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Generalization of Szász-Baskakov operators based on Touchard polynomials

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Abstract. In this study, we present a generalization of the Szász-Baskakov operators using Touchard polynomials. First, we establish the rate of convergence for the newly defined operators. Subsequently, several approximation results are established. Finally, Voronovskaya-type theorem is proved and some graphical results are given to show the rate of convergence of constructed operators to a given function f.

1. Introduction

The Szász operators are obtained by extending the classical Bernstein operators to function defined on the semi infinite interval $[0, \infty)$. The definition of Szász operators in Szász [16] is as follows;

$$S_n(f;x) = e^{-nx}\sum_{j=0}^{\infty}\frac{(nx)^j}{j!}f\left(\frac{j}{n}\right)$$

where $x \geq 0, n \in \mathbb{N}$ and $f \in C[0,\infty)$. Baskakov [4] introduced Baskakov operators as follows,

$$B_n(f;x) = \frac{1}{(1+x)^n}\sum_{j=0}^{\infty}\binom{n+j-1}{j}\frac{x^j}{(1+x)^j}f\left(\frac{j}{n}\right),$$

$x \geq 0, n \in \mathbb{N}^+$. Szász-Mirakyan-Baskakov operators were initially studied by Prasad et al. [14], whose results were later corrected and further developed by Gupta [9]

$$K_n(f;x) = (n-1)\sum_{j=0}^{\infty}e^{-nx}\frac{(nx)^j}{j!}\int_0^{\infty}\binom{n+j-1}{j}\frac{t^j}{(1+t)^{n+j}}f(t)\,dt.$$

Agrawal et al. [2] presented generalized Baskakov–Szász operators with results on statistical and simultaneous approximation. Acu and Gupta [1] introduced hybrid operators involving Phillips and Baskakov–Szász

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types and studied their convergence and Voronovskaya-type behavior. Gupta et al. [10, 12] defined Szász–Baskakov operators and obtained approximation results using the Steklov method. Kanat et al. [13] proposed a generalization of Szász–Baskakov operators via Appell polynomials of class $A^{(2)}$. Sofyalıoğlu and Kanat [15] extended Szász–Baskakov operators using Boas–Buck-type polynomials and analyzed weighted approximation properties.

The main motivation for this study stems from the work presented in reference [11], which extends the structure of Szász type operators based on Touchard polynomials and contributes to the theory of approximation operators.

The kth Touchard polynomial [17] $T_k(x)$ is defined by

$$T_k(x) = \sum_{m=0}^{k} S(k,m)\, x^m,$$

where $S(k,m)$ denotes the classical Stirling numbers of the second kind. The Touchard polynomials are defined by the following generating function

$$e^{x\left(e^t-1\right)} = \sum_{k=0}^{\infty} \frac{T_k(x)}{k!} t^k. \tag{1}$$

Within this framework, we define the Szász-Baskakov operators constructed using Touchard polynomials as follows

$$\Omega_n(f;x) := (n-1)e^{\frac{nx(e-1)}{e}} \sum_{k=0}^{\infty} \frac{T_k\left(\frac{nx}{e}\right)}{k!} \int_0^{\infty} \binom{n+k-1}{k} \frac{t^k}{(1+t)^{n+k}} f(t)\, dt, \tag{2}$$

where $x \geq 0$, $n \in \mathbb{N}$, $n > 1$ and $f \in C[0,\infty)$. It is clear that these operators defined in (2) are linear positive.

2. Approximation properties of Ω_n operators

Lemma 2.1. *The function given in (1) satisfies the following equalities*

1. $\sum\limits_{k=0}^{\infty} \frac{T_{k+1}\left(\frac{nx}{e}\right)}{k!} = e^{\frac{nx(e-1)}{e}} nx,$
2. $\sum\limits_{k=0}^{\infty} \frac{T_{k+2}\left(\frac{nx}{e}\right)}{k!} = e^{\frac{nx(e-1)}{e}} \left[n^2x^2 + nx\right],$
3. $\sum\limits_{k=0}^{\infty} \frac{T_{k+3}\left(\frac{nx}{e}\right)}{k!} = e^{\frac{nx(e-1)}{e}} \left[n^3x^3 + 3n^2x^2 + nx\right],$
4. $\sum\limits_{k=0}^{\infty} \frac{T_{k+4}\left(\frac{nx}{e}\right)}{k!} = e^{\frac{nx(e-1)}{e}} \left[n^4x^4 + 6n^3x^3 + 7n^2x^2 + nx\right].$

Proof. (1) After taking the derivative of both sides of (1) with respect to t, we get

$$e^{x\left(e^t-1\right)} x e^t = \sum_{k=1}^{\infty} \frac{T_k(x)}{(k-1)!} t^{k-1} = \sum_{k=0}^{\infty} \frac{T_{k+1}(x)}{k!} t^k. \tag{3}$$

By setting $t = 1$ and replacing x with $\frac{nx}{e}$ in equation (3), we obtain the desired results.

(2) Applying the second derivative to (1) with respect to t gives

$$e^{x\left(e^t-1\right)} \left[x^2e^{2t} + xe^t\right] = \sum_{k=2}^{\infty} \frac{T_k(x)}{(k-2)!} t^{k-2} = \sum_{k=0}^{\infty} \frac{T_{k+2}(x)}{k!} t^k. \tag{4}$$

By setting $t = 1$ and replacing x with $\frac{nx}{e}$ in equation (4), then we have the desired results.

(3-4) One can find (3) and (4) in a similar way.
□

Lemma 2.2. *Let $\Omega_n(f;x)$ be the operator introduced in (2) and $e_r(t) = t^r$ for $r \in \{0,1,2,3,4\}$ be the test functions. We have the following equalities*

(i) $\Omega_n(e_0;x) = 1$, *(*$n > 1$*),*
(ii) $\Omega_n(e_1;x) = \frac{nx}{n-2} + \frac{1}{n-2}$, *(*$n > 2$*),*
(iii) $\Omega_n(e_2;x) = \frac{n^2x^2}{(n-2)(n-3)} + \frac{5nx}{(n-2)(n-3)} + \frac{2}{(n-2)(n-3)}$, *(*$n > 3$*),*
(iv) $\Omega_n(e_3;x) = \frac{n^3x^3}{(n-2)(n-3)(n-4)} + \frac{12n^2x^2}{(n-2)(n-3)(n-4)} + \frac{28nx}{(n-2)(n-3)(n-4)} + \frac{6}{(n-2)(n-3)(n-4)}$, *(*$n > 4$*),*
(v) $\Omega_n(e_4;x) = \frac{n^4x^4}{(n-2)(n-3)(n-4)(n-5)} + \frac{22n^3x^3}{(n-2)(n-3)(n-4)(n-5)} + \frac{127n^2x^2}{(n-2)(n-3)(n-4)(n-5)} + \frac{185nx}{(n-2)(n-3)(n-4)(n-5)} + \frac{24}{(n-2)(n-3)(n-4)(n-5)}$, *(*$n > 5$*).*

Proof. The Beta-Gamma function is given as $\beta(k,n)$. By taking $f(t) = e_0$ in operator (2), we obtain

$$
\begin{aligned}
\Omega_n(e_0;x) &= (n-1)e^{\frac{nx(e-1)}{e}}\sum_{k=0}^{\infty}\frac{T_k(\frac{nx}{e})}{k!}\int_0^{\infty}\binom{n+k-1}{k}\frac{t^k}{(1+t)^{n+k}}dt \\
&= e^{\frac{-nx(e-1)}{e}}\sum_{k=0}^{\infty}\frac{T_k(\frac{nx}{e})}{k!}\binom{n+k-1}{k}\beta(k+1,n-1) \\
&= 1.
\end{aligned}
$$

By taking $f(t) = e_1$ in operator (2) and by using Lemma 2.1, we get

$$
\begin{aligned}
\Omega_n(e_1;x) &= (n-1)e^{\frac{nx(e-1)}{e}}\sum_{k=0}^{\infty}\frac{T_k(\frac{nx}{e})}{k!}\int_0^{\infty}\binom{n+k-1}{k}\frac{t^{k+1}}{(1+t)^{n+k}}dt \\
&= e^{\frac{-nx(e-1)}{e}}\sum_{k=0}^{\infty}\frac{T_k(\frac{nx}{e})}{k!}\binom{n+k-1}{k}\beta(k+2,n-2) \\
&= \frac{e^{\frac{-nx(e-1)}{e}}}{n-2}\sum_{k=0}^{\infty}\frac{T_k(\frac{nx}{e})}{k!}(k+1) \\
&= \frac{e^{\frac{-nx(e-1)}{e}}}{n-2}\left\{\sum_{k=0}^{\infty}\frac{T_{k+1}(\frac{nx}{e})}{k!} + \sum_{k=0}^{\infty}\frac{T_k(\frac{nx}{e})}{k!}\right\} \\
&= \frac{nx}{n-2} + \frac{1}{n-2}, (n > 2)
\end{aligned}
$$

By taking $f(t) = e_2$ in operator (2) and by using Lemma 2.1, we get

$$
\begin{aligned}
\Omega_n(e_2;x) &= (n-1)e^{\frac{nx(e-1)}{e}}\sum_{k=0}^{\infty}\frac{T_k(\frac{nx}{e})}{k!}\int_0^{\infty}\binom{n+k-1}{k}\frac{t^{k+2}}{(1+t)^{n+k}}dt \\
&= e^{\frac{-nx(e-1)}{e}}\sum_{k=0}^{\infty}\frac{T_k(\frac{nx}{e})}{k!}\binom{n+k-1}{k}\beta(k+3,n-3) \\
&= \frac{e^{\frac{-nx(e-1)}{e}}}{(n-2)(n-3)}\sum_{k=0}^{\infty}\frac{T_k(\frac{nx}{e})}{k!}\{k(k-1)+4k+2)\} \\
&= \frac{e^{\frac{-nx(e-1)}{e}}}{(n-2)(n-3)}\left\{\sum_{k=0}^{\infty}\frac{T_{k+2}(\frac{nx}{e})}{k!} + 2\sum_{k=0}^{\infty}\frac{T_{k+1}(\frac{nx}{e})}{k!} + 2\sum_{k=0}^{\infty}\frac{T_k(\frac{nx}{e})}{k!}\right\} \\
&= \frac{n^2x^2}{(n-2)(n-3)} + \frac{5nx}{(n-2)(n-3)} + \frac{2}{(n-2)(n-3)}, (n > 3).
\end{aligned}
$$

By taking $f(t)=e_3$ in operator (2) and by using Lemma 2.1, we get

$$\begin{aligned}
\Omega_n(e_3;x) &= (n-1)e^{\frac{nx(e-1)}{e}}\sum_{k=0}^{\infty}\frac{T_k(\frac{nx}{e})}{k!}\int_0^{\infty}\binom{n+k-1}{k}\frac{t^{k+3}}{(1+t)^{n+k}}dt\\
&= e^{\frac{-nx(e-1)}{e}}\sum_{k=0}^{\infty}\frac{T_k(\frac{nx}{e})}{k!}\binom{n+k-1}{k}\beta(k+4,n-4)\\
&= \frac{e^{\frac{-nx(e-1)}{e}}}{(n-2)(n-3)(n-4)}\sum_{k=0}^{\infty}\frac{T_k(\frac{nx}{e})}{k!}\{k(k-1)(k-2)+9k(k-1)+18k+6)\}\\
&= \frac{e^{\frac{-nx(e-1)}{e}}}{(n-2)(n-3)(n-4)}\left\{\sum_{k=0}^{\infty}\frac{T_{k+3}(\frac{nx}{e})}{k!}+9\sum_{k=0}^{\infty}\frac{T_{k+2}(\frac{nx}{e})}{k!}+18\sum_{k=0}^{\infty}\frac{T_{k+1}(\frac{nx}{e})}{k!}+6\sum_{k=0}^{\infty}\frac{T_k(\frac{nx}{e})}{k!}\right\}\\
&= \frac{n^3x^3}{(n-2)(n-3)(n-4)}+\frac{12n^2x^2}{(n-2)(n-3)(n-4)}+\frac{28nx}{(n-2)(n-3)(n-4)}+\frac{6}{(n-2)(n-3)(n-4)},
\end{aligned}$$

$(n>4)$.

By taking $f(t)=e_4$ in operator (2) and by using Lemma 2.1, we get

$$\begin{aligned}
\Omega_n(e_4;x) &= (n-1)e^{\frac{nx(e-1)}{e}}\sum_{k=0}^{\infty}\frac{T_k(\frac{nx}{e})}{k!}\int_0^{\infty}\binom{n+k-1}{k}\frac{t^{k+4}}{(1+t)^{n+k}}dt\\
&= e^{\frac{-nx(e-1)}{e}}\sum_{k=0}^{\infty}\frac{T_k(\frac{nx}{e})}{k!}\binom{n+k-1}{k}\beta(k+5,n-5)\\
&= \frac{e^{\frac{-nx(e-1)}{e}}}{(n-2)(n-3)(n-4)(n-5)}\sum_{k=0}^{\infty}\frac{T_k(\frac{nx}{e})}{k!}\{k(k-1)(k-2)(k-3)+16k(k-1)(k-2)\\
&\quad +72k(k-1)+96k+24)\}\\
&= \frac{e^{\frac{-nx(e-1)}{e}}}{(n-2)(n-3)(n-4)(n-5)}\left\{\sum_{k=0}^{\infty}\frac{T_{k+4}(\frac{nx}{e})}{k!}+16\sum_{k=0}^{\infty}\frac{T_{k+3}(\frac{nx}{e})}{k!}+72\sum_{k=0}^{\infty}\frac{T_{k+2}(\frac{nx}{e})}{k!}\right.\\
&\quad \left.+96\sum_{k=0}^{\infty}\frac{T_{k+1}(\frac{nx}{e})}{k!}+24\sum_{k=0}^{\infty}\frac{T_k(\frac{nx}{e})}{k!}\right\}\\
&= \frac{n^4x^4}{(n-2)(n-3)(n-4)(n-5)}+\frac{22n^3x^3}{(n-2)(n-3)(n-4)(n-5)}+\frac{127n^2x^2}{(n-2)(n-3)(n-4)(n-5)}\\
&\quad +\frac{185nx}{(n-2)(n-3)(n-4)(n-5)}+\frac{24}{(n-2)(n-3)(n-4)(n-5)},(n>5).
\end{aligned}$$

□

Lemma 2.3. *By taking advantage of the linearity of the operator Ω_n , the central moments can be expressed as follows*

(i) $\Omega_n(e_1-x;x)=\frac{2x}{n-2}+\frac{1}{n-2}$, *(*$n>2$*),*

(ii) $\Omega_n((e_1-x)^2;x)=x^2\frac{n+6}{(n-2)(n-3)}+x\frac{3n+6}{(n-2)(n-3)}+\frac{2}{(n-2)(n-3)}$, *(*$n>3$*),*

(iii) $\Omega_n((e_1-x)^4;x)=x^4\frac{3n^2+86n+120}{(n-2)(n-3)(n-4)(n-5)}+x^3\frac{18n^2+412n+240}{(n-2)(n-3)(n-4)(n-5)}+x^2\frac{27n^2+452n+240}{(n-2)(n-3)(n-4)(n-5)}+x\frac{161n+120}{(n-2)(n-3)(n-4)(n-5)}$
$+\frac{24}{(n-2)(n-3)(n-4)(n-5)}$, *(*$n>5$*).*

Proof. From the linearity of the operators and Lemma 2.2, we get

$$\Omega_n(e_1-x;x)=\Omega_n(e_1;x)-x\Omega_n(e_0;x),(n>2)$$

$$\Omega_n((e_1-x)^2;x) = \Omega_n(e_2;x) - 2x\Omega_n(e_1;x) + x^2\Omega_n(e_0;x), (n>3)$$
$$\Omega_n((e_1-x)^4;x) = \Omega_n(e_4;x) - 4x\Omega_n(e_3;x) + 6x^2\Omega_n(e_2;x) - 4x^3\Omega_n(e_1;x) + x^4\Omega_n(e_0;x), (n>5).$$

Consequently, the lemma provides the desired result. □

Remark 2.4. *The limit of the central moments results in the following expression;*

1. $\lim\limits_{n\to\infty} n\Omega_n(e_1-x;x) = 2x+1,$
2. $\lim\limits_{n\to\infty} n\Omega_n((e_1-x)^2;x) = x^2+3x,$
3. $\lim\limits_{n\to\infty} n^2\Omega_n((e_1-x)^4;x) = 3x^2(x+3)^2.$

Proof. It is clear from Lemma 2.3. □

Theorem 2.5. *Let f be continuous on $[0,\infty)$ and a member of the class*

$$H = \left\{ f : \frac{f(x)}{1+x^2} \text{ is convergent as } x\to\infty \right\}.$$

Then $\Omega_n(f;x)$ operators converge uniformly on each compact subset of $[0,\infty)$ as

$$\lim_{n\to\infty}\Omega_n(f;x) = f(x).$$

Proof. From Lemma 2.2, we obtain

$$\lim_{n\to\infty}\Omega_n(e_i;x) = x^i, i=0,1,2.$$

Given that the convergence is uniform on compact subsets of $[0,\infty)$, the universal Korovkin theorem [3] ensures the validity of the result. □

Definition 2.6. *[3] Let $f\in C_D[0,\infty)$ and $\delta>0$. Modulus of continuity of the function f, expressed by $\omega(f,\delta)$ is*

$$\omega(f;\delta) := \sup_{\substack{x,y\in[0,\infty)\\ |x-y|\le\delta}} \left|f(x)-f(y)\right|,$$

where $C_D[0,\infty)$ denotes the space of uniformly continuous functions on $[0,\infty)$.

Definition 2.7. *[5] The function $f\in C[a,b]$ has a second modulus of continuity that is defined as*

$$\omega_2(f;\delta) := \sup_{0<t\le\delta}\|f(.+x) - 2f(.+2t) + f(.)\|,$$

where $\|f\| = \max\limits_{t\in[a,b]}\left|f(t)\right|$.

Lemma 2.8. *[8] Let $f\in C^2[0,a]$ and $(\Omega_n)_{n\ge0}$ be a sequence of linear operators with the property $\Omega_n(e_0;x)=1$. Then*

$$\left|\Omega_n(f;x)-f(x)\right| \le \|f'\|\sqrt{\Omega_n((e_1-x)^2;x)} + \frac{1}{2}\|f''\|\Omega_n((e_1-x)^2;x).$$

Definition 2.9. *[18] The function f_μ is called Steklov function if the following holds:*

$$f_\mu(t) = \frac{1}{\mu}\int_{t-\frac{\mu}{2}}^{t+\frac{\mu}{2}} f(u)du = \frac{1}{\mu}\int_{-\frac{\mu}{2}}^{\frac{\mu}{2}} f(t+v)dv,$$

where f is integrable function an implicit and bounded interval $[a, b]$. Derivate of this function at almost every point is given as:

$$f'_\mu(t) = \frac{1}{\mu} f\left(t + \frac{\mu}{2}\right) - f\left(t - \frac{\mu}{2}\right).$$

If the derivative f is uniformly continuous on the real line, the following relationships hold.

$$\sup_{t\in(-\infty,\infty)} \left|f(t) - f_\mu(t)\right| \le \omega\left(\frac{\mu}{2}, f\right),$$

$$\sup_{t\in(-\infty,\infty)} \left|f'_\mu(t)\right| \le \frac{1}{2}\omega\left(\mu, f\right).$$

Lemma 2.10. *[18] Let $f \in C[a, b]$ and $\mu \in \left(0, \frac{b-a}{2}\right)$. Let f_μ be the second-order Steklov function associated with the function f. Then, the following inequalities are satisfied:*

$$\begin{aligned} \|f_\mu - f\| &\le \frac{3}{4}\omega_2(f, \mu), \\ \|f''_\mu\| &\le \frac{3}{2\mu^2}\omega_2(f, \mu). \end{aligned}$$

In general, we minimize the margin of error in linear positive operators by employing the first and second modulus of continuity. In the following two theorems, we establish the rate of convergence by utilizing the previously stated definitions and results.

Theorem 2.11. *Let $f \in C_D[0, \infty) \cap H$. Ω_n operators satisfy the following inequality:*

$$\left|\Omega_n(f; x) - f(x)\right| \le 2\omega(f; \sqrt{\delta_n(x)}),$$

in which

$$\delta := \delta_n(x) = x^2\frac{n+6}{(n-2)(n-3)} + x\frac{3n+6}{(n-2)(n-3)} + \frac{2}{(n-2)(n-3)}, (n > 3).$$

Proof. Let $f \in C_D[0, \infty) \cap H$. Using the property of the modulus of continuity,

$$\left|f(x) - f(t)\right| \le \omega(f; \delta)\left(1 + \frac{|t - x|}{\delta}\right), \tag{5}$$

is provided. Using the above inequality (5), we get

$$
\begin{aligned}
\left|\Omega_n(f;x)-f(x)\right| &\leq \Omega_n\left(\left|f(t)-f(x)\right|;x\right)\\
&\leq \Bigg|(n-1)e^{\frac{nx(e-1)}{e}}\sum_{k=0}^{\infty}\frac{T_k(\frac{nx}{e})}{k!}\int_0^\infty\binom{n+k-1}{k}\frac{t^k}{(1+t)^{n+k}}f(t)\,dt\\
&\quad -(n-1)e^{\frac{nx(e-1)}{e}}\sum_{k=0}^{\infty}\frac{T_k(\frac{nx}{e})}{k!}\int_0^\infty\binom{n+k-1}{k}\frac{t^k}{(1+t)^{n+k}}f(x)\,dt\Bigg|\\
&\leq (n-1)e^{\frac{nx(e-1)}{e}}\sum_{k=0}^{\infty}\frac{T_k(\frac{nx}{e})}{k!}\int_0^\infty\binom{n+k-1}{k}\frac{t^k}{(1+t)^{n+k}}\left|f(t)-f(x)\right|dt\\
&\leq (n-1)e^{\frac{nx(e-1)}{e}}\sum_{k=0}^{\infty}\frac{T_k(\frac{nx}{e})}{k!}\int_0^\infty\binom{n+k-1}{k}\frac{t^k}{(1+t)^{n+k}}\left(1+\frac{|t-x|}{\delta}\right)\omega(f;\delta)\,dt\\
&\leq \left\{1+\frac{1}{\delta}\left((n-1)e^{\frac{nx(e-1)}{e}}\sum_{k=0}^{\infty}\frac{T_k(\frac{nx}{e})}{k!}\int_0^\infty\binom{n+k-1}{k}\frac{t^k}{(1+t)^{n+k}}|t-x|\,dt\right)\right\}\omega(f;\delta).
\end{aligned}
$$

As a consequence of the Cauchy–Schwarz inequality applied to sums, we arrive at the following conclusion

$$
\begin{aligned}
\left|\Omega_n(f;x)-f(x)\right| &\leq \left\{1+\frac{1}{\delta}\left((n-1)e^{\frac{nx(e-1)}{e}}\sum_{k=0}^{\infty}\frac{T_k(\frac{nx}{e})}{k!}\int_0^\infty\binom{n+k-1}{k}\frac{t^k}{(1+t)^{n+k}}dt\right)^{\frac{1}{2}}\right.\\
&\quad \left.\times\left((n-1)e^{\frac{nx(e-1)}{e}}\sum_{k=0}^{\infty}\frac{T_k(\frac{nx}{e})}{k!}\int_0^\infty\binom{n+k-1}{k}\frac{t^k}{(1+t)^{n+k}}|t-x|^2\,dt\right)^{\frac{1}{2}}\right\}\omega(f;\delta)\\
&= \left\{1+\frac{1}{\delta}\left(\Omega_n(e_0;x)\right)^{\frac{1}{2}}\left(\Omega_n((e_1-x)^2;x)\right)^{\frac{1}{2}}\right\}\omega(f;\delta).
\end{aligned}
$$

This completes the proof.

$$
\left|\Omega_n(f;x)-f(x)\right| \leq \left\{1+\frac{1}{\delta}\left(\Omega_n((e_1-x)^2;x)\right)^{\frac{1}{2}}\right\}\omega(f;\delta)
$$

By choosing $\delta := \delta_n(x) = \sqrt{\Omega_n((e_1-x)^2;x)}$, we obtain the desired result. □

Theorem 2.12. *The following estimate holds for* $f \in C[0,a]$,

$$
\left|\Omega_n(f;x)-f(x)\right| \leq \frac{2\mu^2}{a}\|f'\| + \frac{3}{4}\left(a+2+\mu^2\right)\omega_2(f,\mu),
$$

where

$$
\mu := \mu_n(x) = \sqrt[4]{\Omega_n((e_1-x)^2;x)},\ (n>3). \tag{6}
$$

Proof. By using some simple computations, it becomes

$$
\begin{aligned}
\left|\Omega_n(f;x)-f(x)\right| &\leq \left|\Omega_n(f;x)-f(x)+\Omega_n(f_\mu;x)-\Omega_n(f_\mu;x)+f_\mu(x)+f_\mu(x)\right|\\
&\leq \left|\Omega_n(f;x)-\Omega_n(f_\mu;x)\right|+\left|\Omega_n(f_\mu;x)-f_\mu(x)\right|+\left|f_\mu(x)-f(x)\right|\\
&\leq \left\|f-f_\mu\right\|\Omega_n(e_0;x)+\left|\Omega_n(f_\mu;x)-f_\mu(x)\right|+\left\|f_\mu-f\right\|\\
&\leq 2\left\|f_\mu-f\right\|+\left|\Omega_n(f_\mu;x)-f_\mu(x)\right| \qquad (7)
\end{aligned}
$$

In view of the fact that $f_\mu \in C^2[0,a]$ and by using Lemma 2.8, Lemma 2.10 and Landau inequality [6], we derive

$$\begin{aligned} \left|\Omega_n(f_\mu;x)-f_\mu(x)\right| &\leq \|f'_\mu\|\sqrt{\Omega_n((e_1-x)^2;x)}+\frac{1}{2}\|f''_\mu\|\Omega_n((e_1-x)^2;x) \\ &\leq \left(\frac{2}{a}\|f_\mu\|+\frac{a}{2}\|f'_\mu\|\right)\sqrt{\Omega_n((e_1-x)^2;x)}+\frac{1}{2}\|f''_\mu\|\Omega_n((e_1-x)^2;x) \\ &\leq \left(\frac{2}{a}\|f\|+\frac{3a}{4\mu^2}\omega_2(f,\mu)\right)\sqrt{\Omega_n((e_1-x)^2;x)}+\frac{3}{4\mu^2}\omega_2(f,\mu)\,\Omega_n((e_1-x)^2;x). \end{aligned} \tag{8}$$

With the aid of Lemma 2.10,we obtain

$$\|f_\mu-f\|\leq\frac{3}{4}\omega_2(f,\mu). \tag{9}$$

If we substitute this inequality and (8)-(9) into (7) , we get

$$\begin{aligned} \left|\Omega_n(f;x)-f(x)\right| &\leq \frac{3}{2}\omega_2(f,\mu)+\left(\frac{2}{a}\|f\|+\frac{3a}{4\mu^2}\omega_2(f,\mu)\right)\sqrt{\Omega_n((e_1-x)^2;x)} \\ &+ \frac{3}{4\mu^2}\omega_2(f,\mu)\,\Omega_n((e_1-x)^2;x). \end{aligned}$$

Thus, we obtain the desired result by choosing $\mu=\sqrt[4]{\Omega_n((e_1-x)^2;x)}$. □

3. Weighted approximation

Gadjiev extended Korovkin's theorem, which holds a significant place in approximation theory, to unbounded intervals in the context of weighted function spaces [7]. Let φ be a monotone increasing function such that $\lim\limits_{x\to\infty}\varphi(x)=\infty$ and let $\rho(x)=1+\varphi^2(x)$ be a weight function. M_f and K_f are positive constants that depend on the function f. Accordingly, we consider the following weighted spaces of functions which are defined on $[0,\infty)$,

$$\begin{aligned} B_\rho[0,\infty) &:= \left\{f\in[0,\infty):\left|f(x)\right|\leq M_f\cdot\rho(x)\right\}, \\ C_\rho[0,\infty) &:= \left\{f\in B_\rho[0,\infty): f \text{ is continuous}\right\}, \\ C_\rho^K[0,\infty) &:= \left\{f\in C_\rho[0,\infty):\lim_{n\to\infty}\frac{f(x)}{\rho(x)}=K_f<\infty\right\}. \end{aligned}$$

It is obvious that $C_\rho^K[0,\infty)\subset C_\rho[0,\infty)\subset B_\rho[0,\infty)$. $B_\rho[0,\infty)$ is a linear normed space with the following norm:

$$\left\|f\right\|_\rho=\sup_{x\in[0,\infty)}\frac{\left|f(x)\right|}{\rho(x)}.$$

Theorem 3.1. *([7]) Let $(L_n)_{n\geq1}$ be a sequence of linear positive operators. If $(L_n)_{n\geq1}$ satisfying the following two conditions*

i) The operators L_n act from $C_\rho[0,\infty)$ to $B_\rho[0,\infty)$,

ii) $\lim\limits_{n\to\infty}\left\|L_n(e_i;x)-x^i\right\|_\rho=0$, $i=0,1,2,$

then for any function $f\in C_\rho^K[0,\infty)$

$$\lim_{n\to\infty}\left\|L_nf-f\right\|_\rho=0.$$

Lemma 3.2. *The operators Ω_n defined in (2) satisfy the following inequality*

$$\Omega_n(\rho;x) \leq C \cdot \rho(x),$$

where $C > 0, n > 3$ and $\rho(x) = 1 + x^2$.

Proof.

$$\begin{aligned}
\Omega_n(\rho;x) &= (n-1)e^{\frac{nx(e-1)}{e}} \sum_{k=0}^{\infty} \frac{T_k(\frac{nx}{e})}{k!} \int_0^{\infty} \binom{n+k-1}{k} \frac{t^k}{(1+t)^{n+k}}(1+t^2)dt \\
&= \Omega_n(e_0;x) + \Omega_n(e_2;x) \\
&= 1 + \frac{n^2x^2 + 5nx + 2}{(n-2)(n-3)} \\
&\leq \frac{2n^2}{(n-2)(n-3)}(1+x^2) \\
&\leq C \cdot \rho(x), (n > 3)
\end{aligned}$$

□

Theorem 3.3. *The operators Ω_n given in (2) are such that*

$$\lim_{n\to\infty} \left\| \Omega_n(f;x) - f(x) \right\|_\rho = 0$$

for $f \in C_\rho^K[0,\infty)$ where $\rho(x) = 1 + x^2$.

Proof. *i*) Let $f \in C_\rho[0,\infty)$, from Lemma 3.2, we obtain

$$\Omega_n(f;x) = \Omega_n\left(\frac{f}{\rho}\rho;x\right) \leq \left\|f\right\|_\rho \Omega_n(\rho;x) \leq \left\|f\right\|_\rho C \cdot \rho(x) \leq M_f \cdot \rho(x) \tag{10}$$

where $M_f > 0$. From the inequality (10), it is $\Omega_n \in B_\rho[0,\infty)$. So, we get that the operators Ω_n act from $C_\rho[0,\infty)$ to $B_\rho[0,\infty)$.

ii) From Lemma 2.2 , it is clear that

$$\lim_{n\to\infty} \|\Omega_n(e_0;x) - 1\|_\rho = 0.$$

Also, by using Lemma 2.2, we can write

$$\|\Omega_n(e_1;x) - x\|_\rho \leq \sup_{x\in[0,\infty)} \left|\frac{2}{n-2}\right| \frac{x}{1+x^2} + \sup_{x\in[0,\infty)} \left|\frac{1}{n-2}\right| \frac{1}{1+x^2} \leq \frac{2}{n-2}, (n > 2).$$

Thus, we have the following

$$\lim_{n\to\infty} \|\Omega_n(e_1;x) - x\|_\rho = 0.$$

Then

$$\begin{aligned}
\left\|\Omega_n(e_2;x) - x^2\right\|_\rho &\leq \sup_{x\in[0,\infty)} \left|\frac{10n}{(n-2)(n-3)}\right| \frac{x}{1+x^2} + \sup_{x\in[0,\infty)} \left|\frac{2}{(n-2)(n-3)}\right| \frac{1}{1+x^2} \\
&\leq \frac{5n}{(n-2)(n-3)}, (n > 3).
\end{aligned}$$

Thus, we obtain the following

$$\lim_{n\to\infty} \left\|\Omega_n(e_2;x) - x^2\right\|_\rho = 0.$$

As a result, we get

$$\lim_{n\to\infty} \left\|\Omega_n(e_i;x) - x^i\right\|_\rho = 0,\ i = 0,1,2.$$

If we apply Theorem 3.1, we obtain the desired results. □

4. Voronovskaya type theorem

Theorem 4.1. *Let $f, f', f'' \in C[0,\infty) \cap H$ and $x \in [0,\infty)$. Then*

$$\lim_{n\to\infty} n(\Omega_n(f;x) - f(x)) = (2x+1)f(x) + \frac{1}{2}x(x+3)f''(x)$$

holds uniformly on every compact subset of $[0,\infty)$.

Proof. When we apply the classical Taylor expansion of the function f, we have

$$f(t) = f(x) + f'(x)(t-x) + f''(x)\frac{(t-x)^2}{2!} + (t-x)^2 k(t,x), \tag{11}$$

where $k(t,x) \in C[0,\infty) \cap H$ and $\lim_{t\to x} k(t,x) = 0$. By applying the operator Ω_n to both sides of equation (11), we obtain

$$\Omega_n(f;x) = f(x) + \Omega_n(e_1 - x;x)f'(x) + \Omega_n\left((e_1-x)^2;x)\right)\frac{f''(x)}{2} + \Omega_n\left((e_1-x)^2 m(t,x);x\right).$$

Then

$$\lim_{n\to\infty} n\left(\Omega_n(f;x) - f(x)\right) = f'(x)\lim_{n\to\infty} n(\Omega_n((e_1-x);x)) + \frac{f''(x)}{2}\lim_{n\to\infty} n(\Omega_n((e_1-x)^2;x)) + \lim_{n\to\infty} n(\Omega_n((e_1-x)^2 k(t,x);x)).$$

By applying the Cauchy-Schwarz inequality $\lim_{n\to\infty} n(\Omega_n((e_1-x)^2 k(t,x);x))$ provides

$$n(\Omega_n((e_1-x)^2 k^2(t,x);x)) \le \sqrt{\Omega_n(k^2(t,x);x)}\,\sqrt{n^2(\Omega_n((e_1-x)^4;x))}.$$

Assuming that $k(t,x) \to 0$ as $t \to x$,

$$\lim_{n\to\infty} (\Omega_n(k^2(x,t);x) = k^2(x,x) = 0. \tag{12}$$

Thus, we obtain the desired result from Remark 2.4 and (12). □

5. Graphical Results

Finally, this section includes graphical examples that demonstrate the convergence of Szász-Baskakov operators via Touchard polynomials.These graphs help to clearly illustrate the manner in which our operators approach specific functions.

Example 5.1. *In the first illustration, the convergence of the operators Ω_n is demonstrated as a function of n. Here, approximating function $f(x) = \exp(-x^2) + x^3 + 3$, we illustrate the approximation for $n = 50, 70$ and 100 in Figure 1.*

Example 5.2. *Another illustration, the convergence of the operators Ω_n is demonstrated as a function of n. Here, approximating function $f(x) = \sin(x)\cos(x) + x + 1$, we illustrate the approximation for $n = 50, 70$ and 100 in Figure 2.*

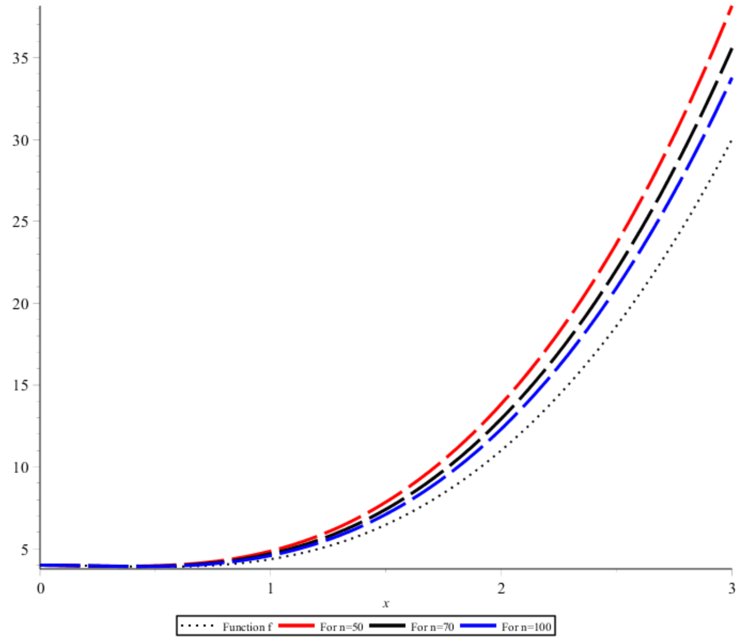

Figure 1: Approximation $\Omega_n(f;x)$ to $f(x) = e^{-x^2} + x^3 + 3$ for $n = 50, n = 70$ and $n = 100$

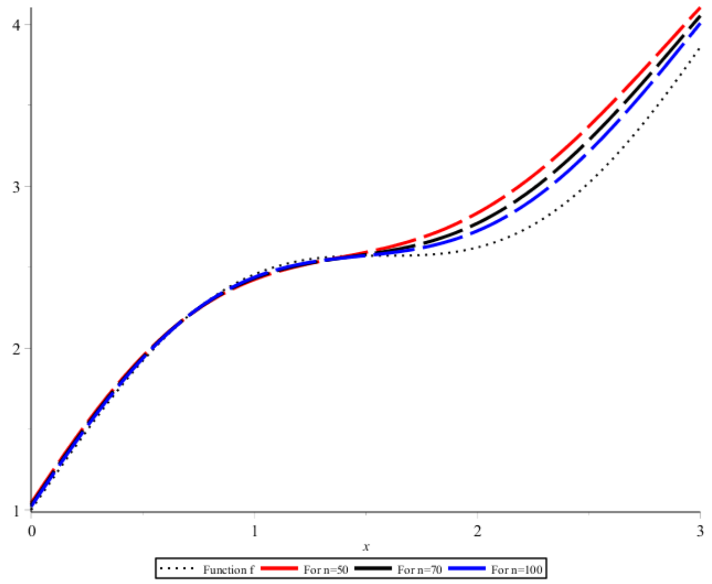

Figure 2: Approximation $\Omega_n(f;x)$ to $f(x) = \sin(x)\cos(x) + x + 1$ for $n = 50, n = 70$ and $n = 100$

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