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Some effect of drift of the generalized Brownian motion process II: Pseudo homomorphism structures of the transforms on function space

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Abstract. In this paper, we analyze pseudo homomorphism structures of the generalized Fourier–Feynman transform (GFFT) on the function space $C_{a,b}[0,T]$ which is induced by a generalized Brownian motion process (GBMP). The Fourier–Feynman transform (FFT) of functionals on classical Wiener space satisfies homomorphism property with a convolution product (CP). Consequently speaking, the GFFTs have no traditional homomorphism structure with their CP, because the stochastic processes defining the GFFT and the CP in this paper are not centered. In order to develop the structure, we modified the definition of the GFFT. We then proceed to investigate pseudo homomorphism structures between the transform and the convolution.

1. Introduction

The theory of the analytic FFT was introduced by Brue [1]. Since then the analytic FFT theory has been developed in many researches, for instance see [3, 5, 16–20]. In the analytic FFT theory, the integrands of the transforms are functionals of the sample paths of standard Brownian motion process (Wiener process). For an elementary survey, we refer the readers to the reference [21]. Usually, the analytic FFT is defined for the functionals on the Wiener space $C_0[0,T]$. In [16–18], Huffman, Park and Skoug defined a CP and established a homomorphism structure between the FFT and the CP as follows: for appropriate functionals F and G on $C_0[0,T]$,

$$T_q^{(1)}\Big((F*G)_q\Big)(y) = T_q^{(1)}(F)\Big(\frac{y}{\sqrt{2}}\Big)T_q^{(1)}(G)\Big(\frac{y}{\sqrt{2}}\Big) \tag{1.1}$$

for scale-invariant almost every $y \in C_0[0,T]$, where $T_q^{(1)}$ and $(\cdot * \cdot)_q$ denote the L_1 analytic FFT and the corresponding CP respectively. The analytic FFT and the CP are defined via the structure of the analytic Feynman integral [2, 4].

In [22], Yeh introduced the function space $C_{a,b}[0,T]$, which is the space of continuous sample paths of the GBMP determined by the continuous functions $a(\cdot)$ and $b(\cdot)$ on the time interval $[0,T]$. The function

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space $C_{a,b}[0,T]$ was used extensively in [8, 11–15]. Furthermore, in [11, 13, 14], the authors used the GBMP to define a generalized Feynman integral and a GFFT for functionals on $C_{a,b}[0,T]$.

Let $(\Omega,\mathcal{F},P)$ be a probability space and let $D=[0,T]$ be a time interval. A GBMP on $\Omega\times D$ is a stochastic process $Y\equiv\{Y_t : t\in D\}$ which satisfies the conditions:

(i) $Y_0=0$ almost surely; and

(ii) for any $0\le s<t\le T$, the random variable Y_t-Y_s is Gaussian distributed with mean $a(t)-a(s)$ and variance $b(t)-b(s)$, where $a(t)$ is a continuous real-valued function and $b(t)$ is a continuous monotonically increasing real-valued function on $[0,T]$.

Thus the functions $a(t)$ and $b(t)$ characterize the GBMP. Choosing $a(t)=0$ and $b(t)=t$ on the time interval $[0,T]$, the GBMP reduces a standard Brownian motion process. In general, the GBMP is not stationary in time (by the function $b(\cdot)$) and is subject to a time drift $a(\cdot)$.

By the effect of the drift function $a(\cdot)$, the generalized Feynman integral has unusual behaviors. In [8], Chang and Choi provided a bounded functionals which are not generalized analytic Feynman integrable on the complete probability space $(C_{a,b}[0,T],\mathcal{W}(C_{a,b}[0,T]),\mu)$. We refer the reader to the reference [8, 11, 12, 15, 23] for detailed construction and illustration of the complete probability space $(C_{a,b}[0,T],\mathcal{W}(C_{a,b}[0,T]),\mu)$ induced by the GBMP associated with the continuous functions $a(\cdot)$ and $b(\cdot)$.

Since the definition of the GFFT is based on the concept of the generalized analytic Feynman integral, the GFFTs have no homomorphism structure with their CP. The purpose of this paper is to characterize the relation such as (1.1), between the GFFT and the CP for functionals on $C_{a,b}[0,T]$. In order to develop the structure between the GFFT and the CP, we in this paper modified the definition of GFFT. We then proceed to investigate pseudo homomorphism structures between the transform and the convolution.

In order to guarantee our assertions, we assume that $a(\cdot)$ is an absolutely continuous real-valued function such that $a(0)=0$, a' is in $L^2[0,T]$, and $\int_0^T |a'(t)|^2 d|a|(t)<+\infty$, where $|a|(\cdot)$ denotes the total variation function of the function $a(\cdot)$, and $b(\cdot)$ is a continuously differentiable real-valued function with $b(0)=0$ and $b'(t)>0$ for each $t\in[0,T]$. We will adopt the notations and terminologies of those papers on the assumption that readers are familiar with the reference [13–15]. The concept of the 'scale-invariant measurability' on $C_{a,b}[0,T]$ may also be found in [8, 11]. However, in order to propose our assertions in this paper, we shall restate the concept of the scale-invariant measurability: A subset S of $C_{a,b}[0,T]$ is called a scale-invariant measurable (s.i.m.) set if ρS is $\mathcal{W}(C_{a,b}[0,T])$-measurable for all $\rho>0$, and an s.i.m. set N is called a scale-invariant null set if $\mu(\rho N)=0$ for all $\rho>0$. A property which holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). A functional F is said to be s.i.m provided F is defined on an s.i.m. set and $F(\rho\,\cdot)$ is $\mathcal{W}(C_{a,b}[0,T])$-measurable for any $\rho>0$. If two functionals F and G on $C_{a,b}[0,T]$ are equal s-a.e., then we write $F\approx G$.

2. Generalized Fourier–Feynman transform and convolution product

We denote the function space integral of a $\mathcal{W}(C_{a,b}[0,T])$-measurable functional F by

$$E[F]\equiv E_x[F(x)]=\int_{C_{a,b}[0,T]} F(x)d\mu(x)$$

whenever the integral exists.

Throughout this paper, we will assume that each functional F (or G) we consider satisfies the conditions;

$$F: C_{a,b}[0,T]\to\mathbb{C} \text{ is defined s-a.e. and s.i.m..} \tag{2.1}$$

and

$$E_x\Big[|F(\rho x)|\Big]<+\infty \text{ for each } \rho>0. \tag{2.2}$$

Let $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$ and let $\widetilde{\mathbb{C}}_+ = \{\lambda \in \mathbb{C} \setminus \{0\} : \mathrm{Re}(\lambda) \geq 0\}$. Let F satisfy conditions (2.1) and (2.2) above, and for $\lambda > 0$, let $J(\lambda) = E_x[F(\lambda^{-1/2}x)]$. If there is a function $J^*(\lambda)$ analytic in $\mathbb{C}_+$ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the analytic function space integral of F over $C_{a,b}[0,T]$ with parameter λ, and for $\lambda \in \mathbb{C}_+$ we write

$$E^{\mathrm{an}_\lambda}[F] \equiv E_x^{\mathrm{an}_\lambda}[F(x)] = J^*(\lambda).$$

Definition 2.1. *Let F satisfy conditions* (2.1) *and* (2.2) *above. For $\lambda \in \mathbb{C}_+$ and $y \in C_{a,b}[0,T]$, let*

$$T_{\lambda,+}(F)(y) = E_x^{\mathrm{an}_\lambda}[F(y+x)].$$

*We define the L_1 analytic forward GFFT (*f*-GFFT), $T_q^{(1,\mathrm{f})}(F)$ of F, by the formula ($\lambda \in \mathbb{C}_+$)*

$$T_q^{(1,\mathrm{f})}(F)(y) = \lim_{\lambda \to -iq} T_{\lambda,+}(F)(y)$$

if it exists for s-a.e. $y \in C_{a,b}[0,T]$.

In fact, the f-GFFT $T_q^{(1,\mathrm{f})}$ for functionals on $C_{a,b}[0,T]$ is the GFFT studied in [13, 14]. By the effect of the drift function $a(\cdot)$ of the GBMP, it follows that

$$E_x[F(x)] \neq E_x[F(-x)]$$

for almost all functionals F on $C_{a,b}[0,T]$. This yields the fact that

$$E_x^{\mathrm{an}_\lambda}[F(x)] \neq E_x^{\mathrm{an}_\lambda}[F(-x)].$$

This result leads us to define a modified transform for functionals on $C_{a,b}[0,T]$.

Definition 2.2. *Let F satisfy condition* (2.1) *and* (2.2) *above. For $\lambda \in \mathbb{C}_+$ and $y \in C_{a,b}[0,T]$, let*

$$T_{\lambda,-}(F)(y) = E_x^{\mathrm{an}_\lambda}[F(y-x)].$$

*We also define the L_1 analytic backward GFFT (*b*-GFFT), $T_q^{(1,\mathrm{b})}(F)$ of F, by the formula ($\lambda \in \mathbb{C}_+$),*

$$T_q^{(1,\mathrm{b})}(F)(y) = \lim_{\lambda \to -iq} T_{\lambda,-}(F)(y),$$

if it exists for s-a.e. $y \in C_{a,b}[0,T]$.

We note that if $T_q^{(1,\mathrm{f})}(F)$ (resp. $T_q^{(1,\mathrm{b})}(F)$) exists and if $F \approx G$, then $T_q^{(1,\mathrm{f})}(G)$ (resp. $T_q^{(1,\mathrm{b})}(G)$) exists and $T_q^{(1,\mathrm{f})}(G) \approx T_q^{(1,\mathrm{f})}(F)$ (resp. $T_q^{(1,\mathrm{b})}(G) \approx T_q^{(1,\mathrm{b})}(F)$).

Next we provide the definition of the CP for functionals on $C_{a,b}[0,T]$.

Definition 2.3. *Let F and G be s.i.m. functionals on $C_{a,b}[0,T]$. For $\lambda \in \widetilde{\mathbb{C}}_+$, we define their CP $(F*G)_\lambda$ (if it exists) by*

$$(F*G)_\lambda(y) = \begin{cases} E_x^{\mathrm{an}_\lambda}\Big[F\Big(\frac{y+x}{\sqrt{2}}\Big)G\Big(\frac{y-x}{\sqrt{2}}\Big)\Big], & \lambda \in \mathbb{C}_+ \\ E_x^{\mathrm{anf}_q}\Big[F\Big(\frac{y+x}{\sqrt{2}}\Big)G\Big(\frac{y-x}{\sqrt{2}}\Big)\Big], & \lambda = -iq, \ q \in \mathbb{R} \setminus \{0\}. \end{cases} \tag{2.3}$$

When $\lambda = -iq$, we denote $(F*G)_\lambda$ by $(F*G)_q$.

In order to verify our main theorems, we provide the following lemma.

Lemma 2.4. *Let stochastic processes $Y_{(+)}, Y_{(-)} : C_{a,b}[0,T] \times C_{a,b}[0,T] \times [0,T] \to \mathbb{R}$ be given by*

$$Y_{(+)}(x_1, x_2; t) = \frac{x_2(t) + x_1(t)}{\sqrt{2}} \quad \text{and} \quad Y_{(-)}(x_1, x_2; t) = \frac{x_2(t) - x_1(t)}{\sqrt{2}}.$$

Then $\{Y_{(+)}(\cdot,\cdot;t) : t \in [0,T]\}$ and $\{Y_{(-)}(\cdot,\cdot;t) : t \in [0,T]\}$ are stochastically independent processes.

Proof. From the definition of the GBMP, it follows that for any $s, t \in [0,T]$,

$$E_{x_k}[x_k(s)] = a(s) \quad \text{and} \quad E_{x_k}[x_k(s)x_k(t)] = \min\{b(s), b(t)\} + a(s)a(t)$$

for each $k = 1, 2$. Using these and a direct calculation, it follows that

$$E_{x_1}[E_{x_2}[Y_{(+)}(x_1, x_2; s)Y_{(-)}(x_1, x_2; t)]] = 0$$

and

$$E_{x_1}[E_{x_2}[Y_{(+)}(x_1, x_2; s)]]E_{x_1}[E_{x_2}[Y_{(-)}(x_1, x_2; t)]] = 0.$$

Since the processes are Gaussian, we conclude the assertion in the lemma. □

As mentioned in Section 1 above, there exists a bounded functional on $C_{a,b}[0,T]$ which is not analytic Feynman integrable (and so the analytic f- and b-GFFTs of the functional do not exist). Thus, in the statements of the theorems presented below, we will assume that the generalized transforms of given functionals exist.

In next theorem, we establish a pseudo homomorphism structure between the f-GFFT and the CP of functionals on $C_[0,T]$.

Theorem 2.5. *Let F and G be s.i.m. functionals on $C_{a,b}[0,T]$. Given a nonzero real number q, assume that the CP and the generalized transforms*

$$(F * G)_q, \; T_{2q}^{(1,\mathrm{f})}\Big(T_{2q}^{(1,\mathrm{f})}(F)\Big), \; T_{2q}^{(1,\mathrm{f})}\Big(T_{2q}^{(1,\mathrm{b})}(G)\Big), \; \text{and} \; T_q^{(1,\mathrm{f})}\Big((F * G)_q\Big)$$

all exist. Then, it follows that

$$T_q^{(1,\mathrm{f})}\Big((F * G)_q\Big)(y) = T_{2q}^{(1,\mathrm{f})}\Big(T_{2q}^{(1,\mathrm{f})}(F)\Big)\Big(\frac{y}{\sqrt{2}}\Big)T_{2q}^{(1,\mathrm{f})}\Big(T_{2q}^{(1,\mathrm{b})}(G)\Big)\Big(\frac{y}{\sqrt{2}}\Big) \tag{2.4}$$

for s-a.e. $y \in C_{a,b}[0,T]$.

Proof. From the assumption that the three GFFT's appearing in equation (2.4) exist, it follows that

$$\begin{aligned} T_{\lambda_2,+}\Big((F * G)_{\lambda_1}\Big)(y) &= E_{x_2}\Big[(F * G)_{\lambda_1}(y + \lambda_2^{-1/2}x_2)\Big] \\ &= E_{x_2}\Big[E_{x_1}\Big[F\Big(\frac{y}{\sqrt{2}} + \frac{x_2}{\sqrt{2\lambda_2}} + \frac{x_1}{\sqrt{2\lambda_1}}\Big)G\Big(\frac{y}{\sqrt{2}} + \frac{x_2}{\sqrt{2\lambda_2}} - \frac{x_1}{\sqrt{2\lambda_1}}\Big)\Big]\Big], \end{aligned}$$

$$\begin{aligned} T_{2\lambda_2,+}\Big(T_{2\lambda_1,+}(F)\Big)\Big(\frac{y}{\sqrt{2}}\Big) &= E_{x_2}\Big[T_{2\lambda_1,+}(F)\Big(\frac{y}{\sqrt{2}} + \frac{x_2}{\sqrt{2\lambda_2}}\Big)\Big] \\ &= E_{x_2}\Big[E_{x_1}\Big[F\Big(\frac{y}{\sqrt{2}} + \frac{x_2}{\sqrt{2\lambda_2}} + \frac{x_1}{\sqrt{2\lambda_1}}\Big)\Big]\Big], \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} T_{2\lambda_2,+}\Big(T_{2\lambda_1,-}(G)\Big)\Big(\frac{y}{\sqrt{2}}\Big) &= E_{x_2}\Big[T_{2\lambda_1,-}(G)\Big(\frac{y}{\sqrt{2}} + \frac{x_2}{\sqrt{2\lambda_2}}\Big)\Big] \\ &= E_{x_2}\Big[E_{x_1}\Big[G\Big(\frac{y}{\sqrt{2}} + \frac{x_2}{\sqrt{2\lambda_2}} - \frac{x_1}{\sqrt{2\lambda_1}}\Big)\Big]\Big] \end{aligned} \tag{2.6}$$

are continuous functions of (λ_1, λ_2) in $\widetilde{\mathbb{C}}_+ \times \widetilde{\mathbb{C}}_+$. Furthermore, these transforms are analytic functions on $\mathbb{C}_+ \times \mathbb{C}_+$. Thus we need only to verify the following equality

$$T_{\lambda,+}\Big((F * G)_\lambda\Big)(y) = T_{2\lambda,+}\Big(T_{2\lambda,+}(F)\Big)\Big(\frac{y}{\sqrt{2}}\Big)T_{2\lambda,+}\Big(T_{2\lambda,-}(G)\Big)\Big(\frac{y}{\sqrt{2}}\Big) \tag{2.7}$$

for all $\lambda \in \mathbb{C}_+$.

But by Lemma 2.4, we have that for any $\lambda > 0$,

$$\begin{aligned} &T_{\lambda,+}\Big((F * G)_\lambda\Big)(y)\\ =&E_{x_2}\Big[E_{x_1}\Big[F\Big(\frac{y}{\sqrt{2}} + \frac{x_2}{\sqrt{2\lambda}} + \frac{x_1}{\sqrt{2\lambda}}\Big)G\Big(\frac{y}{\sqrt{2}} + \frac{x_2}{\sqrt{2\lambda}} - \frac{x_1}{\sqrt{2\lambda}}\Big)\Big]\Big]\\ =&E_{x_2}\Big[E_{x_1}\Big[F\Big(\frac{y}{\sqrt{2}} + \frac{x_2}{\sqrt{2\lambda}} + \frac{x_1}{\sqrt{2\lambda}}\Big)\Big]\Big]E_{x_2}\Big[E_{x_1}\Big[G\Big(\frac{y}{\sqrt{2}} + \frac{x_2}{\sqrt{2\lambda}} - \frac{x_1}{\sqrt{2\lambda}}\Big)\Big]\Big]. \end{aligned}$$

Now, applying equations (2.5) and (2.6) with $\lambda_1 = \lambda_2 = \lambda$, we obtain equation (2.7) for $\lambda > 0$. Next, using the concept of the analytic continuation and letting $\lambda \to -iq$, we complete the proof of Theorem 2.5 in view of Definitions 2.1, 2.2 and 2.3. □

By similar methods used in the proof of Theorem 2.5, we can derive the second pseudo homomorphism structure between the b-GFFT and the CP of functionals on $C_{a,b}[0, T]$.

Theorem 2.6. *Let F and G be s.i.m. functionals on $C_{a,b}[0, T]$. Given a nonzero real number q, assume that the CP and the generalized transforms*

$$(F * G)_q,\ T_{2q}^{(1,\mathrm{b})}\Big(T_{2q}^{(1,\mathrm{f})}(F)\Big),\ T_{2q}^{(1,\mathrm{b})}\Big(T_{2q}^{(1,\mathrm{b})}(G)\Big),\ \textit{and}\ \ T_q^{(1,\mathrm{b})}\Big((F * G)_q\Big)$$

all exist. Then, it follows that

$$T_q^{(1,\mathrm{b})}\Big((F * G)_q\Big)(y) = T_{2q}^{(1,\mathrm{b})}\Big(T_{2q}^{(1,\mathrm{f})}(F)\Big)\Big(\frac{y}{\sqrt{2}}\Big)T_{2q}^{(1,\mathrm{b})}\Big(T_{2q}^{(1,\mathrm{b})}(G)\Big)\Big(\frac{y}{\sqrt{2}}\Big) \tag{2.8}$$

for s-a.e. $y \in C_{a,b}[0, T]$.

3. Examples

In this section, we provide various examples to which equations (2.4) and (2.8) can be applied. For the calculations of the formulas presented below, one can apply the following integration formulas:

(i) For all nonzero complex numbers α and β with $\mathrm{Re}(\alpha) > 0$,

$$\int_{\mathbb{R}} \exp\Big(-\alpha v^2 + \beta v\Big)dv = \sqrt{\frac{\pi}{\alpha}}\exp\Big(\frac{\beta^2}{4\alpha}\Big). \tag{3.1}$$

(ii) For all $\beta \in \mathbb{C}$, $\lambda \in \mathbb{C}_+$ and $w \in C'_{a,b}[0, T]$,

$$E_x[\exp(\beta\lambda^{-1/2}(w, x)^\sim)] = \exp\Big(\frac{\beta^2}{2\lambda}\|w\|^2_{C'_{a,b}} + \beta\lambda^{-1/2}(w, a)_{C'_{a,b}}\Big) \tag{3.2}$$

where $(w, x)^\sim$ denotes the Paley–Wiener–Zygmund stochastic integral and $(\cdot, \cdot)_{C'_{a,b}}$ denotes the inner product on the Cameron–Martin space $C'_{a,b}[0, T]$ in $C_{a,b}[0, T]$. For more detailed definitions of the Paley–Wiener–Zygmund stochastic integral and the Cameron–Martin space $(C'_{a,b}[0, T], (\cdot, \cdot)_{C'_{a,b}}, \|\cdot\|_{C'_{a,b}})$, we refer to the reference [8, 11].

3.1. Simple examples

Let S be a bounded linear operator from $C'_{a,b}[0,T]$ to itself and let S^* be the adjoint operator of S. Let w be a function in $C'_{a,b}[0,T]$. Then the Paley–Wiener–Zygmund stochastic integral $(S^*w,x)^\sim$ is a Gaussian random variable, as a functional of x, with mean

$$E_x[(S^*w,x)^\sim] = (S^*w,a)_{C'_{a,b}} \tag{3.3}$$

and variance

$$\text{Var}[(S^*w,x)^\sim] = E_x[\{(S^*w,x)^\sim - (S^*w,a)_{C'_{a,b}}\}^2] = \|S^*w\|^2_{C'_{a,b}}.$$

Let F and G be functionals on $C_{a,b}[0,T]$ defined by

$$F(x) = (S^*w,x)^\sim \text{ and } G(x) = \exp\{\beta F(x)\} \text{ with } \beta \in \mathbb{C}.$$

Then, the functionals F and G are s.i.m. on the function space $C_{a,b}[0,T]$.

Using equation (3.3), it follows that for all $q \in \mathbb{R} \setminus \{0\}$,

$$T_q^{(1,\text{f})}(F)(y) = (S^*w,y)^\sim + (-iq)^{-1/2}(S^*w,a)_{C'_{a,b}} \tag{3.4}$$

and

$$T_q^{(1,\text{b})}(F)(y) = (S^*w,y)^\sim - (-iq)^{-1/2}(S^*w,a)_{C'_{a,b}} \tag{3.5}$$

for s-a.e. $y \in C_{a,b}[0,T]$, respectively. Also using equation (3.2) with w replaced with S^*w, it follows that for all $q \in \mathbb{R} \setminus \{0\}$,

$$T_q^{(1,\text{f})}(G)(y) = \exp\Big(\beta(S^*w,y)^\sim + i\frac{\beta^2}{2q}\|S^*w\|^2_{C'_{a,b}} + \beta(-iq)^{-1/2}(S^*w,a)_{C'_{a,b}}\Big) \tag{3.6}$$

and

$$T_q^{(1,\text{b})}(G)(y) = \exp\Big(\beta(S^*w,y)^\sim + i\frac{\beta^2}{2q}\|S^*w\|^2_{C'_{a,b}} - \beta(-iq)^{-1/2}(S^*w,a)_{C'_{a,b}}\Big) \tag{3.7}$$

for s-a.e. $y \in C_{a,b}[0,T]$, respectively. Thus, using equations (2.4), (3.4), (3.7), and (3.6), we easily obtain that

$$\begin{aligned}
&T_q^{(1,\text{f})}\big((F*G)_q\big)(y)\\
=&T_{2q}^{(1,\text{f})}\big(T_{2q}^{(1,\text{f})}(F)\big)\Big(\frac{y}{\sqrt{2}}\Big)T_{2q}^{(1,\text{f})}\big(T_{2q}^{(1,\text{b})}(G)\big)\Big(\frac{y}{\sqrt{2}}\Big)\\
=&\Big(\frac{1}{\sqrt{2}}(S^*w,y)^\sim + 2(-2iq)^{-1/2}(S^*w,a)_{C'_{a,b}}\Big)\exp\Big(\frac{\beta}{\sqrt{2}}(S^*w,y)^\sim + i\frac{\beta^2}{2q}\|S^*w\|^2_{C'_{a,b}}\Big)
\end{aligned}$$

for s-a.e. $y \in C_{a,b}[0,T]$, without the calculation of the CP. Also, using equations (2.8), (3.4), (3.5) and (3.7), we obtain that

$$\begin{aligned}
&T_q^{(1,\text{b})}\big((F*G)_q\big)(y)\\
=&T_{2q}^{(1,\text{b})}\big(T_{2q}^{(1,\text{f})}(F)\big)\Big(\frac{y}{\sqrt{2}}\Big)T_{2q}^{(1,\text{b})}\big(T_{2q}^{(1,\text{b})}(G)\big)\Big(\frac{y}{\sqrt{2}}\Big)\\
=&\frac{1}{\sqrt{2}}(S^*w,y)^\sim \exp\Big(\frac{\beta}{\sqrt{2}}(S^*w,y)^\sim + i\frac{\beta^2}{2q}\|S^*w\|^2_{C'_{a,b}} - 2\beta(-2iq)^{-1/2}(S^*w,a)_{C'_{a,b}}\Big)
\end{aligned}$$

for s-a.e. $y \in C_{a,b}[0,T]$.

In particular, let $S : C'_{a,b}[0,T] \to C'_{a,b}[0,T]$ be the linear operator defined by

$$Sw(t) = \int_0^t w(s)db(s).$$

Then we see that the adjoint operator S^* of S is given by

$$S^*w(t) = w(T)b(t) - \int_0^t w(s)db(s) = \int_0^t [w(T) - w(s)]db(s).$$

Moreover, by an integration by parts formula, we have

$$(S^*b, x)^{\sim} = \int_0^T [b(T) - b(t)]dx(t) = \int_0^T x(t)db(t).$$

Hence the functionals F and G with w replaced with $b(\cdot)$ are rewritten by

$$F(x) = \int_0^T x(t)db(t) \ \text{ and } \ G(x) = \exp\Big(\beta \int_0^T x(t)db(t)\Big).$$

These functionals arise naturally in quantum mechanics.

In next subsections, we provide that the assumptions (and hence the conclusions) of Theorems 2.5 and 2.6 are indeed satisfied by several large classes of functionals; we shall very briefly discuss three such classes.

3.2. Banach algebra $\mathcal{F}(C_{a,b}[0,T])$

We will see that the pseudo homomorphism structures (2.4) and (2.8) hold for the GFFTs and the CP of functionals in the Banach algebra $\mathcal{F}(C_{a,b}[0,T])$, which is a generalized class of the Banach algebra $\mathcal{S}$ introduced by Cameron and Storvick [4]. The Banach algebra $\mathcal{F}(C_{a,b}[0,T])$ consists of functionals expressible in the form

$$F(x) = \int_{C'_{a,b}[0,T]} \exp(i(w,x)^{\sim})df(w) \tag{3.8}$$

for s-a.e. $x \in C_{a,b}[0,T]$, where f is an element of $\mathcal{M}(C'_{a,b}[0,T])$, the space of all complex-valued countably additive finite Borel measures on $C'_{a,b}[0,T]$. Further work involving the functionals in $\mathcal{F}(C_{a,b}[0,T])$ and related topics include [6, 8, 11, 12].

(1) Let $F \in \mathcal{F}(C_{a,b}[0,T])$ be given by (3.8). Assume that

$$\int_{C'_{a,b}[0,T]} \exp\Big(\frac{1}{\sqrt{2q_0}}\|w\|_{C'_{a,b}}\|a\|_{C'_{a,b}}\Big)d|f|(w) < +\infty$$

for some positive real number $q_0 > 0$. Then, proceeding as in the proof of [7, Theorem 3.1], one can see that for all real q with $|q| > q_0$, the GFFTs $T_q^{(1,\mathrm{f})}(F)$ and $T_q^{(1,\mathrm{b})}(F)$ of F exist and are given by

$$T_q^{(1,\mathrm{f})}(F)(y) = \int_{C'_{a,b}[0,T]} \exp\Big(i(w,y)^{\sim} - \frac{i}{2q}\|w\|^2_{C'_{a,b}} + i(-iq)^{-1/2}(w,a)_{C'_{a,b}}\Big)df(w)$$

and

$$T_q^{(1,\mathrm{b})}(F)(y) = \int_{C'_{a,b}[0,T]} \exp\Big(i(w,y)^{\sim} - \frac{i}{2q}\|w\|^2_{C'_{a,b}} - i(-iq)^{-1/2}(w,a)_{C'_{a,b}}\Big)df(w)$$

for s-a.e. $y \in C'_{a,b}[0,T]$, respectively.

(2) Let $F \in \mathcal{F}(C_{a,b}[0,T])$ be given by (3.8), and let $G \in \mathcal{F}(C_{a,b}[0,T])$ be given by

$$G(x) = \int_{C'_{a,b}[0,T]} \exp(i(w,x)^{\sim}) dg(w) \tag{3.9}$$

for s-a.e. $x \in C_{a,b}[0,T]$. Assume that

$$\int_{C'_{a,b}[0,T]} \exp\Big(\frac{1}{\sqrt{4q_0}}\|w\|_{C'_{a,b}}\|a\|_{C'_{a,b}}\Big) d\Big[|f| + |g|\Big](w) < +\infty$$

for some positive real number $q_0 > 0$. Then, proceeding as in the proof of [7, Theorem 3.2], one can also see that for all real q with $|q| > q_0$, the CP of F and G, $(F * G)_q$ exists and is given by

$$\begin{aligned}&(F * G)_q(y)\\ &= \int_{C'_{a,b}[0,T]} \int_{C'_{a,b}[0,T]} \exp\Big(\frac{i}{\sqrt{2}}(w_1 + w_2, y)^{\sim} - \frac{i}{4q}\|w_1 - w_2\|^2_{C'_{a,b}} + i(-2iq)^{-1/2}(w_1 - w_2, a)_{C'_{a,b}}\Big) df(w_1) dg(w_2)\end{aligned}$$

for s-a.e. $y \in C'_{a,b}[0,T]$.

Theorem 3.1. *Let F and G in $\mathcal{F}(C_{a,b}[0,T])$ be given by* (3.8) *and* (3.9) *respectively. Assume that*

$$\int_{C'_{a,b}[0,T]} \exp\Big(\frac{1}{\sqrt{q_0}}\|w\|_{C'_{a,b}}\|a\|_{C'_{a,b}}\Big) d\Big[|f| + |g|\Big](w) < +\infty$$

for some positive real $q_0 > 0$. Then for all real q with $|q| > q_0$, the CP $(F * G)_q$ and the GFFTs $T^{(1,\mathrm{f})}_{2q}(F)$, $T^{(1,\mathrm{f})}_{2q}(T^{(1,\mathrm{f})}_{2q}(F))$, $T^{(1,\mathrm{f})}_{2q}(T^{(1,\mathrm{b})}_{2q}(F))$, $T^{(1,\mathrm{b})}_{2q}(G)$, $T^{(1,\mathrm{b})}_{2q}(T^{(1,\mathrm{f})}_{2q}(G))$, $T^{(1,\mathrm{b})}_{2q}(T^{(1,\mathrm{b})}_{2q}(G))$, $T^{(1,\mathrm{f})}_{q}((F * G)_q)$ and $T^{(1,\mathrm{b})}_{q}((F * G)_q)$ all exist. Thus, in view of Theorems 2.5 and 2.6, equation (2.4) *and* (2.8) *hold true.*

(3) On the other hand, as similar results (see [7, Theorem 3.3]), one can see that for all real q with $|q| > q_0$,

$$\begin{aligned}&T^{(1,\mathrm{f})}_{q}((F * G)_q)(y)\\ &= \int_{C'_{a,b}[0,T]} \int_{C'_{a,b}[0,T]} \exp\Big(\frac{i}{\sqrt{2}}(w_1 + w_2, y)^{\sim} - \frac{i}{2q}\Big(\|w_1\|^2_{C'_{a,b}} + \|w_2\|^2_{C'_{a,b}}\Big) + 2i(-2iq)^{-1/2}(w_1, a)_{C'_{a,b}}\Big) df(w_1) dg(w_2)\\ &= T^{(1,\mathrm{f})}_{q}((F * 1)_q)(y) T^{(1,\mathrm{f})}_{q}((1 * G)_q)(y)\end{aligned}$$

and

$$\begin{aligned}&T^{(1,\mathrm{b})}_{q}((F * G)_q)(y)\\ &= \int_{C'_{a,b}[0,T]} \int_{C'_{a,b}[0,T]} \exp\Big(\frac{i}{\sqrt{2}}(w_1 + w_2, y)^{\sim} - \frac{i}{2q}\Big(\|w_1\|^2_{C'_{a,b}} + \|w_2\|^2_{C'_{a,b}}\Big) - 2i(-2iq)^{-1/2}(w_2, a)_{C'_{a,b}}\Big) df(w_1) dg(w_2)\\ &= T^{(1,\mathrm{b})}_{q}((F * 1)_q)(y) T^{(1,\mathrm{b})}_{q}((1 * G)_q)(y)\end{aligned}$$

for s-a.e. $y \in C'_{a,b}[0,T]$, respectively.

3.3. Bounded cylinder functionals

Next we want to briefly discuss another class of functionals to which our general relationships between the GFFT and the CP can be applied. Given a complex-valued Borel measure ν on $\mathbb{R}^m$, the Fourier transform $\widehat{\nu}$ of ν is a complex-valued function on $\mathbb{R}^m$ defined by the formula

$$\widehat{\nu}(\vec{u}) = \int_{\mathbb{R}^m} \exp\Big\{i \sum_{k=1}^{m} u_k v_k\Big\} d\nu(\vec{v}).$$

Given a complex Borel measure ν on $\mathbb{R}^m$ and an orthogonal subset $\mathcal{A} = \{e_1, \dots, e_m\}$ of nonzero functions in $C'_{a,b}[0,T]$, define the functional $F_\nu : C_{a,b}[0,T] \to \mathbb{C}$ by

$$F_\nu(x) = \widehat{\nu}((e_1, x)^\sim, \dots, (e_m, x)^\sim) \tag{3.10}$$

for s-a.e. $x \in C_{a,b}[0,T]$.

For the orthogonal set $\mathcal{A} = \{e_1, \dots, e_m\}$, let $\widehat{\mathfrak{T}}_{\mathcal{A}}$ be the space of all functionals F_ν on $C_{a,b}[0,T]$ having the form (3.10). Note that $F_\nu \in \widehat{\mathfrak{T}}_{\mathcal{A}}$ implies that F_ν is s.i.m on $C_{a,b}[0,T]$.

Let F_{ν_1} and F_{ν_2} in $\widehat{\mathfrak{T}}_{\mathcal{A}}$ be given as equation (3.10) with the corresponding complex measures ν_1 and ν_2, respectively, on $\mathbb{R}^m$. Given a positive real q_0, assume that the complex Borel measures ν_1 and ν_2 satisfy the condition

$$\int_{\mathbb{R}^m} \exp\Big(\frac{\|a\|_{C'_{a,b}}}{\sqrt{q_0}} \sum_{k=1}^m \|e_k\|_{C'_{a,b}} |v_k|\Big) d[|\nu_1| + |\nu_2|](\vec{v}) < +\infty. \tag{3.11}$$

Then for all real q with $|q| > q_0$, the CP $(F_{\nu_1} * F_{\nu_2})_q$ and the GFFTs $T_{2q}^{(1,\mathrm{f})}(F_{\nu_1})$, $T_{2q}^{(1,\mathrm{f})}(T_{2q}^{(1,\mathrm{f})}(F_{\nu_1}))$, $T_{2q}^{(1,\mathrm{f})}(T_{2q}^{(1,\mathrm{b})}(F_{\nu_1}))$, $T_{2q}^{(1,\mathrm{b})}(F_{\nu_2})$, $T_{2q}^{(1,\mathrm{b})}(T_{2q}^{(1,\mathrm{f})}(F_{\nu_2}))$, $T_{2q}^{(1,\mathrm{b})}(T_{2q}^{(1,\mathrm{b})}(F_{\nu_2}))$, $T_q^{(1,\mathrm{f})}((F_{\nu_1} * F_{\nu_2})_q)$ and $T_q^{(1,\mathrm{b})}((F_{\nu_1} * F_{\nu_2})_q)$ all exist.

For instance, applying the change of variables theorem, the Fubini theorem, and equation (3.1), one can verify easily that for all real q with $|q| > q_0$, the GFFTs $T_q^{(1,\mathrm{f})}(F_{\nu_1})$ and $T_q^{(1,\mathrm{b})}(F_{\nu_1})$ and the CP $(F_{\nu_1} * F_{\nu_2})_q$ are given by

$$T_q^{(1,\mathrm{f})}(F_{\nu_1})(y) = \int_{\mathbb{R}^m} \exp\Big(i\sum_{j=1}^m (e_j, y)^\sim v_j - \frac{i}{2q}\sum_{j=1}^m \|e_j\|^2_{C'_{a,b}} v_j^2 + i(-iq)^{-1/2}\sum_{j=1}^m (e_j, a)_{C'_{a,b}} v_j\Big) d\nu_1(\vec{v}),$$

$$T_q^{(1,\mathrm{b})}(F_{\nu_1})(y) = \int_{\mathbb{R}^m} \exp\Big(i\sum_{j=1}^m (e_j, y)^\sim v_j - \frac{i}{2q}\sum_{j=1}^m \|e_j\|^2_{C'_{a,b}} v_j^2 - i(-iq)^{-1/2}\sum_{j=1}^m (e_j, a)_{C'_{a,b}} v_j\Big) d\nu_1(\vec{v}),$$

and

$$\begin{aligned}&(F_{\nu_1} * F_{\nu_2})_q(y)\\ &= \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \exp\Big(i\sum_{j=1}^m (e_j, y)^\sim \Big(\frac{u_j + v_j}{\sqrt{2}}\Big) - \frac{i}{4q}\sum_{j=1}^m \|e_j\|^2_{C'_{a,b}} v_j^2 + i(-2iq)^{-1/2}\sum_{j=1}^m (e_j, a)_{C'_{a,b}} (u_j - v_j)\Big) d\nu_1(\vec{v})\end{aligned}$$

for s-a.e. $y \in C'_{a,b}[0,T]$, respectively.

In view of Theorems 2.5 and 2.6, we have the following theorem.

Theorem 3.2. *Let F_{ν_1} and F_{ν_2} in $\widehat{\mathfrak{T}}_{\mathcal{A}}$ be as above. Given a positive real q_0, assume that the complex Borel measures ν_1 and ν_2 corresponding to the functionals F_{ν_1} and F_{ν_2}, respectively, by* (3.10) *satisfies condition* (3.11)*. Then for all real q with $|q| > q_0$, equation* (2.4) *and* (2.8) *hold true.*

3.4. Exponential-type functionals

Let $\mathcal{E}$ be the class of all functionals which have the form

$$\Psi_w(x) = \exp((w, x)^\sim) \tag{3.12}$$

for some $w \in C'_{a,b}[0,T]$ and for s-a.e. $x \in C_{a,b}[0,T]$. More precisely, since we shall identify functionals which coincide s-a.e. on $C_{a,b}[0,T]$, the class $\mathcal{E}$ can be regarded as the space of all s-equivalence classes of functionals of the form (3.12). The functionals given by equation (3.12) and linear combinations (with complex coefficients) of the $\Psi_w(x)$'s are called partially exponential-type functionals on $C_{a,b}[0,T]$.

The linear space $\mathcal{E}(C_{a,b}[0,T]) = \mathrm{Span}\mathcal{E}$ of partially exponential-type functionals is a commutative (complex) algebra under the pointwise multiplication and with identity $\Psi_0 \equiv 1$. For more details see, [9, 10, 15].

Theorem 3.3. *Let F and G be a partially exponential-type functionals in $\mathcal{E}(C_{a,b}[0,T])$. Then for any nonzero real number q, equation* (2.4) *and* (2.8) *hold true.*

4. Corollaries

We finally list various relationships among the f- and b-GFFTs, and the CP for functionals on $C_{a,b}[0,T]$. For a functional $F : C_{a,b}[0,T] \to \mathbb{C}$, we will write $\overline{F}(x) = F(-x)$. Then we have following five equalities:

$$\overline{T_q^{(1,\mathrm{b})}(F)} \approx T_q^{(1,\mathrm{f})}(\overline{F}) \tag{4.1}$$

$$\overline{T_q^{(1,\mathrm{f})}\Big(T_q^{(1,\mathrm{f})}(F)\Big)} \approx T_q^{(1,\mathrm{b})}\Big(T_q^{(1,\mathrm{b})}(\overline{F})\Big), \tag{4.2}$$

$$\overline{T_q^{(1,\mathrm{b})}\Big(T_q^{(1,\mathrm{b})}(F)\Big)} \approx T_q^{(1,\mathrm{f})}\Big(T_q^{(1,\mathrm{f})}(\overline{F})\Big), \tag{4.3}$$

$$\overline{T_q^{(1,\mathrm{f})}\Big(T_q^{(1,\mathrm{b})}(F)\Big)} \approx T_q^{(1,\mathrm{b})}\Big(T_q^{(1,\mathrm{f})}(\overline{F})\Big), \tag{4.4}$$

and

$$\overline{T_q^{(1,\mathrm{b})}\Big(T_q^{(1,\mathrm{f})}(F)\Big)} \approx T_q^{(1,\mathrm{f})}\Big(T_q^{(1,\mathrm{b})}(\overline{F})\Big). \tag{4.5}$$

Then, using equations (2.4), (2.8), (4.1) through (4.5), we obtain the following four equalities:

$$T_q^{(1,\mathrm{f})}\Big((\overline{F}*G)_q\Big)(y) = \overline{T_{2q}^{(1,\mathrm{b})}\Big(T_{2q}^{(1,\mathrm{b})}(F)\Big)}\Big(\frac{y}{\sqrt{2}}\Big)T_{2q}^{(1,\mathrm{f})}\Big(T_{2q}^{(1,\mathrm{b})}(G)\Big)\Big(\frac{y}{\sqrt{2}}\Big),$$

$$T_q^{(1,\mathrm{f})}\Big((F*\overline{G})_q\Big)(y) = T_{2q}^{(1,\mathrm{f})}\Big(T_{2q}^{(1,\mathrm{f})}(F)\Big)\Big(\frac{y}{\sqrt{2}}\Big)\overline{T_{2q}^{(1,\mathrm{b})}\Big(T_{2q}^{(1,\mathrm{f})}(G)\Big)}\Big(\frac{y}{\sqrt{2}}\Big),$$

$$T_q^{(1,\mathrm{b})}\Big((\overline{F}*G)_q\Big)(y) = \overline{T_{2q}^{(1,\mathrm{f})}\Big(T_{2q}^{(1,\mathrm{b})}(F)\Big)}\Big(\frac{y}{\sqrt{2}}\Big)T_{2q}^{(1,\mathrm{b})}\Big(T_{2q}^{(1,\mathrm{b})}(G)\Big)\Big(\frac{y}{\sqrt{2}}\Big),$$

and

$$T_q^{(1,\mathrm{b})}\Big((F*\overline{G})_q\Big)(y) = T_{2q}^{(1,\mathrm{b})}\Big(T_{2q}^{(1,\mathrm{f})}(F)\Big)\Big(\frac{y}{\sqrt{2}}\Big)\overline{T_{2q}^{(1,\mathrm{f})}\Big(T_{2q}^{(1,\mathrm{f})}(G)\Big)}\Big(\frac{y}{\sqrt{2}}\Big)$$

for s-a.e. $y \in C_{a,b}[0,T]$ respectively. From these, we also obtain the following six equalities:

$$T_q^{(1,\mathrm{f})}\Big((\overline{F}*G)_q\Big)(y) = \overline{T_q^{(1,\mathrm{b})}\Big((\overline{G}*F)_q\Big)}(y),$$

$$T_q^{(1,\mathrm{f})}\Big((F*\overline{G})_q\Big)(y) = \overline{T_q^{(1,\mathrm{b})}\Big((G*\overline{F})_q\Big)}(y),$$

$$T_q^{(1,\mathrm{b})}\Big((\overline{F}*G)_q\Big)(y) = \overline{T_q^{(1,\mathrm{f})}\Big((\overline{G}*F)_q\Big)}(y),$$

$$T_q^{(1,\mathrm{b})}\Big((F*\overline{G})_q\Big)(y) = \overline{T_q^{(1,\mathrm{f})}\Big((G*\overline{F})_q\Big)}(y),$$

$$T_q^{(1,\mathrm{f})}\Big((\overline{F}*\overline{G})_q\Big)(y) = \overline{T_q^{(1,\mathrm{b})}\Big((G*F)_q\Big)}(y),$$

and

$$T_q^{(1,\mathrm{b})}\Big((\overline{F}*\overline{G})_q\Big)(y) = \overline{T_q^{(1,\mathrm{f})}\Big((G*F)_q\Big)}(y)$$

for s-a.e. $y \in C_{a,b}[0,T]$ respectively.

5. Concluding remark

When $a(t) \equiv 0$ and $b(t) = t$ on $[0, T]$, then the GBMP associated with the functions $a(\cdot)$ and $b(\cdot)$ reduces to a standard Brownian motion, and so the function space $C_{a,b}[0, T]$ reduces to the classical Wiener space $C_0[0, T]$. In this case, it follows that

$$T_q^{(1)}(F) \approx T_q^{(1,\mathrm{f})}(F) \approx T_q^{(1,\mathrm{b})}(F)$$

for all s.i.m. functionals F on $C_0[0, T]$, where $T_q^{(1)}(F)$ means the L_1 analytic FFT of functionals F on $C_0[0, T]$, see [16–21], and thus equations (2.4) and (2.8) are rewritten as

$$T_q^{(1)}\Big((F * G)_q\Big)(y) = T_{2q}^{(1)}\Big(T_{2q}^{(1)}(F)\Big)\Big(\frac{y}{\sqrt{2}}\Big)T_{2q}^{(1)}\Big(T_{2q}^{(1)}(G)\Big)\Big(\frac{y}{\sqrt{2}}\Big) \tag{5.1}$$

for s-a.e. $y \in C_0[0, T]$. Furthermore, applying [19, Theorem 3], equation (1.1) follows from equation (5.1). This result subsumes similar known results obtained in [5, 16–18].

However, if the GBMP has a nonzero time drift $a(t)$ on $[0, T]$, i.e., the GBMP is not a centered Gaussian process, the GFFTs $T_q^{(1,\mathrm{f})}$ and $T_q^{(1,\mathrm{b})}$ for functionals on $C_{a,b}[0, T]$ do not have traditional homomorphism structures, such as (1.1) with the CP defined by (2.3).

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