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Nonlinear maps preserving semi-Fredholm operators with finite ascent

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Abstract. Let $\mathcal{X}$ be an infinite-dimensional complex Banach space, and $\mathcal{B}(\mathcal{X})$ the algebra of all bounded linear operators on $\mathcal{X}$. For any given positive integer m, we characterize the nonlinear maps on $\mathcal{B}(\mathcal{X})$ that preserve semi-Fredholm operators with ascent at most m in both directions. We completely describe their structure.

1. Introduction

Over the past few decades, several authors interested in linear or additive preserver problems have sought to characterize linear or additive maps on operator algebras that preserve certain properties, subsets, or relations(cf.[8, 11, 12, 15]). Among other things, linear or additive maps that preserve the classes of semi-Fredholm operators, Fredholm operators, and related operators in both directions are considered(cf.[16]). Recently, certain nonlinear preserver problems concern with those operator classes have attracted interest. For example, in [13], the authors demonstrate the structure of maps that preserve Drazin operators with bounded ascent in both directions. In [5], Ji, Jiao, and Shi investigated nonlinear maps preserving semi-Fredholm operators with nullity at most a given positive integer n. On the other hand, it is known that the ascent of an operator is a fundamental and important ingredient in Fredholm theory. Additionally, there is also a lot of research on the properties of the ascent(cf.[2, 3, 16]). The ascent, as an important index of operators, makes the study of its preserver problems highly significant; see, for example, [3, 7, 16]. In this paper, we focus on nonlinear maps on the algebra of all bounded linear operators on an infinite dimensional Banach space preserving semi-Fredholm operators with finite ascent. We recall some notions.

Throughout this paper, let $\mathcal{X}$ be an infinite-dimensional complex Banach space and $\mathcal{B}(\mathcal{X})$ the algebra of all bounded linear operators on $\mathcal{X}$. $\mathcal{X}^*$ denotes the dual space of $\mathcal{X}$. For any subset $S \subseteq \mathcal{X}$, $[S]$ is the closed subspace generated by S. Let $T \in \mathcal{B}(\mathcal{X})$. We denote $\ker(T)$ as its nullspace, $\operatorname{ran}(T)$ as its range and T^* as its conjugate operator. For a closed subspace L of $\mathcal{X}$, $\dim L$ is the dimension of L, and $\operatorname{codim} L$ is the dimension of the quotient space $\mathcal{X}/L$. T is called a semi-Fredholm operator if $\operatorname{ran}(T)$ is closed and either $\dim\ker(T)<\infty$ or $\operatorname{codim}\operatorname{ran}(T)<\infty$. In this case, the index of T is defined by $\operatorname{ind}(T) := \dim\ker(T) - \operatorname{codim}\operatorname{ran}(T)$. We denote by I the identity on any Banach space without any confusion.

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Recall that the ascent asc(T) of an operator $T \in B(\mathcal{X})$ is defined by

$$\mathrm{asc}(T) = \inf\{k \geq 0 : \ker(T^k) = \ker(T^{k+1})\},$$

where the infimum over the empty set is taken to be infinite (see [10], [17]). From [3, Lemma 1.1], given a non-negative integer k, we have

$$\mathrm{asc}(T) \leq k \Longleftrightarrow \ker(T^n) \cap \mathrm{ran}(T^k) = \{0\} \tag{1}$$

for some (equivalently, all) $n \geq 1$. Given any positive integer m, we introduce the following classes of operators:
(i) $\mathcal{F}(\mathcal{X}) = \{T \in \mathcal{B}(\mathcal{X}) : T$ is of finite rank$\}$;
(ii) $\Phi_{sF}(\mathcal{X}) = \{T \in \mathcal{B}(\mathcal{X}) : T$ is a semi-Fredholm operator$\}$;
(iii) $\mathcal{A}_m(\mathcal{X}) = \{T \in \Phi_{sF}(\mathcal{X}) : \mathrm{asc}(T) \leq m\}$;
(iv) $\mathcal{B}_m(\mathcal{X}) = \{T \in \mathcal{B}(\mathcal{X}) : \dim\ker(T^{m+1}) \leq m\}$;
(v) $\mathcal{B}_m^1(\mathcal{X}) = \Phi_{sF}(\mathcal{X}) \cap \mathcal{B}_m(\mathcal{X})$.

Let $\mathcal{S}$ be a subset of $\mathcal{B}(\mathcal{X})$. A map $\phi : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ is said to preserve $\mathcal{S}$ in both directions if $\phi(T) \in \mathcal{S}$ if and only if $T \in \mathcal{S}$. In this paper, we will analyze the structure of nonlinear maps on $\mathcal{B}(\mathcal{X})$ preserving $\mathcal{A}_m(\mathcal{X})$ in both directions.

2. $\mathcal{A}_m(\mathcal{X})$ and its finite rank perturbations

An operator T is said to be algebraic if there exists a nonzero complex polynomial p such that $p(T) = 0$. The minimal polynomial of T is a monic polynomial of smallest degree annihilating T. Let $p(\lambda) = (\lambda - \lambda_1)^{i_1}(\lambda - \lambda_2)^{i_2} \cdots (\lambda - \lambda_m)^{i_m}$ be the minimal polynomial of T, where $\lambda_1, \lambda_2, \cdots, \lambda_m \in \mathbb{C}$ are distinct. Then it follows from [10] and [17] that

$$\mathcal{X} = \ker(T - \lambda_1 I)^{i_1} \oplus \ker(T - \lambda_2 I)^{i_2} \oplus \cdots \oplus \ker(T - \lambda_m I)^{i_m} \tag{2}$$

and $\dim\ker(T - \lambda_k I)^{i_k} \leq i_k \dim\ker(T - \lambda_k I)$ for any $1 \leq k \leq m$. In this case, $\sigma(T) = \sigma_p(T) = \{\lambda_j : 1 \leq j \leq m\}$, where $\sigma_p(T)$ is the point spectrum of S.

Lemma 2.1. *Let $A, B \in \mathcal{B}(\mathcal{X})$ with $\dim\ker(A) = \infty$. If B is not algebraic, then for any integer $n \geq 1$, there exist two vectors x_0, x in $\mathcal{X}$ such that the set $\{x_0, Ax_0, x, Bx, \cdots, B^{n+1}x\}$ is linearly independent.*

Proof. Since B is not algebraic, it follows from [1, Theorem 4.2.7] that there exists $x \in \mathcal{X}$ such that the set $\{x, Bx, \cdots, B^{2n+3}x\}$ is linearly independent. If there exists $x_0 \in \mathcal{X}$ such that the set $\{Ax_0, x, \cdots, Bx, B^{2n+3}x\}$ is linearly independent, then so is the set $\{Ax_0, x, Bx, \cdots, B^{n+1}x\}$. Since $\dim\ker(A) = \infty$, there exists $y_0 \in \ker(A)$ such that the set $\{x_0 + y_0, A(x_0 + y_0), x, Bx, \cdots, B^{n+1}x\}$ is linearly independent.

Next we assume that $\mathrm{ran}(A) \subseteq [x, Bx, \cdots, B^{2n+3}x]$. In this case, for any nonzero $Ax_0 \in \mathrm{ran}(A)$, we have $Ax_0 = k_0x + k_1Bx + \cdots + k_{2n+3}B^{2n+3}x$ for some $k_0, k_1, \cdots, k_{2n+3} \in \mathbb{C}$. Put $m = \max\{j : k_j \neq 0\}$. If $m \geq n+2$, then the set $\{Ax_0, x, Bx, \cdots, B^{n+1}x\}$ is linearly independent. If $m < n+2$, then so is the set $\{Ax_0, B^{n+2}x, \cdots, B^{2n+3}x\}$. By replacing x with $B^{n+2}x$, we can conclude that the set $\{Ax_0, x, Bx, \cdots, B^{n+1}x\}$ is linearly independent. Using the same method as previous proof, we can get the desired x_0 and x. □

Lemma 2.2. *Let M be a subspace of $\mathcal{X}$ with $\dim M = n < \infty$. If $A \in \mathcal{B}(\mathcal{X})$ is not algebraic, then there exists a nonzero vector $x \in \mathcal{X}$ such that $M \cap [x, Ax, \cdots, A^{n-1}x] = \{0\}$.*

Proof. Suppose $M = [x_1, x_2, \cdots, x_n]$ generated by linearly independent vectors $\{x_1, x_2, \cdots, x_n\}$. Since A is not an algebraic operator, by [1, Corollary 4.2.8] , there exists an $x \in \mathcal{X}$ such that the set $\{x, Ax, \cdots, A^kx, \cdots\}$ is linearly independent. Put $N = [x, Ax, \cdots, A^kx, \cdots]$. If $M \cap N = \{0\}$, then $M \cap [x, Ax, \cdots, A^{n-1}x] = \{0\}$ and the desired result follows.

Next we assume that $M \cap N \neq \{0\}$. We firstly claim that for any closed subspace $N_0 = [y_1, y_2, \cdots, y_p] \subseteq N$ generated by linearly independent vectors $\{y_1, y_2, \cdots, y_p\}$, there is an integer $k \geq 0$ such that $N_0 \cap [A^{k+1}x :$

$k \geq 1] = \{0\}$. In fact, for any $1 \leq i \leq p$, there exists an integer $j_i \geq 1$ and complex numbers $\{c_{i0}, c_{i1}, \cdots, c_{ij_i}\}$ with $c_{ij_i} \neq 0$ such that

$$y_i = c_{i0}x + c_{i1}Ax + \cdots + c_{ij_i}A^{j_i}x.$$

Put $k = \max\{j_i : 1 \leq i \leq p\}$. Then $N_0 \subseteq [x, Ax, \cdots, A^k x]$. Thus $N_0 \cap [A^{k+l}x : l \geq 1] = \{0\}$.

Put $N_0 = M \cap N$ and $M = N_0 \oplus M_0$, where $N_0 \subseteq N$ is a closed subspace and $M_0 \cap N = \{0\}$. Note that $\dim N_0 \leq \dim M \leq n$. By previous claim, $N_0 \cap [A^{k+1}x, \cdots, A^{k+n-1}x] = \{0\}$ for some $k \geq 1$. It follows that

$$M \cap [A^{k+1}x, A^{k+2}x \cdots, A^{k+n-1}x] = \{0\}.$$

By replacing x with $A^k x$, we get that $M \cap [x, Ax \cdots, A^{n-1}x] = \{0\}$. □

Let $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$ be non-zero. We denote by $x \otimes f$ the rank one operator given by $(x \otimes f)(x) = f(z)x$ for all $z \in \mathcal{X}$. For an integer $k \geq 2$, we denote by J_k the $k \times k$ nilpotent matrix of order k whose superdiagonal entries are 1 and all other entries are 0.

Proposition 2.3. *Let R and F be two different nonzero operators in $\mathcal{B}(\mathcal{X})$ with $\dim \operatorname{ran}(F) \geq 2$. Then there exists $T \in \mathcal{B}(\mathcal{X})$ such that $T - R \in \mathcal{B}_m^1(\mathcal{X})$, $T \notin \mathcal{A}_m(\mathcal{X})$ and $T - F \notin \mathcal{A}_m(\mathcal{X})$.*

Proof. We shall divide the proof into four cases.

Case 1. $\dim \ker(F - R) = \infty$ and $\dim \ker(R) = \infty$.

First we claim that there exist nonzero $x_0, y_0 \in \mathcal{X}$ such that the set $\{(F-R)x_0, Ry_0\}$ is linearly independent. Otherwise, $\operatorname{ran}(F - R) = \operatorname{ran}(R) = [x]$ for some $x \in \mathcal{X}$. It follows that there exist f and g in $\mathcal{X}^*$ such that $R = x \otimes f$ while $F - R = x \otimes g$. Thus $F = x \otimes (f + g)$ is of rank 1, a contradiction.

By perturbing x_0 by an element of $\ker(F - R)$ and y_0 by an element of $\ker(R)$, we can assume that $\{x_0, y_0, (F - R)x_0, Ry_0\}$ is a linearly independent set. Take $x_i \in \ker(F - R)$, $y_i \in \ker(R)(1 \leq i \leq m)$ such that $\{(F - R)x_0, Ry_0, x_i, y_i : 0 \leq i \leq m\}$ is a linearly independent set. Put

$$\begin{aligned} \mathcal{X} &= [x_i, y_i : 0 \leq i \leq m] \oplus X_1 \\ &= [(F - R)x_0, Ry_0, x_i, y_i : 0 \leq i \leq m - 1] \oplus X_2, \end{aligned}$$

where X_1 and X_2 are two closed subspaces. Let $S : X_1 \to X_2$ be an invertible bounded linear operator. We define $T \in \mathcal{B}(\mathcal{X})$ by

$$\begin{cases} Ty_0 = 0 \text{ and } Tx_0 = Fx_0, \\ Ty_i = y_{i-1} \text{ and } Tx_i = x_{i-1} + Fx_i \text{ for } 1 \leq i \leq m, \\ Tx = Sx + Rx \text{ for any } x \in X_1. \end{cases}$$

It follows that $T^m y_m = y_0$, which means that $y_0 \in \ker(T) \cap \operatorname{ran}(T^m)$. Consequently, $\operatorname{asc}(T) \geq m + 1$ by (1). Hence $T \notin \mathcal{A}_m(\mathcal{X})$. Similarly, as $(T - F)^m x_m = x_0$, we obtain that $T - F \notin \mathcal{A}_m(\mathcal{X})$. On the other hand,

$$\begin{cases} (T - R)y_0 = -Ry_0 \text{ and } (T - R)x_0 = (F - R)x_0, \\ (T - R)y_i = y_{i-1} \text{ and } (T - R)x_i = x_{i-1} \text{ for } 1 \leq i \leq m, \\ (T - R)x = Sx \text{ for any } x \in X_1, \end{cases}$$

Thus $T - R$ is invertible and $T - R \in \mathcal{B}_m^1(\mathcal{X})$.

Case 2. $\dim \ker(F - R) < \infty$ and $\dim \ker(R) = \infty$.

If $F - R$ is an algebraic operator, then there exists $\lambda \in \mathbb{C} \backslash \{0\}$ such that $\dim \ker(F - R - \lambda I) = \infty$ by (2). Furthermore, there exists $x_0 \in \mathcal{X}$ such that vectors Rx_0 and x_0 are linearly independent by the assumption. We now take any $x_i \in \ker(R)(1 \leq i \leq m)$ and $y_j \in \ker(F - R - \lambda I)(0 \leq j \leq m)$ such that $\{Rx_0, x_i, y_i : 0 \leq i \leq m\}$ is a linearly independent set. Put

$$\mathcal{X} = [Rx_0, x_i : 0 \leq i \leq m] \oplus [y_i : 0 \leq i \leq m] \oplus X_1 \tag{3}$$

again. We may define an operator $T \in \mathcal{B}(\mathcal{X})$ by

$$\begin{cases} Tx_0 = 0 \text{ and } Ty_0 = Fy_0, \\ Tx_i = x_{i-1} \text{ and } Ty_i = y_{i-1} + Fy_i \text{ for } 1 \leq i \leq m, \\ T(Rx_0) = R^2x_0 + x_m, \\ Tx = x + Rx \text{ for any } x \in X_1. \end{cases}$$

It follows that $T^m x_m = x_0$. So $x_0 \in \ker(T) \cap \operatorname{ran}(T^m)$. Consequently, $\operatorname{asc}(T) \geq m + 1$ by (1), and hence $T \notin \mathcal{A}_m(\mathcal{X})$. Similarly, as $(T - F)^m y_m = y_0$, we obtain that $T - F \notin \mathcal{A}_m(\mathcal{X})$. We also have

$$\begin{cases} (T - R)x_0 = -Rx_0 \text{ and } (T - R)y_0 = \lambda y_0, \\ (T - R)x_i = x_{i-1} \text{ and } (T - R)y_i = y_{i-1} + \lambda y_i \text{ for } 1 \leq i \leq m, \\ (T - R)Rx_0 = x_m, \\ (T - R)x = x \text{ for any } x \in X_1. \end{cases}$$

Then we have

$$T - R = \begin{bmatrix} S_1 & 0 & 0 \\ 0 & J_n + \lambda I & 0 \\ 0 & 0 & I \end{bmatrix}$$

under the decomposition (3), where S_1 is invertible. It follows that $T - R$ is invertible and thus $T - R \in \mathcal{B}^1_m(\mathcal{X})$.

Now we assume that $F - R$ is not algebraic. It follows from Lemma 2.1 that there exist $x_0, x \in \mathcal{X}$ such that the set $\{x_0, Rx_0, x, (F - R)x, \cdots, (F - R)^{m+1}x\}$ is linearly independent. Put $y_i = (-1)^i(F - R)^{m-i}x (0 \leq i \leq m)$ and $u = (F - R)^{m+1}x$. Then we can take $x_i \in \ker(R) (1 \leq i \leq m)$ such that $\{u, Rx_0, x_i, y_i : 0 \leq i \leq m\}$ is a linearly independent set. Similarly we may set

$$\mathcal{X} = [Rx_0, x_i : 0 \leq i \leq m] \oplus [y_i : 1 \leq i \leq m] \oplus [y_0, u] \oplus X_1. \tag{4}$$

and define an operator $T \in \mathcal{B}(\mathcal{X})$ by

$$\begin{cases} Tx_0 = 0 \text{ and } Ty_0 = Fy_0, \\ Tx_i = x_{i-1} \text{ and } Ty_i = y_{i-1} + Fy_i \text{ for } 1 \leq i \leq m, \\ TRx_0 = R^2x_0 + x_m, \\ Tu = Ru + y_0, \\ Tx = x + Rx \text{ for any } x \in X_1. \end{cases}$$

It follows that $T^m x_m = x_0$, and so $x_0 \in \ker(T) \cap \operatorname{ran}(T^m)$. Consequently, $\operatorname{asc}(T) \geq m + 1$ by (1), and hence $T \notin \mathcal{A}_m(\mathcal{X})$. Similarly, as $(T - F)^m y_m = y_0$, we obtain that $T - F \notin \mathcal{A}_m(\mathcal{X})$. Furthermore, we also have

$$\begin{cases} (T - R)x_0 = -Rx_0 \text{ and } (T - R)y_0 = u, \\ (T - R)x_i = x_{i-1} \text{ and } (T - R)y_i = 0 \text{ for } 1 \leq i \leq m, \\ (T - R)Rx_0 = x_m, \\ (T - R)u = y_0, \\ (T - R)x = x \text{ for any } x \in X_1. \end{cases}$$

Then we have

$$T - R = \begin{bmatrix} S_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & S_3 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

under the decomposition (4), where S_2, S_3 are invertible. It follows that $\ker(T-R)^{m+1} = [y_i : 1 \le i \le m]$, and hence $\dim\ker(T-R)^{m+1} = m$. Thus $T - R \in \mathcal{B}_m^1(\mathcal{X})$.

Case 3. $\dim\ker(F-R) = \infty$ and $\dim\ker(R) < \infty$.

Let $R_0 = F - R$. Then we have $\dim\ker(F - R_0) < \infty$ and $\dim\ker(R_0) = \infty$. It follows from the second case that there exists a $K \in \mathcal{B}(\mathcal{X})$ such that $K - R_0 \in \mathcal{B}_m^1(\mathcal{X})$, $K \notin \mathcal{A}_m(\mathcal{X})$ and $K - F \notin \mathcal{A}_m(\mathcal{X})$. Take $T = -K + F$. We can get that $T - R = -K + R_0 \in \mathcal{B}_m^1(\mathcal{X})$, $T - F = -K \notin \mathcal{A}_m(\mathcal{X})$ and $T = -K + F \notin \mathcal{A}_m(\mathcal{X})$.

Case 4. $\dim\ker(F-R) < \infty$ and $\dim\ker(R) < \infty$.

1) $F - R$ and R are not algebraic operators.

Since $F - R$ is not algebraic, there exists $z_1 \in \mathcal{X}$ such that $\{z_1, (F-R)z_1, \cdots, (F-R)^{m+1}z_1\}$ is linearly independent. It follows from Lemma 2.2 that there exists $z_2 \in \mathcal{X}$ such that

$$[z_1, (F-R)z_1, \cdots, (F-R)^{m+1}z_1] \cap [z_2, Rz_2, \cdots, R^{m+1}z_2] = \{0\}.$$

Thus $\{z_1, (F-R)z_1, \cdots, (F-R)^{m+1}z_1, z_2, Rz_2, \cdots, R^{m+1}z_2\}$ is linearly independent. Let $y_i = (-1)^i(F-R)^{m-i}z_1,\ x_i = (-1)^i R^{m-i}z_2, (0 \le i \le m)$, $v = (F-R)^{m+1}z_1,\ u = R^{(m+1)}z_2$. We can write

$$\mathcal{X} = [u, x_i : 0 \le i \le m] \oplus [y_i : 1 \le i \le m] \oplus [y_0, v] \oplus X_1. \tag{5}$$

Define $T \in \mathcal{B}(\mathcal{X})$ by

$$\begin{cases} Tx_0 = 0 \text{ and } Ty_0 = Fy_0, \\ Tx_i = x_{i-1} \text{ and } Ty_i = y_{i-1} + Fy_i \text{ for } 1 \le i \le m, \\ Tu = Ru + x_m, \\ Tv = Rv + y_0, \\ Tx = x + Rx \text{ for any } x \in X_1. \end{cases}$$

It follows that $T^m x_n = x_0$, and so $x_0 \in \ker(T) \cap \operatorname{ran}(T^m)$. Consequently, $\operatorname{asc}(T) \ge m+1$ by (1), and hence $T \notin \mathcal{A}_m(\mathcal{X})$. Similarly, as $(T-F)^m y_m = y_0$, we obtain that $T - F \notin \mathcal{A}_m(\mathcal{X})$. Since

$$\begin{cases} (T-R)x_0 = -u \text{ and } (T-R)y_0 = v, \\ (T-R)x_i = 2x_{i-1} \text{ and } (T-R)y_i = 0 \text{ for } 1 \le i \le m, \\ (T-R)u = x_m, \\ (T-R)v = y_0, \\ (T-R)x = x \text{ for any } x \in X_1. \end{cases}$$

Under the decomposition (5), we have

$$T - R = \begin{bmatrix} S_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & S_5 & 0 \\ 0 & 0 & 0 & I \end{bmatrix},$$

where S_4, S_5 are invertible. It follows that $\ker(T-R)^{m+1} = [y_i : 1 \le i \le m]$, and hence $\dim\ker(T-R)^{m+1} = m$. Thus $T - R \in \mathcal{B}_m^1(\mathcal{X})$.

2) Both $F - R$ and R are algebraic operators.

Since $\dim\ker(F-R) < \infty$ and $\dim\ker(R) < \infty$, there exist $\lambda, \mu \in \mathbb{C}\backslash\{0\}$ such that $\dim\ker(F - R - \lambda I) = \infty$ and $\dim\ker(R - \mu I) = \infty$ by (2). So we can take $x_i \in \ker(R - \mu I),\ y_i \in \ker(F - R - \lambda I)(0 \le i \le m)$ such that $\{x_i, y_i : 0 \le i \le m\}$ is linearly independent. We write

$$\mathcal{X} = [x_i : 0 \le i \le m] \oplus [y_i : 0 \le i \le m] \oplus X_1 \tag{6}$$

and define $T \in \mathcal{B}(\mathcal{X})$ by

$$\begin{cases} Tx_0 = 0 \text{ and } Ty_0 = Fy_0, \\ Tx_i = x_{i-1} \text{ and } Ty_i = y_{i-1} + Fy_i \text{ for } 1 \le i \le m, \\ Tx = x + Rx \text{ for any } x \in X_1. \end{cases}$$

It follows that $T^m x_m = x_0$, and so $x_0 \in \ker(T) \cap \operatorname{ran}(T^m)$. Consequently, $\operatorname{asc}(T) \geq m+1$ by (1), and hence $T \notin \mathcal{A}_m(\mathcal{X})$. Similarly, as $(T-F)^m y_m = y_0$, we obtain that $T - F \notin \mathcal{A}_m(\mathcal{X})$. Since

$$\begin{cases} (T-R)x_0 = -\mu x_0 \text{ and } (T-R)y_0 = \lambda y_0, \\ (T-R)x_i = x_{i-1} - \mu x_i \text{ and } (T-R)y_i = y_{i-1} + \lambda y_i \text{ for } 1 \leq i \leq m, \\ (T-R)x = x \text{ for any } x \in X_1, \end{cases}$$

Under the decomposition (6), we have

$$T - R = \begin{bmatrix} J_n - \mu I & 0 & 0 \\ 0 & J_n + \lambda I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

It follows that $T - R$ is invertible, thus $T - R \in \mathcal{B}_m^1(\mathcal{X})$.

3) R is algebraic but $F - R$ is not.

Since $\dim\ker(R) < \infty$, there exists $\lambda \in \mathbb{C}\backslash\{0\}$ such that $\dim\ker(R - \lambda I) = \infty$ by (2) again. Since $F - R$ is not algebraic, it follows from [1, Theorem 4.2.7] that there exists $x \in \mathcal{X}$ such that $\{x, (F-R)x, \cdots, (F-R)^{m+1}x\}$ is linearly independent. Then we can take $x_i \in \ker(R - \lambda)(0 \leq i \leq m)$ such that $\{x, (F-R)x, \cdots, (F-R)^{m+1}x, x_0, x_1, \cdots, x_m\}$ is linearly independent. Let $y_i = (-1)^i (F-R)^{m-i} (0 \leq i \leq m)$ and $u = (F-R)^{m+1}x$. We can write

$$\mathcal{X} = [x_i : 0 \leq i \leq m] \oplus [y_i : 1 \leq i \leq m] \oplus [y_0, u] \oplus X_1. \tag{7}$$

Define $T \in \mathcal{B}(\mathcal{X})$ by

$$\begin{cases} Tx_0 = 0 \text{ and } Ty_0 = Fy_0, \\ Tx_i = x_{i-1} \text{ and } Ty_i = y_{i-1} + Fy_i \text{ for } 1 \leq i \leq m, \\ Tu = Ru + y_0, \\ Tx = x + Rx \text{ for any } x \in X_1. \end{cases}$$

It follows that $T^m x_m = x_0$, and so $x_0 \in \ker(T) \cap \operatorname{ran}(T^m)$. Consequently, $\operatorname{asc}(T) \geq m+1$ by (1), and hence $T \notin \mathcal{A}_m(\mathcal{X})$. Similarly, as $(T-F)^m y_m = y_0$, we obtain that $T - F \notin \mathcal{A}_m(\mathcal{X})$. Since

$$\begin{cases} (T-R)x_0 = -\lambda x_0 \text{ and } (T-R)y_0 = u, \\ (T-R)x_i = x_{i-1} - \lambda x_i \text{ and } (T-R)y_i = 0 \text{ for } 1 \leq i \leq m, \\ (T-R)u = y_0, \\ (T-R)x = x \text{ for any } x \in X_1. \end{cases}$$

Under the decomposition (7) we have

$$T - R = \begin{bmatrix} J_n - \lambda I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & S_6 & 0 \\ 0 & 0 & 0 & I \end{bmatrix},$$

where S_6 is invertible. It follows that $\ker(T-R)^{m+1} = [y_i : 1 \leq i \leq m]$, and hence $\dim\ker(T-R)^{m+1} = m$. Thus $T - R \in \mathcal{B}_m^1(\mathcal{X})$.

4) $F - R$ is algebraic but R is not.

Let $R_0 = F - R$. It follows that R_0 is an algebraic operators, $F - R_0$ is not an algebraic operator. Then we can conclude that there exists $K \in \mathcal{B}(\mathcal{X})$ such that $K - R_0 \in \mathcal{B}_m^1(\mathcal{X})$, $K \notin \mathcal{A}_m(\mathcal{X})$ and $K - F \notin \mathcal{A}_m(\mathcal{X})$ from the analysis of 3). If we let $T = -K + F$, then we can get that $T - R = -K + R_0 \in \mathcal{B}_m^1(\mathcal{X})$, $T - F = -K \notin \mathcal{A}_m(\mathcal{X})$ and $T = -K + F \notin \mathcal{A}_m(\mathcal{X})$. □

Proposition 2.4. *$\mathcal{B}_m^1(\mathcal{X})$ is an open subset of $\mathcal{A}_m(\mathcal{X})$.*

Proof. By definition, $\mathcal{B}_m^1(\mathcal{X}) \subset \mathcal{A}_m(\mathcal{X})$ where $\mathcal{B}_m^1(\mathcal{X})$ is the intersection of $\mathcal{B}_m(\mathcal{X})$ and $\Phi_{sF}(\mathcal{X})$. Since $\Phi_{sF}(\mathcal{X})$ is open, it suffices to show that $\mathcal{B}_m(\mathcal{X})$ is an open set. Let $S \in \mathcal{B}_m(\mathcal{X})$. It follows from [10, Theorem 16.11] that there exists $\eta>0$ such that for all $R \in \mathcal{B}(\mathcal{X})$ with $\|R - S^{m+1}\|<\eta$, we have

$$\dim\ker(R) \leq \dim\ker(S^{m+1}) \leq m. \tag{8}$$

On the other hand, since the map $T \longmapsto T^{m+1}$ is continuous on $\mathcal{B}(\mathcal{X})$, there exists $\varepsilon>0$ such that

$$\|T^{m+1} - S^{m+1}\|<\eta \text{ for all } T \in \mathcal{B}(\mathcal{X}) \text{ with } \|T - S\|< \varepsilon. \tag{9}$$

Combining (8) and (9) we obtain that $\dim\ker(T^{m+1}) \leq \dim\ker(S^{m+1}) \leq m$ for all $T \in \mathcal{B}(\mathcal{X})$ with $\|T - S\|<\varepsilon$. This shows that $\mathcal{B}_m(\mathcal{X})$ is open. Thus $\mathcal{B}_m^1(\mathcal{X})$ is an open set. □

For a subset $\Gamma \subseteq \mathcal{B}(\mathcal{X})$, we write Int($\Gamma$) for its interior. We recall that A and B in $\mathcal{B}(\mathcal{X})$ is said to be adjacent if rank$(A - B) = 1$.

Corollary 2.5. *Let $F \in \mathcal{B}(\mathcal{X})$ with* $\dim\text{ran}(F) \geq 2$. *Then for any $R \in \mathcal{B}(\mathcal{X})\backslash\{0, F\}$, there exists $T \in \mathcal{B}(\mathcal{X})$ such that $T - R \in Int(\mathcal{A}_m(\mathcal{X}))$, $T \notin \mathcal{A}_m(\mathcal{X})$, and $T - F \notin \mathcal{A}_m(\mathcal{X})$.*

Proposition 2.6. *Let A and B be two different operators in $\mathcal{B}(\mathcal{X})$. Then the following assertions are equivalent.*

(*i*) *A and B are adjacent.*

(*ii*) *There exists a $R \in \mathcal{B}(\mathcal{X}) \setminus \{A, B\}$ such that for all $T \in \mathcal{B}(\mathcal{X})$, the relation $T - R \in Int(\mathcal{A}_m(\mathcal{X}))$ implies that either $T - A \in \mathcal{A}_m(\mathcal{X})$ or $T - B \in \mathcal{A}_m(\mathcal{X})$.*

Proof. (*i*) ⇒ (*ii*). Put $R = \frac{1}{2}(A + B)$ and $K = \frac{1}{4}(A - B)$. Then $R \in \mathcal{B}(\mathcal{X})\backslash\{A, B\}$, and K is a rank one operator. Take any $T \in \mathcal{B}(\mathcal{X})$ such that $T - R \in Int(\mathcal{A}_m(\mathcal{X}))$. Then there exists $\varepsilon>0$ such that $T_1 + T - R \in \mathcal{A}_m(\mathcal{X})$ for all $\|T_1\|<\varepsilon$. For any $\alpha \in \{z : 0 < |z| < \min\{\frac{\varepsilon}{\|K\|}, 2\}\}$. We have that $\|\alpha K\|<\varepsilon$. Thus $T - R + \alpha K \in \mathcal{A}_m(\mathcal{X})$. It follows from [9, Proposition 2.2] that either $T - A = T - R - 2K \in \mathcal{A}_m(\mathcal{X})$ or $T - B = T - R + 2K \in \mathcal{A}_m(\mathcal{X})$.

(*ii*) ⇒ (*i*). Suppose that $\dim\text{ran}(A - B) \geq 2$ and $R \in \mathcal{B}(\mathcal{X})\backslash\{A, B\}$ satisfies the conditions of (*ii*). Put $F = A - B$ and $S = R - B$. We have $S \in \mathcal{B}(\mathcal{X})\backslash\{F, 0\}$. It follows from Corollary 2.5 that there exists $C \in \mathcal{B}(\mathcal{X})$ such that $C - S \in Int(\mathcal{A}_m(\mathcal{X}))$, $C \notin \mathcal{A}_m(\mathcal{X})$ and $C - F \notin \mathcal{A}_m(\mathcal{X})$. Put $T = C + B$. We obtain that $T - R = C - S \in Int(\mathcal{A}_m(\mathcal{X}))$, $T - B = C \notin \mathcal{A}_m(\mathcal{X})$ and $T - A = C - F \notin \mathcal{A}_m(\mathcal{X})$, a contradiction. Thus $A - B$ is of rank 1. □

3. Maps preserving semi-Fredholm operators with finite ascent

Let ϕ be a map on $\mathcal{B}(\mathcal{X})$, A and B are two operators in $\mathcal{B}(\mathcal{X})$. We say that ϕ preserves the difference of $\mathcal{A}_m(\mathcal{X})$ in both directions if $\phi(A) - \phi(B) \in \mathcal{S}$ if and only if $A - B \in \mathcal{S}$. If ϕ is a bijection such that both ϕ and ϕ^{-1} are continuous, then we say that ϕ is a bicontinuous map.

Lemma 3.1. *Let ϕ be a bicontinuous map on $\mathcal{B}(\mathcal{X})$. If ϕ preserves the difference of $\mathcal{A}_m(\mathcal{X})$ in both directions, then ϕ preserves the difference of $Int(\mathcal{A}_m(\mathcal{X}))$ in both directions.*

Proof. Suppose that ϕ preserves the difference of $\mathcal{A}_m(\mathcal{X})$ in both directions. Let $A, B \in \mathcal{B}(\mathcal{X})$ such that $A - B \in Int(\mathcal{A}_m(\mathcal{X}))$. Then there exists $\varepsilon>0$ such that $T - (A - B) \in \mathcal{A}_m(\mathcal{X})$ for all $T \in \mathcal{B}(\mathcal{X})$ with $\|T\|< \varepsilon$. Thus

$$S - A \in \mathcal{A}_m(\mathcal{X}) \text{ for all } S \in \mathcal{B}(\mathcal{X}) \text{ with } \|S - B\|<\varepsilon. \tag{10}$$

Put $C = \phi(B)$. Since ϕ^{-1} is continuous, there exists a $\delta>0$ such that

$$\|\phi^{-1}(R) - B\|< \varepsilon \text{ for all } R \in \mathcal{B}(\mathcal{X}) \text{ with } \|R - C\|<\delta. \tag{11}$$

Combining (10) and (11) we obtain that $\phi^{-1}(R) - A \in \mathcal{A}_m(\mathcal{X})$ for all $R \in \mathcal{B}(\mathcal{X})$ with $\|R - \phi(B)\|<\delta$. Hence $R - \phi(A) \in \mathcal{A}_m(\mathcal{X})$ for all $R \in \mathcal{B}(\mathcal{X})$ with $\|R - \phi(B)\|<\delta$. Thus $T - (\phi(A) - \phi(B)) \in \mathcal{A}_m(\mathcal{X})$ for all $T \in \mathcal{B}(\mathcal{X})$ with $\|T\|<\delta$. This means that $\phi(A) - \phi(B)$ belongs to $Int(\mathcal{A}_m(x))$. Since ϕ^{-1} has the same properties as ϕ, we can conclude that ϕ preserves the difference of $Int(\mathcal{A}_m(\mathcal{X}))$ in both directions. □

Lemma 3.2. *Let $\phi : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ be a bicontinuous map. If ϕ preserves the difference of $\mathcal{A}_m(\mathcal{X})$ in both directions, then ϕ preserves adjacency of operators in both directions.*

Proof. By Lemma 3.1, ϕ preserves the difference of $Int(\mathcal{A}_m(\mathcal{X}))$. Let $A, B \in \mathcal{B}(\mathcal{X})$ be adjacent. It follows from Proposition 2.6 that there exists $R \in \mathcal{B}(\mathcal{X})\backslash\{A, B\}$ such that for all $T \in \mathcal{B}(\mathcal{X})$, the relation $T - R \in Int(\mathcal{A}_m(\mathcal{X}))$ implies either $T - A \in \mathcal{A}_m(\mathcal{X})$ or $T - B \in \mathcal{A}_m(\mathcal{X})$. Since ϕ is bijective, we can conclude that there exists $\phi(R) \in \mathcal{B}(\mathcal{X})\backslash\{\phi(A), \phi(B)\}$ such that for all $\phi(T) \in \mathcal{B}(\mathcal{X})$, the relation $\phi(T) - \phi(R) \in Int(\mathcal{A}_m(\mathcal{X}))$ implies either $\phi(T) - \phi(A) \in \mathcal{A}_m(\mathcal{X})$ or $\phi(T) - \phi(B) \in \mathcal{A}_m(\mathcal{X})$. Therefore, $\phi(A)$ and $\phi(B)$ are adjacent by Proposition 2.6. Since ϕ^{-1} has the same properties as ϕ, we can get that ϕ preserves adjacency in both directions. □

Lemma 3.3. *Let $\phi : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ be a bicontinuous map preserving the difference of $\mathcal{A}_m(\mathcal{X})$ in both directions. If $\phi(0) = 0$, then ϕ preserves rank one operators in both directions, $\phi(\mathcal{F}(\mathcal{X})) \subseteq \mathcal{F}(\mathcal{X})$ and $\phi|_{\mathcal{F}(\mathcal{X})}$ is an additive map.*

Proof. It follows from Lemma 3.2 that ϕ preserves adjacency in both directions. Since $\phi(0) = 0$, ϕ preserves rank one operators in both directions. Note that any rank two operator F is adjacent to some rank one operator, it follows that ϕ preserves rank two operators in both directions. Repeat the same argument, we deduce that ϕ maps the subspace $\mathcal{F}(\mathcal{X})$ onto itself. By [14, Theorem 1.5], $\phi|_{\mathcal{F}(\mathcal{X})}$ is an additive map. □

Lemma 3.4. *Let $\phi : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ be a bicontinuous map preserving the difference of $\mathcal{A}_m(\mathcal{X})$ in both directions. Then for any $\lambda \in \mathbb{C}\backslash\{0\}$, there exists a $\mu \in \mathbb{C}\backslash\{0\}$ such that $\phi(\lambda I) = \mu I + \phi(0)$.*

Proof. After replacing ϕ by $T \longmapsto \phi(T) - \phi(0)$, we may assume that $\phi(0) = 0$. For any $\lambda \in \mathbb{C}\backslash\{0\}$, put $S = \phi(\lambda I)$. It follows that $S \in \mathcal{A}_m(\mathcal{X})$. By Lemma 3.3, for any K with rank at most m, there exists an operator F with rank at most m such that $K = \phi(F)$. Since $\lambda I - F \in \mathcal{A}_m(\mathcal{X})$ and ϕ preserves the difference of $\mathcal{A}_m(\mathcal{X})$ in both directions, it follows that $\phi(\lambda I) - \phi(F) = S - K \in \mathcal{A}_m(\mathcal{X})$. We now claim that S is an invertible algebraic operator. If S is not algebraic, then there exists a $z_0 \in \mathcal{X}$ such that the set $\{z_0, Sz_0, S^2z_0, \cdots, S^{m+1}z_0\}$ is linearly independent. So there exists an $f \in \mathcal{X}^*$ such that

$$f(S^m z_0) = 1,\ f(S^i z_0) = 0 (0 \le i \le m+1 \text{ and } i \ne m).$$

Let $K = S^{m+1}z_0 \otimes f$. Then $(S - K)S^m z_0 = 0$, $(S - K)S^i z_0 = S^{i+1}z_0 (0 \le i \le m-1)$. Thus $(S - K)^m z_0 = S^m z_0$ and $(S - K)^{m+1}z_0 = 0$. It follows that $\text{asc}(S - K) \ge m + 1$. Thus $S - K \notin \mathcal{A}_m(\mathcal{X})$, which is a contradiction. Hence S is algebraic. On the other hand, if S is not invertible, then $0 \in \sigma_p(S)$ by (2). Take any nonzero vector $z_0 \in \ker(S)$. Since S is algebraic, there exists $\lambda \in \sigma_p(S)$ such that $\dim \ker(S - \lambda I) = \infty$. Take any vectors $\{z_i : 1 \le i \le m\} \subseteq \ker(S - \lambda I)$ such that the set $\{z_i : 0 \le i \le m\}$ is linearly independent and put $\mathcal{X} = [z_0] \oplus [z_i : 1 \le i \le m] \oplus X_1$. Define an operator K by:

$$\begin{cases} Kz_0 = 0, \\ Kz_i = -z_{i-1} + \lambda z_i, (1 \le i \le m), \\ Ky = 0 \text{ for any } y \in X_1. \end{cases}$$

It follows that $(S - K)z_0 = 0, (S - K)z_i = z_{i-1} (1 \le i \le m)$. Thus $(S - K)^m z_m = z_0$ and $(S - K)^{m+1}z_m = 0$. This means that $\text{asc}(S - K) \ge m + 1$, a contradiction. Thus S is invertible. By [16, Proposition 2.8], there exists $\mu \in \mathbb{C}\backslash\{0\}$ such that $S = \phi(\lambda I) = \mu I$. This completes the proof. □

Lemma 3.5. *Let $\phi : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ be a bicontinous map preserving the difference of $\mathcal{A}_m(\mathcal{X})$ in both directions. If $\phi(F) = F$ for all $F \in \mathcal{F}(\mathcal{X})$, then $\phi(T) = T$ for all $T \in \mathcal{B}(\mathcal{X})$.*

Proof. We shall divide the proof into three steps.

Step 1. $\phi(\lambda I) = \lambda I$ for all non-zero $\lambda \in \mathbb{C}$. It follows from Lemma 3.4 that $\phi(\lambda I) = \mu I$ for some non-zero $\mu \in \mathbb{C}$. Now, for all $F \in \mathcal{F}(\mathcal{X})$, we have $\lambda I - F \in \mathcal{A}_m(\mathcal{X})$ if and only if $\phi(\lambda I) - \phi(F) = \mu I - F \in \mathcal{A}_m(\mathcal{X})$. Thus $\lambda I = \mu I$ by [16, Proposition 3.4].

Step 2. $\phi(\lambda I + F) = \lambda I + F$ for all nonzero $\lambda \in \mathbb{C}$ and $F \in \mathcal{F}(X)$. Fix any $F \in \mathcal{F}(X)$. Consider the following map

$$\psi_F(S) = \phi(S + F) - F \text{ for all } S \in \mathcal{B}(X).$$

It is elementary that ψ_F is a bicontinuous map preserving the difference of $\mathcal{A}_m(X)$ in both directions, and $\psi_F(K) = \phi(K + F) - F = K + F - F = K$ for all $K \in \mathcal{F}(X)$. By the previous step, we conclude that $\psi_F(\lambda I) = \lambda I$ for all nonzero $\lambda \in \mathbb{C}$. In particular, $\phi(\lambda I + F) = \lambda I + F$.

Step 3. $\phi(T) = T$ for any $T \in \mathcal{B}(X)$. Let $T \in \mathcal{B}(X)$. Choose a non-zero scalar λ such that $T - \lambda I$ and $\phi(T) - \lambda I$ are invertible. Then, for all $F \in \mathcal{F}(X)$, we have

$$T - \lambda I + F \in \mathcal{A}_m(X) \Leftrightarrow \phi(T) - \phi(\lambda I - F) = \phi(T) - \lambda I + F \in \mathcal{A}_m(X).$$

By [16, Proposition 3.4] we can conclude that $\phi(T) - \lambda I = T - \lambda I$. Hence $\phi(T) = T$ as desired. □

Remark 3.6. *Let τ be a field automorphism of $\mathbb{C}$. An additive map $A : X \to X$ defined between two Banach spaces is said to be τ-semilinear if $A(\lambda x) = \tau(\lambda)Ax$ for all $x \in X$ and $\lambda \in \mathbb{C}$. If τ is the complex conjugation, we will say simply that A is conjugate linear. Notice that if A is non-zero and bounded, then τ is continuous, and consequently, τ is either the identity or the complex conjugation, see Theorem 14.4.2 and Lemma 14.5.1 in [6]. Moreover, in this case, the adjoint operator $A^* : X^* \to X^*$ defined by $A^*(g) = \tau^{-1} \circ g \circ A$ for all $g \in Y^*$, is again τ-semilinear.*

Theorem 3.7. *Let $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ be a bicontinuous map. If ϕ preserves the difference of $\mathcal{A}_m(X)$ in both directions, then ϕ takes one of the following forms:*

(*i*) *There exist a nonzero complex number $\alpha \in \mathbb{C}$ and a bounded invertible linear or conjugate linear operator $A : X \to X$ such that*

$$\phi(T) = \alpha ATA^{-1} + \phi(0) \text{ for all } T \in \mathcal{B}(X).$$

(*ii*) *There exist a nonzero complex number $\alpha \in \mathbb{C}$ and a bounded invertible linear, or conjugate linear operator $B : X^* \to X$ such that*

$$\phi(T) = \alpha BT^*B^{-1} + \phi(0) \text{ for all } T \in \mathcal{B}(X).$$

In this case, X must be reflexive.

Proof. By Lemma 3.4, we have $\phi(I) = \alpha I + \phi(0)$ for some nonzero $\alpha \in \mathbb{C}$. Let $\psi = \alpha^{-1}(\phi(T) - \phi(0))$ for all $T \in \mathcal{B}(X)$. It is known that ψ also is a bicontinuous map preserving the difference of $\mathcal{A}_m(X)$ in both directions, $\psi(0) = 0$ and $\psi(I) = I$. It follows from Lemma 3.3 that ψ maps the subspace $\mathcal{F}(X)$ onto itself and $\psi|_{\mathcal{F}(X)}$ is additive. Then by [14, Theorem 1.5], there exists a ring automorphism $\tau : \mathbb{C} \to \mathbb{C}$ and either two bijective τ-*semilinear* mappings $A : X \to X$ and $C : X^* \to X^*$ such that

$$\psi(x \otimes f) = Ax \otimes Cf \text{ for all } x \in X \text{ and } f \in X^*, \tag{12}$$

or two bijective τ-*semilinear* mappings $B : X^* \to X$ and $D : X \to X^*$ such that

$$\psi(x \otimes f) = Bf \otimes Dx \text{ for all } x \in X \text{ and } f \in X^*. \tag{13}$$

Assume that (12) holds. We next show that $C(f)(Ax) = \tau(f(x))$ for all $x \in X$ and $f \in X^*$. Take any linearly independent vectors $x_1, x_2, \cdots, x_m \in \ker(f) \cap \ker(C(f)A)$. By [16, Proposition 2.7] and the fact that $\psi|_{\mathcal{F}(X)}$ is an additive map, we obtain that the following statements are equivalent:

(a) $f(x) = 1$;

(b) there exist linear functionals $f_1, f_2, \cdots, f_m \in X^*$ such that

$$I - x \otimes f + x_1 \otimes f_1 + x_2 \otimes f_2 + \cdots + x_m \otimes f_m \notin \mathcal{A}_m(X);$$

(c) there exist linear functionals $f_1, f_2, \cdots, f_m \in X^*$ such that

$$I - Ax \otimes Cf + Ax_1 \otimes Cf_1 + Ax_2 \otimes Cf_2 + \cdots + Ax_m \otimes Cf_m \notin \mathcal{A}_m(X);$$

(d) $C(f)(Ax) = 1$.
If $f(x) = \alpha \neq 0$, then $f(\alpha^{-1}x) = 1$. It follows that

$$C(f)(A(\alpha^{-1}x)) = \tau(\alpha)^{-1}C(f)(Ax) = 1.$$

Thus $C(f)(Ax) = \tau(\alpha) = \tau(f(x))$. Now, if $f(x) = 0$ and $C(f)(Ax) = \lambda \neq 0$, then $C(f)(A(\tau^{-1}(\lambda^{-1})x)) = \lambda^{-1}C(f)(Ax) = 1$. It follows that $f(\tau^{-1}(\lambda^{-1})x) = \tau^{-1}(\lambda^{-1})f(x) = 1$, which contradicts with $f(x) = 0$. Hence

$$C(f)(Ax) = \tau(f(x)) \text{ for all } x \in \mathcal{X} \text{ and } f \in \mathcal{X}^*.$$

Therefore, for all $u \in \mathcal{X}$,

$$\psi(x \otimes f)u = (Ax \otimes Cf)u = Cf(u)Ax = Cf(AA^{-1}u)Ax = \tau(f(A^{-1}u))Ax =$$

$$A(f(A^{-1}u)x) = A(x \otimes f)(A^{-1}u) = A(x \otimes f)A^{-1}u.$$

Arguing as in the proofs of [4, Theorem 3.1], and the Main Theorem in [11], we have τ and A are bounded, and then τ is either the identity or the complex conjugation. Thus $\psi(x \otimes f) = A(x \otimes f)A^{-1}$ for all $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$. Since $\psi|_{\mathcal{F}(\mathcal{X})}$ is additive, we have $\psi(F) = AFA^{-1}$ for all $F \in \mathcal{F}(\mathcal{X})$. Let $\psi_1(T) = A^{-1}\psi(T)A$ for all $T \in \mathcal{B}(\mathcal{X})$. It follows that ψ_1 has the same properites as ψ, $\psi_1(I) = I$, and $\psi_1(F) = F$ for all $F \in \mathcal{F}(\mathcal{X})$. By Lemma 3.5, we can conclude that $\psi_1(T) = T$ for all $T \in \mathcal{B}(\mathcal{X})$, hence $\phi(T) = \alpha ATA^{-1} + \phi(0)$ for all $T \in \mathcal{B}(\mathcal{X})$.

Suppose ψ satisfies the form of (13). Using a similar argument, we get that

$$D(x)(Bf) = \tau(J(x)f) \text{ for all } x \in \mathcal{X} \text{ and } f \in \mathcal{X}^*,$$

where $J : \mathcal{X} \to \mathcal{X}^{**}$ is the natural embedding. Moreover, $\psi(x \otimes f) = B(x \otimes f)^*B^{-1}$ for all $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$. Hence $\psi(F) = BF^*B^{-1}$ for all $F \in \mathcal{F}(\mathcal{X})$. Let $\psi_2(T) = J^{-1}(B^{-1}\psi(T)B)^*J$ for all $T \in \mathcal{B}(\mathcal{X})$, it follows that ψ_2 has the same properites as ψ, $\psi_2(I) = I$, and $\psi_2(F) = F$ for all $F \in \mathcal{F}(\mathcal{X})$. By Lemma 3.5, we can conclude that $\psi_2(T) = T$ for all $T \in \mathcal{B}(\mathcal{X})$, and thus $\phi(T) = \alpha BT^*B^{-1} + \phi(0)$ for all $T \in \mathcal{B}(\mathcal{X})$. In this case, $\mathcal{X}$ is reflexive. □

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