frac{1}{k} \right\},$$
$$S_k^2 := \left\{ m : D_\sigma(\widetilde{f_m}(z_0), \widetilde{f}(z_0)) < \frac{1}{k} \right\},$$

$k = 1, 2, \ldots$. *Observe that,* $\delta_P\left(S_k^1\right) = 0$ *and* $\delta_P\left(S_k^2\right) = 1$, $k = 1, 2, \ldots$. *Also,* $S_1^2 \supset S_2^2 \supset \ldots$. *If we show that* $\widetilde{f_m}(z_0) \to \widetilde{f}(z_0)\,(\sigma)$ *for every* $m \in S_k^2$ *then we get the desired result. Now, suppose that* $\left(\widetilde{f_m}(z_0)\right)$ *is not convergent to* $\widetilde{f}(z_0)$ *with respect to* $\sigma(z_0)$. *Hence, there is* $\varepsilon > 0$ *such that,* $D_\sigma(\widetilde{f_m}(z_0), \widetilde{f}(z_0)) \geq \varepsilon$ *for infinitely many terms. Put*

$$S_\varepsilon := \left\{ m : D_\sigma(\widetilde{f_m}(z_0), \widetilde{f}(z_0)) < \varepsilon \right\} \text{ and } \varepsilon > \frac{1}{k},$$

$k = 1, 2, \ldots$. *Then* $\delta_P(S_\varepsilon) = 0$ *and* $S_k^2 \subset S_\varepsilon$. *Therefore* $\delta_P\left(S_k^2\right) = 0$ *which is a contradiction. Hence,* $\widetilde{f_m}(z) \to \widetilde{f}(z)\,(\sigma)$ *for each* $z \in [a,b]$.

$(ii) \Longrightarrow (iii)$: *There exists a subset* $S_z \subset \mathbb{N}_0$ *with* $\delta_P(S_z) = 1$ *and* $\widetilde{f_{m_k}}(z) \to \widetilde{f}(z)\,(\sigma)$ *for each* $z \in [a,b]$ *where* $m_k \in S_z$. *Now, let define two sequences of fuzzy-number-valued functions* $(\widetilde{g_m})$ *and* $\left(\widetilde{h_m}\right)$ *as follows:*

$$\widetilde{g_m}(z) = \begin{cases} \widetilde{f_m}(z), & \text{if } m \in S_z \\ \widetilde{f}(z), & \text{otherwise} \end{cases} \text{ and } \widetilde{h_m}(z) + \widetilde{f}(z) = \begin{cases} \widetilde{f}(z), & \text{if } m \in S_z, \\ \widetilde{f_m}(z), & \text{otherwise.} \end{cases}$$

Then it is easy to see that $\widetilde{f_m}(z) + \widetilde{f}(z) = \widetilde{g_m}(z) + \widetilde{h_m}(z) + \widetilde{f}(z)$, *i.e.,* $\widetilde{f_m}(z) = \widetilde{g_m}(z) + \widetilde{h_m}(z)$ *and* $\widetilde{g_m}(z) \to \widetilde{f}(z)\,(\sigma)$.

Finally, we show $\widetilde{h_m}(z) \overset{st_P}{\to} \widetilde{0}\,(\sigma)$ *and* $\widetilde{h_m}(z) + \widetilde{f}(z) \overset{st_P}{\to} \widetilde{f}(z)\,(\sigma)$. *Firstly, thanks to definition of* $\widetilde{h_m}(z) + \widetilde{f}(z)$, *we can easily get that* $\widetilde{h_m}(z) + \widetilde{f}(z) \overset{st_P}{\to} \widetilde{f}(z)\,(\sigma)$. *Then according to Theorem 2.1, for given* $\varepsilon > 0$ *and* $z \in [a,b]$, *there exists* $S_z \subset \mathbb{N}_0$ *with* $\delta_P(S_z) = 0$ *such that, for all* $r \in [0,1]$,

$$\left| \frac{h_m(z)_-^{(r)} + f(z)_-^{(r)} - f(z)_-^{(r)}}{\sigma(z)} \right| < \varepsilon \text{ and } \left| \frac{h_m(z)_+^{(r)} + f(z)_+^{(r)} - f(z)_+^{(r)}}{\sigma(z)} \right| < \varepsilon,$$

for any $m \in \mathbb{N}_0 \backslash S_z$. *Hence, we get*

$$\sup_{r\in[0,1]} \max \left\{ \left| \frac{h_m(z)_-^{(r)} - 0}{\sigma(z)} \right|, \left| \frac{h_m(z)_+^{(r)} - 0}{\sigma(z)} \right| \right\} < \varepsilon,$$

i.e., $\widetilde{h}_m(z) \overset{st_P}{\to} \widetilde{0}\,(\sigma)$.

$(iii) \implies (i)$: *There exist two sequences of fuzzy-number-valued functions* $(\widetilde{g}_m)$ *and* $\left(\widetilde{h}_m\right)$ *such that* $\widetilde{f}_m(z) = \widetilde{g}_m(z) + \widetilde{h}_m(z)$ *with* $\widetilde{g}_m(z) \to \widetilde{f}(z)\,(\sigma)$ *and* $\widetilde{h}_m(z) \overset{st_P}{\to} \widetilde{0}\,(\sigma)$ *for all* $z \in [a,b]$. *For any* $\varepsilon > 0$ *and* $z \in [a,b]$, *let*

$$S_1 = \left\{ m : D_\sigma(\widetilde{g}_m(z), \widetilde{f}(z)) \geq \frac{\varepsilon}{2} \right\} \text{ and}$$
$$S_2 = \left\{ m : D_\sigma\left(\widetilde{h}_m(z), \widetilde{0}\right) \geq \frac{\varepsilon}{2} \right\}.$$

Then, clearly $\delta_P(S_1) = 0$ *and* $\delta_P(S_2) = 0$. *Hence, we have*

$$\delta_P\left(\left\{ m : D_\sigma\left(\widetilde{f}_m(z), \widetilde{f}(z)\right) \geq \varepsilon \right\}\right)$$
$$\leq \delta_P(S_1) + \delta_P(S_2) = 0,$$

whence the result. □

Definition 2.2. *Let* $\left(\widetilde{f}_m\right)$ *be a sequence of fuzzy-number-valued functions defined on a closed interval* $[a,b]$. *We say that* $\left(\widetilde{f}_m\right)$ *converges P-statistically relatively uniformly to* $\widetilde{f}$ *on* $[a,b]$ *if there exists a scale function* $\sigma(z)$ *such that for every* $\varepsilon > 0$, *the following convergence condition is satisfied:*

$$\lim_{0<t\to R_p^-} \frac{1}{p(t)} \sum_{m=0}^{\infty} p_m t^m \chi_{\left\{m : D_\sigma^*(\widetilde{f}_m, \widetilde{f}) \geq \varepsilon\right\}} = 0,$$

where the scaled uniform distance D_σ^* *is defined by* $D_\sigma^*(\widetilde{f}_m, \widetilde{f}) = \sup_{z\in[a,b]} D_\sigma(\widetilde{f}_m(z), \widetilde{f}(z))$. *Equivalently, this means that for every* $\varepsilon > 0$, *there exists a subset* $S_z \subset \mathbb{N}_0$ *with P-density zero such that*

$$\forall m \in \mathbb{N}_0 \backslash S_z, \ \sup_{z\in[a,b]} D_\sigma(\widetilde{f}_m(z), \widetilde{f}(z)) < \varepsilon.$$

In this case, we denote the convergence symbolically by $\widetilde{f}_m \overset{st_P}{\rightrightarrows} \widetilde{f}\,([a,b], \sigma)$.

Proposition 2.1. $\widetilde{f}_m \overset{st_P}{\rightrightarrows} \widetilde{f}\,([a,b], \sigma)$ *if and only if* $D_\sigma^*(\widetilde{f}_m, \widetilde{f}) \overset{st_P}{\to} 0$.

Definition 2.3. *We say that a sequence* $\left(\widetilde{f}_m\right)$ *of fuzzy-number-valued functions defined on a closed interval* $[a,b]$ *is P-equi-statistically relatively convergent to a limiting function* $\widetilde{f}$ *if there exists a scale function* $\sigma(z)$ *such that for every* $\varepsilon > 0$,

$$\lim_{0<t\to R_p^-} \frac{1}{p(t)} \sum_{m=0}^{\infty} p_m t^m \chi_{\left\{m : D_\sigma(\widetilde{f}_m(z), \widetilde{f}(z)) \geq \varepsilon\right\}} = 0,$$

uniformly with respect to $z \in [a,b]$. *In other words, the function*

$$g_\varepsilon(z) = \delta_P\left(\left\{ m : D_\sigma(\widetilde{f}_m(z), \widetilde{f}(z)) \geq \varepsilon \right\}\right)$$

converges uniformly to zero on $[a,b]$ *for every* $\varepsilon > 0$.

We denote this convergence symbolically by $\widetilde{f}_m \overset{st_P}{\twoheadrightarrow} \widetilde{f}\,([a,b], \sigma)$.

The following results follow by applying arguments similar to those used in the proof of Theorem 2.1. For the sake of brevity, we omit the detailed proofs.

Theorem 2.3. *A sequence $\left(\widetilde{f_m}\right)$ converges P-statistically relatively uniformly to $\widetilde{f}$ on $[a,b]$ if and only if $\left[\widetilde{f_m}(z)\right]^r$ is uniformly P-statistically relatively convergent to $\left[\widetilde{f}(z)\right]^r$ uniformly, jointly with respect to both $r \in [0,1]$ and $z \in [a,b]$.*

Theorem 2.4. *A sequence $\left(\widetilde{f_m}\right)$ is P-equi-statistically relatively convergent to $\widetilde{f}$ if and only if $\left[\widetilde{f_m}(z)\right]^r$ is P-equi-statistically relatively convergent to $\left[\widetilde{f}(z)\right]^r$ uniformly for any $r \in [0,1]$ and any $z \in [a,b]$.*

Lemma 2.1. *The following logical chain of implications holds between the three types of convergence:*

$$\widetilde{f_m} \overset{st_P}{\rightrightarrows} \widetilde{f}([a,b],\sigma) \implies \widetilde{f_m} \overset{st_P}{\twoheadrightarrow} \widetilde{f}([a,b],\sigma) \implies \widetilde{f_m}(z) \overset{st_P}{\rightarrow} \widetilde{f}(z)(\sigma), \forall z \in [a,b].$$

That is, P-statistical relative uniform convergence implies P-equi-statistical relative convergence, which in turn implies pointwise convergence under the same scale function σ.

While the direct implications above hold, the converse statements are not valid in general. In fact, it is possible to construct explicit counterexamples illustrating that:

Example 2.1. *For any $r \in [0,1]$ and any $z \in [0,1]$, let the scale function $\sigma(z) = 1$ and the fuzzy-number-valued functions $\widetilde{f}(z)$ and $\widetilde{f_m}(z)$ be given as follows:*

$$\widetilde{f_m}(z)(s) = \begin{cases} \left.\begin{array}{ll} 0, & s \in (-\infty, m-1) \cup (m+1, +\infty) \\ s-m+1, & m-1 \le s \le m \\ -s+m+1, & m < s \le m+1 \end{array}\right\}, m = 2k, \\ \left.\begin{array}{ll} 0, & s \in \left(-\infty, \frac{3m^2z^2}{2+m^3z^3} - 1\right) \cup \left(\frac{3m^2z^2}{2+m^3z^3} + 1, +\infty\right) \\ s - \frac{3m^2z^2}{2+m^3z^3} + 1, & \frac{3m^2z^2}{2+m^3z^3} - 1 \le s \le \frac{3m^2z^2}{2+m^3z^3} \\ \frac{3m^2z^2}{2+m^3z^3} + 1 - s, & \frac{3m^2z^2}{2+m^3z^3} < s \le \frac{3m^2z^2}{2+m^3z^3} + 1 \end{array}\right\}, m = 2k+1, \end{cases}$$

$k = 0, 1, 2, \ldots$ and

$$\widetilde{f}(z)(s) = \begin{cases} 0, & s \in (-\infty, -1) \cup (1, +\infty), \\ s+1, & -1 \le s \le 0, \\ -s+1, & 0 < s \le 1. \end{cases}$$

Also, let

$$p_m = \begin{cases} 0, & m = 2k \\ 1, & m = 2k+1 \end{cases}, k = 0, 1, 2, \ldots.$$

Then, it can be easily seen that $\delta_P(\{m : m = 2k\}) = 0$.

Now, we show that $\widetilde{f_m}(z) \overset{st_P}{\rightarrow} \widetilde{f}(z)(\sigma)$, however $\widetilde{f_m}(z)$ is not P-equi-statistically relatively convergent to $\widetilde{f}(z)$. Let $z \in [a,b]$ be fixed. For $m = 2k+1$, we have

$$\begin{aligned} & D_\sigma(\widetilde{f_m}(z), \widetilde{f}(z)) \\ &= \sup_{r\in[0,1]} \max \left\{ \left| \frac{f_m(z)_-^{(r)} - f(z)_-^{(r)}}{\sigma(z)} \right|, \left| \frac{f_m(z)_+^{(r)} - f(z)_+^{(r)}}{\sigma(z)} \right| \right\} \\ &= \sup_{r\in[0,1]} \max \left\{ \left| \frac{3m^2z^2}{2+m^3z^3} - (1-r) - (r-1) \right|, \left| \frac{3m^2z^2}{2+m^3z^3} + (1-r) - (1-r) \right| \right\} \\ &= \frac{3m^2z^2}{2+m^3z^3} \to 0. \end{aligned}$$

That is, there exists M_1 *such that* $D_\sigma(\widetilde{f_m}(z), \widetilde{f}(z)) < \varepsilon$ *for all* $m > M_1$ *and* $m = 2k+1$, $k = 0,1,2,\ldots$ *Hence,* $\forall z \in [a,b]$, $\forall \varepsilon > 0$, $\exists S_z = \{m : m = 2k\} \cup \{m \le M_1\}$ *with* $\delta_P(S_z) = 0$, $\forall m \in \mathbb{N}_0 \backslash S_z$, $D_\sigma(\widetilde{f_m}(z), \widetilde{f}(z)) < \varepsilon$, *i.e.,* $\widetilde{f_m}(z) \xrightarrow{st_P} \widetilde{f}(z)(\sigma)$. *On the other hand, let* $\varepsilon_0 = \frac{1}{2}$ *and choose* $z = \frac{1}{m}$ *for all* $m = 2k+1$, $k = 0,1,2,\ldots$ *Then*

$$\delta_P\left(\left\{m : \left|\frac{f_m(\frac{1}{m})_-^{(r)} - f(\frac{1}{m})_-^{(r)}}{\sigma\left(\frac{1}{m}\right)}\right| \ge \varepsilon_0\right\}\right)$$
$$= \delta_P\left(\left\{m : \left|\frac{3m^2\left(\frac{1}{m}\right)^2}{2 + m^3\left(\frac{1}{m}\right)^3} - (1-r) - (r-1)\right| = 1 \ge \frac{1}{2}\right\}\right)$$
$$= 1.$$

Similarly, we have $\delta_P\left(\left\{m : \left|\frac{f_m(\frac{1}{m})_+^{(r)} - f(\frac{1}{m})_+^{(r)}}{\sigma(\frac{1}{m})}\right| \ge \varepsilon_0\right\}\right) = 1$. *Therefore,* $f_m(z)_-^{(r)}$ *and* $f_m(z)_+^{(r)}$ *are not uniformly P-equi-statistically relatively convergent and thanks to Theorem 2.4, we get that* $\left(\widetilde{f_m}\right)$ *is not P-equi-statistically relatively convergent to* $\widetilde{f}(z)$.

Example 2.2. *For any* $r \in [0,1]$ *and any* $z \in [0,1]$, *let the scale function* $\sigma(z) = \begin{cases} 1, & z = 0 \\ \frac{1}{z}, & 0 < z \le 1 \end{cases}$, $\widetilde{f}(z) = \widetilde{0}$ *and the fuzzy-number-valued functions* $\widetilde{f_m}(z)$ *be defined as follows:*

$$\widetilde{f_m}(z)(s) = \begin{cases} \mu_m(z)(s), & z \in \left[\frac{1}{2^m}, \frac{3}{2^{m+1}}\right] \\ \nu_m(z)(s), & z \in \left[\frac{3}{2^{m+1}}, \frac{1}{2^{m-1}}\right] \\ 0, & z \notin \left[\frac{1}{2^m}, \frac{1}{2^{m-1}}\right] \end{cases}, \quad m = 2k,$$
$$\widetilde{f_m}(z)(s) = \xi_m(z)(s), \quad m = 2k+1,$$

where

$$\mu_m(z)(s) = \begin{cases} 0, & s \in \left(-\infty, 2^{m+1}z - 3\right) \cup \left(2^{m+1}z - 1, +\infty\right) \\ s - 2^{m+1}z + 3, & 2^{m+1}z - 3 \le s \le 2^{m+1}z - 2 \\ -s + 2^{m+1}z - 1, & 2^{m+1}z - 2 < s \le 2^{m+1}z - 1 \end{cases},$$

$$\nu_m(z)(s) = \begin{cases} 0, & s \in \left(-\infty, -2^{m+1}z + 3\right) \cup \left(-2^{m+1}z + 5, +\infty\right) \\ s + 2^{m+1}z - 3, & -2^{m+1}z + 3 \le s \le -2^{m+1}z + 4 \\ -s - 2^{m+1}z + 5, & -2^{m+1}z + 4 < s \le -2^{m+1}z + 5 \end{cases},$$

$$\xi_m(z)(s) = \begin{cases} 0, & s \in (-\infty, m-1) \cup (m+1, +\infty) \\ s - m + 1, & m - 1 \le s \le m \\ -s + m + 1, & m < s \le m + 1 \end{cases},$$

$k = 0,1,2,\ldots$ *Also, let* $p_m = \begin{cases} 1, & m = 2k \\ 0, & m = 2k+1 \end{cases}$, $k = 0,1,2,\ldots$ *Then, it can be easily seen that* $\delta_P(\{m : m = 2k+1\}) = 0$ *and* $\widetilde{f_m}(z) \overset{st_P}{\rightrightarrows} \widetilde{f}(z)([0,1], \sigma)$, *however* $\left(\widetilde{f_m}\right)$ *is not P-statistically relatively uniformly convergent to* $\widetilde{f}$. *In fact, if* m *is even then, for every* $z \in [0,1]$, *the cardinality of* $\left\{m : \widetilde{f_m}(z) \ne \widetilde{0}\right\}$ *is less than or equal to* 1. *Hence*

$$\frac{1}{p(t)}\sum_{m=0}^{\infty} p_m t^m \chi_{\left\{m : D_\sigma\left(\widetilde{f_m}(z), \widetilde{f}(z)\right) \ge \varepsilon\right\}} \le \frac{1}{p(t)}\sum_{m=0}^{\infty} p_m t^m \chi_{\left\{m : \widetilde{f_m}(z) \ne \widetilde{0}\right\}}$$
$$\le \frac{1}{p(t)} p_{m_0} t^{m_0} \to 0 \ \left(0 < t \to R_p^-\right),$$

$m_0 \in \mathbb{N}_0$. *Thus* $\widetilde{f_m}(z) \overset{st_P}{\rightrightarrows} \widetilde{f}(z)([0,1], \sigma)$. *On the other hand, we get* $D_\sigma^*(\widetilde{f_m}, \widetilde{f}) = m + 1$, *then* $\left(\widetilde{f_m}\right)$ *is not P-statistically relatively uniformly convergent to* $\widetilde{f}$.

3. Application to Fuzzy Korovkin-Type Approximation via P-Equi-Statistical Relative Convergence

In this section, we leverage the framework of P-equi-statistical relative convergence with respect to a scale function to establish a Korovkin-type approximation theorem for sequences of fuzzy-number-valued functions. In addition, the robustness of the primary theorem will be demonstrated, with a view to highlighting its strength in comparison to existing results.

Let $\widetilde{f}:[a,b]\to\mathbb{R}_{\mathcal{F}}$ be a fuzzy-number-valued function. The function $\widetilde{f}$ is said to be fuzzy continuous at a point $z_0\in[a,b]$ if, for every sequence (z_m) satisfying $z_m\to z_0$, it follows that

$$D\left(\widetilde{f}(z_m),\widetilde{f}(z_0)\right)\to 0 \text{ as } m\to\infty.$$

We say $\widetilde{f}$:is fuzzy continuous on $[a,b]$ if it is fuzzy continuous at every point in the interval. The collection of such functions is denoted by $C_{\mathcal{F}}[a,b]$ (see, e.g., [3]). It is important to notice that $C_{\mathcal{F}}[a,b]$ is only a cone not a vector space. Now let $\widetilde{L}:C_{\mathcal{F}}[a,b]\to C_{\mathcal{F}}[a,b]$ be an operator. Then $\widetilde{L}$ is termed fuzzy linear if for all $\lambda_1,\lambda_2\in\mathbb{R}$, $\widetilde{f_1},\widetilde{f_2}\in C_{\mathcal{F}}[a,b]$, and $z\in[a,b]$, the following holds:

$$\widetilde{L}\left(\lambda_1\odot\widetilde{f_1}\oplus\lambda_2\odot\widetilde{f_2};z\right)=\lambda_1\odot\widetilde{L}\left(\widetilde{f_1};z\right)\oplus\lambda_2\odot\widetilde{L}\left(\widetilde{f_2};z\right).$$

Furthermore, $\widetilde{L}$ is called fuzzy positive linear operator if it is fuzzy linear and satisfies the order-preserving condition:

$$\widetilde{f}(z)\preceq\widetilde{g}(z)\Longrightarrow\widetilde{L}\left(\widetilde{f};z\right)\preceq\widetilde{L}\left(\widetilde{g};z\right),\text{ for all } z\in[a,b].$$

Throughout the paper, we employ the standard Korovkin test functions:

$$e_i(z)=z^i\ \ (i=0,1,2),$$

as well as a scale function $\sigma(z)$ with $|\sigma(z)|>0$ for all $z\in[a,b]$. The norm $\left\|f\right\|$ denotes the usual supremum norm of f.

The subsequent Korovkin type theorems in fuzzy setting has been given by authors:

Theorem 3.1. *[5]Let $\left(\widetilde{L}_m\right)$ be a sequence of fuzzy positive linear operators from $C_{\mathcal{F}}[a,b]$ into itself. Assume that there exists a corresponding sequence (L_m) of positive linear operators from $C[a,b]$ into itself with the property*

$$\left\{\widetilde{L}_m\left(\widetilde{f};z\right)\right\}_{\pm}^{(r)}=L_m\left(f_{\pm}^{(r)};z\right)\tag{1}$$

for all $z\in[a,b]$, $r\in[a,b]$, $m\in\mathbb{N}$ and $\widetilde{f}\in C_{\mathcal{F}}[a,b]$. Assume further that

$$st-\lim_m\|L_m(e_i)-e_i\|=0 \text{ for each } i=0,1,2.$$

Then, for all $\widetilde{f}\in C_{\mathcal{F}}[a,b]$, we have

$$st-\lim_m D^*(\widetilde{L}_m\left(\widetilde{f}\right),\widetilde{f})=0.$$

Theorem 3.2. *[31] Let $\widetilde{L}_m$ be fuzzy positive linear operators for every $m\in\mathbb{N}_0$ from $C_{\mathcal{F}}[a,b]$ into itself. Suppose that there exists a corresponding positive linear operators L_m from $C[a,b]$ into itself with the property (1). If*

$$st_P-\lim\|L_m(e_i)-e_i\|=0 \text{ for each } i=0,1,2.$$

then for all $\widetilde{f}\in C_{\mathcal{F}}[a,b]$, we have

$$st_P-\lim D^*(\widetilde{L}_m\left(\widetilde{f}\right),\widetilde{f})=0.$$

We now state our main theorem.

Theorem 3.3. *Let $\left(\widetilde{L}_m\right)$ be a sequence of fuzzy positive linear operators from $C_{\mathcal{F}}[a,b]$ into itself. Suppose there exists a corresponding sequence of classical positive linear operators (L_m) from $C[a,b]$ into itself satisfying the assumption (1). Assume further that for each $i = 0,1,2$ and for every $\varepsilon > 0$, the following P-equi-statistical relative convergence condition holds:*

$$\lim_{0<t\to R_p^-} \frac{1}{p(t)} \sum_{m=0}^{\infty} p_m t^m \chi_{\left\{m: \left|\frac{L_m(e_i;z)-e_i(z)}{\sigma_i(z)}\right| \geq \varepsilon\right\}} = 0, \text{ uniformly in } z, \tag{2}$$

where $|\sigma_i(z)| > 0$ for all $z \in [a,b]$, and $\sigma_i(z)$ may be unbounded. Then, for all $\widetilde{f} \in C_{\mathcal{F}}[a,b]$, we have

$$\widetilde{L}_m\left(\widetilde{f}\right) \overset{st_P}{\twoheadrightarrow} \widetilde{f}\left([a,b],\sigma\right),$$

where the scale function is defined by $\sigma(z) = \max\{|\sigma_i(z)|; i = 0,1,2\}$.

Proof. *Let $\widetilde{f} \in C_{\mathcal{F}}[a,b]$, with $z \in [a,b]$ and $r \in [0,1]$. By assumption, the real-valued functions $f_{\pm}^{(r)} \in C[a,b]$. Therefore, for every $\varepsilon > 0$, there exists a positive number $\delta > 0$ such that the inequality $\left|f_{\pm}^{(r)}(y) - f_{\pm}^{(r)}(z)\right| < \varepsilon$ holds for every $y \in [a,b]$ satisfying $\left|y-z\right| < \delta$. Using this fact, it follows that for all $y \in [a,b]$,*

$$\left|f_{\pm}^{(r)}(y) - f_{\pm}^{(r)}(z)\right| \leq \varepsilon + 2M_{\pm}^{(r)} \frac{(y-z)^2}{\delta^2}, \tag{3}$$

where $M_{\pm}^{(r)} := \left\|f_{\pm}^{(r)}\right\|$. This is

$$-\varepsilon - 2M_{\pm}^{(r)} \frac{(y-z)^2}{\delta^2} \leq f_{\pm}^{(r)}(y) - f_{\pm}^{(r)}(z) \leq \varepsilon + 2M_{\pm}^{(r)} \frac{(y-z)^2}{\delta^2}.$$

Now operating $L_m(e_0;z)$ to this inequality since $L_m\left(f_{\pm}^{(r)};z\right)$ is monotone and linear, we can write

$$\begin{aligned}
&\left|L_m\left(f_{\pm}^{(r)};z\right) - f_{\pm}^{(r)}(z)\right| \\
&\leq \varepsilon + \varepsilon\left|L_m(e_0;z) - e_0(z)\right| + \frac{2M_{\pm}^{(r)}}{\delta^2} L_m\left((y-z)^2;z\right) + M_{\pm}^{(r)}\left|L_m(e_0;z) - e_0(z)\right| \\
&\leq \varepsilon + \left(\varepsilon + M_{\pm}^{(r)}\right)\left|L_m(e_0;z) - e_0(z)\right| + \frac{2M_{\pm}^{(r)}}{\delta^2}\left[L_m(e_2;z)\right. \\
&\left. -2zL_m(e_1;z) + z^2 L_m(e_0;z)\right] \\
&\leq \varepsilon + \left(\varepsilon + M_{\pm}^{(r)}\right)\left|L_m(e_0;z) - e_0(z)\right| + \frac{2M_{\pm}^{(r)}}{\delta^2}\left[\left|L_m(e_2;z) - e_2(z)\right|\right. \\
&\left. +2|z|\left|L_m(e_1;z) - e_1(z)\right| + z^2\left|L_m(e_0;z) - e_0(z)\right|\right] \\
&\leq \varepsilon + \left(\varepsilon + M_{\pm}^{(r)} + \frac{2M_{\pm}^{(r)}c^2}{\delta^2}\right)\left|L_m(e_0;z) - e_0(z)\right| \\
&+ \frac{4M_{\pm}^{(r)}c}{\delta^2}\left|L_m(e_1;z) - e_1(z)\right| + \frac{2M_{\pm}^{(r)}}{\delta^2}\left|L_m(e_2;z) - e_2(z)\right|
\end{aligned}$$

where $c := \max\{|a|,|b|\}$. If we take

$K_{\pm}^{(r)}(\varepsilon) := \max\left\{\varepsilon + M_{\pm}^{(r)} + \frac{2c^2 M_{\pm}^{(r)}}{\delta^2}, \frac{4cM_{\pm}^{(r)}}{\delta^2}, \frac{2M_{\pm}^{(r)}}{\delta^2}\right\}$, *we get*

$$\left|L_m\left(f_{\pm}^{(r)};z\right) - f_{\pm}^{(r)}(z)\right| \leq \varepsilon + K_{\pm}^{(r)}(\varepsilon)\left\{|L_m(e_0;z) - e_0(z)| + |L_m(e_1;z) - e_1(z)| + |L_m(e_2;z) - e_2(z)|\right\}.$$

Then, we observe that

$$\begin{aligned}&\sup_{r\in[0,1]}\max\left|L_m\left(f_{\pm}^{(r)};z\right) - f_{\pm}^{(r)}(z)\right|\\ &\leq \varepsilon + \sup_{r\in[0,1]}\max K_{\pm}^{(r)}(\varepsilon)\left\{|L_m(e_0;z) - e_0(z)| + |L_m(e_1;z) - e_1(z)|\right.\\ &\left.+ |L_m(e_2;z) - e_2(z)|\right\}.\end{aligned}$$

If we take $K := K(\varepsilon) := \sup_{r\in[0,1]}\max\left\{K_{-}^{(r)}(\varepsilon), K_{+}^{(r)}(\varepsilon)\right\}$, *then we get*

$$\begin{aligned}&\sup_{r\in[0,1]}\max\left|\frac{L_m\left(f_{\pm}^{(r)};z\right) - f_{\pm}^{(r)}(z)}{\sigma(z)}\right|\\ &\leq \frac{\varepsilon}{|\sigma(z)|} + K(\varepsilon)\left\{\left|\frac{L_m(e_0;z) - e_0(z)}{\sigma(z)}\right| + \left|\frac{L_m(e_1;z) - e_1(z)}{\sigma(z)}\right|\right.\\ &\left.+\left|\frac{L_m(e_2;z) - e_2(z)}{\sigma(z)}\right|\right\}.\end{aligned}$$

Hence, we get

$$D_\sigma(\widetilde{L}_m\left(\widetilde{f};z\right), \widetilde{f}(z)) \tag{4}$$
$$\leq \frac{\varepsilon}{|\sigma(z)|} + K\left|\frac{L_m(e_0;z) - e_0(z)}{\sigma_0(z)}\right| \tag{5}$$
$$+ K\left|\frac{L_m(e_1;z) - e_1(z)}{\sigma_1(z)}\right| + K\left|\frac{L_m(e_2;z) - e_2(z)}{\sigma_2(z)}\right|.$$

Now, for a given $r > 0$ *and any* $z \in [a,b]$, *choose* $\varepsilon > 0$ *such that* $0 < \frac{\varepsilon}{|\sigma(z)|} < r$. *Then,*

$$R := \left\{m \in \mathbb{N} : D_\sigma(\widetilde{L}_m\left(\widetilde{f};z\right), \widetilde{f}(z)) \geq r\right\}$$

and

$$R_i := \left\{m \in \mathbb{N} : \left|\frac{L_m(e_i;z) - e_i(z)}{\sigma_i(z)}\right| \geq \frac{r - \frac{\varepsilon}{|\sigma(z)|}}{3K}\right\},\ i = 0,1,2.$$

It follows from (4) that $R \subset \bigcup_{i=0}^{2} R_i$ *which implies that*

$$\frac{1}{p(t)}\sum_{m\in R} p_m t^m \leq \frac{1}{p(t)}\left\{\sum_{m\in R_0} p_m t^m + \sum_{m\in R_1} p_m t^m + \sum_{m\in R_2} p_m t^m\right\}$$

If we take the limit on both sides and use the hypothesis that we have given, we can conclude that

$$\lim_{0<t\to R_p^-}\frac{1}{p(t)}\sum_{m=0}^{\infty} p_m t^m \chi_{\left\{m\in\mathbb{N}: D_\sigma(\widetilde{L}_m\left(\widetilde{f};z\right), \widetilde{f}(z)) \geq r\right\}} = 0,\ \textit{uniformly in } z.$$

So, the proof is completed. □

Now consider (p,q)-Bernstein-Cholodowsky operators are defined as follows [7]:

$$C_m^{p,q}(f;z)=\sum_{k=0}^{m} f\left(\frac{b_m[k]_{p,q}}{p^{k-m}[n]_{p,q}}\right)\frac{p^{k(k-1)/2}}{p^{m(m-1)/2}}\left[\begin{array}{c} m \\ k \end{array}\right]_{p,q}\left(\frac{z}{b_m}\right)^{m-k}\prod_{s=0}^{m-k-1}\left(p^s-q^s\frac{z}{b_m}\right),$$

where (b_m) is a nonnegative non-decreasing sequence such that $\lim\limits_{m\to\infty} b_m=\infty$ and $\lim\limits_{m\to\infty}\frac{b_m}{[m]_{p,q}}=0$, f is a function that takes on real values and is defined for real numbers that are not negative and $z\in[0,b_m]$. Here, the (p,q)-integer $[i]_{p,q}$ is defined as

$$[i]_{p,q}:=\frac{p^i-q^i}{p-q}$$

where $i=0,1,2,...$ and $0<q<p\leq 1$. (p,q)-Factorial $[i]_{p,q}!$ and (p,q)-binomial coefficients are defined by

$$[i]_{p,q}!:=[i]_{p,q}[i-1]_{p,q}...[2]_{p,q}[1]_{p,q},\quad [0]_{p,q}!:=1,$$

and

$$\left[\begin{array}{c} j \\ i \end{array}\right]_{p,q}:=\frac{[j]_{p,q}!}{[i]_{p,q}![j-i]_{p,q}!},$$

respectively. It is well-established that the (p,q)-Bernstein-Cholodowsky operators are positive linear operators from $C[0,\alpha]$ to itself. These operators not only preserve positivity and linearity, but also exhibit approximation behavior that aligns with classical Korovkin-type theorems. In particular, it is known that

(1.1) $C_m^{p,q}(1;z)=1,$
(1.2) $C_m^{p,q}(t;z)=z,$
(1.3) $C_m^{p,q}\left(t^2;z\right)=\frac{q[m-1]_{p,q}}{[m]_{p,q}}z^2+\frac{p^{m-1}b_m}{[m]_{p,q}}z.$

Since $[m]_{p,q}=p^{m-1}+q[m-1]_{p,q}$, (1.3) can be reduced to

(1.4) $C_m^{p,q}\left(t^2;z\right)=\left(1-\frac{p^{m-1}}{[m]_{p,q}}\right)z^2+\frac{p^{m-1}b_m}{[m]_{p,q}}z.$

In this paper, it is more convenient to use (1.4) instead of (1.3).

Definition 3.1. *[27] Let $\alpha<b_m$ for each $m\in\mathbb{N}$ and let $\widetilde{f}$ be a fuzzy continuous function defined on $[0,\alpha]$. We define fuzzy (p,q)-Bernstein-Cholodowsky operators by*

$$C_{m,p,q}^{\mathcal{F}}\left(\widetilde{f};z\right)=\sum_{k=0}^{m}{}^{*}\,\widetilde{f}\left(\tau_{k,m}^{p,q}\right)\odot S_{k,m}^{p,q}(z),\ z\in[0,\alpha],$$

where $\tau_{k,m}^{p,q}:=\frac{b_m[k]_{p,q}}{p^{k-m}[m]_{p,q}}$ and

$$S_{k,m}^{p,q}=\frac{p^{k(k-1)/2}}{p^{m(m-1)/2}}\left[\begin{array}{c} m \\ k \end{array}\right]_{p,q}\left(\frac{z}{b_m}\right)^{m-k}\prod_{s=0}^{m-k-1}\left(p^s-q^s\frac{z}{b_m}\right),$$

for all $k,m\in\mathbb{N}$, $q\in(0,1]$ and $p\in(q,1]$.

In 2022, Özkan [27] established a structural link between the classical (p,q)-Bernstein-Cholodowsky operators and their fuzzy counterparts as follows:

$$\left\{C^{\mathcal{F}}_{m,p,q}\left(\widetilde{f};z\right)\right\}^{(r)}_{\pm} = C^{p,q}_{m}\left(f^{(r)}_{\pm};z\right) \text{ for } \widetilde{f} \in C_{\mathcal{F}}[0,\alpha].$$

It was demonstrated that the fuzzy (p,q)-Bernstein-Cholodowsky operators are fuzzy positive linear operators. The hypotheses of the fuzzy Korovkin-type approximation theorem were also confirmed ([3]). Now, we introduce the following fuzzy (p,q)-Bernstein-Cholodowsky type operators:

$$\widetilde{L}_m\left(\widetilde{f};z\right) = \left(1+\widetilde{f_m}(z)\right) \odot C^{\mathcal{F}}_{m,p,q}\left(\widetilde{f};z\right) \tag{6}$$

where $\widetilde{f_m}$, p_m and σ are given in Example 2.2. Since $\widetilde{f_m}(z) \overset{st_P}{\rightrightarrows} \widetilde{0}\left([0,1],\sigma\right)$, we can easily get that

$$\lim_{0<t\to R_p^-}\frac{1}{p(t)}\sum_{m=0}^{\infty}p_m t^m \chi_{\left\{m:\left|\frac{L_m(e_i;z)-e_i(z)}{\sigma_i(z)}\right|\geq\varepsilon\right\}} = 0, \text{uniformly in } z.$$

Hence by Theorem 3.3, we obtain for all $\widetilde{f} \in C_{\mathcal{F}}[0,1]$ that

$$\widetilde{L}_m\left(\widetilde{f}\right) \overset{st_P}{\rightrightarrows} \widetilde{f}\left([0,1],\sigma\right).$$

However, since $\left(\widetilde{f_m}\right)$ is not P-statistically relatively uniformly convergent to $\widetilde{f}=\widetilde{0}$ on the interval $[0,1]$, we can say that Theorem 3.2 does not work for our operators defined by (6). Furthermore, since $\left(\widetilde{f_m}\right)$ is not uniformly convergent (in the ordinary sense) to the function $\widetilde{f}=\widetilde{0}$ on $[0,1]$, the classical fuzzy Korovkin theorem ([3]) does not work either.

4. Statistical Relative Fuzzy Rates

In this section we study the rates of P-equi-statistical relative convergence in Theorem 3.3.

Definition 4.1. *Let (a_m) be a positive non increasing sequence of real numbers. A sequence $\left(\widetilde{f_m}\right)$ is P-equi-statistically relatively convergent to $\widetilde{f}$ with the rate of $o(a_m)$ if there exists a scale function $\sigma(z)$ such that for every $\varepsilon>0$,*

$$\lim_{0<t\to R_p^-}\frac{1}{p(t)}\sum_{m=0}^{\infty}p_m t^m \chi_{\left\{m:D_\sigma\left(\widetilde{f_m}(z),\widetilde{f}(z)\right)\geq\varepsilon a_m\right\}} = 0, \text{ uniformly in } z.$$

In this case, it is denoted by

$$(e_{st_P}) - \left(\widetilde{f_m}(z) - \widetilde{f}(z)\right) = o(a_m)\ \left([a,b],\sigma\right).$$

Now we recall the modulus of continuity in fuzzy setting.

Let $\widetilde{f}:[a,b]\to\mathbb{R}_{\mathcal{F}}$ be a fuzzy-number-valued function. The first fuzzy modulus of continuity of $\widetilde{f}$, as originally introduced by Gal [17] (see also [3]), is defined for any $0<\delta\leq b-a$ by

$$\omega_1^{(\mathcal{F})}\left(\widetilde{f},\delta\right) := \sup_{z,y\in[a,b];\left|z-y\right|\leq\delta} D(\widetilde{f}(z),\widetilde{f}(y)).$$

Then we have the following.

Theorem 4.1. *Let $\left(\widetilde{L_m}\right)$ be a sequence of fuzzy positive linear operators acting from $C_{\mathcal{F}}[a,b]$ into itself. Suppose there exists an associated sequence of positive linear operators (L_m) defined on $C[a,b]$ satisfying the condition given in (1). Further, assume that the following conditions are satisfied:*

(a) $\left(e_{st_p}\right) - (L_m(e_0;z) - e_0(z)) = o(a_m)([a,b],\sigma_0)$,

(b) $\left(e_{st_p}\right) - \omega_1^{(\mathcal{F})}\left(\widetilde{f},\mu_m\right) = o(b_m)([a,b],\sigma_1)$, *where* $\mu_m = \sqrt{L_m(\varphi;z)}$ *with* $\varphi(.) = (.-z)^2$ *for each* $z \in [a,b]$.

Then we have, for all $\widetilde{f} \in C_{\mathcal{F}}[a,b]$, that

$$\left(e_{st_p}\right) - (\widetilde{L_m}\left(\widetilde{f};z\right) - \widetilde{f}(z)) = o(c_m)([a,b],\sigma),$$

where $c_m = \max\{a_m, b_m, a_m b_m\}$, for every $m \in \mathbb{N}$ and $\sigma(z) = \max\{\sigma_0(z), \sigma_1(z), \sigma_0(z)\sigma_1(z)\}$, $|\sigma_i(z)| > 0$, and $\sigma_i(z)$ is unbounded, $i = 0, 1$.

Proof. From Theorem 3 of [3] and a simple calculation, we can get, for each $m \in \mathbb{N}$ and $\widetilde{f} \in C_{\mathcal{F}}[a,b]$ and $z \in [a,b]$,

$$\begin{aligned} D_\sigma(\widetilde{L_m}\left(\widetilde{f};z\right), \widetilde{f}(z)) \le\ & M\left|\frac{L_m(e_0;z) - e_0(z)}{\sigma_0(z)}\right| \\ & + \left|\frac{L_m(e_0;z) + e_0(z)}{\sigma_0(z)}\right| \omega_1^{(\mathcal{F})}\left(\widetilde{f},\mu_m\right), \end{aligned}$$

where $M := D(\widetilde{f}(z), \widetilde{0})$. Then we may write that

$$\begin{aligned} D_\sigma(\widetilde{L_m}\left(\widetilde{f};z\right), \widetilde{f}(z)) \le\ & M\left|\frac{L_m(e_0;z) - e_0(z)}{\sigma_0(z)}\right| \\ & + \frac{\omega_1^{(\mathcal{F})}\left(\widetilde{f},\mu_m\right)}{|\sigma_1(z)|}\left|\frac{L_m(e_0;z) - e_0(z)}{\sigma_0(z)}\right| \\ & + 2\frac{\omega_1^{(\mathcal{F})}\left(\widetilde{f},\mu_m\right)}{|\sigma_1(z)|}. \end{aligned} \tag{7}$$

Now, for a given $\varepsilon > 0$ and any $z \in [a,b]$,

$$S_m := \left\{m \in \mathbb{N} : D_\sigma(\widetilde{L_m}\left(\widetilde{f};z\right), \widetilde{f}(z)) \ge \varepsilon'\right\},$$

$$S_{1,m} := \left\{m \in \mathbb{N} : \left|\frac{L_m(e_0;z) - e_0(z)}{\sigma_0(z)}\right| \ge \frac{\varepsilon'}{3M}\right\},$$

$$S_{2,m} := \left\{m \in \mathbb{N} : \left|\frac{L_m(e_0;z) - e_0(z)}{\sigma_0(z)}\right| \frac{\omega_1^{(\mathcal{F})}\left(\widetilde{f},\mu_n\right)}{|\sigma_1(z)|} \ge \frac{\varepsilon'}{3}\right\},$$

$$S_{3,m} := \left\{m \in \mathbb{N} : \frac{\omega_1^{(\mathcal{F})}\left(\widetilde{f},\mu_n\right)}{|\sigma_1(z)|} \ge \frac{\varepsilon'}{6}\right\},$$

then we get $S_m \subset S_{1,m} \cup S_{2,m} \cup S_{3,m}$. Then by (7), $S_m \subset \bigcup_{i=0}^{2} S_{i,m}$. Also let

$$S_{4,m} = \left\{m \in \mathbb{N} : \frac{\omega_1^{(\mathcal{F})}\left(\widetilde{f},\mu_n\right)}{|\sigma_1(z)|} \ge \sqrt{\frac{\varepsilon'}{3}}\right\}$$

$$S_{5,m} = \left\{m \in \mathbb{N} : \left|\frac{L_m(e_0;z) - e_0(z)}{\sigma_0(z)}\right| \ge \sqrt{\frac{\varepsilon'}{3}}\right\}$$

we can easily see that $S_{2,m} \subset S_{4,m} \cup S_{5,m}$, *which gives* $S_m \subset S_{1,m} \cup S_{3,m} \cup S_{4,m} \cup S_{5,m}$. *Hence*

$$\frac{1}{p(t)} \sum_{m \in S_m} p_m t^m$$

$$\leq \frac{1}{p(t)} \left\{ \sum_{m \in S_{1,m}} p_m t^m + \sum_{m \in S_{3,m}} p_m t^m + \sum_{m \in S_{4,m}} p_m t^m + \sum_{m \in S_{5,m}} p_m t^m \right\}$$

By taking limit as on the both sides and thanks to the hypotheses (a), (b), *the proof is done.* □

5. Conclusion and Future Directions

In this paper, we have introduced and analyzed new convergence concepts for sequences of fuzzyvalued functions, namely *P-statistical relative pointwise, uniform*, and *P-equi-statistical relative convergence*. These convergence concepts have been examined through their structural properties and interrelationships, as well as through their corresponding r-level sets. As an application, we established Korovkin-type approximation theorems under these convergence notions and illustrated their effectiveness through fuzzy *(p,q)-Bernstein–Chlodowsky type operators*. Furthermore, wehave provided estimates for the rate of convergence, demonstrating the effectiveness and applicability of the developed theory.

Several directions remain open for future research. For instance, the current results could be extended to multivariate fuzzy-number-valued functions, allowing approximation in higher-dimensional settings. One may explore other classes of positive linear operators, including those defined via q-calculus or integral-type operators, under the proposed convergence frameworks. The applicability of these convergence modes to different summability methods, such as matrix summability or Cesàro-type means, may yield additional insights.

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