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Numerical solution Fredholm integral equation of the second kind by the optimal quadrature method

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Abstract. In mathematical modeling and computational mathematics, special attention is paid to the creation of various optimal calculation methods. This article is devoted to the construction of optimal quadrature formulas with derivative in the space of differentiable functions using the Sobolev method. This quadrature formula consists of a linear combination of the values of the interval $[0,1]$ up to the second derivative of the function at all nodes. The error of the quadrature formulas is estimated by the norm of the error function. We obtain the optimal quadrature formula by minimizing the norm of the error functional by the coefficients of the quadrature formula with derivative. The resulting optimal quadrature formulas are exact for all functions which polinomials degree of $m-1$. In addition, some methods for the numerical solution of Fredholm integral equations of the second kind are given. These methods are optimal quadrature formulas and Simpson's method. Numerical examples are provided to demonstrate the effectiveness and accuracy of the work presented.

1. Introduction

Numerical analysis is a branch of mathematics that deals with the development of effective methods for obtaining numerical solutions to complex problems. Due to the development of computing technology, numerical problem solving is developing. As a result, many practical software packages (Mathlab, Wolframalfa, Maple, C++, Python, etc.) were created to solve complex problems effectively and easily. These applications solve problems using numerical methods, where the user can get results by entering the necessary variables without knowing the theoretical side of the numerical method. So, the question arises as to why we need to learn the number of methods when the application package is created. Here are some reasons to have a basic understanding of the theoretical foundations of numerical methods:
1. The study of numerical methods and their analysis allows the development of new numerical methods or, if the existing methods do not provide sufficient accuracy, it is necessary to choose another effective

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method.
2. There are many ways to solve problems, but it is important to choose the right method to achieve a clear result in a short time.
3. It is important to use numerical methods appropriately based on the specifics of the problem, and when the results are not as expected, it is important to understand what is going wrong in the program.

It is known that optimal quadrature formulas with derivative are methods of approximate calculation of definite integrals. They are necessary for calculating the integrals when an antiderivative of the functions under the integral cannot be expressed by elementary functions or the integrand exists only at discrete points. The use of formulas with a high level of algebraic accuracy in the integration of non-smooth functions does not lead to the good results. Therefore, it is important to construct optimal quadrature formulas with derivatives in the space of differentiable functions and estimate their errors. Several studies have been conducted on this approach [1–3, 6, 21, 22]. In [20], Fourier integrals were numerically solved using optimal quadrature formulas and these formulas were used for image reconstruction. Using the values of functions up to the first order derivative at the nodes, quadrature formulas were constructed and proved to be more efficient than classical formulas [7, 8, 23–25].

In this work, optimal quadrature formulas with derivatives are obtained based on the variational approach for the approximate calculation of definite integrals and numerical solutions of integral equations using the values of the given function up to the second order derivative at the nodes. We consider the following quadrature formula with derivatives

$$\int_0^1 u(x)dx \cong \sum_{\beta=0}^{N} C_0\,[\beta]u[\beta] + \frac{h^2}{12}(u'[0] - u'[1]) + \sum_{\beta=0}^{N} C_1\,[\beta]u''[\beta] \tag{1}$$

here $C_0[\beta] = \begin{cases} \frac{h}{2}, & \beta = 0, N, \\ h, & \beta = \overline{1, N-1}, \end{cases}$ $C_1[\beta]$ and $[\beta]$ are the coefficients and the nodes of the quadrature formula (1), respectively.

We suppose that functions $u(x)$ belong to the Hilbert space

$$L_2^{(m)}(0,1) = \left\{u : [0,1] \rightarrow \mathbb{R} \;\middle|\; u^{(m-1)} \textit{ is abs. cont. and } u^{(m)} \in L_2(0,1) \right\},$$

equipped with the norm

$$\| u(x)\|_{L_2^{(m)}(0,1)} = \left\{ \int_0^1 (u^{(m)}(x))^2 dx \right\}^{1/2}$$

and $\int_0^1 (u^{(m)}(x))^2 dx < \infty$. The last equality is the semi-norm and $\| u \|= 0$ if and only if $u(x) = P_{m-1}(x)$, where $P_{m-1}(x)$ is a polynomial of degree $m-1$.

The difference below is called the error of the quadrature formula (1)

$$(\ell_N, u) = \int_0^1 u(x)dx - \sum_{\beta=0}^{N} C_0\,[\beta]u[\beta] - \frac{h^2}{12}(u'[0] - u'[1]) - \sum_{\beta=0}^{N} C_1\,[\beta]u''[\beta]$$

with error functional

$$\ell_N(x) = \varepsilon_{[0,1]}(x) - \sum_{\beta=0}^{N} C_0[\beta]\delta(x-\beta) + \frac{h^2}{12}(\delta'(x) - \delta'(x-1)) - \sum_{\beta=0}^{N} C_1[\beta]\delta''(x-\beta) \tag{2}$$

where $\varepsilon_{[0,1]}(x)$ is the indicator of the interval $[0,1]$, $\delta(x)$ is the Dirac delta-function, $[\beta]=h\beta$, $h=\frac{1}{N}$.

The error functional $\ell_N(x)$ corresponding to the quadrature formula (1) is a linear, continuous functional defined in the conjugate space $L_2^{(m)*}(0,1)$.

According to the definition of the functional norm

$$\left\|\ell_N\left|L_2^{(m)^*}\right.\right\| = \sup_{u,\ \|u\|\neq 0}\frac{|(\ell_N,u)|}{\left\|u|L_2^{(m)}\right\|}$$

from this, using the definition of supremum, we get the Cauchy-Schwarz inequality

$$|(\ell_N,u| \leq \| u|L_2^{(m)} \| \cdot \| \ell_N|L_2^{(m)^*} \|$$

As it can be seen from this inequality, the error of the quadrature formula with derivative (1) is estimated by the product of the norm of the error functional $\ell_N(x)$ obtained from the conjugate space $L_2^{(m)^*}$ and the norm of the function u obtained from the space $L_2^{(m)}(0,1)$

$$\left\|\ell_N\left|L_2^{(m)^*}\right.\right\| = \sup_{u,\ \left\|u|L_2^{(m)}\right\|=1}|(\ell_N,u)| \tag{3}$$

Thus, the evaluation of the error of the quadrature formula (1) on the $L_2^{(m)}(0,1)$ space elements (2) is related to the norm of the error functional ℓ_N in the $L_2^{(m)*}(0,1)$ conjugate space. The norm of the error functional ℓ_N depends on the coefficients and nodes of the quadrature formula. In this work, we solve the problem of minimizing the norm of the ℓ_N error functional only by coefficients when the nodes are fixed. The quadrature formula with derivatives is constructed using the Sobolev method based on the discrete analog of the d^{2m-4}/dx^{2m-4} differential operator in $L_2^{(m)}(0,1)$ space [1]. Since the error function ℓ_N is defined in space $L_2^{(m)}(0,1)$, it satisfies the following conditions

$$(\ell_N,x^\alpha)=0,\quad \alpha=0,1,2,...,m-1. \tag{4}$$

Therefore, the condition $N\geq m-2$ must be fulfilled for quadrature formulas with derivatives of the form (1) to be exist. As noted above by the Cauchy-Schwarz inequality, the error of the quadrature formula (1) is estimated by the norm of the error functional (2). Moreover, the norm of the error functional (2) depends on $C_1[\beta]$ coefficients, and we minimize the norm of the error functional by $C_1[\beta]$ coefficients. In order to construct the optimal quadrature formula with derivatives in the form (1), we need to solve the following problems:

Problem 1. *Finding the general form of the norm of the error functional (2) of the quadrature formula with derivatives of the form (1) in* $L_2^{(m)}(0,1)$ *space.*

Problem 2. *To find the coefficients that give the minimum to the norm of the error functional.*

2. Main results

In this section, we present certain results and the main theorems of this work.

Lemma: (see, [4]). The following equality holds for the roots of the Euler polynomial

$$\sum_{k=1}^{m-3}\sum_{t=0}^{\alpha}\frac{a_kq_k+(-1)^{t+1}b_kq_k^{N+t}}{(q_k-1)^{t+1}}\Delta^t0^\alpha=(-1)^{\alpha+1}\sum_{k=1}^{m-3}\sum_{t=0}^{\alpha}\frac{a_kq_k^t+(-1)^{t+1}b_kq_k^{N+1}}{(1-q_k)^{t+1}}\Delta^t0^\alpha$$

here $\Delta^i\gamma^\alpha$ is the finite difference of order i of γ^j, $\Delta^t0^\alpha=\sum_{l=1}^{t}(-1)^{t-l}C_t^l l^\alpha$, $q_k<1$.

To solve **Problem 1**, according to the Riesz theorem (see, Theorem 2.5.8 (Riesz representation theorem) in [5]) about the general form of a linear continuous functional and the concept of an extremal function. The norm of the error function of the quadrature formula can be written as follows [6–8]:

$$\|\ell_N\|^2 = (\ell_N, U_\ell) = (\ell_N(x), ((-1)^m \ell_N(x) * G_{m-2}(x) + P_{m-1}(x)))$$

where

$$U_\ell(x) = (-1)^m \ell_N(x) * G_{m-2}(x) + P_{m-1}(x),$$

$$G_{m-2}(x) = \frac{|x|^{2m-5}}{2 \cdot (2m-5)!}$$

is a solution of the equation

$$\frac{d^{2m-4}}{dx^{2m-4}} G_{m-2}(x) = \delta(x).$$

The following is true [9].

Theorem 2.1. *An overview of the norm of the error functional (2) corresponding to the quadrature formula with derivative (1) is as follows*

$$\|\ell_N\|^2 = (-1)^m \Bigg[\sum_{\beta=0}^{N} C_1[\beta] \sum_{\gamma=0}^{N} C_1[\gamma] \frac{\big|[\beta]-[\gamma]\big|^{2m-5}}{2(2m-5)!} - 2\sum_{\beta=0}^{N} C_1[\beta] \int_0^1 \frac{\big|x-[\beta]\big|^{2m-3}}{2(2m-3)!} dx$$

$$- \frac{h^2}{6} \sum_{\beta=0}^{N} C_1[\beta] \left[\frac{([\beta])^{2m-4} + (1-[\beta])^{2m-4}}{2(2m-4)!} \right] + 2\sum_{\beta=0}^{N} C_0[\beta] \sum_{\gamma=0}^{N} C_1[\gamma] \frac{\big|[\beta]-[\gamma]\big|^{2m-3}}{2(2m-3)!}$$

$$- \frac{h^2}{6} \sum_{\beta=0}^{N} C_0[\beta] \left[\frac{[\beta]^{2m-2} + (1-[\beta])^{2m-2}}{2(2m-2)!} \right] - 2\sum_{\beta=0}^{N} C_0[\beta] \int_0^1 \frac{\big|x-[\beta]\big|^{2m-1}}{2(2m-1)!} dx$$

$$+ \sum_{\beta=0}^{N} C_0[\beta] \sum_{\gamma=0}^{N} C_0[\gamma] \frac{\big|[\beta]-[\gamma]\big|^{2m-1}}{2(2m-1)!} + \frac{h^2}{6(2m-1)!} + \frac{1}{(2m+1)!} + \frac{h^4}{144(2m-3)!} \Bigg].$$

Theorem 2.2. *Coefficients of the optimal quadrature formula with derivative in $L_2^{(m)}(0,1)$ space are determined as follows*

$$C_1[0] = h^3 \sum_{k=1}^{m-3} d_k \frac{q_k^N - q_k}{1-q_k},$$

$$C_1[\beta] = h^3 \sum_{k=1}^{m-3} d_k \left(q_k^\beta + q_k^{N-\beta} \right), \quad \beta = \overline{1, N-1},$$

$$C_1[N] = h^3 \sum_{k=1}^{m-3} d_k \frac{q_k^N - q_k}{1-q_k}$$

where d_k satisfy the following system of $m-3$ linear equations

$$\sum_{k=1}^{m-3} d_k \sum_{i=1}^{l} \frac{q_k^{N+i} + (-1)^{i+1} q_k}{(1-q_k)^{i+1}} \Delta^i 0^l = \frac{B_{l+3}}{(l+1)(l+2)(l+3)}$$

here $l = \overline{1, m-3}$, B_{l+3} are Bernoulli numbers, $\Delta^i \gamma^l$ is the finite difference of order i of γ^j, q_k are the roots of the Euler-Frobenius polynomial $E_{2m-6}(q)$, $|q_k| < 1$.

Proof. To determine the optimal coefficients, we employ the method of Lagrange multipliers, which is used to find the conditional extremum of multivariable functions. In this context, based on the conditions (4), we formulate the following Lagrange function

$$\Lambda\left(C_1[0], C_1[1], ..., C_1[N-1], C_1[N], P_2, P_3, ..., P_{m-1}\right) = \|\ell_N\|^2$$

$$+(-1)^{m+1}\sum_{\alpha=2}^{m-1} P_\alpha\left(\frac{1}{\alpha+1} - \sum_{\beta=0}^{N} C_0[\beta]([\beta])^\alpha + \frac{\alpha h^2}{12} - \alpha(\alpha-1)\sum_{\beta=0}^{N} C_1[\beta][\beta]^{\alpha-2}\right) \tag{5}$$

From the Lagrange function (5), equating to 0 the partial derivatives of the function $\Lambda(C_1[\beta], P_\alpha)$ by $C_1[\beta]$, $(\beta = 0, ..., N)$ and P_α, $(\alpha = 2, 3, ..., m-1)$, we get the following system of linear equations

$$\sum_{\gamma=0}^{N} C_1[\gamma]\frac{\left|[\beta]-[\gamma]\right|^{2m-5}}{2(2m-5)!} + P_{m-3}[\beta] = f_m[\beta], \qquad \beta = \overline{0, N}, \tag{6}$$

$$\sum_{\beta=0}^{N} C_1[\beta][\beta]^\alpha = -\sum_{j=1}^{\alpha}\frac{\alpha! B_{\alpha+3-j} h^{\alpha+3-j}}{j!(\alpha+3-j)!}, \qquad \alpha = \overline{0, m-3}. \tag{7}$$

$$f_m[\beta] = \sum_{i=0}^{2m-5}\frac{[\beta]^{2m-5-i}}{(2m-5-i)!}\left[-\frac{B_{i+3}h^{i+3}}{(i+3)!} + \sum_{j=1}^{i}\frac{(-1)^i B_{i+3-j}h^{i+3-j}}{2j!(i+3-j)!}\right] + \frac{B_{2m-2}h^{2m-2}}{(2m-2)!}.$$

In the system of equations (6) - (7), there are $N+m-1$ unknowns, and to determine them, system of $N+m-1$ equations is provided. This system of equations has a unique solution at each discrete point [3].

To solve the system of equations (6)-(7), we employ an approach based on the discrete analog $D_{m-2}[\beta]$ of the differential operator d^{2m-4}/dx^{2m-4}. For this purpose, taking $\beta < 0$ and $\beta > N$ as $C_1[\beta] = 0$, we rewrite equation (6) in convolution form

$$G_{m-2}(h\beta) * C_1[\beta] + P_{m-3}[\beta] = f_m[\beta], \quad \beta = 0, 1, ..., N \tag{8}$$

we introduce the following notations

$$v[\beta] = G_{m-2}[\beta] * C_1[\beta], \tag{9}$$

$$u[\beta] = v[\beta] + P_{m-3}[\beta] \tag{10}$$

Discrete analog $D_{m-2}[\beta]$ of the differential operator d^{2m-4}/dx^{2m-4}, which satisfies the equality $hD_{m-2}[\beta] * G_{m-2}[\beta] = \delta[\beta]$, has been constructed, and its properties have been studied [10]. By applying the discrete analog $D_{m-2}[\beta]$ to both sides of equation (10), it is possible to determine the coefficients $C_1[\beta]$

$$C_1[\beta] = hD_{m-2}[\beta] * u[\beta]. \tag{11}$$

To compute the convolution (9), it is necessary to determine the form of the function $u[\beta]$ for all integer values of β. If $h\beta \in [0, 1]$ holds, then $u[\beta] = f_m[\beta]$. When $\beta < 0$ and $\beta > N$, we determine the form of $u(h\beta)$ when $\beta < 0$

$$v[\beta] = \sum_{i=0}^{m-3}\frac{[\beta]^{2m-5-i}(-1)^i}{2(2m-5-i)!\cdot i!}\sum_{j=1}^{i}\frac{i! B_{i+3-j}h^{i+3-j}}{j!(i+3-j)!} - \sum_{i=m-2}^{2m-5}\frac{[\beta]^{2m-5-i}(-1)^i}{2(2m-5-i)!\cdot i!}\sum_{\gamma=0}^{N} C_1[\gamma][\gamma]^i$$

when $\beta > N$

$$v[\beta] = -\sum_{i=0}^{m-3} \frac{[\beta]^{2m-5-i}(-1)^i}{2(2m-5-i)!\cdot i!} \sum_{j=1}^{i} \frac{i!B_{i+3-j}h^{i+3-j}}{j!(i+3-j)!} + \sum_{i=m-2}^{2m-5} \frac{[\beta]^{2m-5-i}(-1)^i}{2(2m-5-i)!\cdot i!} \sum_{\gamma=0}^{N} C_1[\gamma][\gamma]^i$$

we introduce the following notations

$$Q_{m-3}[\beta] = \sum_{i=m-2}^{2m-5} \frac{[\beta]^{2m-5-i}(-1)^i}{2(2m-5-i)!\cdot i!} \sum_{\gamma=0}^{N} C_1[\gamma][\gamma]^i$$

$$R_{2m-5}[\beta] = \sum_{i=0}^{m-3} \frac{[\beta]^{2m-5-i}(-1)^i}{2(2m-5-i)!\cdot i!} \sum_{j=1}^{i} \frac{i!B_{i+3-j}h^{i+3-j}}{j!(i+3-j)!}, \tag{12}$$

then notations

$$v[\beta] = \begin{cases} R_{2m-5}[\beta] - Q_{m-3}[\beta], & \beta < 0, \\ -R_{2m-5}[\beta] + Q_{m-3}[\beta], & \beta > N. \end{cases} \tag{13}$$

So,

$$u[\beta] = \begin{cases} R_{2m-5}[\beta] + Q^{-}_{m-3}[\beta], & \beta < 0, \\ F_m[\beta] \quad , & 0 \le \beta \le N, \\ -R_{2m-5}[\beta] + Q^{+}_{m-3}[\beta], & \beta < 0\,, \end{cases} \tag{14}$$

where

$$\begin{aligned} Q^{-}_{m-3}[\beta] &= P_{m-3}[\beta] - Q_{m-3}[\beta], \\ Q^{+}_{m-3}[\beta] &= P_{m-3}[\beta] + Q_{m-3}[\beta]. \end{aligned} \tag{15}$$

$Q^{-}_{m-3}[\beta]$ and $Q^{+}_{m-3}[\beta]$ are unknown polynomials of degree $(m-3)$.

If we determine $Q^{-}_{m-3}[\beta]$ and $Q^{+}_{m-3}[\beta]$, then from (15), we can find the unknown polynomials $P_{m-3}[\beta]$ and $Q_{m-3}[\beta]$

$$\begin{aligned} P_{m-3}[\beta] &= \tfrac{1}{2}\left(Q^{+}_{m-3}[\beta] + Q^{-}_{m-3}[\beta]\right), \\ Q_{m-3}[\beta] &= \tfrac{1}{2}\left(Q^{+}_{m-3}[\beta] - Q^{(-)}_{m-3}[\beta]\right). \end{aligned}$$

Using the explicit form of the functions $D_{m-2}[\beta]$ and the previously determined $u[\beta]$ with discrete arguments, we determine the optimal coefficients $C_1[\beta]$, when $\beta = \overline{1, N-1}$. To simplify the calculations, we introduce the following notation

$$\begin{aligned} a_k &= \tfrac{(2m-5)!(1-q_k)^{2m-3}}{h^{2m-6}q_kE_{2m-5}(q_k)} \sum_{\gamma=1}^{\infty} q_k^{\gamma}\left(R_{2m-5}[-\gamma] + Q^{-}_{m-3}[-\gamma] - F_m[-\gamma]\right), \\ b_k &= \tfrac{(2m-5)!(1-q_k)^{2m-3}}{h^{2m-6}q_kE_{2m-5}(q_k)} \sum_{\gamma=1}^{\infty} q_k^{\gamma}\left(-R_{2m-5}(1+[\gamma]) + Q^{+}_{m-3}(1+[\gamma]) - F_m(1+[\gamma])\right) \end{aligned} \tag{16}$$

where $k = 1, 2, ..., m-3$, $E_{2m-5}(q)$ is the Euler-Frobenius polynomial of degree $2m-5$, $|q_k| < 1$, h is a small positive parameter. Note that because of $|q_k| < 1$ the series in (16) are convergent.

Theorem 2.3. *In $L_2^{(m)}(0,1)$ space for $m \ge 4$, the coefficients of the optimal quadrature formula with derivatives have the following form*

$$C_1[\beta] = h^3 \sum_{k=1}^{m-3} \left(a_k q_k^{\beta} + b_k q_k^{N-\beta}\right), \qquad \beta = 1, 2, ..., N-1$$

where a_k, b_k are determined by (16).

Proof. When $\beta = 1, 2, ..., N-1$ holds, using (11) and (14), it can be written for $C_1[\beta]$

$$C_1[\beta] = hD_{m-2}[\beta] * u[\beta] = h\{D_{m-2}[\beta] * F_m[\beta]$$

$$+\sum_{k=1}^{m-3} q_k^{\beta} \frac{(2m-5)!}{h^{2m-4}} \frac{(1-q_k)^{2m-3}}{q_k E_{2m-5}(q_k)} \sum_{\gamma=1}^{\infty} q_k^{\gamma} \left(R_{2m-5}[-\gamma] + Q_{m-3}^{-}[-\gamma] - F_m[-\gamma]\right)$$

$$+\sum_{k=1}^{m-3} q_k^{N-\beta} \frac{(2m-5)!}{h^{2m-4}} \frac{(1-q_k)^{2m-3}}{q_k E_{2m-5}(q_k)} \sum_{\gamma=1}^{\infty} q_k^{\gamma} \left(-R_{2m-5}(1+[\gamma]) + Q_{m-3}^{+}(1+[\gamma]) -\right.$$

$$\left.-f_m(1+[\gamma])\right)\} = h^3 \sum_{k=1}^{m-3} \left(a_k q_k^{\beta} + b_k q_k^{N-\beta}\right)$$

□

Thus, the optimal coefficients $C_1[\beta]$ depend on $2m-6$ unknowns a_k and b_k ($k = \overline{1, m-3}$), and to determine them, a system of $2m-6$ equations is required. First, from equality (7), we determine the form of the coefficients $C_1[0]$, $C_1[N]$ for $\alpha = 0$ and $\alpha = 1$

$$C_1[0] = -\sum_{\beta=1}^{N-1} C_1[\beta][\beta] - \sum_{\beta=1}^{N-1} C_1[\beta], \tag{17}$$

$$C_1[N] = -\sum_{\beta=1}^{N-1} C_1[\beta][\beta]. \tag{18}$$

From equalities (17) and (18), it can be seen that the coefficients $C_1[0]$ and $C_1[N]$ are expressed in terms of the coefficients $C_1[\beta]$ ($\beta = \overline{1, N-1}$). We determine the coefficient $C_1[0]$ from the system of equations (6)

$$S = \sum_{\gamma=0}^{N} C_1[\gamma] G_{m-2}([\beta] - [\gamma])$$

$$C_1[0] \frac{[\beta]^{2m-5}}{(2m-5)!} + \sum_{\gamma=1}^{\beta-1} C_1[\gamma] \frac{([\beta]-[\gamma])^{2m-5}}{(2m-5)!} - \sum_{\gamma=0}^{N} C_1[\gamma] \frac{([\beta]-[\gamma])^{2m-5}}{2(2m-5)!}$$

$$= C_1[0] \frac{[\beta]^{2m-5}}{(2m-5)!} + S_1 - S_2 \tag{19}$$

We simplify the sums S_1 and S_2 in equation (19) using Theorem 2.3.

$$S_1 = \sum_{\gamma=1}^{\beta-1} C_1[\gamma] \frac{([\beta]-[\gamma])^{2m-5}}{(2m-5)!} = h^3 \sum_{\gamma=1}^{\beta-1} \left(a_k q_k^{\gamma} + b_k q_k^{N-\gamma}\right) \frac{([\beta]-[\gamma])^{2m-5}}{(2m-5)!}$$

$$= \frac{h^{2m-2}}{(2m-5)!} \left\{ \sum_{k=1}^{m-3} \left[\frac{a_k q_k^{\beta}}{1-q_k^{-1}} \sum_{i=0}^{2m-5} \left(\frac{q_k^{-1}}{1-q_k^{-1}}\right)^i \Delta^i 0^{2m-5} - \frac{a_k}{1-q_k^{-1}} \sum_{i=0}^{2m-5} \left(\frac{q_k^{-1}}{1-q_k^{-1}}\right)^i \Delta^i \beta^{2m-5} \right] \right.$$

$$+\sum_{k=1}^{m-3}\left[\frac{b_k q_k^{N-\beta}}{1-q_k}\sum_{i=0}^{2m-5}\left(\frac{q_k}{1-q_k}\right)^i\Delta^i 0^{2m-5}-\frac{b_k q_k^N}{1-q_k}\sum_{i=0}^{2m-5}\left(\frac{q_k}{1-q_k}\right)^i\Delta^i\beta^{2m-5}\right]\Bigg\}$$

here, considering that q_k is a root of the Euler-Frobenius polynomial of degree $(2m-6)$

$\sum_{i=0}^{2m-5}\left(\frac{q_k}{1-q_k}\right)^i\Delta^i 0^{2m-5}=0$ and $\sum_{i=0}^{2m-5}\left(\frac{q_k^{-1}}{1-q_k^{-1}}\right)^i\Delta^i 0^{2m-5}=0$

So,

$$S_1=-\sum_{j=0}^{2m-5}\frac{h^{j+3}[\beta]^{2m-5-j}}{j!(2m-5-j)!}\left[\sum_{k=1}^{m-3}\frac{a_k q_k}{q_k-1}\sum_{i=0}^{2m-5}\left(\frac{1}{q_k-1}\right)^i\Delta^i 0^j+\sum_{k=1}^{m-3}\frac{b_k q_k^N}{1-q_k}\sum_{i=0}^{2m-5}\left(\frac{q_k}{1-q_k}\right)^i\Delta^i 0^j\right]. \tag{20}$$

Now, we simplify the sum S_2. To do this, using the orthogonality condition in equation (7), we obtain the following

$$S_2=\sum_{\gamma=0}^{N}C_1[\gamma]\sum_{j=0}^{2m-5}\frac{[\beta]^{2m-5-j}[-\gamma]^j}{2j!(2m-5-j)!}$$

$$=\sum_{j=m-2}^{2m-5}\frac{[\beta]^{2m-5-j}(-1)^j}{2j!(2m-5-j)!}\sum_{\gamma=0}^{N}C_1[\gamma][\gamma]^j-\sum_{j=0}^{m-3}\frac{[\beta]^{2m-5-j}(-1)^j}{2j!(2m-5-j)!}\sum_{i=0}^{j}\frac{j!B_{j+3-i}h^{j+3-i}}{i!(j+3-i)!} \tag{21}$$

Substituting equations (20) and (21) into equation (19), we obtain the following sum

$$S=C_1[0]\frac{[\beta]^{2m-5}}{(2m-5)!}-\sum_{j=0}^{2m-5}\frac{h^{j+3}[\beta]^{2m-5-j}}{j!(2m-5-j)!}\left[\sum_{k=1}^{m-3}\frac{a_k q_k}{q_k-1}\sum_{i=0}^{2m-5}\left(\frac{1}{q_k-1}\right)^i\Delta^i 0^j\right.$$

$$\left.+\sum_{k=1}^{m-3}\frac{b_k q_k^N}{1-q_k}\sum_{i=0}^{2m-5}\left(\frac{q_k}{1-q_k}\right)^i\Delta^i 0^j\right]-\sum_{j=m-2}^{2m-5}\frac{[\beta]^{2m-5-j}(-1)^j}{2j!(2m-5-j)!}\sum_{\gamma=0}^{N}C_1[\gamma][\gamma]^j$$

$$+\sum_{j=0}^{m-3}\frac{[\beta]^{2m-5-j}(-1)^j}{2j!(2m-5-j)!}\sum_{i=0}^{j}\frac{j!B_{j+3-i}h^{j+3-i}}{i!(j+3-i)!} \tag{22}$$

We group the terms of equation (6) by the powers of $[\beta]$ using (22)

$$\frac{[\beta]^{2m-5}}{(2m-5)!}\left[C_1[0]-h^3\sum_{k=1}^{m-3}\frac{a_k q_k-b_k q_k^N}{q_k-1}\right]$$

$$-\sum_{j=1}^{2m-5}\frac{[\beta]^{2m-5-j}h^{j+3}}{j!(2m-5-j)!}\left[\sum_{k=1}^{m-3}\frac{a_k q_k}{q_k-1}\sum_{i=0}^{j}\left(\frac{1}{q_k-1}\right)^i\Delta^i 0^j+\sum_{k=1}^{m-3}\frac{b_k q_k^N}{1-q_k}\sum_{i=0}^{j}\left(\frac{q_k}{1-q_k}\right)^i\Delta^i 0^j\right]$$

$$-\sum_{j=m-2}^{2m-5}\frac{[\beta]^{2m-5-j}(-1)^j}{2j!(2m-5-j)!}\sum_{\gamma=0}^{N}C_1[\gamma][\gamma]^j+P_{m-3}[\beta]=-\frac{[\beta]^{2m-5}B_3h^3}{3!(2m-5)!}-\sum_{j=1}^{2m-5}\frac{[\beta]^{2m-5-j}B_{j+3}h^{j+3}}{(2m-5-j)!(j+3)!}$$

$$+\sum_{j=m-2}^{2m-5}\frac{[\beta]^{2m-5-j}}{(2m-5-j)!}\sum_{i=1}^{j}\frac{(-1)^j B_{j+3-i}h^{j+3-i}}{2i!(j+3-i)!}+\frac{B_{2m-2}h^{2m-2}}{(2m-2)!}. \tag{23}$$

From equation (23), by equating the degrees of $(2m-5)$ of $[\beta]$, we determine the expression for the coefficient $C_1[0]$

$$C_1[0]=h^3\sum_{k=1}^{m-3}\frac{a_k q_k-b_k q_k^N}{q_k-1} \tag{24}$$

To determine the unknowns a_k and b_k $(k=\overline{1,m-3})$, we equate the powers of $[\beta]$ from the $m-2$-th degree to the $2m-6$-th degree in (23) and obtain the following system of $m-3$ equations

$$\sum_{k=1}^{m-3}\left[\sum_{i=0}^{j}\frac{a_k q_k}{(q_k-1)^{i+1}}\Delta^i 0^j+\sum_{i=0}^{j}\frac{b_k q_k^{N+i}}{(1-q_k)^{i+1}}\Delta^i 0^j\right]=\frac{B_{j+3}}{(j+1)(j+2)(j+3)} \tag{25}$$

From the orthogonality condition in equation (7), we obtain $\alpha=0$ and determine the coefficient $C_1\,[N]$

$$C_1[N]=h^3\sum_{k=1}^{m-3}\frac{a_k q_k^N-b_k q_k}{1-q_k} \tag{26}$$

we write the remaining part of equation (7)

$$\sum_{\beta=0}^{N}C_1\,[\beta]\,[\beta]^\alpha=-\sum_{j=1}^{\alpha}\frac{\alpha!B_{\alpha+3-j}h^{\alpha+3-j}}{j!(\alpha+3-j)!},\qquad \alpha=\overline{1,\,m-3} \tag{27}$$

For the left-hand side of equation (27), using Theorem 2.2 and equation (24), we obtain the following

$$\sum_{\beta=0}^{N}C_1[\beta][\beta]^\alpha=h^3\sum_{k=1}^{m-3}\frac{a_k q_k^N-b_k q_k}{1-q_k}+h^{\alpha+3}\sum_{k=1}^{m-3}\left[a_k\sum_{t=0}^{\alpha}\frac{q_k^t\Delta^t 0^\alpha-q_k^{N+t}\Delta^t N^\alpha}{(1-q_k)^{t+1}}\ +b_k\sum_{t=0}^{\alpha}\frac{q_k^{N+1}\Delta^t 0^\alpha-q_k\Delta^t N^\alpha}{(q_k-1)^{t+1}}\right] \tag{28}$$

We substitute the final equation (28) into (27) and, after some intermediate calculations, obtain the following

$$h^{\alpha+3}\sum_{k=1}^{m-3}\sum_{t=0}^{\alpha}\frac{a_k q_k^t+(-1)^{t+1}b_k q_k^{N+1}}{(1-q_k)^{t+1}}\Delta^t 0^\alpha-h^{\alpha+3}\sum_{k=1}^{m-3}\sum_{t=0}^{\alpha}\frac{a_k q_k^{N+t}+(-1)^{t+1}b_k q_k}{(1-q_k)^{t+1}}\Delta^t 0^\alpha$$

$$-\sum_{j=1}^{\alpha-1}\frac{\alpha!h^{j+3}}{j!(\alpha-j)!}\sum_{k=1}^{m-3}\sum_{t=0}^{\alpha}\frac{a_k q_k^{N+t}+(-1)^{t+1}b_k q_k}{(1-q_k)^{t+1}}\Delta^t 0^j=-\sum_{j=1}^{\alpha-1}\frac{\alpha!B_{\alpha+3-j}h^{\alpha+3-j}}{j!(\alpha+3-j)!},\quad \alpha=\overline{1,m-3} \tag{29}$$

from (29), we equate the coefficients preceding $h^{\alpha+3}$

$$\sum_{k=1}^{m-3}\sum_{t=0}^{\alpha}\frac{a_k q_k^t+(-1)^{t+1}b_k q_k^{N+1}}{(1-q_k)^{t+1}}\Delta^t 0^\alpha=\sum_{k=1}^{m-3}\sum_{t=0}^{\alpha}\frac{a_k q_k^{N+t}+(-1)^{t+1}b_k q_k}{(1-q_k)^{t+1}}\Delta^t 0^\alpha \tag{30}$$

where $\alpha=\overline{1,m-3}$. For the right-hand side of equation (29), after performing the substitution of $\alpha+3-j=\bar{j}+3 \;\Rightarrow\; \begin{cases} j=1, & \bar{j}=\alpha-1\\ j=\alpha-1, & \bar{j}=1,\end{cases}$, we obtain the following equality

$$\sum_{j=1}^{\alpha-1}\frac{\alpha!h^{j+3}}{j!(\alpha-j)!}\sum_{k=1}^{m-3}\sum_{t=0}^{\alpha}\frac{a_kq_k^{N+t}+(-1)^{t+1}b_kq_k}{(1-q_k)^{t+1}}\Delta^t0^j=\sum_{j=1}^{\alpha-1}\frac{\alpha!B_{j+3}h^{j+3}}{(j+3)!(\alpha-j)!},\quad \begin{matrix} j=\overline{1,\alpha-1},\\ \alpha=\overline{1,m-3}.\end{matrix} \tag{31}$$

After some simplifications in equation (21)

$$\sum_{k=1}^{m-3}\sum_{t=0}^{\alpha}\frac{a_kq_k^{N+t}+(-1)^{t+1}b_kq_k}{(1-q_k)^{t+1}}\Delta^t0^j=\frac{B_{j+3}}{(j+1)(j+2)(j+3)} \tag{32}$$

here $\begin{matrix} j=\overline{1,\alpha-1},\\ \alpha=\overline{1,m-3}\end{matrix}$. We rewrite equation (25) as follows

$$\sum_{k=1}^{m-3}\sum_{i=0}^{\alpha}\frac{a_kq_k+(-1)^{i+1}b_kq_k^{N+i}}{(q_k-1)^{i+1}}\Delta^i0^\alpha=\frac{B_{\alpha+3}}{(\alpha+1)(\alpha+2)(\alpha+3)} \tag{33}$$

Based on **Lemma**, equation (30) can be written as follows

$$\sum_{k=1}^{m-3}\sum_{i=0}^{\alpha}\frac{a_kq_k^{N+i}+(-1)^{i+1}b_kq_k}{(1-q_k)^{i+1}}\Delta^i0^\alpha=\frac{(-1)^{\alpha+1}B_{\alpha+3}}{(\alpha+1)(\alpha+2)(\alpha+3)} \tag{34}$$

Taking into account that $B_{2N+1}=0$ holds for Bernoulli numbers, we subtract the system of equations (34) from (33)

$$\sum_{k=1}^{m-3}\sum_{i=0}^{\alpha}\frac{(a_k-b_k)(q_k+q_k^{N+i}(-1)^i)}{(q_k-1)^{i+1}}\Delta^i0^\alpha=0,\quad \alpha=\overline{1,m-3}. \tag{35}$$

From equation (35), it follows that $a_k=b_k$ holds. Thus, to determine the unknowns a_k, $k=\overline{1,m-3}$, we obtain the following new system of equations

$$\sum_{k=1}^{m-3}a_k\sum_{i=0}^{\alpha}\frac{(q_k^{N+i}+(-1)^{i+1}q_k)}{(1-q_k)^{i+1}}\Delta^i0^\alpha=\frac{B_{\alpha+3}}{(\alpha+1)(\alpha+2)(\alpha+3)} \tag{36}$$

here $\alpha=\overline{1,m-3}$. The **Theorem 2** has been proven. □

In the next section, we analyze the obtained results by numerically solving Fredholm's integral equation of the second kind and comparing it with Simpson's rule.

3. Numerical results

It is known that finding an analytical solution to integral equations is quite complicated. If the kernel of the integral equation consists of complex functions, solving the integral equation analytically requires a lot of work [11, 12]. In [13] finite difference-composite trapezoidal discretization schemes were employed for the numerical solution of first-order linear Fredholm integro-differential equations. In [14], the method of expanding Bernstein polynomials was utilized for the numerical solution of systems of Fredholm and Volterra integral equations of the second kind. In [15, 16] study numerical integration formulas for hyper-singular integrals. We consider here the numerical solution of Fredholm's integral equation of the second kind, and in this connection, we recall the following method [17, 18, 26, 27].

We are given the integral equation

$$y(x) = \lambda \int_0^1 K(x,t)y(t)dt + f(x),$$

we replace the integral with the following sum

$$\int_0^1 K(x,t)y(t)dt \cong \sum_{j=0}^{N} \alpha_j K(x_i,t_j)y(t_j), \quad i=\overline{0,N},$$

where α_j are the coefficients of the quadrature formula. By inserting the last approximate equation into the given integral equation, we take $x = x_i, (i = 0, 1, 2..., N)$ and following form a system of linear equations.

$$y_i - \lambda \sum_{j=0}^{N} \alpha_j K_{ij} y_j = f_i, i = \overline{0,N}$$

where $x_i = t_i, \quad f(x_i) = f_i, \quad K(x_i,t_j) = K_{ij}$. We find $y_i, \ (i = 0, 1, 2...N)$ from the system of linear equations.

We apply the derived optimal quadrature formula constructed in this work to the numerical solution of Fredholm's integral equation of the second kind. To do this, we first replace the first and second derivatives in the formula with fourth order finite differences

$$\varphi^{'}(0) = \frac{-25\varphi(0) + 48\varphi(h) - 36\varphi(2h) + 16\varphi(3h) - 3\varphi(4h)}{12h}$$

$$\varphi^{'}(1) = \frac{25\varphi(1) - 48\varphi((N-1)h) + 36\varphi((N-2)h) - 16\varphi((N-3)h) + 3\varphi((N-4)h)}{12h}$$

$$\varphi''(x_i) = \frac{-\varphi(x_{i-2}) + 16\varphi(x_{i-1}) - 30\varphi(x_i) + 16\varphi(x_{i+1}) - \varphi(x_{i+2})}{12h^2}, \quad i = \overline{2, N-2}$$

$$\varphi''(0) = \frac{45\varphi(0) - 154\varphi(h) + 214\varphi(2h) - 156\varphi(3h) + 61\varphi(4h) - 10\varphi(5h)}{12h^2}$$

$$\varphi''(1) = \frac{1}{12h^2}\Big[45\varphi(1) - 154\varphi((N-1)h) + 214\varphi((N-2)h) - 156\varphi((N-3)h) + 61\varphi((N-4)h) - 10\varphi((N-5)h)\Big].$$

We numerically solve the following integral equations using the optimal quadrature formula constructed in $L_2^{(4)}(0,1)$ space.

Example 1.

$$u(x) = e^x + 2\int_0^1 e^{x+t}u(t)dt.$$

We numerically analyze the given integral equation with the help of the optimal quadrature formula with derivative and Simpson's formula with $N = 6$ nodes. For this, we write down the optimal quadrature formula derived from (1) using finite differences as follows

$$\int_0^1 \varphi(x)dx \cong \sum_{\beta=0}^{6} C_0\,[\beta]\varphi[\beta]) + \frac{h^2}{12}(\varphi'[0] - \varphi'[1]) + \sum_{\beta=0}^{6} C_1\,[\beta]\varphi''[\beta]$$

$$= \varphi(0)\left[\frac{47h^3 + 540C_1[0] - 12C_1[2] - 120C_1[5]}{144h^2}\right]$$

$$+\varphi(\frac{1}{6})\left[\frac{16h^3 - 154C_1[0] + 45C_1[1] + 16C_1[2] - C_1[3] + 61C_1[5] - 10C_1[6]}{12h^2}\right]$$

$$+\varphi(\frac{2}{6})\left[\frac{35h}{48} + \frac{214C_1[0] - 154C_1[1] - 30C_1[2] + 16C_1[3] - C_1[4] - 156C_1[5] + 61C_1[6]}{12h^2}\right]$$

$$+\varphi(\frac{3}{6})\left[\frac{11h}{9} - \frac{78C_1[0] + 107C_1[1] + 8C_1[2] - 15C_1[3] + 8C_1[4] + 107C_1[5] - 78C_1[6]}{6h^2}\right]$$

$$+\varphi(\frac{4}{6})\left[\frac{35h}{48} + \frac{61C_1[0] - 156C_1[1] - C_1[2] + 16C_1[3] - 30C_1[4] - 154C_1[5] + 214C_1[6]}{12h^2}\right]$$

$$+\varphi(\frac{5}{6})\left[\frac{16h^3 - 10C_1[0] + 61C_1[1] - C_1[3] + 16C_1[4] + 45C_1[5] - 154C_1[6]}{12h^2}\right]$$

$$+\varphi(1)\left[\frac{47h^3 - 120C_1[1] - 12C_1[4] + 576C_1[6]}{144h^2}\right]$$

where $C_1[\beta]$, $\beta = \overline{0,6}$ are given in Theorem 2.

We write the given integral equation in the form below

$$u(x_i) = e^{x_i} + 2e^{x_i}\sum_{j=0}^{6} A_j e^{t_j} u(t_j), \qquad i = \overline{0,6}$$

and we find the unknown $u(x_i)$ by solving the system of linear algebraic equations. The exact solution of the integral equation is $u(x) = \frac{e^x}{2-e^2}$. In **Table 1** gives a numerical analysis of the solution of the integral equation. **Table 1** shows the error of the Simpson formula through **Error(Simpson)**, and the error of the optimal quadrature formula with derivatives through **Error(OQF)**.

Table 1: The exact solution and the errors

x_i	**The exact solution**	**Error (Simpson)**	**Error (OQF)**
0	-0.1855612526	1.48×10^{-5}	1.33×10^{-6}
1/6	-0.2192147180	1.75×10^{-5}	1.58×10^{-6}
2/6	-0.2589715897	2.07×10^{-5}	1.86×10^{-6}
3/6	-0.3059387842	2.45×10^{-5}	2.20×10^{-6}
4/6	-0.3614239684	2.90×10^{-5}	2.60×10^{-6}
5/6	-0.4269719685	3.42×10^{-5}	3.07×10^{-6}
1	-0.5044077809	4.04×10^{-5}	3.63×10^{-6}

Example 2.

$$u(x) = x^4 + 1 - \frac{14x}{45} + \int_0^1 xt^4 u(t)dt$$

The exact solution of the integral equation is $u(x) = x^4 + 1$.

Table 2: The exact solution and the errors

x_i	**The exact solution**	**Error (Simpson)**	**Error (OQF)**
0	1	0	0
1/6	1.000771605	2.90×10^{-4}	1.61×10^{-4}
2/6	1.012345679	5.80×10^{-4}	3.22×10^{-4}
3/6	1.062500000	8.70×10^{-4}	4.84×10^{-4}
4/6	1.197530864	1.16×10^{-3}	6.45×10^{-4}
5/6	1.482253086	1.45×10^{-3}	8.07×10^{-4}
1	2	1.74×10^{-3}	9.68×10^{-4}

In Table 3, for the approximate computation of definite integrals using $N = 10$ nodal points, the error of Simpson's rule **(Error1)** and the error of the derivative-based quadrature formulas constructed in this work **(Error2)** are defined. The numerical computation errors for the definite integrals of the following five functions are provided $I = \int_0^1 u(x)dx$

a) $u(x) = e^{-\cos x} + 2\sin x$ b) $u(x) = \ln(1 + x^2) + \sqrt{1 + x^3}$
c) $u(x) = \cos 2x - \frac{1}{1+x^2}$ d) $u(x) = e^{x^2}$

Table 3: Errors

Error	**Example 1**	**Example 2**	**Example 3**	**Example 4**
Error1	8.75×10^{-7}	2.70×10^{-6}	4.07×10^{-6}	2.96×10^{-5}
Error2	1.05×10^{-8}	4.12×10^{-8}	4.77×10^{-8}	8.36×10^{-7}

In Table 4, the error of the derived formula from [19] **(I-MSONC4)** and the error of the formula constructed in this work **(Error2)** are numerically analyzed using the following five functions

a) $f(x) = e^{-x}$ b) $f(x) = \frac{1}{1+x}$
c) $f(x) = \sqrt{1 + x^2}$ d) $f(x) = \frac{\ln(1+x)}{1+x^2}$

Table 4: Errors

Error	**Example 1**	**Example 2**	**Example 3**	**Example 4**
I - MSONC4	9.60×10^{-4}	7.96×10^{-3}	1.27×10^{-3}	1.42×10^{-2}
Error2	8.50×10^{-4}	5.64×10^{-3}	3.87×10^{-4}	7.53×10^{-3}

In Table 5, the square of the error functional norm of the derivative-based optimal quadrature formula constructed in [3] **(Error1)** and the square of the error functional norm of the optimal quadrature formula with derivative in this work **(Error2)** are numerically analyzed at $N = 10$, $N = 100$, $N = 500$, $N = 1000$ node points

Table 5: Errors

$\|\ell_N\|^2$	**N=10**	**N=100**	**N=500**	**N=1000**
Error1	9.93×10^{-15}	8.42×10^{-23}	2.12×10^{-28}	2.29×10^{-31}
Error2	9.38×10^{-15}	8.37×10^{-23}	2.02×10^{-28}	1.77×10^{-31}

Table 6 and Table 7 present the errors of numerical computations of definite integrals using Simpson's rule and optimal quadrature formulas with derivative.

Table 6: The exact solution and the errors $u(x) = \sin x + e^x$

N	**The exact solution**	**Error (Simpson)**	**Error (OQF)**
10	2.177979522	2.56×10^{-8}	1.82×10^{-8}
50	2.177979522	5.62×10^{-10}	5.86×10^{-12}
100	2.177979522	5.76×10^{-10}	2.50×10^{-13}
150	2.177979522	5.92×10^{-10}	1.00×10^{-14}

Table 7: The exact solution and the errors $u(x) = \ln(1 + x) - e^{-x}$

N	**The exact solution**	**Error (Simpson)**	**Error (OQF)**
10	-0.245826197	4.17×10^{-8}	2.99×10^{-8}
50	-0.245826197	1.21×10^{-10}	9.87×10^{-12}
100	-0.245826197	9.51×10^{-11}	3.11×10^{-13}
150	-0.245826197	1.09×10^{-11}	4.20×10^{-14}

From **Table 6** and **Table 7** it can be seen that the error of the optimal quadrature formula is smaller than the error of the Simpson's rule.

4. Conclusion

In conclusion, this research is related to the construction of optimal quadrature formulas with derivatives using the second-order derivative of a function at nodal points for the approximate calculation of definite integrals and the numerical solution of integral equations in the $L_2^{(m)}(0, 1)$ space. As a result of the study, the square of the norm of the error functional of the derived optimal quadrature formulas was determined. A system of algebraic equations was derived to minimize this norm through coefficients. By solving the system of equations, the optimal coefficients were obtained. The efficiency and effectiveness of the constructed formula in the numerical solution of definite integrals and integral equations are demonstrated in the tables.

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