Filomat 39:32 (2025), 11637–11652
https://doi.org/10.2298/FIL2532637J

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some novelty integral inequalities on fractal sets via generalized convexity and the generalized Beta function with applications

Iram Javed[a], Adrian Naço[b], Arslan Razzaq[c], Artion Kashuri[b,*]

[a]School of Mathematical Sciences, Beihang University, Beijing, 100191, P.R. China
[b]Department of Mathematical Engineering, Polytechnic University of Tirana, 1001 Tirana, Albania
[c]School of Mathematics and Physics, China University of Geosciences, Wuhan, 430074, P. R. China

Abstract. Classical Hermite-Hadamard inequalities, which provide bounds for the integral means of convex functions, encounter limitations when applied to fractal domains due to the non-differentiable and irregular nature of such spaces. This work addresses these limitations by extending Hermite-Hadamard-type inequalities using Yang's local fractional calculus in conjunction with the generalized Beta function. We derive new bounds for integral means via fractal-specific integration techniques and convex function theory, presenting trapezoidal and midpoint inequalities that generalize classical Hölder and Young inequalities within a fractal framework. Notably, the proposed results reduce to their classical counterparts when the fractal parameter $\varpi = 1$, ensuring consistency with standard calculus. Applications to special means and Mittag-Leffler functions highlight the effectiveness of the results in modeling anomalous diffusion, material heterogeneity, and complex dynamical behavior. Graphical illustrations confirm the adaptability and robustness of the proposed inequalities under varying fractal dimensions, showcasing practical implications for modeling non-smooth phenomena. Furthermore, the integration of fractal calculus and convex analysis offers a robust mathematical foundation for analyzing irregular systems across disciplines such as physics, engineering, and data science.

1. Introduction and Preliminaries

For many years, mathematical inequalities have been indispensable tools in applied sciences, analysis, and optimization, as they provide critical bounds for integrals, functions, and stochastic processes. Recent advances have significantly broadened their scope by incorporating majorization principles, convexity theory, and fractional calculus to address complex problems in contemporary domains such as engineering, machine learning, and quantum physics. In particular, fractional operators and generalized convex functions, such as s-convex and quasi-convex functions, have been instrumental in extending classical inequalities like those of Ostrowski, Hermite-Hadamard, and Jensen, allowing sharper error estimates and broader

2020 *Mathematics Subject Classification*. Primary 26D07; Secondary 26D10, 26D15, 26A33, 15A42, 52A41.
Keywords. Fractal Beta function; Generalized convex functions; Hermite-Hadamard inequality; Mittag-Leffler functions; Hölder's inequality; Young's inequality; Special means.
Received: 11 June 2025; Revised: 09 July 2025; Accepted: 28 September 2025
Communicated by Miodrag Spalević
* Corresponding author: Artion Kashuri
Email addresses: iram.javed.412@gmail.com (Iram Javed), nacoad@yahoo.com (Adrian Naço), arslanrazzaq0125@gmail.com (Arslan Razzaq), a.kashuri@fimif.edu.al (Artion Kashuri)
ORCID iDs: https://orcid.org/0000-0002-5212-4422 (Iram Javed), https://orcid.org/0009-0004-0918-0236 (Adrian Naço), https://orcid.org/0009-0001-9843-3300 (Arslan Razzaq), https://orcid.org/0000-0003-0115-3079 (Artion Kashuri)

theoretical applicability. One can see several recent studies in the literature [1, 2, 3]. The concept of convex functions, originally formalized over a century ago by Jensen, has since been subjected to numerous generalizations. The continuous exploration of convexity has revealed not only its structural elegance but also its profound utility across disciplines. Mathematicians continue to uncover valuable results in this area, both of theoretical significance and of practical utility. These developments have led to a diverse array of inequality formulations useful in fields ranging from statistical mechanics to information theory. As a result, convex functions have remained a focal point in mathematical research, particularly in the study of inequality theory and its applications [4, 5, 6, 7].

Definition 1.1. *A function $\Upsilon : [\vartheta_1, \vartheta_2] \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex, if*

$$\Upsilon(\mathrm{t}\varkappa + (1-\mathrm{t})\varkappa_1) \leq \mathrm{t}\Upsilon(\varkappa) + (1-\mathrm{t})\Upsilon(\varkappa_1) \tag{1}$$

holds for all $\varkappa, \varkappa_1 \in [\vartheta_1, \vartheta_2]$ and $\mathrm{t} \in [0,1]$. Further details on the various forms of convexity and their contributions to inequality theory can be found in [8, 9, 10].

The most significant and often applied result involving convex functions is the Hermite-Hadamard inequality (H-H) [11, 12, 13]. The (H-H) inequality is the most well-known inequality pertaining to the integral mean of a convex function and defined as:

$$\Upsilon\left(\frac{\vartheta_1+\vartheta_2}{2}\right) \leq \frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} \Upsilon(\varkappa)d\varkappa \leq \frac{\Upsilon(\vartheta_1)+\Upsilon(\vartheta_2)}{2}, \tag{2}$$

where $\Upsilon : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $\vartheta_1, \vartheta_2 \in I$ with $\vartheta_1 < \vartheta_2$.

In Yang's calculus, differentiation and integration rules generated on fractal sets are summarised. The essence of Yang's calculus has captured the imagination of not only intellectuals but also physicists and technicians [14, 15]. Many researchers studied the properties of functions in fractal space and developed various fractional calculus concepts using various methods. In the idea of local fractional calculus, Mo et al. [16] defined the generalized convex function on the fractal space $\mathbb{R}^{\varpi}(0 < \varpi \leq 1)$ of a real number and obtained the "generalized Jensen's inequality" and "generalized Hermite-inequality" Hadamard's for a generalised convex function. More latest results in this area can be found in [17, 18, 19, 20, 21, 22, 23, 24, 25, 26]. The non-differentiable behaviours of physical laws are described by the local fractional partial differential equations that arise in mathematical physics. The key challenge these days is finding non-differentiable solutions. On local fractional differential equations, useful techniques have been successfully applied. The main techniques include the decomposition method and Laplace transform method with a local fractional operator [27, 28]. Now we use $\mathbb{R}^{\varpi}$ to categorize different definitions in the Yang's calculus and so on, recalling the Gao-Yang-Kang notion.
Yang's fractional sets theory [29] can be stated as.

For $0 < \varpi \leq 1$, the ϖ-type set of element set are given below:
$\mathbb{Z}^{\varpi}$: The ϖ-type set of integer numbers is used to define
the set $\{0^{\varpi}, \pm 1^{\varpi}, \pm 2^{\varpi}, \ldots, \pm n^{\varpi}, \ldots\}$.
$\mathbb{Q}^{\varpi}$: The ϖ-type set of rational numbers is used to define
the set $\left\{\mathrm{m}^{\varpi} = (u/v)^{\varpi} : u, v \in \mathbb{Z}, v \neq 0\right\}$.
$\mathbb{J}^{\varpi}$: The ϖ-type set of irrational numbers is used to define
the set $\left\{\mathrm{m}^{\varpi} \neq (u/v)^{\varpi} : u, v \in \mathbb{Z}, v \neq 0\right\}$.
$\mathbb{R}^{\varpi}$: The ϖ-type set of real numbers is used to define
the set $\mathbb{R}^{\varpi}=\mathbb{Q}^{\varpi} \cup \mathbb{J}^{\varpi}$.
The following operations hold for u^{ϖ}, v^{ϖ} and τ^{ϖ} belong to the set $\mathbb{R}^{\varpi}$ of real line numbers:
(i) $u^{\varpi} + v^{\varpi}$ and $u^{\varpi}v^{\varpi}$ belong to the set $\mathbb{R}^{\varpi}$;
(ii) $u^{\varpi} + v^{\varpi} = v^{\varpi} + u^{\varpi} = (u+v)^{\varpi} = (v+u)^{\varpi}$;
(iii) $u^{\varpi} + (v^{\varpi} + \tau^{\varpi}) = (u^{\varpi} + v^{\varpi}) + \tau^{\varpi}$;
(iv) $u^{\varpi}v^{\varpi} = v^{\varpi}u^{\varpi} = (uv)^{\varpi} = (vu)^{\varpi}$;

(v) $u^{\varpi}(v^{\varpi}\tau^{\varpi}) = (u^{\varpi}v^{\varpi})\tau^{\varpi}$;
(vi) $u^{\varpi}(v^{\varpi} + \tau^{\varpi}) = u^{\varpi}v^{\varpi} + u^{\varpi}\tau^{\varpi}$;
(vii) $u^{\varpi} + 0^{\varpi} = 0^{\varpi} + u^{\varpi} = u^{\varpi}$ and $u^{\varpi}1^{\varpi} = 1^{\varpi}u^{\varpi} = u^{\varpi}$.

Definition 1.2. *[29] The local fractional integral of Υ on the interval $[\vartheta_1, \vartheta_2]$ of order ϖ is defined by*

$$ {}_{\vartheta_1}I_{\vartheta_2}^{(\varpi)}\Upsilon = \frac{1}{\Gamma(\varpi+1)}\int_{\vartheta_1}^{\vartheta_2}\Upsilon(\eta)(d\eta)^{\varpi} := \frac{1}{\Gamma(\varpi+1)}\lim_{\Delta\eta\to 0}\Sigma_{j=0}^{N-1}\Upsilon(\eta_j)(\Delta\eta_j)^{\varpi}, \tag{3}$$

provided that the limit exists. Here, it follows that ${}_{\vartheta_1}I_{\vartheta_2}^{(\varpi)}\Upsilon = 0$ if $\vartheta_1 = \vartheta_2$ and ${}_{\vartheta_1}I_{\vartheta_2}^{(\varpi)}\Upsilon = -{}_{\vartheta_2}I_{\vartheta_1}^{(\varpi)}\Upsilon$ if $\vartheta_1 < \vartheta_2$.

Lemma 1.3. *[29] The aforementioned arguments are valid:*

1. *(Fractal derivative of $\varkappa^{k\varpi}$):*

$$\frac{d^{\varpi}\varkappa^{k\varpi}}{d\varkappa^{\varpi}} = \frac{\Gamma(1+k\varpi)}{\Gamma(1+(k-1)\varpi)}\varkappa^{(k-1)\varpi}. \tag{4}$$

2. *(Local fractional integration is anti-differentiation). Suppose that $\Upsilon = \mathfrak{g}^{\varpi} \in C_{\varpi}[\vartheta_1, \vartheta_2]$. Then, we have*

$$ {}_{\vartheta_1}I_{\vartheta_2}^{(\varpi)}\Upsilon = \mathfrak{g}(\vartheta_2) - \mathfrak{g}(\vartheta_1). \tag{5}$$

3. *(Local fractional integration by parts). Suppose that $\Upsilon, \mathfrak{g} \in D_{\varpi}[\vartheta_1, \vartheta_2]$ and $\Upsilon^{\varpi}, \mathfrak{g}^{\varpi} \in C_{\varpi}[\vartheta_1, \vartheta_2]$. Then, we have*

$$ {}_{\vartheta_1}I_{\vartheta_2}^{(\varpi)}\Upsilon\mathfrak{g}^{\varpi} = \Upsilon\mathfrak{g}\,|_{\vartheta_1}^{\vartheta_2} - {}_{\vartheta_1}I_{\vartheta_2}^{(\varpi)}\Upsilon^{\varpi}\mathfrak{g}. \tag{6}$$

4. *(Fractal definite integrals of $\varkappa^{k\varpi}$):*

$$\frac{1}{\Gamma(1+\varpi)}\int_{\vartheta_1}^{\vartheta_2}\varkappa^{k\varpi}(d\varkappa)^{\varpi} = \frac{\Gamma(1+k\varpi)}{\Gamma(1+(k+1)\varpi)}({\vartheta_2}^{(k+1)\varpi} - {\vartheta_1}^{(k+1)\varpi}),\ k \in \mathbb{R}. \tag{7}$$

Definition 1.4. *[16] For any $\vartheta_1, \vartheta_2 \in I$ and $\mathrm{t} \in [0, 1]$, if the following inequality*

$$\Upsilon(\mathrm{t}\vartheta_1 + (1-\mathrm{t})\vartheta_2) \le \mathrm{t}^{\varpi}\Upsilon(\vartheta_1) + (1-\mathrm{t})^{\varpi}\Upsilon(\vartheta_2), \tag{8}$$

holds, then Υ is called generalized convex function on I, where $\Upsilon : I \subseteq \mathbb{R} \to \mathbb{R}^{\varpi}$. This definition generalizes classical convexity to fractal domains by incorporating the fractal dimension parameter. For $\varpi = 1$, it reduces to the standard convexity inequality (1)*, ensuring consistency with classical calculus. The exponent ϖ accounts for non-differentiable structures in fractal spaces.*

Theorem 1.5. *[16] Let $\Upsilon : I \subseteq \mathbb{R} \to \mathbb{R}^{\varpi}$ be a generalized convex function defined on the interval I of real numbers and $\vartheta_1, \vartheta_2 \in I$ with $\vartheta_1 < \vartheta_2$. Then, we have*

$$\Upsilon\Big(\frac{\vartheta_1+\vartheta_2}{2}\Big) \le \frac{\Gamma(1+\varpi)}{(\vartheta_2-\vartheta_1)^{\varpi}}\,{}_{\vartheta_1}I_{\vartheta_2}^{(\varpi)}\Upsilon \le \frac{\Upsilon(\vartheta_1)+\Upsilon(\vartheta_2)}{2^{\varpi}}. \tag{9}$$

Further information can be found in the published publications ([30, 31] and references therein).

Theorem 1.6. *[32] (Generalized Young's inequality) Let $\Upsilon, \mathfrak{g} \in C_{\varpi}[\vartheta_1, \vartheta_2]$ and $p, q > 1$ are conjugate exponents, then we have*

$$\begin{aligned}&\frac{1}{\Gamma(1+\varpi)}\int_{\vartheta_1}^{\vartheta_2}|\Upsilon(\mathrm{t})\mathfrak{g}(\mathrm{t})|(d\mathrm{t})^{\varpi}\\ &\le \frac{1}{p}\Big(\frac{1}{\Gamma(1+\varpi)}\int_{\vartheta_1}^{\vartheta_2}|\Upsilon(\mathrm{t})|^{p}(d\mathrm{t})^{\varpi}\Big) + \frac{1}{q}\Big(\frac{1}{\Gamma(1+\varpi)}\int_{\vartheta_1}^{\vartheta_2}|\mathfrak{g}(\mathrm{t})|^{q}(d\mathrm{t})^{\varpi}\Big).\end{aligned} \tag{10}$$

Definition 1.7. *[33] The fractal Gamma function is expressed in the following form:*

$$\Gamma_{\varpi}(\varkappa) := \frac{1}{\varpi!}\int_0^{\infty} \mathbb{E}_{\varpi}(-\mathrm{t}^{\varpi})\mathrm{t}^{((\varkappa-1)\varpi)}(d\mathrm{t})^{\varpi}, \tag{11}$$

where $0 < \varpi \leq 1$ *and* $\varkappa \in \mathbb{R}^+$. *For* $\varpi = 1$, *we have* $\Gamma_{\varpi}(\varkappa) = \Gamma(\varkappa)$.

Definition 1.8. *[33] The fractal Beta function is stated as:*

$$\beta_{\varpi}(\varkappa, \varkappa_1) = \frac{\Gamma_{\varpi}(\varkappa)\Gamma_{\varpi}(\varkappa_1)}{\Gamma_{\varpi}(\varkappa + \varkappa_1)} := \frac{1}{\Gamma(1+\varpi)}\int_0^1 \mathrm{t}^{(\varkappa-1)\varpi}(1-\mathrm{t})^{(\varkappa_1-1)\varpi}(d\mathrm{t})^{\varpi}, \tag{12}$$

where $0 < \varpi \leq 1$ *and* $\varkappa, \varkappa_1 \in \mathbb{R}^+$ *with* $\varkappa, \varkappa_1 > 0$. *For* $\varpi = 1$, *we have* $\beta_{\varpi}(\varkappa, \varkappa_1) = \beta(\varkappa, \varkappa_1)$.

Definition 1.9. *[33] The fractal incomplete Beta function is stated as:*

$$\beta_{\mathrm{t}\varpi}(\varkappa, \varkappa_1) := \frac{1}{\Gamma(1+\varpi)}\int_0^{\mathrm{t}} \mathrm{s}^{(\varkappa-1)\varpi}(1-\mathrm{s})^{(\varkappa_1-1)\varpi}(d\mathrm{s})^{\varpi}, \tag{13}$$

where $0 < \varpi \leq 1$ *and* $\varkappa, \varkappa_1 \in \mathbb{R}^+$ *with* $\varkappa, \varkappa_1 > 0$. *For* $\mathrm{t} = 1$, *we have* $\beta_{\mathrm{t}\varpi}(\varkappa, \varkappa_1) = \beta_{\varpi}(\varkappa, \varkappa_1)$.

Corollary 1.10. *[33] If* $\Upsilon : [\vartheta_1, \vartheta_2] \to \mathbb{R}$ *is* ϖ*-fractional analytic function on* $[\vartheta_1, \vartheta_2]$, *then by fundamental theorem of local fractional calculus, we have*

$$\frac{d^{\varpi}}{d\varkappa^{\varpi}}\int_{\vartheta_1}^{\varkappa} \Upsilon(\mathrm{t})d(\mathrm{t})^{\varpi} = \Gamma(1+\varpi)\Upsilon(\varkappa), \tag{14}$$

for all $(\vartheta_1, \vartheta_2)$. *On the other hand, If* $\Upsilon : [\vartheta_1, \vartheta_2] \to \mathbb{R}$ *is an* ϖ*-fractional analytic on* $[\vartheta_1, \vartheta_2]$, *then*

$$\int_{\vartheta_1}^{\vartheta_2} \Upsilon^{\varpi}(\varkappa)d(\varkappa)^{\varpi} = \Gamma(1+\varpi)\Big(\Upsilon(\vartheta_2) - \Upsilon(\vartheta_1)\Big). \tag{15}$$

This paper addresses the significant challenge of extending classical Hermite-Hadamard inequalities-central to convex analysis and integral mean estimation-to fractal domains, where classical calculus is inadequate due to the presence of non-differentiable and irregular structures. By incorporating Yang's local fractional calculus and the generalized Beta function, we propose a unified framework for deriving novel Hermite-Hadamard type inequalities adapted to fractal settings. The principal aim is to bridge the gap between classical convexity theory and fractal geometry, facilitating precise bounds for integral means in fractal systems. A key contribution of this study is the formulation of fractal Beta function-based inequalities, including Trapezoidal and Midpoint variants, which generalize Hölder's and Young's inequalities under non-smooth conditions. These inequalities are rigorously established using tools from local fractional calculus and are validated through applications to special means and Mittag-Leffler functions-important in modeling anomalous diffusion and complex dynamics. Significantly, the results reduce to classical inequalities when the fractal dimension parameter $\varpi = 1$, thereby preserving consistency with standard analysis.

Beyond theoretical development, the results presented offer practical analytical tools for real-world systems characterized by irregularity, such as fractured materials, financial time series, and stochastic processes. By integrating fractal calculus with convex function theory, this work contributes scalable techniques for optimization and modeling across physics, engineering, and data science. The methodologies introduced here not only enhance the mathematical framework of fractal inequalities but also provide a foundation for addressing interdisciplinary challenges driven by non-differentiability. The paper is organized to systematically develop the theory of Hermite-Hadamard type inequalities within Yang's fractal calculus. Section 2 introduces the main results, presenting new inequalities derived via the generalized Beta function. Key supporting lemmas and theorems establish fractal analogues of Trapezoidal and Midpoint rules, founded on classical inequalities such as Hölder's and Young's. Section 3 illustrates the applicability of these results through examples involving special means and Mittag-Leffler functions. Section 4 transitions to graphical illustration of our newly derived results. Section 5 summarizes the theoretical contributions and highlights directions for future research, including potential extensions to multidimensional fractal systems and variable-order operators.

2. Main Results

In this section, we provide novel Hermite-Hadamard type inequality via fractal Beta function.

Theorem 2.1. *Consider a generalized convex function* $\Upsilon : I^o \subset \mathbb{R} \to \mathbb{R}^{\varpi}$ *(*I^o *is the interior of* $I \subset \mathbb{R}$*) such that* $\Upsilon \in D_{\varpi}(I^o)$ *and* $\Upsilon^{\varpi} \in C_{\varpi}[\vartheta_1, \vartheta_2]$ *for* $\vartheta_1, \vartheta_2 \in I^o$ *with* $\vartheta_1 < \vartheta_2$*, then we have the following inequality:*

$$\Upsilon\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\beta_{\varpi}(\mathrm{m}_1,\mathrm{m}_2) \le \frac{\Gamma(\varpi+1)}{2^{\varpi}}\frac{1}{(\vartheta_2-\vartheta_1)^{(\mathrm{m}_1+\mathrm{m}_2-1)\varpi}}\,{}_{\vartheta_1}I^{(\varpi)}_{\vartheta_2}\Omega\Upsilon \le \beta_{\varpi}(\mathrm{m}_1,\mathrm{m}_2)\frac{\Upsilon(\vartheta_1)+\Upsilon(\vartheta_2)}{2^{\varpi}},$$

where $\beta_{\varpi}(\mathrm{m}_1,\mathrm{m}_2)$ *is fractal Beta function and*

$$\Omega(\varkappa) := (\vartheta_2-\varkappa)^{(\mathrm{m}_1-1)\varpi}(\varkappa-\vartheta_1)^{(\mathrm{m}_2-1)\varpi} + (\varkappa-\vartheta_1)^{(\mathrm{m}_1-1)\varpi}(\vartheta_2-\varkappa)^{(\mathrm{m}_2-1)\varpi},$$

for $\mathrm{m}_1, \mathrm{m}_2 > 0$*.*

Proof. For $\mathrm{t} \in [0,1]$, define

$$\varkappa = \mathrm{t}\vartheta_2 + (1-\mathrm{t})\vartheta_1, \qquad \varkappa_1 = \mathrm{t}\vartheta_1 + (1-\mathrm{t})\vartheta_2.$$

By generalized convexity of Υ, we obtain

$$\Upsilon\left(\frac{\varkappa+\varkappa_1}{2}\right) \le \frac{\Upsilon(\varkappa)+\Upsilon(\varkappa_1)}{2^{\varpi}}$$

substituting $\frac{\varkappa+\varkappa_1}{2} = \frac{\vartheta_1+\vartheta_2}{2}$, we get

$$2^{\varpi}\Upsilon\left(\frac{\vartheta_1+\vartheta_2}{2}\right) \le \Upsilon(\mathrm{t}\vartheta_2+(1-\mathrm{t})\vartheta_1) + \Upsilon(\mathrm{t}\vartheta_1+(1-\mathrm{t})\vartheta_2). \tag{16}$$

Multiplying both sides of inequality (16) by $\frac{1}{\Gamma(1+\varpi)}\mathrm{t}^{(\mathrm{m}_1-1)\varpi}(1-\mathrm{t})^{(\mathrm{m}_2-1)\varpi}$, then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}
&2^{\varpi}\Upsilon\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\frac{1}{\Gamma(1+\varpi)}\int_0^1 \mathrm{t}^{(\mathrm{m}_1-1)\varpi}(1-\mathrm{t})^{(\mathrm{m}_2-1)\varpi}(d\mathrm{t})^{\varpi}\\
&\le \frac{1}{\Gamma(1+\varpi)}\int_0^1 \mathrm{t}^{(\mathrm{m}_1-1)\varpi}(1-\mathrm{t})^{(\mathrm{m}_2-1)\varpi}\Upsilon(\mathrm{t}\vartheta_2+(1-\mathrm{t})\vartheta_1)(d\mathrm{t})^{\varpi}\\
&+ \frac{1}{\Gamma(1+\varpi)}\int_0^1 \mathrm{t}^{(\mathrm{m}_1-1)\varpi}(1-\mathrm{t})^{(\mathrm{m}_2-1)\varpi}\Upsilon(\mathrm{t}\vartheta_1+(1-\mathrm{t})\vartheta_2)(d\mathrm{t})^{\varpi}.
\end{aligned}$$

As a consequence, we obtain

$$\Upsilon\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\beta_{\varpi}(\mathrm{m}_1,\mathrm{m}_2) \le \frac{\Gamma(\varpi+1)}{2^{\varpi}}\frac{1}{(\vartheta_2-\vartheta_1)^{(\mathrm{m}_1+\mathrm{m}_2-1)\varpi}}\,{}_{\vartheta_1}I^{(\varpi)}_{\vartheta_2}\Omega\Upsilon.$$

Since Υ is a generalized convex function, for every $\mathrm{t} \in [0,1]$, we have

$$\Upsilon(\mathrm{t}\vartheta_2+(1-\mathrm{t})\vartheta_1) + \Upsilon(\mathrm{t}\vartheta_1+(1-\mathrm{t})\vartheta_2) \le \Upsilon(\vartheta_1) + \Upsilon(\vartheta_2). \tag{17}$$

Multiplying both sides of inequality (17) by $\frac{1}{\Gamma(1+\varpi)}t^{(\mathfrak{m}_1-1)\varpi}(1-t)^{(\mathfrak{m}_2-1)\varpi}$ and integrating with respect to t over $[0,1]$,

$$\frac{\Gamma(\varpi+1)}{2^{\varpi}}\frac{1}{(\vartheta_2-\vartheta_1)^{(\mathfrak{m}_1+\mathfrak{m}_2-1)\varpi}}\,{}_{\vartheta_1}I^{(\varpi)}_{\vartheta_2}\Omega\Upsilon \leq \beta_{\varpi}(\mathfrak{m}_1,\mathfrak{m}_2)\frac{\Upsilon(\vartheta_1)+\Upsilon(\vartheta_2)}{2^{\varpi}}, \tag{18}$$

and the second inequality is deduced. Hence, the proof is completed. □

Remark 2.2. *If we substitute* $\mathfrak{m}_1 = 1 = \mathfrak{m}_2$ *in Theorem 2.1, then we obtain Theorem 1.5.*

Remark 2.3. *If we set* $\varpi = 1$ *in Theorem 2.1, then we have*

$$\begin{aligned}&\Upsilon\left(\frac{\varkappa+\varkappa_1}{2}\right)\beta(\mathfrak{m}_1,\mathfrak{m}_2)\\&\leq \frac{1}{2(\varkappa_1-\varkappa)^{(\mathfrak{m}_1+\mathfrak{m}_2-1)}}\int_{\varkappa}^{\varkappa_1}\Omega(\varkappa)\Upsilon(\varkappa)d\varkappa\\&\leq \beta(\mathfrak{m}_1,\mathfrak{m}_2)\frac{\Upsilon(\varkappa)+\Upsilon(\varkappa_1)}{2},\end{aligned} \tag{19}$$

which is proved by M. Z. Sarikaya and A. T. A. Fatih in [34].

This leads to different scenarios as:

- *If we get* $\mathfrak{m}_1 = \mathfrak{m}_2 = 1$ *in inequality* (19)*, then it reduced to inequality* (2)*.*
- *If we take* $\mathfrak{m}_1 = 1$, $\mathfrak{m}_2 = \varpi$, *(or* $\mathfrak{m}_2 = 1$, $\mathfrak{m}_1 = \varpi$*) in inequality* (19)*, then it reduced to Theorem 2 proved by Sarikaya et al. in [35].*

2.1. Trapezoid type inequalities via fractal Beta function

In this section, we present a novel Hermite-Hadamard-type identity involving the fractal Beta function. Using this identity, we derive several related results.

Lemma 2.4. *Consider* $\Upsilon : I^o \subset \mathbb{R} \to \mathbb{R}^{\varpi}$ *(*I^o *is the interior of I) such that* $\Upsilon \in D_{\varpi}(I^o)$ *and* $\Upsilon^{\varpi} \in C_{\varpi}[\vartheta_1,\vartheta_2]$ *for* $\vartheta_1,\vartheta_2 \in I^o$ *with* $\vartheta_1 < \vartheta_2$*, then we have the identity:*

$$\begin{aligned}&\frac{\Upsilon(\vartheta_1)+\Upsilon(\vartheta_2)}{2^{\varpi}}\beta_{\varpi}(\mathfrak{m}_1,\mathfrak{m}_2)-\frac{\Gamma(\varpi+1)}{2^{\varpi}}\frac{1}{(\vartheta_2-\vartheta_1)^{(\mathfrak{m}_1+\mathfrak{m}_2-1)\varpi}}\,{}_{\vartheta_1}I^{(\varpi)}_{\vartheta_2}\Omega\Upsilon\\&=\frac{(\vartheta_2-\vartheta_1)^{\varpi}}{2^{\varpi}}\frac{1}{\Gamma(1+\varpi)}\int_0^1\beta_{t\varpi}(\mathfrak{m}_1,\mathfrak{m}_2)\Big[\Upsilon^{(\varpi)}(t\vartheta_2+(1-t)\vartheta_1)-\Upsilon^{(\varpi)}(t\vartheta_1+(1-t)\vartheta_2)\Big](dt)^{\varpi},\end{aligned} \tag{20}$$

where $\beta_{t\varpi}(\mathfrak{m}_1,\mathfrak{m}_2)$ *is fractal incomplete Beta function.*

Proof. By applying integration by parts, we have

$$\begin{aligned}\Upsilon_1 &= \frac{1}{\Gamma(1+\varpi)}\int_0^1\beta_{t\varpi}(\mathfrak{m}_1,\mathfrak{m}_2)\Upsilon^{(\varpi)}(t\vartheta_2+(1-t)\vartheta_1)(dt)^{\varpi}\\&=\frac{\beta_{\varpi}(\mathfrak{m}_1,\mathfrak{m}_2)\Upsilon(\vartheta_2)}{(\vartheta_2-\vartheta_1)^{\varpi}}-\frac{\Gamma(1+\varpi)}{(\vartheta_2-\vartheta_1)^{\varpi}}\\&\times\left(\frac{1}{\Gamma(1+\varpi)}\int_0^1\Upsilon(t\vartheta_2+(1-t)\vartheta_1)\,t^{(\mathfrak{m}_1-1)\varpi}(1-t)^{(\mathfrak{m}_2-1)\varpi}(dt)^{\varpi}\right)\\&=\frac{\beta_{\varpi}(\mathfrak{m}_1,\mathfrak{m}_2)\Upsilon(\vartheta_2)}{(\vartheta_2-\vartheta_1)^{\varpi}}-\frac{\Gamma(1+\varpi)}{(\vartheta_2-\vartheta_1)^{\varpi}}\frac{1}{(\vartheta_2-\vartheta_1)^{(\mathfrak{m}_1+\mathfrak{m}_2-1)\varpi}}\\&\times\left(\frac{1}{\Gamma(1+\varpi)}\int_{\vartheta_1}^{\vartheta_2}(\varkappa-\vartheta_1)^{(\mathfrak{m}_1-1)\varpi}(\vartheta_2-\varkappa)^{(\mathfrak{m}_2-1)\varpi}\Upsilon(\varkappa)(d\varkappa)^{\varpi}\right).\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
\Upsilon_2 &= \frac{1}{\Gamma(1+\varpi)}\int_0^1 \beta_{t\varpi}(m_1,m_2)\Upsilon^{(\varpi)}(t\vartheta_1+(1-t)\vartheta_2)(dt)^{\varpi}\\
&= -\frac{\beta_{\varpi}(m_1,m_2)\Upsilon(\vartheta_1)}{(\vartheta_2-\vartheta_1)^{\varpi}} + \frac{\Gamma(1+\varpi)}{(\vartheta_2-\vartheta_1)^{\varpi}}\\
&\times\left(\frac{1}{\Gamma(1+\varpi)}\int_0^1 \Upsilon\left(t\vartheta_1+(1-t)\vartheta_2\right) t^{(m_1-1)\varpi}(1-t)^{(m_2-1)\varpi}(dt)^{\varpi}\right)\\
&= -\frac{\beta_{\varpi}(m_1,m_2)\Upsilon(\vartheta_1)}{(\vartheta_2-\vartheta_1)^{\varpi}} + \frac{\Gamma(1+\varpi)}{(\vartheta_2-\vartheta_1)^{\varpi}}\frac{1}{(\vartheta_2-\vartheta_1)^{(m_1+m_2-1)\varpi}}\\
&\times\left(\frac{1}{\Gamma(1+\varpi)}\int_{\vartheta_1}^{\vartheta_2} (\varkappa-\vartheta_1)^{(m_2-1)\varpi}\,(\vartheta_2-\varkappa)^{(m_1-1)\varpi}\,\Upsilon(\varkappa)(d\varkappa)^{\varpi}\right).
\end{aligned}$$

By subtracting Υ_2 from Υ_1 and multiplying by $\frac{(\vartheta_2-\vartheta_1)^{\varpi}}{2^{\varpi}}$. Hence, the proof is completed. □

Remark 2.5. *If we set $\varpi = 1$ in Lemma 2.4, then the following equality holds:*

$$\begin{aligned}
&\frac{\Upsilon(\vartheta_1)+\Upsilon(\vartheta_2)}{2}\beta(m_1,m_2) - \frac{1}{2}\frac{1}{(\vartheta_2-\vartheta_1)^{(m_1+m_2-1)}}\int_{\vartheta_1}^{\vartheta_2}\Omega(\varkappa)\Upsilon(\varkappa)d\varkappa\\
&= \frac{(\vartheta_2-\vartheta_1)}{2}\int_0^1 \beta_t(m_1,m_2)[\Upsilon'\left(t\vartheta_2+(1-t)\vartheta_1\right)-\Upsilon'\left(t\vartheta_1+(1-t)\vartheta_2\right)]dt,
\end{aligned} \tag{21}$$

which is proved by M. Z. Sarikaya and A. T. A. Fatih in [34].

This leads to different scenarios as:

- *If in equation (21), we set $m_1 = 1$, $m_2 = \varpi$, (or $m_2 = 1$, $m_1 = \varpi$), then we have Lemma 2 proved by Sarikaya et al. in [35].*
- *If in equation (21), we set $m_1 = m_2 = 1$, then we have Lemma 2.1 proved by Dragomir et al. in [36].*

Remark 2.6. *In Lemma 2.4, when we change the variable, then the equality (20) reduces to*

$$\begin{aligned}
&\frac{\Upsilon(\vartheta_1)+\Upsilon(\vartheta_2)}{2^{\varpi}}\beta_{\varpi}(m_1,m_2) - \frac{\Gamma(\varpi+1)}{2^{\varpi}}\frac{1}{(\vartheta_2-\vartheta_1)^{(m_1+m_2-1)\varpi}}\,{}_{\vartheta_1}I_{\vartheta_2}^{(\varpi)}\Omega\Upsilon\\
&= \frac{(\vartheta_2-\vartheta_1)^{\varpi}}{2^{\varpi}}\frac{1}{\Gamma(1+\varpi)}\int_0^1 [\beta_{t\varpi}(m_1,m_2)-\beta_{(1-t)\varpi}(m_1,m_2)]\Upsilon^{\varpi}\left(t\vartheta_2+(1-t)\vartheta_1\right)(dt)^{\varpi}.
\end{aligned}$$

Theorem 2.7. *Let Υ be defined as in Lemma 2.4 and if the function $|\Upsilon^{(\varpi)}|$ is generalized convex on $[\vartheta_1, \vartheta_2]$, then we have the following inequality:*

$$\begin{aligned}
&\left|\frac{\Upsilon(\vartheta_1)+\Upsilon(\vartheta_2)}{2^{\varpi}}\beta_{\varpi}(m_1,m_2) - \frac{\Gamma(\varpi+1)}{2^{\varpi}}\frac{1}{(\vartheta_2-\vartheta_1)^{(m_1+m_2-1)\varpi}}\,{}_{\vartheta_1}I_{\vartheta_2}^{\varpi}\Omega\Upsilon\right|\\
&\leq \frac{(\vartheta_2-\vartheta_1)^{\varpi}}{2^{\varpi}}\left[|\Upsilon^{(\varpi)}(\vartheta_1)|+|\Upsilon^{(\varpi)}(\vartheta_2)|\right]\left(\frac{1}{\Gamma(1+\varpi)}\int_0^{\frac{1}{2}}|\beta_{t\varpi}(m_1,m_2)-\beta_{(1-t)\varpi}(m_1,m_2)|(dt)^{\varpi}\right),
\end{aligned}$$

for $m_1, m_2 > 0$.

Proof. Utilizing Lemma 2.4 with generalized convexity of $|\Upsilon^{(\varpi)}|$, we get

$$\begin{aligned}
&\left|\frac{\Upsilon(\vartheta_1)+\Upsilon(\vartheta_2)}{2^{\varpi}}\beta_{\varpi}(m_1,m_2)-\frac{\Gamma(\varpi+1)}{2^{\varpi}}\frac{1}{(\vartheta_2-\vartheta_1)^{(m_1+m_2-1)\varpi}}\,{}_{\vartheta_1}I^{\varpi}_{\vartheta_2}\Omega\Upsilon\right|\\
&\leq \frac{(\vartheta_2-\vartheta_1)^{\varpi}}{2^{\varpi}}\left(\frac{1}{\Gamma(1+\varpi)}\int_0^1\left|\beta_{t\varpi}(m_1,m_2)-\beta_{(1-t)\varpi}(m_1,m_2)\right|\left|\Upsilon^{(\varpi)}(t\vartheta_2+(1-t)\vartheta_1\right|(dt)^{\varpi}\right)\\
&\leq \frac{(\vartheta_2-\vartheta_1)^{\varpi}}{2^{\varpi}}\left(\frac{1}{\Gamma(1+\varpi)}\int_0^{\frac{1}{2}}\left[\beta_{t\varpi}(m_1,m_2)-\beta_{(1-t)\varpi}(m_1,m_2)\right]\right.\\
&\left.\times\left[t^{\varpi}|\Upsilon^{(\varpi)}(\vartheta_2)|+(1-t)^{\varpi}|\Upsilon^{(\varpi)}(\vartheta_1)|\right](dt)^{\varpi}\right)+\frac{(\vartheta_2-\vartheta_1)^{\varpi}}{2^{\varpi}}\\
&\times\left(\frac{1}{\Gamma(1+\varpi)}\int_{\frac{1}{2}}^1\left[\beta_{(1-t)\varpi}(m_1,m_2)-\beta_{t\varpi}(m_1,m_2)\right]\left[t^{\varpi}|\Upsilon^{(\varpi)}(\vartheta_2)|+(1-t)^{\varpi}|\Upsilon^{(\varpi)}(\vartheta_1)|\right](dt)^{\varpi}\right)\\
&\leq \frac{(\vartheta_2-\vartheta_1)^{\varpi}}{2^{\varpi}}\left[|\Upsilon^{(\varpi)}(\vartheta_1)|+|\Upsilon^{(\varpi)}(\vartheta_2)|\right]\left(\frac{1}{\Gamma(1+\varpi)}\int_0^{\frac{1}{2}}\left|\beta_{t\varpi}(m_1,m_2)-\beta_{(1-t)\varpi}(m_1,m_2)\right|(dt)^{\varpi}\right).
\end{aligned}$$

Hence, this completes the proof. □

Theorem 2.8. *Let Υ be defined as in Lemma 2.4 and if the function $|\Upsilon^{(\varpi)}|^{q}$, $q>1$ is generalized convex on $[\vartheta_1,\vartheta_2]$, then we have the following inequality:*

$$\begin{aligned}
&\left|\frac{\Upsilon(\vartheta_1)+\Upsilon(\vartheta_2)}{2^{\varpi}}\beta_{\varpi}(m_1,m_2)-\frac{\Gamma(\varpi+1)}{2^{\varpi}}\frac{1}{(\vartheta_2-\vartheta_1)^{(m_1+m_2-1)\varpi}}\,{}_{\vartheta_1}I^{\varpi}_{\vartheta_2}\Omega\Upsilon\right|\\
&\leq \frac{(\vartheta_2-\vartheta_1)^{\varpi}}{2^{\varpi}}\left[\frac{1}{p}\left(\frac{1}{\Gamma(1+\varpi)}\int_0^1|\beta_{t\varpi}(m_1,m_2)-\beta_{(1-t)\varpi}(m_1,m_2)|^{p}(dt)^{\varpi}\right)\right.\\
&\left.+\frac{1}{q}\left(\frac{\Gamma(1+\varpi)(|\Upsilon^{(\varpi)}(\vartheta_2)|^{q}-|\Upsilon^{(\varpi)}(\vartheta_1)|^{q})}{\Gamma(1+2\varpi)}+\frac{|\Upsilon^{(\varpi)}(\vartheta_1)|^{q}}{\Gamma(\varpi+1)}\right)\right],
\end{aligned}$$

for $m_1,m_2>0$, where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Utilizing Lemma 2.4 and Young's inequality with generalized convexity of $|\Upsilon^{(\varpi)}|^{q}$, then we have

$$\begin{aligned}
&\left|\frac{\Upsilon(\vartheta_1)+\Upsilon(\vartheta_2)}{2^{\varpi}}\beta_{\varpi}(m_1,m_2)-\frac{\Gamma(\varpi+1)}{2^{\varpi}}\frac{1}{(\vartheta_2-\vartheta_1)^{(m_1+m_2-1)\varpi}}\,{}_{\vartheta_1}I^{\varpi}_{\vartheta_2}\Omega\Upsilon\right|\\
&\leq \frac{(\vartheta_2-\vartheta_1)^{\varpi}}{2^{\varpi}}\left[\frac{1}{p}\left(\frac{1}{\Gamma(1+\varpi)}\int_0^1\left|\beta_{t\varpi}(m_1,m_2)-\beta_{(1-t)\varpi}(m_1,m_2)\right|^{p}(dt)^{\varpi}\right)\right.\\
&\left.+\frac{1}{q}\left(\frac{1}{\Gamma(1+\varpi)}\int_0^1\left|\Upsilon^{(\varpi)}\,(t\vartheta_2+(1-t)\vartheta_1)\right|^{q}(dt)^{\varpi}\right)\right]\\
&\leq \frac{(\vartheta_2-\vartheta_1)^{\varpi}}{2^{\varpi}}\left[\frac{1}{p}\left(\frac{1}{\Gamma(1+\varpi)}\int_0^1|\beta_{t\varpi}(m_1,m_2)-\beta_{(1-t)\varpi}(m_1,m_2)|^{p}(dt)^{\varpi}\right)\right.\\
&\left.+\frac{1}{q}\left(\frac{1}{\Gamma(1+\varpi)}\int_0^1\left(t^{\varpi}|\Upsilon^{(\varpi)}(\vartheta_2)|^{q}+(1-t)^{\varpi}|\Upsilon^{(\varpi)}(\vartheta_1)|^{q}\right)(dt)^{\varpi}\right)\right]\\
&= \frac{(\vartheta_2-\vartheta_1)^{\varpi}}{2^{\varpi}}\left[\frac{1}{p}\left(\frac{1}{\Gamma(1+\varpi)}\int_0^1|\beta_{t\varpi}(m_1,m_2)-\beta_{(1-t)\varpi}(m_1,m_2)|^{p}(dt)^{\varpi}\right)\right.\\
&\left.+\frac{1}{q}\left(\frac{\Gamma(1+\varpi)(|\Upsilon^{(\varpi)}(\vartheta_2)|^{q}-|\Upsilon^{(\varpi)}(\vartheta_1)|^{q})}{\Gamma(1+2\varpi)}+\frac{|\Upsilon^{(\varpi)}(\vartheta_1)|^{q}}{\Gamma(1+\varpi)}\right)\right].
\end{aligned}$$

Hence, this completes the proof. □

Remark 2.9. *If we set $\varpi = 1$ in Theorem 2.8, then the following inequality holds:*

$$\left|\frac{\Upsilon(\vartheta_1)+\Upsilon(\vartheta_2)}{2}\beta(m_1,m_2)-\frac{1}{2}\frac{1}{(\vartheta_2-\vartheta_1)^{(m_1+m_2-1)}}\int_{\vartheta_1}^{\vartheta_2}\Omega(\varkappa)\Upsilon(\varkappa)d\varkappa\right|$$
$$\leq \frac{(\vartheta_2-\vartheta_1)}{2}\left[\frac{1}{p}\left(\int_0^1|\beta_t(m_1,m_2)-\beta_{(1-t)}(m_1,m_2)|^p dt\right)+\frac{1}{q}\left(\frac{|\Upsilon'(\vartheta_1)|^q+|\Upsilon'(\vartheta_2)|^q}{2}\right)\right],$$

which is a new inequality in literature.

Theorem 2.10. *Let Υ be defined as in Lemma 2.4 and if the function $|\Upsilon^{(\varpi)}|^q$, $q>1$ is generalized convex on $[\vartheta_1,\vartheta_2]$, then we have the following inequality:*

$$\left|\frac{\Upsilon(\vartheta_1)+\Upsilon(\vartheta_2)}{2^\varpi}\beta_\varpi(m_1,m_2)-\frac{\Gamma(\varpi+1)}{2^\varpi}\frac{1}{(\vartheta_2-\vartheta_1)^{(m_1+m_2-1)\varpi}}\,{}_{\vartheta_1}I^{\varpi}_{\vartheta_2}\Omega\Upsilon\right|$$
$$\leq \frac{(\vartheta_2-\vartheta_1)^\varpi}{2^\varpi}\left(\frac{1}{\Gamma(1+\varpi)}\int_0^1|\beta_{t\varpi}(m_1,m_2)-\beta_{(1-t)\varpi}(m_1,m_2)|^p(dt)^\varpi\right)^{\frac{1}{p}}$$
$$\times\left(\frac{\Gamma(1+\varpi)(|\Upsilon^{(\varpi)}(\vartheta_2)|^q-|\Upsilon^{(\varpi)}(\vartheta_1)|^q)}{\Gamma(1+2\varpi)}+\frac{|\Upsilon^{(\varpi)}(\vartheta_1)|^q}{\Gamma(\varpi+1)}\right)^{\frac{1}{q}},$$

for $m_1,m_2>0$*, where* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Utilizing Lemma 2.4 and Hölder's inequality with generalized convexity of $|\Upsilon^{(\varpi)}|^q$, then we have

$$\left|\frac{\Upsilon(\vartheta_1)+\Upsilon(\vartheta_2)}{2^\varpi}\beta_\varpi(m_1,m_2)-\frac{\Gamma(\varpi+1)}{2^\varpi}\frac{1}{(\vartheta_2-\vartheta_1)^{(m_1+m_2-1)\varpi}}\,{}_{\vartheta_1}I^{\varpi}_{\vartheta_2}\Omega\Upsilon\right|$$
$$\leq \frac{(\vartheta_2-\vartheta_1)^\varpi}{2^\varpi}\left(\frac{1}{\Gamma(1+\varpi)}\int_0^1|\beta_{t\varpi}(m_1,m_2)-\beta_{(1-t)\varpi}(m_1,m_2)|^p(dt)^\varpi\right)^{\frac{1}{p}}$$
$$\times\left(\frac{1}{\Gamma(1+\varpi)}\int_0^1\left|\Upsilon^{(\varpi)}(t\vartheta_2+(1-t)\vartheta_1)\right|^q(dt)^\varpi\right)^{\frac{1}{q}}$$
$$\leq \frac{(\vartheta_2-\vartheta_1)^\varpi}{2^\varpi}\left(\frac{1}{\Gamma(1+\varpi)}\int_0^1|\beta_{t\varpi}(m_1,m_2)-\beta_{(1-t)\varpi}(m_1,m_2)|^p(dt)^\varpi\right)^{\frac{1}{p}}$$
$$\times\left(\frac{1}{\Gamma(1+\varpi)}\int_0^1\left(t^\varpi|\Upsilon^{(\varpi)}(\vartheta_2)|^q+(1-t)^\varpi|\Upsilon^{(\varpi)}(\vartheta_1)|^q\right)(dt)^\varpi\right)^{\frac{1}{q}}$$
$$=\frac{(\vartheta_2-\vartheta_1)^\varpi}{2^\varpi}\left(\frac{1}{\Gamma(1+\varpi)}\int_0^1|\beta_{t\varpi}(m_1,m_2)-\beta_{(1-t)\varpi}(m_1,m_2)|^p(dt)^\varpi\right)^{\frac{1}{p}}$$
$$\times\left(\frac{\Gamma(1+\varpi)(|\Upsilon^{(\varpi)}(\vartheta_2)|^q-|\Upsilon^{(\varpi)}(\vartheta_1)|^q)}{\Gamma(1+2\varpi)}+\frac{|\Upsilon^{(\varpi)}(\vartheta_1)|^q}{\Gamma(1+\varpi)}\right)^{\frac{1}{q}}.$$

Hence, this completes the proof. □

2.2. *Midpoint type Inequalities via fractal Beta function*

New midpoint Hermite-Hadamard type identity involving fractal Beta function is presented in this section.

Lemma 2.11. *Consider $\Upsilon : I^o\subset\mathbb{R}\to\mathbb{R}^\varpi$ (I^o is the interior of $I\subset\mathbb{R}$) such that $\Upsilon\in D_\varpi(I^o)$ and $\Upsilon^\varpi\in C_\varpi[\vartheta_1,\vartheta_2]$ for $\vartheta_1,\vartheta_2\in I^o$ with $\vartheta_1<\vartheta_2$, then we have the identity:*

$$\Upsilon\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\beta_\varpi(m_1,m_2)-\frac{\Gamma(\varpi+1)}{2^\varpi}\frac{1}{(\vartheta_2-\vartheta_1)^{(m_1+m_2-1)\varpi}}\,{}_{\vartheta_1}I^{(\varpi)}_{\vartheta_2}\Omega\Upsilon=\frac{(\vartheta_2-\vartheta_1)^\varpi}{2^\varpi}\sum_{k=1}^{4}g_k, \tag{22}$$

where

$$g_1 := \frac{1}{\Gamma(1+\varpi)} \int_0^{\frac{1}{2}} [\beta_{t\varpi}(m_1, m_2)] \Upsilon^{(\varpi)} (t\vartheta_2 + (1-t)\vartheta_1) (dt)^{\varpi},$$
$$g_2 := \frac{1}{\Gamma(1+\varpi)} \int_0^{\frac{1}{2}} [-\beta_{t\varpi}(m_1, m_2)] \Upsilon^{(\varpi)} (t\vartheta_1 + (1-t)\vartheta_2) (dt)^{\varpi},$$
$$g_3 := \frac{1}{\Gamma(1+\varpi)} \int_{\frac{1}{2}}^{1} [-\beta_{(1-t)\varpi}(m_2, m_1)] \Upsilon^{(\varpi)} (t\vartheta_2 + (1-t)\vartheta_1) (dt)^{\varpi},$$
$$g_4 := \frac{1}{\Gamma(1+\varpi)} \int_{\frac{1}{2}}^{1} [\beta_{(1-t)\varpi}(m_2, m_1)] \Upsilon^{(\varpi)} (t\vartheta_1 + (1-t)\vartheta_2) (dt)^{\varpi},$$

for $m_1, m_2 > 0$.

Proof. *Here using integration by parts, we have*

$$\begin{aligned} g_1 &= \frac{1}{\Gamma(1+\varpi)} \int_0^{\frac{1}{2}} [\beta_{t\varpi}(m_1, m_2)] \Upsilon^{(\varpi)} (t\vartheta_2 + (1-t)\vartheta_1) (dt)^{\varpi} \\ &= \frac{\beta_{\frac{\varpi}{2}}(m_1, m_2)}{(\vartheta_2 - \vartheta_1)^{\varpi}} \Upsilon\left(\frac{\vartheta_1 + \vartheta_2}{2}\right) - \frac{\Gamma(1+\varpi)}{(\vartheta_2 - \vartheta_1)^{\varpi}} \\ &\times \left(\frac{1}{\Gamma(1+\varpi)} \int_0^{\frac{1}{2}} t^{(m_1-1)\varpi}(1-t)^{(m_2-1)\varpi} \Upsilon (t\vartheta_2 + (1-t)\vartheta_1) (dt)^{\varpi} \right), \end{aligned}$$

$$\begin{aligned} g_2 &= \frac{1}{\Gamma(1+\varpi)} \int_0^{\frac{1}{2}} [-\beta_{t\varpi}(m_1, m_2)] \Upsilon^{(\varpi)} (t\vartheta_1 + (1-t)\vartheta_2) (dt)^{\varpi} \\ &= \frac{\beta_{\frac{\varpi}{2}}(m_1, m_2)}{(\vartheta_2 - \vartheta_1)^{\varpi}} \Upsilon\left(\frac{\vartheta_1 + \vartheta_2}{2}\right) - \frac{\Gamma(1+\varpi)}{(\vartheta_2 - \vartheta_1)^{\varpi}} \\ &\times \left(\frac{1}{\Gamma(1+\varpi)} \int_0^{\frac{1}{2}} t^{(m_1-1)\varpi}(1-t)^{(m_2-1)\varpi} \Upsilon (t\vartheta_1 + (1-t)\vartheta_2) (dt)^{\varpi} \right), \end{aligned}$$

$$\begin{aligned} g_3 &= \frac{1}{\Gamma(1+\varpi)} \int_{\frac{1}{2}}^{1} [-\beta_{(1-t)\varpi}(m_2, m_1)] \Upsilon^{(\varpi)} (t\vartheta_2 + (1-t)\vartheta_1) (dt)^{\varpi} \\ &= \frac{\beta_{\frac{\varpi}{2}}(m_2, m_1)}{(\vartheta_2 - \vartheta_1)^{\varpi}} \Upsilon\left(\frac{\vartheta_1 + \vartheta_2}{2}\right) - \frac{\Gamma(1+\varpi)}{(\vartheta_2 - \vartheta_1)^{\varpi}} \\ &\times \left(\frac{1}{\Gamma(1+\varpi)} \int_0^{\frac{1}{2}} t^{(m_2-1)\varpi}(1-t)^{(m_1-1)\varpi} \Upsilon (t\vartheta_2 + (1-t)\vartheta_1) (dt)^{\varpi} \right), \end{aligned}$$

$$\begin{aligned} g_4 &= \frac{1}{\Gamma(1+\varpi)} \int_{\frac{1}{2}}^{1} [\beta_{(1-t)\varpi}(m_2, m_1)] \Upsilon^{(\varpi)} (t\vartheta_1 + (1-t)\vartheta_2) (dt)^{\varpi} \\ &= \frac{\beta_{\frac{\varpi}{2}}(m_2, m_1)}{(\vartheta_2 - \vartheta_1)^{\varpi}} \Upsilon\left(\frac{\vartheta_1 + \vartheta_2}{2}\right) - \frac{\Gamma(1+\varpi)}{(\vartheta_2 - \vartheta_1)^{\varpi}} \\ &\times \left(\frac{1}{\Gamma(1+\varpi)} \int_0^{\frac{1}{2}} t^{(m_2-1)\varpi}(1-t)^{(m_1-1)\varpi} \Upsilon (t\vartheta_1 + (1-t)\vartheta_2) (dt)^{\varpi} \right). \end{aligned}$$

Thus, by the above expressions, the desired equality (22) *is obtained.* □

Remark 2.12. *The identity* (22) *reduces to following result by changing the variables:*

$$\Upsilon\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\beta_{\varpi}(\mathrm{m}_1,\mathrm{m}_2)-\frac{\Gamma(\varpi+1)}{2^{\varpi}}\frac{1}{(\vartheta_2-\vartheta_1)^{(\mathrm{m}_1+\mathrm{m}_2-1)\varpi}}\,{}_{\vartheta_1}I^{(\varpi)}_{\vartheta_2}\Omega\Upsilon$$
$$=\frac{(\vartheta_2-\vartheta_1)^{\varpi}}{2^{\varpi}}\frac{1}{\Gamma(1+\varpi)}\int_0^{\frac{1}{2}}\left[\beta_{\mathrm{t}\varpi}(\mathrm{m}_1,\mathrm{m}_2)+\beta_{\mathrm{t}\varpi}(\mathrm{m}_2,\mathrm{m}_1)\right]$$
$$\times\left[\Upsilon^{(\varpi)}\left(\mathrm{t}\vartheta_2+(1-\mathrm{t})\vartheta_1\right)-\Upsilon^{(\varpi)}\left(\mathrm{t}\vartheta_1+(1-\mathrm{t})\vartheta_2\right)\right](d\mathrm{t})^{\varpi}.$$

Theorem 2.13. *Let* $\Upsilon:[\vartheta_1,\vartheta_2]\to\mathbb{R}$ *be a differential function on* $(\vartheta_1,\vartheta_2)$ *with* $\vartheta_1<\vartheta_2$. *If* $|\Upsilon^{(\varpi)}|$ *is generalized convex function on* $[\vartheta_1,\vartheta_2]$, *then the following inequality holds:*

$$\left|\Upsilon\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\beta_{\varpi}(\mathrm{m}_1,\mathrm{m}_2)-\frac{\Gamma(\varpi+1)}{2^{\varpi}}\frac{1}{(\vartheta_2-\vartheta_1)^{(\mathrm{m}_1+\mathrm{m}_2-1)\varpi}}\,{}_{\vartheta_1}I^{(\varpi)}_{\vartheta_2}\Omega\Upsilon\right|$$
$$\leq\frac{(\vartheta_2-\vartheta_1)^{\varpi}}{2^{\varpi}}\left(\left(\frac{1}{2}\right)^{\varpi}\beta_{\varpi}(\mathrm{m}_1,\mathrm{m}_2)-\beta_{\frac{\varpi}{2}}(\mathrm{m}_1+1,\mathrm{m}_2)-\beta_{\frac{\varpi}{2}}(\mathrm{m}_2+1,\mathrm{m}_1)\right)$$
$$\times\left[|\Upsilon^{(\varpi)}(\vartheta_1)|+|\Upsilon^{(\varpi)}(\vartheta_2)|\right],\tag{23}$$

for $\mathrm{m}_1,\mathrm{m}_2>0$.

Proof. By using Remark 2.12 with generalized convexity of $|\Upsilon^{(\varpi)}|$, then we have

$$\left|\Upsilon\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\beta_{\varpi}(\mathrm{m}_1,\mathrm{m}_2)-\frac{\Gamma(\varpi+1)}{2^{\varpi}}\frac{1}{(\vartheta_2-\vartheta_1)^{(\mathrm{m}_1+\mathrm{m}_2-1)\varpi}}\,{}_{\vartheta_1}I^{(\varpi)}_{\vartheta_2}\Omega\Upsilon\right|$$
$$\leq\frac{(\vartheta_2-\vartheta_1)^{\varpi}}{2^{\varpi}}\left(\frac{1}{\Gamma(1+\varpi)}\int_0^{\frac{1}{2}}\left[\beta_{\mathrm{t}\varpi}(\mathrm{m}_1,\mathrm{m}_2)+\beta_{\mathrm{t}\varpi}(\mathrm{m}_2,\mathrm{m}_1)\right]\right)\left[|\Upsilon^{(\varpi)}(\vartheta_1)|+|\Upsilon^{(\varpi)}(\vartheta_2)|\right].\tag{24}$$

Alternating variable in the integrals, we get

$$\frac{1}{\Gamma(1+\varpi)}\int_0^{\frac{1}{2}}[\beta_{\mathrm{t}\varpi}(\mathrm{m}_1,\mathrm{m}_2)+\beta_{\mathrm{t}\varpi}(\mathrm{m}_2,\mathrm{m}_1)](d\mathrm{t})^{\varpi}$$
$$=\frac{1}{\Gamma(1+\varpi)}\int_0^{\frac{1}{2}}\int_0^{\mathrm{t}}\mathfrak{s}^{(\mathrm{m}_1-1)\varpi}(1-\mathfrak{s})^{(\mathrm{m}_2-1)\varpi}(d\mathfrak{s})^{\varpi}(d\mathrm{t})^{\varpi}$$
$$+\frac{1}{\Gamma(1+\varpi)}\int_0^{\frac{1}{2}}\int_0^{\mathrm{t}}\mathfrak{s}^{(\mathrm{m}_2-1)\varpi}(1-\mathfrak{s})^{(\mathrm{m}_1-1)\varpi}(d\mathfrak{s})^{\varpi}(d\mathrm{t})^{\varpi}$$
$$=\left(\frac{1}{2}\right)^{\varpi}\beta_{\varpi}(\mathrm{m}_1,\mathrm{m}_2)-\beta_{\frac{\varpi}{2}}(\mathrm{m}_1+1,\mathrm{m}_2)-\beta_{\frac{\varpi}{2}}(\mathrm{m}_2+1,\mathrm{m}_1).\tag{25}$$

By writing (25) in (24), we obtain the required result. This concludes the proof. □

Remark 2.14. *If in Theorem 2.13, we get* $\varpi=1$, *then the inequality* (23) *becomes as follows:*

$$\left|\Upsilon\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\beta(\mathrm{m}_1,\mathrm{m}_2)-\frac{1}{2(\vartheta_2-\vartheta_1)^{(\mathrm{m}_1+\mathrm{m}_2-1)}}\int_{\vartheta_1}^{\vartheta_2}\Omega(\varkappa)\Upsilon(\varkappa)d\varkappa\right|$$
$$\leq\frac{(\vartheta_2-\vartheta_1)}{2}\left(\frac{1}{2}\beta(\mathrm{m}_1,\mathrm{m}_2)-\beta_{\frac{1}{2}}(\mathrm{m}_1+1,\mathrm{m}_2)-\beta_{\frac{1}{2}}(\mathrm{m}_2+1,\mathrm{m}_1)\right)\left[|\Upsilon'(\vartheta_1)|+|\Upsilon'(\vartheta_2)|\right],\tag{26}$$

which is proved by M. Z. Sarikaya and A. T. A. Fatih in [34].

This leads to different scenarios as:

- *If in equation* (26), *we get* $m_1 = 1 = m_2$, *then the inequality reduces to the inequality (2.3) of Theorem 2.3 by Kirmaci in [37].*
- *If in equation* (26), *we get* $m_1 = 1, m_2 = \varpi$, *(or* $m_1 = \varpi, m_2 = 1$*), then the inequality reduces to the inequality (3) of Theorem 2 by Iqbal et al. in [38].*
- *If in equation* (26), *we get* $m_1 = 1, m_2 = \frac{\varpi}{k}$, *(or* $m_2 = 1, m_1 = \frac{\varpi}{k}$*), then the inequality reduces to the inequality of Corollary 9 by Sarikaya and Ertugral in [39].*

3. Applications

In this section, applications are given to show the effectiveness of the newly established results.

3.1. Application to special mean

As in [40], let us recall the following generalized arithmetic mean:

$$\mathbb{A}_\varpi(\vartheta_1, \vartheta_2) := \frac{{\vartheta_1}^\varpi + {\vartheta_2}^\varpi}{2^\varpi}.$$

Proposition 3.1. *Let* $0 \le \vartheta_1 < \vartheta_2$ *and* $m_2 \in \mathbb{N}, m_2 \ge 2$, *then the following inequality holds:*

$$\begin{aligned}
&[\mathbb{A}_\varpi(\vartheta_1, \vartheta_2)]^{m_2}\beta_\varpi(m_1, m_2) \\
&\le \frac{\Gamma(\varpi+1)}{2^\varpi}\frac{1}{(\vartheta_2-\vartheta_1)^{(m_1+m_2-1)\varpi}}\left(\frac{1}{\Gamma(\varpi+1)}\int_{\vartheta_1}^{\vartheta_2}\Omega(\varkappa)\varkappa^{m_2\varpi}d(\varkappa)^\varpi\right) \\
&\le \beta_\varpi(m_1, m_2)\mathbb{A}_\varpi({\vartheta_1}^{m_2}, {\vartheta_2}^{m_2}). \qquad (27)
\end{aligned}$$

Proof. The required inequality can be obtained by applying Theorem 2.1 to the generalized convex function $\Upsilon(\varkappa) = \varkappa^{m_2\varpi}$, where $\varkappa > 0$.

$$\Upsilon\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\beta_\varpi(m_1, m_2) \le \frac{\Gamma(\varpi+1)}{2^\varpi}\frac{1}{(\vartheta_2-\vartheta_1)^{(m_1+m_2-1)\varpi}}\,{}_{\vartheta_1}I_{\vartheta_2}^{(\varpi)}\Omega\Upsilon \le \beta_\varpi(m_1, m_2)\frac{\Upsilon(\vartheta_1)+\Upsilon(\vartheta_2)}{2^\varpi}. \qquad (28)$$

Consider the function $\Upsilon : (0, \infty) \to \mathbb{R}^\varpi$, $\Upsilon(\varkappa) = \varkappa^{m_2\varpi}$, $m_2 \in \mathbb{Z}\backslash\{-1, 0\}$. Then for $0 \le \vartheta_1 < \vartheta_2$, we have

$$\begin{aligned}
&\Upsilon\left(\frac{\vartheta_1+\vartheta_2}{2}\right) = [\mathbb{A}_\varpi(\vartheta_1, \vartheta_2)]^{m_2}, \\
&\frac{\Upsilon(\vartheta_1)+\Upsilon(\vartheta_2)}{2^\varpi} = \mathbb{A}_\varpi({\vartheta_1}^{m_2}, {\vartheta_2}^{m_2}).
\end{aligned}$$

Hence, the required result is obtained. □

Proposition 3.2. *Let* $0 \le \vartheta_1 < \vartheta_2$ *and* $m_2 \in \mathbb{N}, m_2 \ge 2$, *then the following inequality holds:*

$$\begin{aligned}
&\left|\left[\mathbb{A}_\varpi({\vartheta_1}^{m_2}, {\vartheta_2}^{m_2})\right]\beta_\varpi(m_1, m_2) - \frac{\Gamma(\varpi+1)}{2^\varpi(\vartheta_2-\vartheta_1)^{(m_1+m_2-1)\varpi}}\left(\frac{1}{\Gamma(\varpi+1)}\int_{\vartheta_1}^{\vartheta_2}\Omega(\varkappa)\varkappa^{m_2\varpi}(d\varkappa)^\varpi\right)\right| \\
&\quad \le \frac{(\vartheta_2-\vartheta_1)^\varpi}{2^\varpi}\left[\frac{\Gamma(1+m_2\varpi)}{\Gamma(1+(m_2-1)\varpi)}\mathbb{A}_\varpi({\vartheta_1}^{m_2-1}, {\vartheta_2}^{m_2-1})\right] \\
&\quad \times\left(\frac{1}{\Gamma(1+\varpi)}\int_0^{\frac{1}{2}}\left|\beta_{t\varpi}(m_1, m_2) - \beta_{(1-t)\varpi}(m_1, m_2)\right|(dt)^\varpi\right). \qquad (29)
\end{aligned}$$

Proof. The required inequality can be obtained by applying Theorem 2.7 to the generalized convex function $\Upsilon(\varkappa) = \varkappa^{m_2\varpi}$, where $\varkappa > 0$.

$$\left|\frac{\Upsilon(\vartheta_1)+\Upsilon(\vartheta_2)}{2^{\varpi}}\beta_{\varpi}(m_1,m_2) - \frac{\Gamma(\varpi+1)}{2^{\varpi}}\frac{1}{(\vartheta_2-\vartheta_1)^{(m_1+m_2-1)\varpi}}\,{}_{\vartheta_1}I^{\varpi}_{\vartheta_2}\Omega\Upsilon\right|$$
$$\leq \frac{(\vartheta_2-\vartheta_1)^{\varpi}}{2^{\varpi}}\left[|\Upsilon^{(\varpi)}(\vartheta_1)| + |\Upsilon^{(\varpi)}(\vartheta_2)|\right]\left(\frac{1}{\Gamma(1+\varpi)}\int_0^{\frac{1}{2}}|\beta_{t\varpi}(m_1,m_2) - \beta_{(1-t)\varpi}(m_1,m_2)|(dt)^{\varpi}\right).$$

Consider the function $\Upsilon : (0,\infty) \to \mathbb{R}^{\varpi}$, $\Upsilon(\varkappa) = \varkappa^{m_2\varpi}$, $m_2 \in \mathbb{Z}\backslash\{-1,0\}$. Then for $0 \leq \vartheta_1 < \vartheta_2$, we have

$$\Upsilon\left(\frac{\vartheta_1+\vartheta_2}{2}\right) = [\mathbb{A}_{\varpi}(\vartheta_1,\vartheta_2)]^{m_2},$$
$$\frac{\Upsilon(\vartheta_1)+\Upsilon(\vartheta_2)}{2^{\varpi}} = \mathbb{A}_{\varpi}({\vartheta_1}^{m_2},{\vartheta_2}^{m_2}).$$

Hence, we obtain the required inequality. □

3.2. Application to Mittag-Leffler function

The special Mittag-Leffler function is defined as follows:

$$\mathbb{E}_{\varpi}(\varkappa) := \sum_{\mathfrak{k}=0}^{\infty}\frac{\varkappa^{\mathfrak{k}}}{\Gamma(1+\varpi\mathfrak{k})},\ \ \varpi \in C,\ \ \mathbb{R}(\varpi) > 0,\ \ \varkappa \in C,$$

where

$$\frac{d^{\varpi}}{(d\varkappa)^{\varpi}}(\mathbb{E}_{\varpi}(\varkappa^{\varpi})) = \mathbb{E}_{\varpi}(\varkappa^{\varpi}). \tag{30}$$

Proposition 3.3. *Let $\vartheta_1, \vartheta_2 \in \mathbb{R}$ and $0 \leq \vartheta_1 < \vartheta_2$, then we have*

$$\mathbb{E}_{\varpi}\left(\left(\frac{\vartheta_1+\vartheta_2}{2}\right)^{\varpi}\right)\cdot\beta_{\varpi}(m_1,m_2)$$
$$\leq \left(\frac{1}{2}\right)^{\varpi}\frac{1}{(\vartheta_2-\vartheta_1)^{(m_1+m_2-1)\varpi}}\int_{\vartheta_1}^{\vartheta_2}\Omega(\varkappa)\mathbb{E}_{\varpi}(\varkappa^{\varpi})(d\varkappa)^{(\varpi)}$$
$$\leq \beta_{\varpi}(m_1,m_2)\cdot\frac{\mathbb{E}_{\varpi}({\vartheta_1}^{\varpi})+\mathbb{E}_{\varpi}({\vartheta_2}^{\varpi})}{2^{\varpi}}. \tag{31}$$

Proof. Let $\Upsilon(\varkappa) := \mathbb{E}_{\varpi}(\varkappa^{\varpi})$. It is easy to see that Υ is generalized convex function and proof can be obtained by employing this function in Theorem 2.1. □

Proposition 3.4. *Let $\vartheta_1, \vartheta_2 \in \mathbb{R}$ and $0 \leq \vartheta_1 < \vartheta_2$, then we get*

$$\left|\frac{\mathbb{E}_{\varpi}((\vartheta_1)^{\varpi})+\mathbb{E}_{\varpi}((\vartheta_2)^{\varpi})}{2^{\varpi}}\cdot\beta_{\varpi}(m_1,m_2) - \left(\frac{1}{2}\right)^{\varpi}\frac{1}{(\vartheta_2-\vartheta_1)^{(m_1+m_2-1)\varpi}}\int_{\vartheta_1}^{\vartheta_2}\Omega(\varkappa)\mathbb{E}_{\varpi}(\varkappa^{\varpi})(d\varkappa)^{\varpi}\right|$$
$$\leq \frac{(\vartheta_2-\vartheta_1)^{\varpi}}{2^{\varpi}}\left[|\mathbb{E}_{\varpi}({\vartheta_1}^{\varpi})| + |\mathbb{E}_{\varpi}({\vartheta_2}^{\varpi})|\right]\left(\frac{1}{\Gamma(1+\varpi)}\int_0^{\frac{1}{2}}|\beta_{t\varpi}(m_1,m_2) - \beta_{(1-t)\varpi}(m_1,m_2)|(dt)^{\varpi}\right).$$

Proof. Let $\Upsilon(\varkappa) := \mathbb{E}_{\varpi}(\varkappa^{\varpi})$. It is easy to see that Υ is generalized convex function and proof can be obtained by employing this function in Theorem 2.7. □

4. Discussion

To validate the theoretical contributions of Theorem 2.1, we perform a numerical investigation of the fractal Hermite-Hadamard inequalities using two distinct generalized convex functions: a polynomial $\Upsilon(\varkappa) = \varkappa^{2\varpi}$ and an exponential $\Upsilon(\varkappa) = e^{\varkappa\varpi}$. Parameters $m_1 = 1.5$ and $m_2 = 2$ were chosen to illustrate non-integer fractal orders, while $\varpi \in (0, 1]$ spans the range from highly irregular ($\varpi \to 0$) to smooth ($\varpi = 1$) domains. Figures 1 and 2 visualize the interplay between the fractal Beta function $\beta_\varpi(m_1, m_2)$, the kernel $\Omega(\varkappa)$, and the integral mean, demonstrating consistency with classical calculus at $\varpi = 1$ and adaptability to non-differentiable systems for $\varpi < 1$.

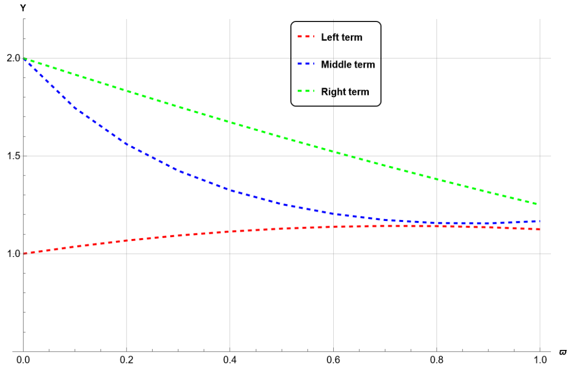

Figure 1: Bounds and integral mean for $\Upsilon(\varkappa) = \varkappa^{2\varpi}$ under Theorem 2.1.

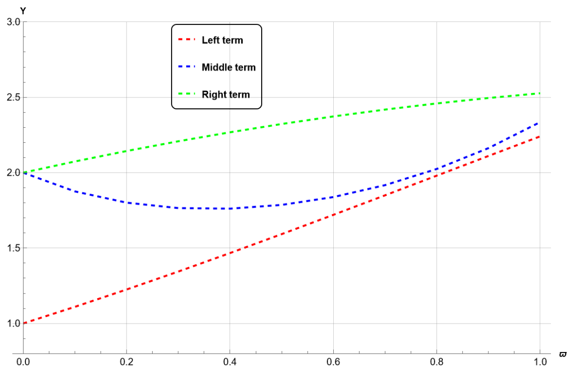

Figure 2: Bounds and integral mean for $\Upsilon(\varkappa) = e^{\varkappa\varpi}$ under Theorem 2.1.

Figure 1 illustrates the bounds for $\Upsilon(\varkappa) = \varkappa^{2\varpi}$. The lower bound $\Upsilon\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\beta_\varpi(m_1, m_2)$ and upper bound $\beta_\varpi(m_1, m_2) \cdot \frac{\Upsilon(\vartheta_1)+\Upsilon(\vartheta_2)}{2^\varpi}$ exhibit monotonic growth with ϖ, governed by the scaling behavior of $\beta_\varpi(m_1, m_2)$. The integral mean given in Theorem 2.1 as $\frac{\Gamma(1+\varpi)}{2^\varpi(\vartheta_2-\vartheta_1)^{(m_1+m_2-1)\varpi}} \, {}_{\vartheta_1}I_{\vartheta_2}^{(\varpi)}\Omega\Upsilon$ remains confined between these bounds, with the shaded region narrowing as $\varpi \to 1$. This reflects the regularization effect of increasing fractal dimensionality, where smoother integration reduces uncertainty in the mean estimate. For $\Upsilon(\varkappa) = e^{\varkappa\varpi}$ (Figure 2), the rapid growth of the exponential function amplifies the disparity between bounds, particularly for $\varpi < 0.5$. Despite this divergence, the integral mean remains strictly bounded, confirming the inequality's robustness even under pronounced irregularity. The convergence of bounds to classical results at $\varpi = 1$ underscores the framework's backward compatibility, while the widening gap at lower ϖ quantifies the computational cost of non-differentiability-a critical insight for modeling fractal phenomena like anomalous diffusion. The numerical results affirm Theorem 2.1's capacity to unify classical and fractal analysis. The polynomial case highlights the role of $\beta_{\varpi}(\mathrm{m}_1, \mathrm{m}_2)$ in scaling bounds proportionally to ϖ, while the exponential case demonstrates adaptability to sharp functional growth. Together, they validate the generalized Beta function's utility in bounding integrals over irregular domains, offering a mathematical toolkit for interdisciplinary applications-from characterizing heterogeneous materials to optimizing stochastic processes. The parameter-driven design ensures flexibility, enabling tailored bounds for specific fractal regimes without sacrificing theoretical rigor.

5. Conclusion

This study advances the development of Hermite-Hadamard type inequalities within Yang's fractal calculus, utilizing the generalized Beta function to address the mathematical challenges of non-differentiable systems. By unifying fractal integration with convex function theory, we establish robust bounds for integral means in irregular domains, offering tools to model phenomena such as anomalous diffusion, material fractures, and chaotic dynamics. The derived trapezoidal and midpoint inequalities, validated through applications to special means and Mittag-Leffler functions, bridge theoretical fractal mathematics with practical interdisciplinary problems in physics, engineering, and data science. Computational visualizations corroborate the theoretical findings, illustrating the sensitivity of the derived bounds to fractal parameters and underscoring their relevance for complex real-world systems. Future work should extend these results to broader function classes, multidimensional fractal systems, and advanced operators like variable-order calculus, while fostering computational methods to solve fractal integrals efficiently. This research not only enriches fractal theory but also equips scientists and engineers with scalable frameworks to analyze and optimize complex, real-world systems governed by irregularity.

References

[1] T. Tunç, Í. Demir, *Some trapezoid-type inequalities for newly defined proportional Caputo-hybrid operator*, J. Inequal. Math. Anal. **1** (2025), 65–78.

[2] A. Lakhdari, H. Budak, N. Mlaiki, B. Meftah, T. Abdeljawad, *New insights on fractal–fractional integral inequalities: Hermite–Hadamard and Milne estimates*, Chaos Solitons Fractals **193** (2025), Article 116087.

[3] C. Chesneau, *Theoretical results on a special two-parameter trivariate Hilbert-type integral inequality*, J. Inequal. Math. Anal. **1** (2025), 1–14.

[4] S. S. Dragomir, C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.

[5] C. P. Niculescu, L. E. Persson, *Convex functions and their applications: A contemporary approach*, 2nd ed., CMS Books in Mathematics, Springer, Cham, 2018.

[6] S. I. Butt, M. K. Bakula, D. Pečarić, J. Pečarić, *Jensen-Grüss inequality and its applications for the Zipf-Mandelbrot law*, Math. Methods Appl. Sci. **44** (2021), 1664–1673.

[7] C. Liang, S. Shaokat, A. Razzaq, K. H. Hakami, *Some new integral inequalities for F-convex functions via ABK-fractional operator*, J. Math. Anal. Appl. **542** (2024), Article 128876.

[8] P. Agarwal, S. S. Dragomir, M. Jleli, B. Samet, *Advances in Mathematical Inequalities and Applications*, Springer, Singapore, 2018.

[9] M. E. Ozdemir, S. I. Butt, B. Bayraktar, J. Nasir, *Several integral inequalities for (ϖ, s, m)-convex functions*, AIMS Math. **5** (2020), 3906–3921.

[10] J. E. N. Valdés, F. Rabossi, A. D. Samaniego, *Convex functions: Ariadne's thread or Charlotte's Spiderweb?*, Adv. Math. Models Appl. **5** (2020).

[11] J. E. N. Valdés, *A Note on the Hermite-Hadamard Inequalities via Fourier Transform*, (2025).

[12] C. P. Niculescu, L. E. Persson, *Old and new on the Hermite-Hadamard inequality*, Real Anal. Exch. **29** (2003), 663–686.

[13] S. I. Butt, A. Kashuri, M. Tariq, J. Nasir, A. Aslam, W. Gao, *Hermite-Hadamard type inequalities via n-polynomial exponential-type convexity and their applications*, Adv. Difference Equ. **2020** (2020), 1–25.

[14] A. Carpinteri, P. Cornetti, *A Fractional calculus approach to the description of stress and strain localization in fractal media*, Chaos Solitons Fractals **13** (2002), 85–94.

[15] X. Yang, F. Gao, *Fundamentals of Local Fractional Iteration of the Continuously Non-Differentiable Functions Derived from Local Fractional Calculus*, Springer, 2011.

[16] H. Mo, X. Sui, D. Yu, *Generalized convex functions and some inequalities on fractal sets*, arXiv:1404.3964.

[17] S. Ivelić Bradanović, N. Lovričević, *Generalized Jensen and Jensen–Mercer inequalities for strongly convex functions with applications*, J. Inequal. Appl. **2024** (2024), Article 112.

[18] T. Du, H. Wang, M. A. Khan, Y. Zhang, *Certain integral inequalities considering generalized m-convexity on fractal sets and their applications*, Fractals **27** (2019), Article 1950117.

[19] A. Lakhdari, H. Budak, N. Mlaiki, B. Meftah, T. Abdeljawad, *New insights on fractal-fractional integral inequalities: Hermite-Hadamard and Milne estimates*, Chaos Solitons Fractals **193** (2025), Article 116087.

[20] S. I. Butt, P. Agarwal, S. Yousaf, J. L. Guirao, *Generalized Fractal Jensen and Jensen-Mercer inequalities for harmonic convex function with applications*, J. Inequal. Appl. **2022** (2022), 1–18.
[21] A. Razzaq, I. Javed, J. E. Napoles Valdés, F. M. González, *Hermite-Hadamard inequalities for generalized ($\mathfrak{m}$-$\mathfrak{F}$)-convex function in the framework of local fractional integrals*, Ann. Univ. Craiova Math. Comp. Sci. Ser. **51** (2024), 198–222.
[22] M. Sarikaya, H. Budak, *Generalized Ostrowski type inequalities for local fractional integrals*, Proc. Am. Math. Soc. **145** (2017), 1527–1538.
[23] M. Z. Sarikaya, T. Tunç, H. Budak, *On generalized some integral inequalities for local fractional integrals*, Appl. Math. Comput. **276** (2016), 316–323.
[24] H. Budak, M. Z. Sarikaya, H. Yildirim, *New inequalities for local fractional integrals*, Iran. J. Sci. Technol. Trans. A Sci. **41** (2017), 1039–1046.
[25] A. Akkurt, M. Z. Sarikaya, H. Budak, H. Yildirim, *Generalized Ostrowski type integral inequalities involving generalized moments via local fractional integrals*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. **111** (2017), 797–807.
[26] M. Sarikaya, H. Budak, *Generalized Ostrowski type inequalities for local fractional integrals*, Proc. Am. Math. Soc. **145** (2017), 1527–1538.
[27] S. P. Yan, H. Jafari, H. K. Jassim, *Local fractional Adomain decomposition and function decomposition methods for Laplace equation within local fractional operators*, Adv. Math. Phys. **7** (2014), Article 161580.
[28] Y. Z. Zhang, A. M. Yang, Y. Long, *Initial boundary value problem for fractal heat equation in the semi-infinite region by Yang-Laplace transform*, Therm. Sci. **18** (2014), 677–681.
[29] X. J. Yang, *Advanced Local Fractional Calculus and Its Applications*, World Science Publisher, New York, 2012.
[30] T. Tunç, H. Budak, F. Usta, M. Z. Sarikaya, *New Hermite-Hadamard type inequalities on fractal set*, Int. J. Nonlinear Anal. Appl. **12** (2021), 782–789.
[31] M. Vivas, J. Hernández, N. Merentes, *New Hermite-Hadamard and Jensen type inequalities for h-convex functions on fractal sets*, Rev. Colomb. Mat. **50** (2016), 145–164.
[32] R. Hussain, A. Ali, G. Gulshan, A. Latif, M. Muddassar, *Generalized co-ordinated integral inequalities for convex functions by way of k-fractional derivatives*, J. Comput. Anal. Appl. **22** (2017), 1208–1219.
[33] X. J. Yang, *Advanced Local Fractional Calculus and Its Applications*, World Science Publisher, 2012.
[34] M. Z. Sarikaya, A. T. A. Fatih, *On the Generalized Hermite-Hadamard inequalities involving Beta function*, Konuralp J. Math. **9** (2021), 112–118.
[35] M. Z. Sarikaya, E. Set, H. Yaldiz, N. Basak, *Hermite-Hadamard's Inequalities for fractional integrals and related fractional inequalities*, Math. Comput. Modelling **57** (2013), 2403–2407.
[36] S. S. Dragomir, R. P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. Lett. **11** (1998), 91–95.
[37] U. S. Kirmaci, *Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula*, Appl. Math. Comput. **147** (2004), 137–146.
[38] M. Iqbal, M. I. Bhatti, K. Nazeer, *Generalization of inequalities analogous to Hermite-Hadamard inequality via fractional integrals*, Bull. Korean Math. Soc. **52** (2015), 707–716.
[39] M. Z. Sarikaya, F. Ertugral, *On the Generalized Hermite-Hadamard inequalities*, Ann. Univ. Craiova Math. Comp. Sci. Ser. **47** (2020), 193–213.
[40] C. E. M. Pearce, J. Pečarić, *Inequalities for differentiable mappings with application to special means and quadrature formulae*, Appl. Math. Lett. **13** (2000), 51–55.